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GLOBAL BEHAVIOUR OF SOLUTIONS TO A PARABOLIC MEAN CURVATURE EQUATION

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Abstract. Consider (1.1) for a domain Ω for which there is no classical nonparametric solution of the stationary problem. We study viscosity solutions of (1.1). In general they fail to satisfy Dirichlet data on the boundary and "detach."In fact the solution tends to infinity with finite speed. The velocity stabilizes as $t \to \infty$, and we give some results on asymptotic growth. These new effects can be reconciled with the notion of viscosity solutions. The free boundary data are shown to be Lipschitzian for special domains Ω. Problem (1.1) is related to some isoperimetric geometric problem.

1. Introduction. The aim of this paper is to investigate the global behaviour of solutions to the time-dependent constant-mean-curvature-type equation

$$u_t - \operatorname{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) = 1 \qquad \text{in } \ Q = \Omega \times (0,\infty), \tag{1.1}$$

under vanishing Cauchy and Dirichlet data; i.e.,

$$u = 0 \qquad \text{on the parabolic boundary } \ (\partial\Omega \times [0,\infty)) \cup (\Omega \times \{0\}), \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is bounded and has smooth boundary $\partial\Omega$, and where div and D denote spatial derivatives. It is for the sake of simplicity that we choose the right-hand side equal to 1; many of our arguments carry over to a nonconstant right-hand side $f(x,t,u)$; i.e.,

$$u_t - \operatorname{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) = f(x,t,u) \qquad \text{in } Q = \Omega \times (0,\infty). \tag{1.3}$$

Equation (1.1) should not be confused with the equation

$$u_t - \sqrt{1+|Du|^2}\,\operatorname{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) = 0, \qquad \text{in } \ Q = \Omega \times (0,\infty), \tag{1.4}$$

which is encountered in the modeling of phase transitions and mean curvature flow; see [2, 8].

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The motivation to consider equation (1.1) comes from its apparent relation to the following isoperimetric problem (see [28, 21]):

Given Ω, find a set $\Omega_* \subset \Omega$ which minimizes the ratio of perimeter over volume; in short

$$\min_{G\subset\Omega} \frac{|\partial G|}{|G|} = \frac{|\partial\Omega_*|}{|\Omega_*|}\,, \tag{1.5}$$

where $|\partial G|$ is shorthand for the perimeter of G in $\mathbb{R}^n$ in the sense of De Giorgi, and $|G|$ denotes n-dimensional Lebesgue measure. Notice that the existence question for (1.5) has been settled for instance in [17], so that a minimum actually exists.

What does (1.5) have to do with (1.1)–(1.2)? Marcellini and Miller observed in numerical calculations that a solution of (1.1)–(1.2) can blow up as time goes to infinity. Those points in Ω in which the speed u_t tends to a maximum (as $t \to \infty$) seem to constitute the set Ω_* which solves (1.5) and vice versa. In other words, solving the parabolic problem (1.1)–(1.2) could, according to their conjecture, be a way of constructing the set Ω_*, which is a solution of a geometric variational problem. For the reader's convenience we repeat their heuristic arguments which supported this conjecture. Let $MV(u, G)$ denote the mean value of u on $G \subset \Omega$. Integration yields

$$(MV(u,G))_t = \frac{1}{|G|}\int_{\partial G} \frac{Du\cdot n(x)}{\sqrt{1+|Du|^2}}\,dx + 1 \geq -\frac{|\partial G|}{|G|} + 1,$$

or shorter,

$$(MV(u,G))_t \geq 1 - \frac{|\partial G|}{|G|},$$

and equality holds if and only if $\partial u/\partial n = -\infty$ on ∂G.

Let Ω_* be the subset of Ω on which u attains its maximal asymptotic speed c. Then $\partial u/\partial n \to -\infty$ on $\partial\Omega_*$ as $t \to \infty$ and Ω_* is a domain for which equality holds above. Moreover, Ω_* maximizes the functional $f(G) = 1 - |\partial G|/|G|$ and $c = 1 - |\partial\Omega_*|/|\Omega_*|$.

Maximizing $f(G)$ is equivalent to solving (1.5), and this is how (1.1)–(1.2) relates to (1.5).

Incidentally, problem (1.5) is interesting in itself. As a geometric problem it has intrigued people like J. Steiner, but it also has applications in simple models of plate failure and elasticity; see [20, 22, 37] and references therein.

While the heuristic considerations above were confirmed by numerical calculations of [28], they ran into anticipated problems computing solutions of (1.1)–(1.2), since those solutions (and their derivatives) can get very large in a finite time. In particular, solutions of (1.1)–(1.2) can "detach from the boundary," i.e., fail to satisfy the boundary condition $u = 0$ after some time. This effect is not limited to a radial setting; in fact it occurs also on numerical calculations for squares, as we have learned from V. Oliker in private communication.

The failure to satisfy Dirichlet data in the classical sense is a commonly known phenomenon for quasilinear elliptic equations such as nonparametric minimal surfaces; see [32, 33, 29, 30, 14, 16, 10, 11] and references therein. These solutions have been obtained through variational or continuity methods.

Let us mention, in passing, that parabolic equations can be interpreted as degenerate elliptic ones, and that the failure to satisfy general Dirichlet boundary conditions is illustrated by the limit $\varepsilon \to 0$ in the following degenerate elliptic problem:

$$\begin{aligned} u^\varepsilon_{xx} + \varepsilon u^\varepsilon_{yy} - u^\varepsilon_y = 0 \qquad &\text{in } G := [0,1] \times [0,T] \\ u^\varepsilon(x,y) = \phi \qquad &\text{on } \partial G. \end{aligned}$$

In [38] the mean curvature operator was regularized by a viscosity term $-\varepsilon \Delta u$, and Temam's generalized pseudosolution is in fact a weak viscosity solution in the sense of Section 2, Definition 2.2 of this paper.

Nowadays we know a fair amount about existence and nonexistence of classical solutions (continuous up to the boundary) or of generalized pseudosolutions (which satisfy the Dirichlet problem only in a generalized sense) for elliptic mean curvature equations.

As for the nonstationary case of equation (1.1), relatively little seems to be known, in particular if Ω is "too large" to admit a stationary solution of (1.1). Lichnewsky and Temam introduced the notion of pseudosolutions [24] for evolutionary minimal surface equations. Those are of class $L^\infty_{\text{loc}}(0,\infty; H^{1,2}_{\text{loc}}(\Omega))$ and satisfy the Dirichlet condition in a generalized sense. To be precise, they first investigated a viscosity limit, i.e., the limit u, as $\varepsilon \to 0$, of a family of functions u^ε which in turn solve

$$u^\varepsilon_t - \operatorname{div}\Big(\frac{Du^\varepsilon}{\sqrt{1+|Du^\varepsilon|^2}}\Big) - \varepsilon \Delta u^\varepsilon = 0 \qquad \text{in } Q = \Omega \times (0,\infty) \tag{1.6}$$

under suitable initial and boundary conditions. This limit was then shown to coincide with the gradient semiflow generated by the Lyapunov functional

$$e(v) = \int_\Omega \sqrt{1+|Dv|^2}\, dx + \int_{\partial\Omega} |v-\phi|\, ds,$$

where ϕ were the prescribed Dirichlet data. Their solution tended to a stationary point of e, as $t \to \infty$.

In terms of dynamical systems, our problem (1.1)–(1.2) can be interpreted as the gradient semiflow associated with the functional

$$J(v) = \int_\Omega \sqrt{1+|Dv|^2} - v\, dx + \int_{\partial\Omega} |v-\phi|\, ds. \tag{1.7}$$

If we denote the epigraph of $v(t,\cdot)$ by $E(t) := \{(x,z) \in (\Omega \times \mathbb{R}^+) : 0 \le z \le v(t,x)\}$ we can interpret $J(v)$ in turn as

$$J(v) = |\partial E(t)| - |E(t)| - |\Omega|, \tag{1.8}$$

where $|\cdot|$ denotes the corresponding $(n+1)$- or n-dimensional Hausdorff measure. So the epigraph of u tries to minimize surface area minus volume (as $t \to \infty$), a problem that was investigated in detail in [20]. In this context problem (1.5) shows up again.

In fact, if $E(t)$ has constant horizontal cross section G, then $G = \Omega_*$ is the solution of (1.5).

In the present paper we are interested in the case that $J(v)$ has no infimum. If J has an infimum, then v converges to the steady state of (1.1)–(1.2). We shall not discuss this easy case any further.

It should be mentioned that Gerhard [13] studied equation (1.3) in the case of nonlinear Neumann boundary conditions

$$-\frac{\partial u}{\partial n} \in \beta(x,u) \qquad \text{with } \ |\beta| \leq 1.$$

Finally, problem (1.3)–(1.2) was investigated by Marcellini and Miller ([28]) for the case that Ω is a ball and $f = f(r)$ is radially symmetric. They pointed out some pathologies of the pseudosolutions, i.e., limits of solutions to

$$u_t^\varepsilon - \operatorname{div}\Big(\frac{Du^\varepsilon}{\sqrt{1+|Du^\varepsilon|^2}}\Big) - \varepsilon\Delta u^\varepsilon = f \qquad \text{in } \ Q = \Omega \times (0,\infty)$$

in large domains Ω or for large right-hand sides f, namely "detachment" of the solution from its boundary data, nonexistence of stationary solutions and development of a "rising elliptic cap" as $t \to \infty$. Moreover, they conjectured and justified the above-mentioned relation between problems (1.1)–(1.2) and (1.5).

Unlike the above papers, in the present one we start with the investigation of new effects. As a consequence we prove the conjecture of Marcellini and Miller on the relation between (1.1)–(1.2) and (1.5) for arbitrary domains Ω. The new phenomena are caused by blow up of the gradient on the boundary, as well as by amplitude blow up of u when Ω is sufficiently large. The combination of these two effects leads to a traveling wave phenomenon or detachment of solutions on part of the boundary, and to the development of a "rising elliptic cap," depending on the mean curvature of the minimizing set Ω_*. After finishing this manuscript, Y. Giga kindly pointed out to us that traveling wave phenomena have also been observed in a paper of Altschuler and Wu ([1, and references therein]), however under conditions which lead to solutions with bounded gradients. In contrast to these results we deal with solutions which do not satisfy boundary conditions in the classical sense.

In order to investigate the global behaviour of solutions after the time at which the gradient blows up and the equation becomes degenerate, we use the technique of viscosity solutions. This method was developed by many authors; see [6] for first-order equations or [27] for second-order equations. In fact, existence and uniqueness of continuous or Hölder continuous viscosity solutions to fully nonlinear, even degenerate, parabolic equations was developed in [3, 15, 7, 4, 9, 18, 19]. However, our analysis of (1.1) shows that the solutions are classical in the interior, i.e., of class $C^\infty(Q)$, and satisfy the boundary condition in a viscosity sense (see the definition in Section 2 or in [5, Definition 7.4]).

As for equations (1.1) and (1.2). we shall prove the existence of a viscosity solution. For fixed $\varepsilon > 0$ there exists a classical solution to the regularized problem

$$\begin{aligned} u_t^\varepsilon - \operatorname{div}\Big(\frac{Du^\varepsilon}{\sqrt{1+|Du^\varepsilon|^2}}\Big) - \varepsilon\Delta u^\varepsilon &= 1 \qquad \text{in } \ Q = \Omega \times (0,\infty) \\ u^\varepsilon &= 0 \qquad \text{on the parabolic boundary.} \end{aligned} \tag{1.9}$$

Under the viscosity limit [5, Section 7.A] we obtain a continuous viscosity solution of problem (1.1)–(1.2) which can fail to satisfy the Dirichlet condition in the classical sense, but which does satisfy it in a viscosity sense. This solution is *classical in the interior*, because Gerhardt ([12]) (see also [25] for more general equations) have derived interior gradient bounds, uniform with respect to ε, which in turn permit us to use interior Schauder estimates for uniformly parabolic equations as in [23]. The solution is unique. This follows from the maximum principle for viscosity solutions (see for example [19, 4, 15]) or from the corresponding uniqueness result for pseudosolutions (see [24, 28]).

The paper is organized as follows. In Section 2 we state some definitions and the main results. We follow mainly the notation and definitions in [5]. Section 3 deals with the particular case that Ω is a ball in $\mathbb{R}^n$ with radius $R > n$. We show the existence of a unique viscosity solution. Moreover, this solution exhibits the behaviour of a traveling wave. It travels in a vertical direction upward. For special nonzero initial data it does not even change its shape. In Section 4 we investigate the precise behaviour of the viscosity solution for the case that Ω is a sufficiently large square in $\mathbb{R}^2$. These two model problems illustrate all the necessary ideas which are needed to give proofs for general domains in a forthcoming paper.

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2. Definitions and main results. For completeness we recall the definition of viscosity solutions and of Dirichlet conditions in the viscosity sense. Let S^n be the space of all $n \times n$ symmetric matrices and let G be a locally compact subset of $\mathbb{R}^n$. Then for every finite $T > 0$ we denote the cylinder $G \times (0, T]$ by $G(T)$ and define the *parabolic second-order superjets* $\mathcal{P}^{2,+}_{G(T)} u(x,t)$ and $\mathcal{P}^{2,-}_{G(T)} u(x,t)$ as follows:

$$\begin{aligned} \mathcal{P}^{2,+}_{G(T)} u(x,t) = \big\{ &(\tau, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S^n : u(y,s) \le u(x,t) \\ &+ \tau(s-t) + \langle p, y-x \rangle + \frac{1}{2} \langle X(y-x), y-x \rangle \\ &+ o(|s-t| + |y-x|^2) \ \text{ as } \ (y,s) \to (x,t) \in G(T) \big\}, \end{aligned} \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. Analogously, the second-order superjet $\mathcal{P}^{2,-}_{G(T)} u(x,t)$ can be introduced by means of the opposite inequality in (2.1) or by $\mathcal{P}^{2,-}_{G(T)} u = -\mathcal{P}^{2,+}_{G(T)}(-u)$. The superjet $\mathcal{P}^{2,+}_{G(T)} u(x,t)$ can also be defined for $(x,t) \in G \times (0,T]$ in the following equivalent way, which goes back to the original definition of viscosity solutions in [6]:

$$\begin{aligned} \mathcal{P}^{2,+}_{G(T)} u(x,t) = \big\{ &(D_t\phi(x,t), D_x\phi(x,t), D_x^2\phi(x,t)) : \phi \in C^2(G(T)) \\ &\text{and } \ u - \phi \text{ has a local maximum at } (x,t) \in G \times (0,T] \big\}. \end{aligned} \tag{2.2}$$

Since (2.2) is more complicated in the case of a domain with boundary (see the comments in Remark 2.7 or (2.15) in [5]) we shall use (2.1) in order to define *viscosity sub- and supersolutions of* (1.3) (see [5, Section 8; 15, Section 2] or [4]).

Definition 2.1. An upper semicontinuous function $u : G(T) \mapsto \mathbb{R}$ is a *viscosity subsolution of* (1.3) on $G(T)$, i.e.,

$$u_t - \frac{(1+|Du|^2)\Delta u - D_iuD_juD_{ij}u}{(1+|Du|^2)^{3/2}} - f(x,t,u) = u_t + F(x,t,u,Du,D^2u) = 0$$

on $G(T)$ if and only if

$$\tau + F(x,t,u(x,t),p,X) \le 0 \quad \text{for } (x,t;\tau,p,X) \in G(T) \times \mathcal{P}^{2,+}_{G(T)}u(x,t). \tag{2.3}$$

Analogously, a *viscosity supersolution of* (1.3) on $G(T)$, i.e., $u_t+F(x,t,u,Du,D^2u) = 0$, is a lower semicontinuous function $u : G(T) \mapsto \mathbb{R}$ such that

$$\tau + F(x,t,u(x,t),p,X) \ge 0 \quad \text{for } (x,t;\tau,p,X) \in G(T) \times \mathcal{P}^{2,-}_{G(T)}u(x,t). \tag{2.4}$$

Finally, u is a viscosity solution of (1.3) on $G(T)$ if it is both a viscosity subsolution and a viscosity supersolution of (1.3) on $G(T)$.

Our solutions of (1.1) will be classical ones, and consequently they will also be viscosity solutions, but we shall use only the stability of viscosity solutions under viscosity limits as defined below in (2.5)–(2.6).

Notice that so far we have not yet mentioned any boundary conditions. A reasonable definition of subsolutions of (1.3)–(1.2) could be the requirement that $u \le 0$ on the parabolic boundary. This is in fact a commonly used definition; see e.g. the definition of strong viscosity subsolutions of boundary value problems in [5, Definition 7.1].

As discussed in [5, Section 7.A], this notion coincides with the classical definition only in the case of nondegenerate (elliptic or parabolic) equations. However, unfortunately the limits of subsolutions,

$$\lim_{n\to\infty} \sup{}^* u_n(z) = \lim_{j\to\infty} \sup\{u_n(y) : n \ge j,\ y \in G(T) \text{ and } |z-y| < \frac{1}{j}\}, \tag{2.5}$$

respectively supersolutions,

$$\lim_{n\to\infty} \inf{}_* u_n(z) = \lim_{j\to\infty} \inf\{u_n(y) : n \ge j,\ y \in G(T) \text{ and } |z-y| < \frac{1}{j}\}, \tag{2.6}$$

are not necessarily subsolutions, respectively supersolutions, of the limiting initial boundary value problem in the strong viscosity sense, even though they are viscosity subsolutions, respectively supersolutions, of the corresponding differential equation in $G(T)$. This is the motivation in [5] to relax the interpretation of boundary conditions in the strong viscosity sense in such a way that it is stable under taking limits. As we shall see, this relaxation is not artificial and abstract but the natural way to define the Dirichlet problem for mean curvature equations and even for wide classes of elliptic and parabolic problems.

Definition 2.2. An upper semicontinuous function u on $\overline{G(T)}$ is called a *weak (viscosity) subsolution of* (1.3)–(1.2) if and only if (2.3) holds for $x, t \in G(T)$, $(\tau, p, X) \in \overline{\mathcal{P}}^{2,+}_{G(T)} u(x,t)$ (i.e., if u is a viscosity subsolution of (1.3) on $\overline{G(T)}$) and if it satisfies

$$\begin{aligned} \min\{\tau + F(x, t, u(x,t), p, X), u(x,t)\} \leq 0 \\ \text{for } \; t \in (0,T],\; x \in \partial G,\; (\tau, p, X) \in \overline{\mathcal{P}}^{2,+}_{G(T)} u(x,t). \end{aligned} \tag{2.7}$$

Here (see [5, page 11]),

$$\begin{aligned} \overline{\mathcal{P}}^{2,+}_{G(T)} u(x,t) = \{ & (\tau, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S^n : \\ & \exists (x_n, t_n, \tau_n, p_n, X_n) \in \overline{G(T)} \times \mathbb{R} \times \mathbb{R}^n \times S^n \\ & \text{with } (\tau_n, p_n, X_n) \in \mathcal{P}^{2,+}_{G(T)} u(x_n, t_n) \\ & \text{and } (x_n, t_n, u(x_n, t_n), \tau_n, p_n, X_n) \to (x, t, u(x,t), \tau, p, X)\}. \end{aligned}$$

Analogously, a *weak (viscosity) supersolution of* (1.3)–(1.2) is a lower semicontinuous function on $\overline{G(T)}$, which is a viscosity supersolution of (1.3) on $\overline{G(T)}$ and which satisfies

$$\begin{aligned} \max\{\tau + F(x, t, u(x,t), p, X), u(x,t)\} \geq 0 \\ \text{for } t \in (0,T],\; x \in \partial G,\; (\tau, p, X) \in \overline{\mathcal{P}}^{2,-}_{G(T)} u(x,t). \end{aligned} \tag{2.8}$$

The set $\overline{\mathcal{P}}^{2,-}_{G(T)} u(x,t)$ in (2.8) is defined just like $\overline{\mathcal{P}}^{2,+}_{G(T)} u(x,t)$, except that the plus sign in the superjet $\overline{\mathcal{P}}^{2,+}_{G(T)} u(x_n,t_n)$ is replaced by a minus. Finally, $u(x,t)$ is a *weak (viscosity) solution* if and only if it is both a weak (viscosity) sub- and supersolution.

Using the above definitions, let us now formulate the main results of our paper.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a ball with radius $R > n$. Then problem* (1.1)–(1.2) *has a unique solution $u \in C^\infty(\Omega \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty)$ which solves the Dirichlet condition of* (1.2) *in the weak viscosity sense. Moreover, the spatial gradient of u blows up on the boundary after a finite time T_* and for $t < T_*$ the Dirichlet condition is satisfied in the classical sense. Finally $u_t(x,t)$ approaches $(1 - n/R)$ as $t \to \infty$ everywhere in Ω.*

Remark 2.4. For a ball of radius $R > n$ the solution of problem (1.5) is easily detected as $\Omega_* = \Omega$, and $|\partial\Omega_*|/|\Omega_*| = n/R$. Therefore Theorem 1 confirms the conjecture of Marcellini and Miller.

Next we consider the case that Ω is a square, $\Omega = K := \{0 < x_i < a,\; i = 1,2\}$.

Theorem 2.5. *Let K be the square with length $a > 2 + \sqrt{\pi}$. Then*

i) *Problem* (1.1)–(1.2) *has a unique solution $u \in C^\infty(K \times (0,\infty)) \cap C(\overline{K} \times [0,\infty))$ which solves the Dirichlet condition of* (1.2) *in the weak viscosity sense. Moreover, the trace of u on the boundary is Lipschitz continuous.*
ii) *The gradient of the solution blows up on $\partial K_1 \cap \partial K$, after a finite time $T_*(x)$. Here K_1 is obtained from K by rounding off the corners with circular arcs of*

radius 1. *Until this blow up occurs,* (1.2) *holds in the classical sense. After the blow up time* $T_*(x)$ *the solution detaches from the boundary data on* $\partial K_1 \cap \partial K$ *with infinite slope.*

iii) *If* K_R *denotes "the square with rounded corners of radius* R*," then for* $R \in [1, a/(2+\sqrt{\pi})]$ *the following sharp estimates hold for large time*

$$u(x,t) \geq \big(1 - \frac{1}{R}\big)t + \underline{w}_R(x) \quad \text{for } \; x \in K_R, \; t >> 0, \tag{2.9}$$

$$u(x,t) \leq \big(1 - \frac{1}{R}\big)t + \overline{w}_R(x) \quad \text{for } \; x \in K \setminus K_R, \; t >> 0, \tag{2.10}$$

where $\underline{w}_R$ *and* $\overline{w}_R$ *are independent of* t *and locally finite.*

Remark 2.6. Theorem 2.5 confirms the conjectures of Marcellini and Miller in several ways. On the set Ω_* defined in (1.5) the solution u grows with maximal speed, and off the set Ω_* it grows less than maximal in time.

3. Traveling wave phenomena in a ball. The aim of this section is to illustrate the concept of viscosity solutions in the simplest case of a ball. In fact, problem (1.1) and even equations with more general right-hand sides $f(|x|)$ were investigated in [28] by Marcellini and Miller. They used the radial symmetry of solutions in an essential way in their analysis, and they proved existence and uniqueness of a "generalized pseudosolution" by taking limits in the regularized problem (1.9)–(1.2). Therefore, we shall merely sketch certain parts of the proof of Theorem 2.3, namely those which give more general (nonradial) arguments to already known results in the radial setting.

The uniqueness of weak viscosity solutions to problem (1.1)–(1.2) follows with minor changes from uniqueness results for viscosity solutions in [19,5]. The only difference is that we consider Dirichlet's boundary condition in the weak viscosity sense and not in the strong viscosity sense, comparing sub- and supersolutions which are continuous up to the boundary instead of semicontinuous ones.

In contrast to the comparison theorem for strong viscosity solutions (e.g., [5, Theorem 3.3], it may happen for weak viscosity solutions that an (upper semicontinuous weak) subsolution lies above a (lower semicontinuous weak) supersolution of (1.1)–(1.2) on the lateral surface $\partial\Omega \times (0,T)$. An example for this is given below by $\overline{u}$ and $\underline{u}$.

Fortunately the generalized comparison principle for pseudosolutions suggested by Lichnewsky and Temam ([24]; see also Lemma 3.1 in [28]) is exactly a comparison principle for solutions of (1.1) which are smooth in the interior of the space-time cylinder and which satisfy the Dirichlet condition in the weak viscosity sense. Due to the interior regularity of solutions this comparison principle is a direct consequence of the classical one. For the reader's convenience we will repeat it here.

Lemma 3.1 (Marcellini and Miller). *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with smooth boundary, and let* u*,* v *be of class* $C^2(\Omega' \times [0,T])$ *for any compact subdomain* Ω' *of* Ω*. Let us assume that* u_t*,* v_t *are bounded, that* $v - u$ *is bounded from above, and that* u, v, Du, Dv *are defined on* $\partial\Omega$ *as a limit* (*possibly infinite*) *of their respective values in* Ω*. Then the following conditions* (i), (ii) *and* (iii) *imply that* $v \leq u$ *in* $\overline{\Omega} \times [0,T]$*:*

(i) $v_t - div(Dv/\sqrt{1+|Dv|^2}) \leq u_t - div(Du/\sqrt{1+|Du|^2})$ *in* $\Omega \times (0,T)$*;*

(ii) *$v(x,0) \le u(x,0)$ in Ω;*

(iii) *for every $x \in \partial\Omega$ and $t \in (0,T)$ we have $v \le u$ or $(Dv,n)/\sqrt{1+|Dv|^2} \le (Du,n)/\sqrt{1+|Du|^2}$, where n is the unit outward normal to $\partial\Omega$.*

The proof of this Lemma is omitted.

The statements on existence in Theorems 1, 2 and 3 are based on the investigation of the regularized problem (1.9) in the cylinder $Q = \Omega \times (0,\infty)$. It follows from [23] and [36] that for every $\varepsilon > 0$ problem (1.9)–(1.2) has a unique solution $u^\varepsilon(x,t) \in C^\infty(\overline{\Omega} \times (0,\infty)) \cap C^{1,\beta}(\overline{\Omega} \times [0,\infty))$ with $\beta \in (0,1)$ arbitrary. Let us recall some well-known or easily proved a priori estimates for the solutions u^ε of (1.9)–(1.2) in general domains Ω.

Lemma 3.2. *The functions 0 and t are classical sub- and supersolutions of the regularized equation* (1.9) *in the cylinder $Q = \Omega \times (0,\infty)$. Moreover, they are weak viscosity sub- and supersolutions of* (1.1)–(1.2), *and for every $\varepsilon > 0$ the following estimate holds:*

$$0 \le u^\varepsilon(x,t) \le t \qquad \text{in } \overline{\Omega} \times [0,\infty). \tag{3.1}$$

Proof. Estimate (3.1) is a simple consequence of the classical comparison principle ([31]) for the functions 0, u^ε and t. The second statement of Lemma 3.2 follows directly from Definitions 2.1 and 2.2. To check, for example, that t is a viscosity supersolution of (1.1), let us examine the superjet $\mathcal{P}^{2,-}_{\Omega(T)}t$, where $\Omega(T) = \Omega \times (0,T]$. Easy calculations give us, from (2.1), that

$$\mathcal{P}^{2,-}_{\Omega(T)}t = \begin{cases} (1,0,X),\ X \le 0, & \text{for } \ 0 < t < T, \\ (\tau,0,X),\ \tau \ \ge 1,\ X \le 0, & \text{for } \ t = T. \end{cases}$$

Indeed, from the definition of $\mathcal{P}^{2,-}_{\Omega(T)}$ we have that

$$s \ge t + \tau(s-t) + \langle p, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + o(|s-t| + |y-x|^2) \tag{3.2}$$

as $(s,y) \to (t,x) \in \Omega(T)$. If $(t,x) \in \Omega(T)$, then for $(s,y) = (s_n,x)$ and $s_n \nearrow t$ we obtain the inequality $\tau \ge 1$. Moreover, if $t < T$ and $s_n \searrow t$ we get the opposite inequality $\tau \le 1$. Therefore, $\tau = 1$ if $t < T$. Analogously, for $(s,y) = (t,y_n)$ and $y_n \to x$ with $(y_n - x)/|y_n - x| \to q \in \mathbb{R}^n$ we may put in (3.2) $s = t$, $y = y_n$, divide the result by $|y_n - x|$ and pass to the limit to conclude that

$$\langle p, q\rangle \le 0 \quad \text{for every } \ q \in \mathbb{R}^n; \text{i.e., } p = 0.$$

In the same way, after division by $|y_n - x|^2$, we have the inequality $\langle Xq, q\rangle \le 0$ for $q \in \mathbb{R}^n$; i.e., $X \le 0$. Now inequality (2.4) follows trivially, since $\tau - \sum_{i=1}^n X_{ii} - 1 \ge 0$ for $(t,x,\tau,p,X) \in \Omega(T) \times \mathcal{P}^{2,-}_{\Omega(T)}$.

To check that t satisfies the corresponding boundary condition in the weak viscosity sense we need the superjet $\overline{\mathcal{P}}^{2,-}_{\overline{\Omega(T)}}$ on the boundary points $(t,x) \in (0,T] \times \partial\Omega$; see (2.8). Compared to the superjet in interior points this is more complicated (see for example the comments in Remark 2.7 in [5]). Fortunately, the function t is nonnegative on the parabolic boundary, so that (2.8) follows trivially, independent of the structure of the superjet $\overline{\mathcal{P}}^{2,-}_{\Omega(T)}$.

The verification that 0 is a subsolution of (1.1)–(1.2) in the weak viscosity sense is the same, using (2.3) and (2.7) instead of (2.4) and (2.8). Therefore we omit the proof.

Lemma 3.3. *Suppose that Ω is a bounded domain with $\partial\Omega$ of class C^2. Then there exist positive constants δ and C, such that* (1.1)–(1.2) *has a unique classical solution for $t < \delta$ and such that the estimate*

$$\sup_{\overline{\Omega}\times[0,\delta]} |Du(x,t)| \leq C \tag{3.3}$$

holds.

The proof follows from the classical result of Sobolevskii-Tanabe (see for example Section 3.4 in [35]), according to which one obtains uniqueness and existence of classical solutions to (1.1)–(1.2) locally in time through analytic semigroup theory. Equation (1.1) is parabolic as long as $|Du|$ remains bounded, and from Lemma 3.2 the amplitude $\sup|u|$ of the solution is bounded in bounded time cylinders $\overline{\Omega}\times(O,T]$. Moreover, from the results e.g. in [25] or [34, Section 6], one can globally bound $|Du|$ in terms of gradient estimates on the lateral boundary $\partial\Omega \times [0,T]$. Thus problem (1.1) has a classical solution up to the first blow-up time of the spatial gradient on the boundary $\partial\Omega \times [0,T]$.

For the interior regularity of the viscosity solution to (1.1)–(1.2) we shall use the following result of Gerhardt ([13]).

Lemma 3.4 (Gerhardt)**.** *Suppose Ω' is a compact subdomain of Ω. Then there exists a positive constant C depending on $\Omega', n, T, \sup|u^\varepsilon|$ and dist$(\Omega',\partial\Omega)$, but otherwise independent of ε such that the following estimate holds uniformly in ε:*

$$\sup_{\overline{\Omega'}\times[0,T]} |Du^\varepsilon(x,t)| \leq C(\Omega', n, T, \sup|u^\varepsilon|, \operatorname{dist}(\Omega',\partial\Omega)). \tag{3.4}$$

Combining (3.4) with standard Schauder estimates yields higher-order a priori estimates for the solutions u^ε of (1.9) in every compact subdomain $\Omega' \times [0,T]$ of $\Omega(T)$, with constant independent of ε.

Let us now assume that Ω is a ball. It was proved in [28] that the limit $\lim_{\varepsilon\to 0} u^\varepsilon(x,t) = u(x,t)$ exists and that $u(x,t)$ is smooth in the interior of Q and satisfies (1.1) in the classical sense. The continuous extension of $u(x,t)$ up to the boundary $\partial\Omega \times (0,\infty)$ exists and satisfies the Dirichlet condition (1.2) in a generalized but not in a classical sense, because $u(x,t)$ detaches from zero boundary data. Moreover, $u(x,t)$ is nondecreasing in t, concave in x and the exterior normal derivative $u_n = -\infty$ on $\partial\Omega$, where $u(x,t)$ detaches. We shall reprove these results by pointing out how they follow from the theory of viscosity solutions. Existence of a continuous viscosity solution u in $\overline{\Omega} \times [0,\infty)$ follows from the stability of weak viscosity solutions to (1.9)–(1.2) under the limit $\varepsilon \to 0$ as follows; see Section 7A in [5] and Lemma 3.2. Since 0 and t are sub- and supersolutions of (1.9)–(1.2), which are independent of ε, then by [5, Lemma 6.1] the functions $\overline{u} = \lim_{\varepsilon\to 0} \sup^* u_\varepsilon(x,t) \le t$ and $\underline{u} = \lim_{\varepsilon\to 0} \inf_* u_\varepsilon(z) \ge 0$ defined in (2.5) and (2.6) are an upper-semicontinuous viscosity subsolution and a lower-semicontinuous viscosity supersolution of equation (1.1). By definition $\underline{u}(x,t) \le \overline{u}(x,t)$. Since Lemma 3.4 and the Arzela-Ascoli theorem imply that the limit is locally uniform in Q, we obtain that $\underline{u}(x,t) = \overline{u}(x,t)$ in $\Omega \times [0,\infty)$. As for the boundary data on the lateral surface $\partial\Omega \times [0,\infty)$, it is clear from definition (2.6) that $\underline{u}(x,t) = 0$ there while $0 \le \overline{u}(x,t) \le t$.

We observe that $\overline{u}$ satisfies (2.7) and that $\overline{u}$ ($= \underline{u}$ in Q) is a viscosity subsolution of (1.1) in Q, as well as a weak viscosity supersolution of (1.1) in Q by definition. Thus $\overline{u}$ is a viscosity solution of (1.1) in Q, but not necessarily yet on $\overline{Q}$. In other words, we do not know yet if it is a viscosity solution of (1.1)–(1.2). What can be said about the continuity of $\overline{u}$ on the lateral boundary $\partial\Omega \times [0,\infty)$? In those points $(x_0,t_0) \in \partial\Omega \times [0,\infty)$, where $\overline{u}(x_0,t_0) = 0$ we have continuity for the following reason: $\overline{u} \ge 0$ in Q and $\overline{u}$ is upper semicontinuous so that for $(x_n,t_n) \to (x_0,t_0)$ we have the inequalities $0 \le \liminf \overline{u}(x_n,t_n) \le \limsup \overline{u}(x_n,t_n) \le \overline{u}(x_0,t_0) = 0$.

At any other point $(x_0,t_0) \in \partial\Omega \times [0,\infty)$ we have $\overline{u}(x_0,t_0) > 0$. Let us show that now (2.7) implies that the exterior normal derivative $\overline{u}_\nu(x_0,t_0) = -\infty$. This will be an instructive exercise on how to interpret (2.7). Clearly, since (2.7) holds for (x_0,t_0), and $\overline{u}(x_0,t_0) > 0$, we must have

$$\tau - \frac{(1+|p|^2)X_{ii} - p_i X_{ij} p_j}{(1+|p|^2)^{3/2}} - 1 \le 0 \quad \text{for every} \quad (\tau,p,X) \in \overline{\mathcal{P}}^{2,+}_{\Omega(t_0)} u(t_0,x_0).$$

Note that this relation trivially holds if the superjet $\overline{\mathcal{P}}^{2,+}_{\Omega(t_0)} u(t_0,x_0)$ is empty. So suppose that it is not empty and that there exists some $(\tau,p,X) \in \overline{\mathcal{P}}^{2,+}_{\Omega(t_0)} u(t_0,x_0)$. Then [5, Remark 2.7(iii), page 14] implies that $(\tau, p-\lambda\nu, X+\mu\nu\otimes\nu) \in \overline{\mathcal{P}}^{2,+}_{\Omega(t_0)} u(t_0,x_0)$ for positive λ and any real μ. Therefore,

$$\tau - \frac{(1+|p-\lambda\nu|^2)(X_{ii}+\mu) - (p_i-\lambda\nu_i)(X_{ij}+\mu\,\nu_i\nu_j)(p_j-\lambda\nu_j)}{(1+|p-\lambda\,\nu|^2)^{3/2}} - 1 \le 0$$

for every $\lambda > 0$ and $\mu \in \mathbb{R}$, or equivalently,

$$\tau - \frac{(1+|p-\lambda\nu|^2)X_{ii} - (p_i-\lambda\nu_i)(X_{ij}(p_j-\lambda\nu_j))}{(1+|p-\lambda\nu|^2)^{3/2}} - 1 - \mu\frac{1+|p|^2-(p,\nu)^2}{(1+|p-\lambda\nu|^2)^{3/2}} \le 0.$$

But if λ is fixed and $\mu \to -\infty$, we obtain a contradiction to (2.7). Therefore, the superjet must be empty. We claim that this leads to unbounded $\overline{u}_\nu(t_0,x_0)$. (In Remark 3.9 we show that the converse holds as well: unbounded gradient on the boundary implies emptiness of the superjet.) If $\overline{u}_\nu(t_0,x_0)$ were bounded, then, since second derivatives of $\overline{u}$ were bounded, (2.2) would imply that $\mathcal{P}^{2,+}_{\Omega(t_0)} u(t_0,x_0)$ is nonempty, a contradiction. Therefore, $\overline{u}_\nu(t_0,x_0) = -\infty$. Now upper semicontinuity and monotonicity of $\overline{u}$ in the normal direction implies continuity.

As for the time variable t, the function $\overline{u}(x,t)$ is Lipschitz continuous on the boundary. The proof follows directly from the definition of $\overline{u}$ and the monotonicity of $u^\varepsilon(x,t)$ with respect to t, as well as from the uniform bound $u_t^\varepsilon \leq 1$. This is proved for example in Proposition 4.3 for the square, but the proof is valid without changes for arbitrary domains.

Marcellini and Miller obtained a continuous extension to the boundary through the observation that u is concave in x, if Ω is a ball and u is radial.

It is interesting to note that problem (1.1)–(1.2) admits lower-semicontinuous weak viscosity supersolutions, e.g. $\underline{u}$, which are less than a weak viscosity subsolution, $\overline{u}$. In fact, any function v which coincides with $\overline{u}$ in Q and which has boundary data between $\underline{u}$ and u is such a weak viscosity supersolution that lies under the weak viscosity solution u of (1.1)–(1.2).

Moreover we should point out that the failure of u to satisfy (1.2) in the classical sense can only occur in those boundary points where $|Du|$ becomes infinite. Otherwise the operator in (1.1) is nondegenerate, and then the boundary condition is known to hold in the strong sense of Definition 2.2; see [5, Proposition 7.1].

Now that we have existence of a weak viscosity solution of (1.1)–(1.2) we can prove its properties described in Theorem 1. We need the following global estimates which are based on Giusti's extremal solutions ([16]) for the ball, namely the semispheres $\sqrt{R^2-|x|^2}$ under suitable translations.

Lemma 3.6. *For the solutions $u^\varepsilon(x,t)$ and $u(x,t)$ of* (1.9)–(1.2) *and* (1.1)–(1.2) *the following estimates hold on $\overline{\Omega}\times[0,\infty)$:*

$$u^\varepsilon(x,t) \leq \sqrt{R^2-|x|^2} + (1-\frac{n}{R})t, \tag{3.6}$$

$$\sqrt{R^2-|x|^2} + (1-\frac{n}{R})t - R \leq u(x,t). \tag{3.7}$$

Remark 3.7. Note that in estimates (3.6)–(3.7) the quantity $n/R = |\partial\Omega|/|\Omega|$ is the minimizer of problem (1.5).

Remark 3.8. Lemma 3.6 describes a traveling wave phenomenon. Since (3.6) holds also for u, one can see that u stays between two traveling waves in the space-time cylinder, which move with speed $(1-\frac{n}{R})$ upward in time.

Proof of Lemma 3.6. The function $v=\sqrt{R^2-|x|^2}+(1-\frac{n}{R})t$ is a classical solution of (1.1) and a classical supersolution of (1.9)–(1.2); i.e.,

$$v_t - \varepsilon\Delta v - \operatorname{div}\Big(\frac{Dv}{\sqrt{1+|Dv|^2}}\Big) - 1 = \varepsilon\frac{n(R^2-|x|^2)+|x|^2}{(R^2-|x|^2)^{3/2}} \geq 0 \quad \text{in } \; Q,$$
$$v(x,t) = (1-\frac{n}{R})t \geq 0 \quad \text{for } \; x\in\partial\Omega,\ t\geq 0,$$
$$v(x,0) = \sqrt{R^2-|x|^2} \geq 0 \quad \text{for } \; x\in\Omega.$$

Therefore, estimate (3.6) follows from the classical comparison principle. To prove (3.7) one has to note that the function $w=\sqrt{R^2-|x|^2}+(1-\frac{n}{R})t-R$ is a weak viscosity subsolution of (1.1)–(1.2). $\square$

Remark 3.9. It is instructive to check relation (2.7) for w. We claim that the set $\mathcal{P}^{2,+}_{G(T)}w(x,t)$ in (2.7) is empty for $x \in \partial\Omega$, so that (2.7) is trivially satisfied. Indeed $\mathcal{P}^{2,+}_{G(T)}w(x,t) = \{(\tau,p,X) \in \mathbb{R}\times\mathbb{R}^n\times S^n : \sqrt{R^2-|y|^2}+(1-\frac{n}{R})s-R \le \sqrt{R^2-|x|^2}+(1-\frac{n}{R})t-R+\tau(s-t)+\langle p,y-x\rangle+\frac{1}{2}\langle X(y-x),y-x\rangle+o(|s-t|+|y-x|^2)$ as $(y,s)\to(x,t)\in\partial\Omega\times(0,T]\}$. In particular, for $s=t$ and $y_n\to x$ we can divide by $|y_n-x|$ and pass to the limit to obtain that the above inequality fails for every $p\in\mathbb{R}^n$.

As a consequence of Lemma 3.6 it follows that the weak viscosity solution u of (1.1)–(1.2) detaches from the homogeneous Dirichlet data after a finite time $T^* \le R^2/(R-n)$ and that hence the gradient of u blows up on the boundary after a finite time. Let $T_* \le T^*$ be the first blow-up time for the gradient of u on the boundary. Since $u_{rt}(x,t)\le 0$ in Q (see [28, Lemma 6.1]), u_r is nonincreasing in t. This and the equality $u_r(x,T_*)=-\infty$ on $\partial\Omega$ implies immediately that u_r remains unbounded on $\partial\Omega$ for every $t\ge T_*$. This completes the proof of Theorem 1. □

4. The case of a square. In this section we suppose that Ω is a square, $\Omega = K := \{0<x_i<a,\ i=1,2\}$. When the side of the square is sufficiently large (and a detailed calculation using Lemma 4.1 yields $a > 2+\sqrt{\pi}$) there is no stationary solution of (1.1)–(1.2) according to [16, Theorem 1.1], and in contrast to the radially symmetric case of Section 3 several new effects occur. In particular, the unique weak viscosity solution u detaches on part of the boundary, but not on the entire boundary of K.

Moreover, the solution Ω_* of problem (1.5) does no longer coincide with Ω, as in the case of a large ball. Let us therefore recall a result on the free boundary $\Gamma_0 = \partial\Omega_*\cap\Omega$ for general Ω.

Lemma 4.1. *Suppose Ω is a bounded domain in $\mathbb{R}^n$, $n\ge 2$, and Ω_* is a nontrivial solution of the isoperimetric problem* (1.5). *Then the free boundary $\Gamma_0=\partial\Omega_*\cap\Omega$ has constant mean curvature equal to $|\partial\Omega_*|/|\Omega_*|$.*

For the proof we refer to [17] or 22].

Using Lemma 4.1 it is easy to determine the minimizing set Ω_* for a square K: it is a "square with rounded corners," i.e., arcs of radius $a/(2+\sqrt{\pi})$.

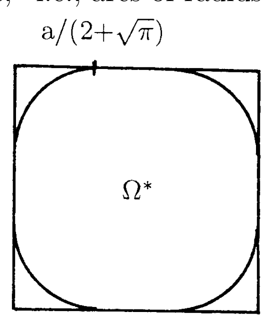

Figure 4.1. Shape of $\Omega = K$ and Ω_*.

Determining the curvature of these corners is a straightforward one-dimensional optimization problem, which renders the curvature $(2+\sqrt{\pi})/a = |\partial\Omega_*|/|\Omega_*|$; see Figure 4.1.

As in Section 3 we can prove existence and uniqueness of a continuous weak viscosity solution u to problem (1.1)–(1.2). In order to prove the remaining statements of Theorem 2 we shall prove some precise estimates from above for the approximating solutions u^ε of (1.9)–(1.2) as well as estimates from below for the limiting solution $u(x,t)$.

To prove the estimates from above, let $\omega(R)$ be the difference of the square $[0,R]\times[0,R]$ minus the quarter disk with radius R and center at (R,R), where $R < a/2$. Since u^ε and u have all the symmetries of K, it suffices to study the behaviour of u^ε and u in the southwest corner.

Lemma 4.2. *For every $\delta > 0$ and for every $R \in [1, a/(2+\sqrt{\pi}))$ the following estimates hold for the solution u of* (1.1)–(1.2):

$$u(x,t) \le (1-\frac{1}{R})\ t + \overline{w}_R(x) \qquad \text{for } \ x\in\omega(R),\ t\ge 0,$$

$$u(x,t) \le (1-\frac{2+\sqrt{\pi}}{a}+\delta)\ t + C_\delta \qquad \text{for } \ x\in K,\ t\ge 0, \tag{4.1}$$

where C_δ is a positive constant depending on K and δ, and $\overline{w}_R$ is finite in $\omega(R)$ and on the straight parts of its boundary, but $\overline{w}_R$ can be infinite on the circular part γ_R of $\partial\omega(R)$.

Lemma 4.2 guarantees global boundedness of the solution $u(x,t)$ near the corners of K, i.e., in $\omega(1)$. Moreover, in $\omega(R)$ with $R\in(1,a/(2+\sqrt{\pi}))$, the maximal speed of growth for $u(x,t)$ as $t\to\infty$ is $(1-1/R)$, which is strictly less than $(1-(2+\sqrt{\pi})/a)$, the speed of growth in the extremal domain Ω_*. The proof of Lemma 4.2 is based on several simple propositions.

Proposition 4.3. *For every $\varepsilon > 0$ the following inequalities hold*:

$$u^\varepsilon_{x_i}(x,t) > 0 \quad \text{for } \ x_i\in(0,a/2),\ t>0,\ i=1,2,$$

$$1 > u^\varepsilon_t(x,t) > 0 \qquad\qquad \text{for } \ x\in K,\ t\ge 0. \tag{4.2}$$

Proof of Proposition 4.3. From (3.1) and (1.2) we have $u^\varepsilon_{x_1}(x,t)\ge 0$ for $x\in\partial K$, $x_2\le a/2$ and $t>0$. Moreover, by symmetry, $u^\varepsilon_{x_1}=0$ on the line $x_1=a/2$. On the left half K_1 of the square we may thus consider the function $z(x,t)=u^\varepsilon_{x_1}(x,t)$. It satisfies the differential equation

$$\begin{aligned} z_t - \varepsilon\Delta z &- \frac{(1+|Du^\varepsilon|^2)\Delta z - D_i u^\varepsilon D_j u^\varepsilon D_{ij} z}{(1+|Du^\varepsilon|^2)^{3/2}} \\ &- \frac{2\Delta u^\varepsilon Du^\varepsilon Dz - D_j u^\varepsilon D_{ij} u^\varepsilon D_i z - D_i u^\varepsilon D_{ij} u^\varepsilon D_j z}{(1+|Du^\varepsilon|^2)^{3/2}} \\ &+ \frac{3(u^\varepsilon_t - 1 - \varepsilon\Delta u^\varepsilon)}{(1+|Du^\varepsilon|^2)} Du^\varepsilon Dz = 0 \quad \text{in } \ K_1\times(0,\infty) \end{aligned} \tag{4.3}$$

under nonnegative initial and boundary conditions. Thus $z>0$ due to the strong maximum principle. In a similar way one shows that $u^\varepsilon_{x_2}$ and u^ε_t as well as $1-u^\varepsilon_t$ are positive. □

Proposition 4.4. *For every $\varepsilon > 0$ and $T > 0$ the following estimate,*

$$\sup\ |Du^\varepsilon| \le \sqrt{2T/\varepsilon}, \tag{4.4}$$

holds uniformly on $K \times [0,T]$.

Proof. Since the modulus of the gradient $|Du^\varepsilon|$ attains its maximum on the boundary, it is enough to prove (4.4) only on the boundary $\partial K \times [0,T]$. For this purpose we use the barrier function $w(x,t) = (x_2/\sqrt{\varepsilon})(\sqrt{2T} - x_2/2\sqrt{\varepsilon})$ in the cylinder $\tilde{Q} = \{(x_1,x_2,t) \in (0,a) \times (0,\sqrt{2T\varepsilon}) \times (0,T)\}$. From the inequality

$$w_t - \varepsilon\Delta w - \operatorname{div}\Big(\frac{Dw}{\sqrt{1+|Dw|^2}}\Big) \ge 1 \qquad \text{in } \ \tilde{Q},$$

Lemma 3.2 and the initial data, it follows that $u^\varepsilon - w$ attains its maximum when $x_2 = 0$ and $0 \le t \le T$. Hence, from the standard boundary maximum principle we have the estimate $u^\varepsilon_{x_2}(x_1,0,t) \le w_{x_2}(x_1,0,t) = \sqrt{2T/\varepsilon}$, which combined with the symmetry of the square gives us the proof of (4.4). $\square$

Now we are in a situation to prove Lemma 4.2. To derive its first estimate, for every fixed $T > 0$ we consider the solution $v(x,t) = u^\varepsilon(x,t) - (1-1/R)t$ of the following initial value problem:

$$v_t - \operatorname{div}\Big(\frac{Dv}{\sqrt{1+|Dv|^2}}\Big) - \varepsilon\Delta v = \frac{1}{R} \qquad \text{in } \ \omega(R) \times (0,T],$$
$$v(x,0) = 0 \qquad \text{in } \ \omega(R),$$

with boundary conditions on the straight part of the boundary

$$v(x,t) = -(1-1/R)t \quad \text{for } x_1 = 0,\ x_2 \in [0,R] \text{ or } x_2 = 0,\ x_1 \in [0,R] \text{ and } t \in [0,T].$$

As stated in Lemma 4.2 we have to estimate v in $\omega(\gamma R)$ for every $\gamma \in (0,1)$. $\omega(\gamma R)$ will be partitioned into a corner piece $\omega(R_0)$ with $R_0 = \gamma R\sqrt{2}/2$ and into slices $w_k(R_k) = w(R_k) \setminus \omega(R_{k+1})$ with $R_{k+1} = (\sqrt{2}R\gamma + (2-\sqrt{2})R_k)/2$, $k \in \mathbb{N}$. Note that $R_{k+1} = R\gamma[1-((2-\sqrt{2})/2)^{k+2}]$ tends to $R\gamma$ as $k \to \infty$. Our estimates of v on $\omega(R_0)$ and $w_k(R_k)$ will not depend on ε and T, but depend on R,γ and k.

Let us first estimate v on $\omega(R_0)$. We shall compare v with a suitable function $w(x) = h(r(x) - R_0)$, where the function $h(s)$ depends only on the distance r of x to (R_0,R_0), and where h is a nonnegative solution of the ordinary differential equation

$$\begin{aligned} &h''\big[\varepsilon + (1+{h'}^2)^{-3/2}\big] = \frac{1-\gamma}{2\gamma R} \qquad \text{and} \\ &h \ge 0 \quad \text{ for } \ s \in (0, \frac{2-\sqrt{2}}{2}R\gamma), \qquad h'(\frac{\gamma R\sqrt{2}}{2}\varepsilon^{1/4}) = -1/\sqrt{\varepsilon}. \end{aligned} \tag{4.5}$$

The idea to use this barrier function is the following. Imagine a radially decreasing function of constant mean curvature on an annulus with radii R and $\sqrt{2}R$, which is nonnegative on the outer boundary and has infinite slope on the inner boundary. Such a function would be a good comparison function for the unperturbed equation (1.1). To treat the perturbed equation (1.9), we modify it slightly to make the calculations somewhat easier. This is how we arrive at (4.5).

The function h is defined as

$$h(s) = \int_{\gamma R(2-\sqrt{2}/2)}^{s} G^{-1}\Big(\frac{1-\gamma}{2\gamma R}\big(z - \frac{\gamma R\sqrt{2}\varepsilon^{1/4}}{2}\big)\Big)dz,$$

where $G^{-1}(s)$ is the inverse function of

$$G : [-A/\varepsilon^{3/4}, -B] \mapsto [-(1-\gamma)\sqrt{2}\varepsilon^{1/4}/4, (1-\gamma)(2-\sqrt{2}-\sqrt{2}\varepsilon^{1/4})/4]$$

defined by

$$G(z) = \int_{-1/\sqrt{\varepsilon}}^{z} \Big[\varepsilon + (1+\tau^2)^{-3/2}\Big]\, d\tau.$$

Here $A(\varepsilon), B(\varepsilon)$ will be chosen in such a way that G is invertible, i.e.,

$$\begin{aligned} &\frac{1}{(B+\sqrt{1+B^2})\sqrt{1+B^2}} - \varepsilon B \\ &\qquad = -\sqrt{\varepsilon} + \frac{\varepsilon}{(1+\sqrt{1+\varepsilon})\sqrt{1+\varepsilon}} + \frac{1-\gamma}{4}(2-\sqrt{2}-\sqrt{2}\varepsilon^{1/4}) \end{aligned} \tag{4.6}$$

$$A - \frac{\varepsilon^{5/4}}{(A+\sqrt{A^2+\varepsilon^{3/2}})\sqrt{A^2+\varepsilon^{3/2}}} = \varepsilon^{1/4} + \frac{(1-\gamma)\sqrt{2}}{4} - \frac{\varepsilon^{3/4}}{(1+\sqrt{1+\varepsilon})\sqrt{1+\varepsilon}}.$$

The function h is well defined, $h \in C^2[0, \gamma R(2-\sqrt{2})/2]$, and has the following properties: $h \geq 0$, $h' \leq 0$ and $h'' \geq 0$ in $[0, \gamma R(2-\sqrt{2})/2]$.

Now let us prove that $v(x,t) \leq w(x) = h(r(x) - R_0)$ in $\omega(R_0) \times [0,T]$. From (4.5) and (4.6) we have the inequalities

$$\begin{aligned} &w_t - \varepsilon\Delta w - \operatorname{div}\Big(\frac{Dw}{\sqrt{1+|Dw|^2}}\Big) \\ &= -h''(1+h'^2)^{-3/2} - \frac{h'}{r}(1+h'^2)^{-1/2} - \varepsilon h'' - \varepsilon\frac{h'}{r} \\ &= -h''[\varepsilon + (1+h'^2)^{-3/2}] - \frac{1}{r}\big[\varepsilon h' - 1 + (\sqrt{1+h'^2} - h')^{-1}(1+h'^2)^{-1/2}\big] \\ &\geq \frac{\gamma-1}{2\gamma R} + \frac{1+\gamma}{2\gamma R} - \frac{1}{r}\big[\varepsilon h' - \frac{1-\gamma}{2} + (\sqrt{1+h'^2} - h')^{-1}(1+h'^2)^{-1/2}\big] \\ &\geq \frac{1}{R} - \frac{1}{r}\big[\varepsilon h' + \varepsilon B - \sqrt{\varepsilon} + \frac{\varepsilon}{(1+\sqrt{1+\varepsilon})\sqrt{1+\varepsilon}} - \frac{1-\gamma}{2} \\ &\qquad + \frac{1-\gamma}{4}(2-\sqrt{2}-\sqrt{2}\varepsilon^{1/4})\big] \\ &\geq \frac{1}{R} = v_t - \varepsilon\Delta v - \operatorname{div}\Big(\frac{Dv}{\sqrt{1+|Dv|^2}}\Big). \end{aligned}$$

Therefore, $w(x) - v(x,t)$ does not attain a negative minimum at a point in $\omega(R_0) \times (0,T]$. Since $w(x) - v(x,t) \geq 0$ on the flat part of the boundary; i.e., for $x_1 = 0$, $x_2 \in [0, R_0]$, $x_2 = 0$, $x_1 \in [0, R_0]$ and $t \in [0,T]$ as well as initially, it follows that $w - v$ can attain a negative minimum only on $\gamma(R_0) \times [0,T]$, where $\gamma(R_0) = \Omega \cap \partial\omega(R_0)$ denotes the circular part of the boundary of $\omega(R_0)$. But then Hopf's Lemma implies that

$$w_r(P_0) - v_r(P_0) = -A\varepsilon^{-3/4} - v_r(P_0) > 0$$

for some $P_0 \in \gamma(R_0) \times [0,T]$. Note that the subscript r denotes radial differentiation from (R_0, R_0) into $\omega(R_0)$. The last inequality contradicts (4.4). Therefore,

$$v(x,t) \leq w(x) = h(r(x) - R_0) \qquad \text{in } \ \omega(R_0) \times [0,T].$$

Unfortunately the barrier function $h(s)$ is not uniformly bounded in ε on the whole interval $[0, \gamma R(2-\sqrt{2})/2]$. This is true only in a slightly shorter interval

$$I_\varepsilon := [\gamma R\sqrt{2}\varepsilon^{1/4}/2, \gamma R(2-\sqrt{2})/2],$$

which is enough for our proof. Indeed, simple calculations show that in I_ε we have

$$\begin{aligned} h(s) \leq h(\gamma R\sqrt{2}\varepsilon^{1/4}/2) &= \int_{\gamma R(2-\sqrt{2})/2}^{\gamma R\sqrt{2}\varepsilon^{1/4}/2} G^{-1}\Big(\frac{1-\gamma}{2\gamma R}\big(\tau - \frac{\gamma R\sqrt{2}\varepsilon^{1/4}}{2}\big)\Big)\, d\tau \\ &= \frac{2\gamma R}{1-\gamma}\int_{-B}^{-1/\sqrt{\varepsilon}} s[\varepsilon + (1+s^2)^{-3/2}]\, ds \\ &= \frac{2\gamma R}{1-\gamma}\Big[\frac{1}{2} - \frac{\varepsilon B^2}{2} - \frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon+1}} + \frac{1}{\sqrt{1+B^2}}\Big] \leq \frac{3\gamma R}{1-\gamma}. \end{aligned}$$

Therefore,

$$v(x,t) \leq \frac{3\gamma R}{1-\gamma} \quad \text{in } \ \{\omega(R_0) \times [0,T]\} \cap \{r \geq R_0(1+\varepsilon^{1/4})\}. \tag{4.7}$$

Now we take the viscosity limit $\varepsilon \to 0$ in (4.7) and obtain the estimate

$$u(x,t) - \big(1 - \frac{1}{R}\big)t \leq \frac{3\gamma R}{1-\gamma} \tag{4.8}$$

in $\omega(R_0) \times [0,T]$. Noting that the constant is independent of T, we conclude that (4.8) holds even in $\omega(R_0) \times [0,\infty)$.

Using (4.7) we can repeat the above calculations for $v(x,t)$ and $w(x) = h(r(x) - R_1) + 3\gamma R/(1-\gamma)$ in the domain $\omega_1(R_1)$. Now $r(x)$ is the distance to (R_1, R_1), and the only other change is that the barrier function h is used in a shorter interval $[0, R\gamma((2-\sqrt{2})/2)^2]$. Therefore, we obtain the estimate

$$v(x,t) = u^\varepsilon(x,t) - \big(1 - \frac{1}{R}\big) \leq \frac{6\gamma R}{1-\gamma} \quad \text{in } \ \omega_1(R_1) \times [0,T].$$

Analogously, for $k \in \mathbb{N}$, we can use $h(r(x) - R_k) + 3k\gamma R/(1-\gamma)$ in $\omega_k(R_k)$, with the argument of h in $[0, R\gamma((2-\sqrt{2})/2)^{k+1}]$ and obtain

$$v(x,t) = u^\varepsilon(x,t) - \big(1 - \frac{1}{R}\big)t \le 3(k+1)\frac{\gamma R}{1-\gamma} \quad \text{in } \ \omega_k(R_k) \times [0,T].$$

After the viscosity limit $\varepsilon \to 0$ we arrive at

$$u(x,t) \le \big(1 - \frac{1}{R}\big)t + \frac{3(k+1)\gamma R}{(1-\gamma)} \quad \text{in } \ \omega_k(R_k) \times [0,\infty). \tag{4.9}$$

Now we observe that any point $x \in \omega(R)$ can be caught in a finite union $\cup_{k=0}^N \omega_k(R_k)$, because γ can be made arbitrarily close to 1. So the function $\overline{w}_R$ in Lemma 4.2 will depend on γ and R, but this proves the first statement of Lemma 4.2.

To be precise, in the iteration process above we have to use the radii

$$\begin{aligned} R_{k+1} &= \frac{\gamma R\sqrt{2}}{2} + R_k \frac{2-\sqrt{2}-\sqrt{2}\varepsilon^{1/4}}{2} \\ &= \gamma R(1+\varepsilon^{1/4})^{-1}\big[1 - \big(\frac{2-\sqrt{2}-\sqrt{2}\varepsilon^{1/4}}{2}\big)^{k+2}\big], \end{aligned}$$

but the small ε perturbation is insignificant and so we omit it.

The proof of (4.1), the second statement of Lemma 4.2, follows immediately from Theorem 1.1 in [16]. Indeed, the function $u^\varepsilon(x,t) - (1 - \frac{2+\sqrt{\pi}}{a} + \delta)t$ solves the boundary value problem

$$\begin{aligned} v_t - \varepsilon\Delta v - \operatorname{div}\big(\frac{Dv}{\sqrt{1+|Dv|^2}}\big) &= \frac{2+\sqrt{\pi}}{a} - \delta \qquad \text{in } \ K \times (0,\infty), \\ v(x,0) &= 0 \qquad \text{in } \ K, \\ v(x,t) &= -(1 - \frac{2+\sqrt{\pi}}{a} + \delta)t \quad \text{on } \ \partial K \times (0,\infty), \end{aligned} \tag{4.10}$$

for every fixed positive constant δ. According to [16, Theorem 1.1] the equation

$$\operatorname{div}\big(\frac{Dw_\delta}{\sqrt{1+|Dw_\delta|^2}}\big) = \frac{\delta}{2} - \frac{2+\sqrt{\pi}}{a} \qquad \text{in } \ K \tag{4.11}$$

(without boundary condition) has a classical bounded solution $w_\delta(x)$, since

$$\Big|\int_A \big(\frac{\delta}{2} - \frac{2+\sqrt{\pi}}{a}\big)\,dx\Big| = \big(\frac{2+\sqrt{\pi}}{a} - \frac{\delta}{2}\big)|A| < |\partial A|$$

for every Cacciopoli set $A \subset K$. To realize this, we just have to recall the isoperimetric problem (1.5) in the case of a square. There we know that

$$\min_{A\subset K} \frac{|\partial A|}{|A|} = \frac{2+\sqrt{\pi}}{a} > \frac{2+\sqrt{\pi}}{a} - \frac{\delta}{2}$$

for every Caccioppoli set $A \subset K$. So for $\varepsilon < \varepsilon_0$ sufficiently small, such that $\varepsilon_0 |\Delta w_\delta| < \delta/2$, the function $w_\delta(x) - \min_{\overline{K}} w_\delta$ is a classical supersolution of Problem (4.10), and hence

$$u^\varepsilon(x,t) - (1 - \frac{2+\sqrt{\pi}}{a} + \delta)\ t \leq \operatorname{osc}\ w_\delta(x) = C_\delta \qquad \text{for} \ \ x \in K,\ t \in [0,\infty)$$

implies (4.1) after the viscosity limit. This completes the proof of Lemma 4.2. □

In the following lemma we shall prove that the growth estimates of Lemma 4.2 are optimal, since the solution $u(x,t)$ of (1.1)–(1.2) has maximal growth $(1-(2+\sqrt{\pi})/a)\ t$ for fixed $x \in \Omega_*$, moderate growth $(1-1/R)\ t$ for $x \in \Omega(R)$, $1 \leq R \leq \frac{a}{2+\sqrt{\pi}}$ (where $\Omega(R)$ is the "square with round corners" of radius R), and $u(x,t)$ remains uniformly bounded in t for $x \in \omega(1)$.

Lemma 4.5. *For every $R \in (1, a/(2+\sqrt{\pi})]$ the following estimate holds for the solution $u(x,t)$ of* (1.1)–(1.2):

$$u(x,t) \geq \big(1 - \frac{1}{R}\big)t + \underline{w}_R(x) \qquad \text{for} \ \ x \in \overline{\Omega(R)},\ t \geq 0, \tag{4.12}$$

with a function $\underline{w}_R$ that is locally finite in $\Omega(R)$.

Remark 4.6. As predicted by Marcellini and Miller, the maximal speed is $(1-(2+\sqrt{\pi})/a)$, and it is only attained on the solution $\Omega_* = \Omega(a/(2+\sqrt{\pi}))$ of (1.5).

Remark 4.7. It is interesting to note that, in contrast to the radial case, $u(x,t)$ is no longer concave in x for every fixed t.

For the proof of (4.12) we shall first treat the case $R = a/(2+\sqrt{\pi})$, where $\Omega(R) = \Omega_*$. Simple computations give us that the function $v(x,t) = u(x,t) - (1-(2+\sqrt{\pi})/a)t$ is a solution of the following problem in $\Omega_* \times (0,\infty)$:

$$\begin{aligned}
v_t - \operatorname{div}\left(\frac{Dv}{\sqrt{1+|Dv|^2}}\right) &= \frac{2+\sqrt{\pi}}{a} && \text{in} \ \ \Omega_* \times (0,\infty),\\
v(x,0) &= 0 && \text{in} \ \ \Omega_*,\\
v(x,t) &= \frac{2+\sqrt{\pi}-a}{a}\ t && \text{on} \ \ (\partial\Omega_* \cap \partial K) \times (0,\infty).
\end{aligned} \tag{4.13}$$

Notice that no Dirichlet data are given on $\partial\Omega_* \cap K$. What can we say about solutions of (4.13)? We can construct such a solution more or less explicitly by choosing $\delta = 0$ in (4.11). In this case a solution w of (4.11) still exists in Ω_*, but it is no longer smooth everywhere on the boundary. In fact, it follows from Theorems 1.1, 2.1, 3.1 and 3.2 in [16], that $w(x) \in C^\infty(\Omega_*)$, that w is vertical on the fixed boundary $\partial K \cap \partial\Omega_*$ and $w \to -\infty$ on the free boundary $K \cap \partial\Omega_*$.

Now we shall prove that $W(x,t) = w(x) - \sup_{x\in K} w(x)$ is a weak viscosity subsolution of (4.13). In fact, $W(x,t)$ is a classical solution of the differential equation and is nonpositive at time $t = 0$. Moreover, on the free boundary $(K \cap \partial\Omega_*) \times [0,\infty)$ we have the inequality $W(x,t) = -\infty < (2+\sqrt{\pi}-a)t/a = v(x,t)$. As in the proof of Lemma

3.6 the verification of boundary conditions on the fixed boundary $(\partial K \cap \partial\Omega_*) \times [0,\infty)$ becomes superfluous, since there the superjet $\mathcal{P}^{2,+}_{\Omega_* \times [0,\infty)} W(x,t)$ is empty and there is nothing to check in (2.7). Another way to get the desired comparison is via Lemma 3.1 above.

The proof of (4.12) for the remaining values of $R \in [1, a/(2+\sqrt{\pi}))$ is essentially the same, the difference being the domain $\Omega_*(R)$ on which we consider Giusti's extremal solution w. In this case $\Omega_*(R)$ will be constructed as follows. First we modify Ω_* by changing the circular arc in the southwest corner to an arc of radius R and center (R,R). Let A be the square with rounded corners of radius R near the origin, and with radius $a/(2+\sqrt{\pi})$ otherwise; see Figure 4.2.

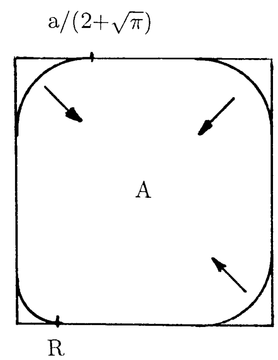

Figure 4.2: The set A.

Then we know

$$\min_{G\subset A} \frac{|\partial G|}{|G|} = \frac{|\partial\Omega_*|}{|\Omega_*|} = \frac{2+\sqrt{\pi}}{a} < \frac{1}{R}\,.$$

Let B be the ball of radius R and center (R,R). Then from Theorem 2.3 we have

$$\min_{G\in B} \frac{|\partial G|}{|G|} = \frac{|\partial B|}{|B|} = \frac{2}{R} > \frac{1}{R}\,.$$

If we connect A and B in a monotone way, then there will be a domain $\Omega_*(R)$ such that

$$\inf_{G\subset\Omega_*(R)} \frac{|\partial G|}{|G|} = \frac{1}{R}\,. \tag{4.14}$$

Let us indicate how to get from A to B. The boundaries of A and B coincide in the circular arc near the origin. Starting from A let us replace the other circular arcs of radius $a/(a+\sqrt{\pi})$ monotonically by those of radius $r \in (a/(a+\sqrt{\pi}), a/2)$ first. This way we arrive in a monotonic way at a domain C whose boundary consists of a 3/4-circle with radius $a/2$, two straight segments $\{0\} \times (R, \frac{a}{2})$ and $(R, \frac{a}{2}) \times \{0\}$, and the circular arc of radius R; see Figure 4.3.

Now we move the center of the 3/4-circle along the diagonal and shrink its radius, so that it remains tangent to to the coordinate axes, until we reach B. This procedure is illustrated in Figure 4.4.

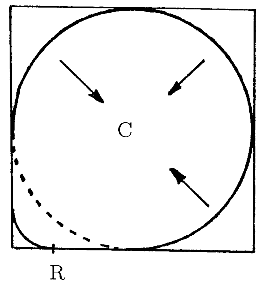

Figure 4.3: The set C.

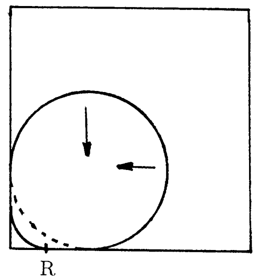

Figure 4.4: Approaching the set B.

Now that we have implicitly constructed $\Omega_*(R)$, let us look for the solution of problem (4.14). We claim that $G = \Omega_*(R)$ is itself a solution. If not, a minimizing domain would have part of its boundary in the interior of $\Omega_*(R)$ with constant curvature $1/R$ and this free boundary would tangentially touch $\partial\Omega_*(R)$, a contradiction of the fact that $\partial\Omega_*(R)\backslash\gamma(R)$ has curvature zero or $1/r < 1/R$.

Let us now consider the solution $v(x,t) = u(x,t) - (1 - \frac{1}{R})t$ of the problem

$$\begin{aligned} v_t - \operatorname{div}\Big(\frac{Dv}{\sqrt{1+|Dv|^2}}\Big) &= \frac{1}{R} && \text{in } \Omega_*(R) \times (0,\infty), \\ v(x,0) &= 0 && \text{in } \Omega_*(R) \\ v(x,t) &= \frac{1-R}{R}\, t && \text{on } (\partial\Omega_*(R) \cap \partial K) \times (0,\infty). \end{aligned}$$

We can then repeat all the arguments that were already used for a discussion of (4.13) to complete the proof of Lemma 4.5. $\square$

Finally we are in a situation to prove Theorem 2.5 from Lemmata 3.1, 3.4 and 4.2–4.5. The existence part of Theorem 2.5 can be obtained after passing to the limit $\limsup^* u^\varepsilon$ (see (2.5)) as $\varepsilon \to 0$ with solutions u^ε of the regularized problem (1.9). Since from Lemma 3.4 the convergence of u^ε is uniform on compact subdomains of $K \times [0,\infty)$ we obtain that the limit function $u(x,t) = \lim_{\varepsilon\to 0} \sup^* u^\varepsilon(x,t)$ is of class C^∞ (in fact even real analytic) and satisfies (1.1) at every interior point of $K \times [0,\infty)$.

Once we have shown that u is continuous, the uniqueness follows from Lemma 3.1, which is a comparison lemma. Thus it remains to prove inequalities (2.9) and (2.10) up to the boundary and the continuity of $u(x,t)$ in $\overline{K} \times [0,\infty)$. Then (2.9) will guarantee the detachment of the solution u from the classical zero boundary data.

To prove continuity of u we use monotonicity properties of u^ε and show that they are preserved under the limit $\varepsilon \to 0$. This way Proposition 4.3 holds for $u(x,t)$. In the interior of K the preservation of monotonicity follows from the uniform convergence of u^ε to u. So let $P = (p, 0, T)$ and $Q = (q, 0, T)$ be two points on the boundary, $0 < p < q < a/2$, $T > 0$. We shall prove that $u(P) \le u(Q)$. From Proposition 4.3 we know that $u^\varepsilon(p, x_2, t) \le u^\varepsilon(q, x_2, t)$ and that the function $u^\varepsilon(x,t)$ attains its maximum over the cylinder $[p - \varepsilon/2, p + \varepsilon/2] \times [0, \varepsilon] \times [0, T]$ at the point $P^\varepsilon = (p + \varepsilon/2, \varepsilon, T)$. Correspondingly the maximum of u^ε on $[q - \varepsilon/2, q + \varepsilon/2] \times [0, \varepsilon] \times [0, T]$ is attained in $Q^\varepsilon = (q + \varepsilon/2, \varepsilon, T)$. Hence, if the neighborhoods in (2.5) are taken to be cylinders, we have that $u(P) = \lim_{\varepsilon \to 0} \sup_{0<\delta<\varepsilon} u^\delta(P^\varepsilon) \le \lim_{\varepsilon \to 0} \sup_{0<\delta<\varepsilon} u^\delta(Q^\varepsilon) = u(Q)$. In a similar way one can prove monotonicity in time, so that Proposition 4.3 holds for $u(x,t)$ as well.

Now we can prove continuity of u. By definition $u(x,t)$ is upper semicontinuous in $\overline{K} \times [0, \infty)$, and since it is decreasing along outer normals to the boundary, its boundary value is well defined. Moreover $u(x,t)$ is continuous in Ω and on $\partial\Omega$ it is continuous in directions normal to the boundary.

On the boundary $u(x,t)$ is also Lipschitz continuous in t since u_t^ε is uniformly bounded in t and $u^\varepsilon(x,t)$ is monotone increasing in t. It remains to discuss continuity of u in space directions tangent to the boundary. Let $P = (p, 0, T)$ and $Q = (q, 0, T)$ be two boundary points with $0 < p < q \le \frac{a}{2}$ and let R be the point $(p, q - p, T)$; see Figure 4.5.

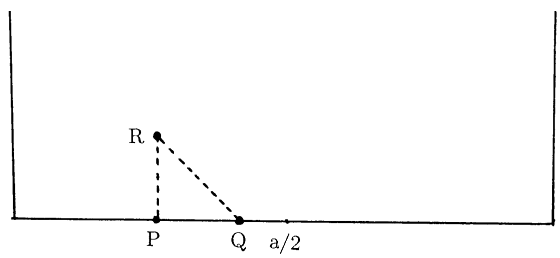

Figure 4.5. On continuity along $\partial\Omega$.

For reasons of symmetry it suffices to study u on this part of the boundary. If $\tau(\cdot)$ is the modulus of continuity of $u(p, \cdot, T)$ in the direction normal to $\partial\Omega$, from the monotonicity of u^ε in diagonal directions (which follows from Proposition 4.3 after rotation by $\pi/2$) we can argue

$$0 \le u(Q) - u(P) \le u(R) - u(P) \le (\tau|R - P|).$$

This proves the continuity of $u(x,t)$.

In order to complete the proof of our second main result let us finally prove local Lipschitz continuity of the trace of a solution on the boundary. We consider in the cylinder $Q_1 = (0, a/2) \times (0, a) \times (0, T]$ the auxiliary function

$$w = x_1^2 D_1 u^\varepsilon(x) - (16 + 2a)t.$$

Easy calculations give us from (4.3) that w satisfies the equation

$$w_t - \varepsilon \Delta w - \frac{(1 + |Du^\varepsilon|^2)\Delta w - D_i u^\varepsilon D_j u^\varepsilon D_{ij} w}{(1 + |Du^\varepsilon|^2)^{3/2}} + A_i \; D_i w = F \quad \text{in} \;\; Q_1,$$

where A_i are smooth bounded functions. The right-hand side F can be estimated in the following way at the point $P = (y_1, y_2, t)$ of possible interior maximum of w over Q_1 as follows, using Proposition 4.4 and $\varepsilon < 1/(8T)$. For brevity of notation we write u_i for $D_i u^\varepsilon$ in the next formula.

$$\begin{aligned} F(P) &= 2\varepsilon u_1 - 16 - 2a + \frac{6u_1(1+u_2^2)(u_1^2-u_2^2-1)}{(1+u_1^2+u_2^2)^{5/2}} + \frac{2u_1^2(2u_2^2-u_1^2-1)}{(1+u_1^2+u_2^2)^{5/2}} \\ &\quad \times \frac{2u_1(1+u_2^2) + 2\varepsilon u_1(1+u_1^2+u_2^2)^{3/2} + y_1(u_t-1)(1+u_1^2+u_2^2)^{3/2}}{1+u_1^2+\varepsilon(1+u_1^2+u_2^2)^{3/2}} \\ &\leq 2\varepsilon u_1 - 1 \leq 2\varepsilon\sqrt{2T/\varepsilon} - 1 < 0. \end{aligned}$$

But this contradicts the parabolic maximum principle, and hence w attains its maximum on the parabolic boundary of Q_1, where $w \leq 0$. Thus $w \leq 0$ in Q_1. But now, by definition of w and using the monotonicity of u^ε, we have $0 \leq D_1 u^\varepsilon(Q) \leq (16+2a)T/(q^2)$ for $Q = (q, x_2, t)$ and $q \in (o, a/2]$, $x_2 \in (0, a]$. Therefore the tangential derivative of the viscosity solution u is locally bounded as well.

Remark 4.8. Numerical experiments suggest that u satisfies the boundary condition $u = 0$ near the corners of K in a classical sense. This question as well as the regularity of the boundary data in points where u fails to satisfy (1.2) in the classical sense will be addressed in a forthcoming paper.

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