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ON MULTIPLE SOLUTIONS FOR THE NONHOMOGENEOUS p-LAPLACIAN WITH A CRITICAL SOBOLEV EXPONENT

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Abstract. We prove the existence of at least two distinct solutions of problem (1), (2) under some restrictions on Q, f and λ. One solution has been obtained by a local minimization and the second one by applying the mountain pass theorem.

1. Introduction. In this paper we consider the Dirichlet problem

$$-\Delta_p u = |u|^{p^*-2}u + \lambda|u|^{q-2}u + f(x) \text{ in } Q, \tag{1}$$

$$u(x) = 0 \text{ on } \partial Q, \tag{2}$$

where $Q \subset \mathbb{R}_n$ is a bounded domain, $1 < p, q < p^* = \frac{np}{n-p}$, $1 < p < n$ and

$$-\Delta_p u = -D_i\big(|\nabla u(x)|^{p-2} D_i u\big).$$

The assumptions on f will be formulated later. Since p^* is a critical Sobolev exponent for which the embedding $W^{1,2}(Q) \subset L^{p^*}(Q)$ is not compact, we encounter serious difficulties in applying variational methods to problem (1), (2).

The case $f \equiv 0$ on Q has an extensive literature. Using the so-called generalized Pohozaev identity, it is possible to prove that, if the domain Q is starshaped, then problem (1), (2), with $f \equiv 0$, cannot have any nontrivial solution in $\overset{\circ}{W}{}^{1,p}(Q)$ if $\lambda \leq 0$. Therefore, in this case we are reduced to considering positive λ. In particular, if $p = q = 2$, Brézis and Nirenberg [4] obtained the following result: let λ_1 be the first eigenvalue for Δ; then (i) if $n \geq 4$, then for every $\lambda \in (0, \lambda_1)$ there exists a positive solution; (ii) if $n = 3$ and Q is a ball, then there exists a positive solution if and only if $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$. These results have been extended by J. Garcia Azorero and I. Peral Alonso [2, 3] to the p-Laplacian. Namely, they proved that (i) if $p = q$ and $p^2 \leq n$, then there exists a nontrivial solution for $0 < \lambda < \lambda_1$; (ii) if $p < q < p^*$, there exists $\lambda_\circ > 0$ such that problem (1), (2) has a nontrivial solution for each $\lambda \geq \lambda_\circ$; (iii) if $(*)$ $\max\big(p, p^* - \frac{p}{p-1}\big) < q < p^*$, then there exists a nontrivial solution of problem (1), (2) for each $\lambda > 0$. We observe that if $n \geq p^2$ and $p < q < p^*$, then $q > p^* - \frac{p}{p-1}$ and $(*)$ holds. On the other hand, if $n < p^2$, then it is easy to check that $(*)$ does not hold.

However, less attention has been given to the nonhomogeneous problem (1), (2). In case $p = q = 2$ and $\lambda = 0$, Tarantello [11] proved the existence of two distinct solutions. A similar result has been obtained by Deng Yingbing [6] for equation (1) in $\mathbb{R}_n$ with $\|f\|_2$ small. In [7] Deng Yin-Bing established the existence of at least

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two solutions extending the Ambrosetti-Prodi result [1] for nonlinearities with the limiting Sobolev exponent and crossing the first eigenvalue.

The main objective of this work is to establish the existence of at least two distinct solutions of problem (1), (2) with $f \not\equiv 0$. Using P.L. Lions [10] concentration-compactness principle we prove that a local version of the Palais-Smale condition holds. One solution is obtained by a local minimization and the second solution is a consequence of the mountain pass-theorem without the Palais-Smale condition [4]. We distinguish two cases: (i) $p \le q < p^*$ and (ii) $q < p$. If $p < q < p^*$, then the local minimization leads to a solution for every $\lambda > 0$ provided norm of f is small. In case $q \le p$ the situation is different and we obtain a solution either for $\lambda > 0$ large or for a domain Q of small measure. A second solution in case $p < q$ is obtained for $\lambda > 0$ large and in case $q \le p$ we show that our method is inconclusive.

Finally, we point out that methods adopted in this work intertwine with methods used in papers [6], [7] and [11]. However, unlike in these papers we make a significant use of the concentration-compactness principle [10].

2. Local (restricted) version of the Palais–Smale condition. By S we denote the best Sobolev constant

$$S = \inf\{\|u\|_{\overset{\circ}{W}{}^{1,p}(Q)} : u \in \overset{\circ}{W}{}^{1,p}(Q),\ \|u\|_{p^*} = 1\}.$$

To establish a local version of the Palais-Smale condition we need the concentration-compactness principle due to P.L. Lions [10].

Lemma 1. *Let $\{u_j\}$ be a weakly convergent sequence in $\overset{\circ}{W}{}^{1,p}(Q)$ to a function u and such that*

(i) $|\nabla u_j|^p$ converges weak-$*$ to a measure μ,
(ii) $|u_j|^{p^*}$ converges weak-$*$ to measure ν.

Then there exist an at most countable index set J, positive constants $\{\nu_j\}$, $\{\mu_j\}$, $j \in J$, and a collection of points $\{x_j\}$, $j \in J$, in $\bar{Q}$ such that for all $j \in J$,

$$\nu = |u|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j}, \tag{3}$$

$$\mu \ge |\nabla u|^p + \sum_{j\in J} \mu_j \delta_{x_j}, \tag{4}$$

$$\nu_j^{\frac{p}{p^*}} \le \frac{\mu_j}{S}. \tag{5}$$

To proceed further, we denote by $F : \overset{\circ}{W}{}^{1,p}(Q) \to \mathbb{R}$ a functional defined by

$$F(u) = \frac{1}{p}\int_Q |\nabla u|^p\,dx - \frac{1}{p^*}\int_Q |u|^{p^*}\,dx - \frac{\lambda}{q}\int_Q |u|^q\,dx - \int_Q fu\,dx.$$

Since $\overset{\circ}{W}{}^{1,p}(Q) \subset L^{p^*}(Q)$ continuously and $\overset{\circ}{W}{}^{1,p}(Q) \subset L^q(Q)$ compactly, this functional is well defined. Critical points of the functional F are solutions of problem (1), (2).

To formulate a local version of the Palais-Smale condition we shall use constants $K_1 \geq 0$, $K_2 > 0$ and $K_3 > 0$ determined in the following manner:

$$K_1 = K_1(n, p, \|f\|_{p^{*\prime}}) = \frac{n^{\frac{p^{*\prime}}{p^*}}}{(p')^{p^{*\prime}} p^{*\prime} (p^*)^{\frac{p^{*\prime}}{p^*}}} \|f\|_{p^{*\prime}}^{p^{*\prime}},$$

where $\frac{1}{p^{*\prime}} + \frac{1}{p^*} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \not\equiv 0$. To define constants K_2 and K_3 we introduce a function $g : [0, \infty) \to \mathbb{R}$ defined by

$$g(t) = \frac{1}{n} t^{p^*} + \lambda\big(\frac{1}{p} - \frac{1}{q}\big)|Q|^{\frac{p^*-q}{p^*}} t^q - \big(1 - \frac{1}{p}\big)\|f\|_{p^{*\prime}} t,$$

where $|Q|$ denotes the Lebesgue measure of Q. If $\lambda > 0$ and $1 < q < p$ we set

$$-K_2 = -K_2(\lambda, n, p, q, |Q|, \|f\|_{p^{*\prime}}) = \inf_{t \geq 0} g(t) < 0.$$

Similarly, if $\lambda < 0$ and $p < q < p^*$ we set

$$-K_3 = -K_3(\lambda, n, p, q, |Q|, \|f\|_{p^{*\prime}}) = \inf_{t \geq 0} g(t) < 0.$$

It is clear that for $\lambda > 0$ and $p < q$ we have

$$K_2(\lambda, n, p, q, |Q|, \|f\|_{p^{*\prime}}) = K_3(-\lambda, n, q, p, |Q|, \|f\|_{p^{*\prime}}).$$

We observe that for a fixed $\lambda > 0$ we have

$$\lim_{|Q| \to 0, \|f\|_{p^{*\prime}} \to 0} K_2(\lambda, n, p, q, |Q|, \|f\|_{p^{*\prime}}) = 0. \tag{6}$$

Also, for a fixed bounded domain $Q \subset \mathbb{R}_n$, we have

$$\lim_{\lambda \to 0, \|f\|_{p^{*\prime}} \to 0} K_2(\lambda, n, p, q, |Q|, \|f\|_{p^{*\prime}}) = 0. \tag{7}$$

Proposition 1. *Let $f \in L^{p^{*\prime}}(Q)$; then*

(*i*) ($1 < p \leq q < p^*$, $\lambda > 0$) *Every sequence $\{u_j\} \subset \mathring{W}^{1,p}(Q)$ satisfying*

$$F(u_j) \to C < \frac{1}{n} S^{\frac{n}{p}} - K_1 \text{ and } F'(u_j) \to 0 \text{ in } W^{-1,p'}(Q)$$

as $j \to \infty$, has a subsequence converging in $\mathring{W}^{1,p}(Q)$.

(*ii*) ($1 < q < p$, $\lambda > 0$) *Every sequence $\{u_j\} \subset \mathring{W}^{1,p}(Q)$ satisfying*

$$F(u_j) \to C < \frac{1}{n} S^{\frac{n}{p}} - K_2 \text{ and } F'(u_j) \to 0 \text{ in } W^{-1,p'}(Q)$$

as $j \to \infty$, has a subsequence converging in $\mathring{W}^{1,p}(Q)$.

(*iii*) ($1 < p < q < p^*$, $\lambda < 0$) *Every sequence* $\{u_j\} \subset \overset{\circ}{W}{}^{1,p}(Q)$ *satisfying*

$$F(u_j) \to C < \frac{1}{n} S^{\frac{n}{p}} - K_3 \ \ and \ \ F'(u_j) \to 0 \ \ in \ \ W^{-1,p'}(Q)$$

as $j \to \infty$, *has a subsequence converging in* $\overset{\circ}{W}{}^{1,p}(Q)$.

(*iv*) ($1 < q < p$, $\lambda < 0$) *Every sequence* $\{u_j\} \subset \overset{\circ}{W}{}^{1,p}(Q)$ *satisfying*

$$F(u_j) \to C < \frac{1}{n} S^{\frac{n}{p}} - K_1 \ \ and \ \ F'(u_j) \to 0 \ \ in \ \ W^{-1,p'}(Q)$$

as $j \to \infty$, *has a subsequence converging in* $\overset{\circ}{W}{}^{1,p}(Q)$.

(*v*) ($1 < p, q < p^*$, $\lambda = 0$) *Every sequence* $\{u_j\} \subset \overset{\circ}{W}{}^{1,p}(Q)$ *satisfying*

$$F(u_j) \to C < \frac{1}{n} S^{\frac{n}{p}} - K_1 \ \ and \ \ F'(u_j) \to 0 \ \ in \ \ W^{-1,p'}(Q)$$

as $j \to \infty$, *has a subsequence converging in* $\overset{\circ}{W}{}^{1,p}(Q)$.

Proof. We follow the ideas from the paper [3]. We only show (*i*) and (*ii*), since the proofs of the remaining cases are identical.
(*i*)$1 < p \leq q < p^*$.

It is easy to show that $\{u_m\}$ is bounded in $\overset{\circ}{W}{}^{1,p}(Q)$. Therefore, we may assume that $u_j \to u$ weakly in $\overset{\circ}{W}{}^{1,p}(Q)$, $u_j \to u$ in $L^p(Q)$ and $L^q(Q)$ and almost everywhere on Q. According to Lemma 1, we may assume that (3), (4) and (5) hold. We show that $\nu_k = 0$ for all $k \in J$. To show this, let $x_k \in \bar{Q}$ be in the support of the singular part of μ and ν. We define a function $\phi \in C^1_\circ(\mathbb{R}_n)$ such that $\phi(x) \equiv 1$ on $B(x_k, \epsilon)$, $\phi(x) = 0$ on $\mathbb{R}_n - B(x_k, 2\epsilon)$ and $|\nabla\phi(x)| \leq \frac{2}{\epsilon}$ on $\mathbb{R}_n$. Since

$$\lim_{j\to\infty} \langle F'(u_j), \phi u_j \rangle = 0,$$

$$\int_Q \phi \, d\nu + \lambda \int_Q |u|^q \phi \, dx + \int_Q f\phi u \, dx - \int_Q \phi \, d\mu = \lim_{j\to\infty} \int_Q |\nabla u_j|^{p-2} u_j (\nabla u_j, \nabla\phi) \, dx.$$

Applying the Hölder inequality we get

$$\lim_{j\to\infty} \int_Q |\nabla u_j|^{p-2} |u_j (\nabla u_j, \nabla\phi)| \, dx \leq C\Big(\int_{B(x_k, 2\epsilon)} |u|^{p^*} \, dx \Big)^{\frac{1}{p^*}} \to 0$$

as $\epsilon \to 0$ and hence

$$0 = \lim_{\epsilon\to 0} \Big(\int_Q \phi \, d\nu + \lambda \int_Q |u|^q \, dx + \int_Q f\phi u \, dx - \int_Q \phi \, d\mu \Big) = \nu_k - \mu(\{x_k\}).$$

By (5) of Lemma 1 we have $\mu(\{x_k\}) \geq \mu_k \geq S\nu_k^{\frac{p}{p^*}}$, that is, $\nu_k \geq S\nu_k^{\frac{p}{p^*}}$. This means that either (*i*) $\nu_k = 0$ or (*ii*) $\nu_k \geq S^{\frac{n}{p}}$. Also, since ν is a bounded measure there exists a finite number of singular points $\{x_k\}$, $k \in J$. We show that (ii) is not possible. Assume that $\nu_{k_\circ} \neq 0$ for some $k_\circ$, that is $\nu_{k_\circ} \geq S^{\frac{n}{p}}$. It follows from (3) and (4) that

$$C = \lim_{j\to\infty} F(u_j) \geq F(u) + \big(\frac{1}{p} - \frac{1}{p^*}\big) \sum \nu_k \geq F(u) + \frac{1}{n} S^{\frac{n}{p}},$$

so

$$F(u) \leq C - \frac{1}{n} S^{\frac{n}{p}} < \frac{1}{n} S^{\frac{n}{p}} - K_1 - \frac{1}{n} S^{\frac{n}{p}} < 0,$$

and this implies that $u \not\equiv 0$. We now observe that

$$\begin{aligned} C = \lim_{j\to\infty} \big[F(u_j) - \frac{1}{p}\langle F'(u_j), u_j\rangle\big] &\geq \frac{1}{n}\int_Q |u|^{p^*}\,dx + \frac{1}{n} S^{\frac{n}{p}} \\ &+ \lambda\big(\frac{1}{p} - \frac{1}{q}\big)\int_Q |u|^q\,dx - \big(1 - \frac{1}{p}\big)\int_Q fu\,dx. \end{aligned} \tag{8}$$

Given $\epsilon > 0$ satisfying $\frac{\epsilon^{p^*}}{p^*} = \frac{1}{n}$, we get, by the Hölder inequality, that

$$\big(1 - \frac{1}{p}\big)\int_Q fu\,dx \leq \frac{1}{(p')^{p^{*\prime}} p^{*\prime} \epsilon^{p^{*\prime}}} \int_Q |f|^{p^{*\prime}}\,dx + \frac{\epsilon^{p^*}}{p^*}\int_Q |u|^{p^*}\,dx. \tag{9}$$

Since $u \not\equiv 0$, we deduce from (8) and (9) that

$$C > \frac{1}{n} S^{\frac{n}{2}} - \frac{1}{(p')^{p^{*\prime}} \epsilon^{p^{*\prime}} p^{*\prime}} \|f\|_{p^{*\prime}}^{p^{*\prime}}$$

and we arrive at a contradiction. This means that $\nu_k = 0$ for each k and

$$\lim_{j\to\infty}\int_Q |u_j|^{p^*}\,dx = \int_Q |u|^{p^*}\,dx.$$

Consequently, by the uniform convexity of the space $L^{p^*}(Q)$ we get that $\lim_{j\to\infty} u_j = u$ in $L^{p^*}(Q)$. Now the continuity of the operator Δ_p^{-1} implies that $u_j \to u$ in $\overset{\circ}{W}{}^{1,p}(Q)$.
$(ii) 1 < q < p$.
We proceed as before to arrive at inequality (8). Applying the Hölder inequality we get

$$\begin{aligned} C &\geq \frac{1}{n}\int_Q |u|^{p^*}\,dx + \frac{1}{n} S^{\frac{n}{p}} + \lambda\big(\frac{1}{p} - \frac{1}{q}\big)|Q|^{\frac{p^*-q}{p^*}}\Big(\int_Q |u|^{p^*}\,dx\Big)^{\frac{q}{p^*}} \\ &\quad - \big(1 - \frac{1}{p}\big)\Big(\int_Q |f|^{p^{*\prime}}\,dx\Big)^{\frac{1}{p^{*\prime}}}\Big(\int_Q |u|^{p^*}\,dx\Big)^{\frac{1}{p^*}} \\ &= \frac{1}{n} S^{\frac{n}{p}} + g(\|u\|_{p^*}). \end{aligned}$$

Hence,

$$C \geq \frac{1}{n} S^{\frac{n}{p}} - K_2$$

and we arrive at a contradiction. The rest of the proof is the same as in (i).

3. Local minimization. The aim of this section is to show that F has a negative infimum on a small ball in $\overset{\circ}{W}{}^{1,p}(Q)$. This will allow us, using the Ekeland's variational principle, to deduce the existence of a solution to problem (1), (2).

To achieve this we need the following inequality from the monograph by Ladyzhenskaya [9] (p. 45); namely, there exists a constant $c > 0$ depending only on n, p and q such that

$$\|u\|_q \le c|Q|^{\frac{1}{q}-\frac{1}{p^*}}\|Du\|_p \tag{10}$$

for $1 \le q \le \frac{np}{n-p} = p^*$. If $q = p^*$, we set $c = S^{-1}$ (the best Sobolev constant).

Lemma 2 $(1 < p < q < p^*)$**.** *For every $\lambda \in \mathbb{R}$ there exist constants*

$$\rho_\circ = \rho_\circ(\lambda, p, q, n, S, c, |Q|) > 0 \quad \textit{and} \quad M = M(\lambda, p, q, n, S, c, |Q|) > 0$$

such that

$$\textit{if } \|f\|_{p'} \le M, \ \textit{ then } F(u) \ge 0 \ \textit{ for all } \|u\|_{W^{1,p}} = \rho_\circ.$$

The constants $\rho_\circ$ and M depend on λ only for $\lambda \in (0, \infty)$ and are independent of q for $\lambda \le 0$.

Proof. Let $t^p = \int_Q |Du(x)|^p\,dx$ and $\lambda > 0$. It follows from (10) and the Hölder inequality that

$$\begin{aligned} F(u) &\ge \frac{1}{p}\int_Q |Du|^p\,dx - \frac{1}{p^*}S^{-p^*}\Big(\int_Q |Du|^p\,dx\Big)^{\frac{p^*}{p}} \\ &\quad - \frac{\lambda}{q}c^q|Q|^{1-\frac{q}{p^*}}\Big(\int_Q |Du|^p\,dx\Big)^{\frac{q}{p}} - \|f\|_{p'}c|Q|^{\frac{1}{p}-\frac{1}{p^*}}\Big(\int_Q |Du|^p\,dx\Big)^{\frac{1}{p}} \\ &= t\big\{t^{p-1}\big[\frac{1}{p} - \frac{1}{p^*}S^{-p^*}t^{p^*-p} - \frac{\lambda}{q}c^q|Q|^{1-\frac{q}{p^*}}t^{q-p}\big] - \|f\|_{p'}c|Q|^{\frac{1}{p}-\frac{1}{p^*}}\big\} \\ &= t\big(t^{p-1}h(t) - \|f\|_{p'}c|Q|^{\frac{1}{p}-\frac{1}{p^*}}\big). \end{aligned} \tag{11}$$

We now choose $\rho_\circ > 0$ so that $h(\rho_\circ) > 0$ and the result follows with

$$M = \rho_\circ^{p-1}h(\rho_\circ)c^{-1}|Q|^{\frac{1}{p^*}-\frac{1}{p}}.$$

If $\lambda < 0$, then for $t^p = \int_Q |Du(x)|^p\,dx$ we have

$$\begin{aligned} F(u) &\ge \frac{1}{p}\int_Q |Du|^p\,dx - \frac{1}{p^*}S^{-p^*}\Big(\int_Q |Du|^p\,dx\Big)^{\frac{p^*}{p}} - c\|f\|_{p'}|Q|^{\frac{1}{p}-\frac{1}{p^*}}\Big(\int_Q |Du|^p\,dx\Big)^{\frac{1}{p}} \\ &= t\big\{t^{p-1}\big[\frac{1}{p} - \frac{1}{p^*}S^{-p^*}t^{p^*-p}\big] - c\|f\|_{p'}|Q|^{\frac{1}{p}-\frac{1}{p^*}}\big\} \end{aligned}$$

and the second claim is obvious.

Lemma 3 $(1 < p = q < p^*)$**.** *(i) For $\lambda > 0$ satisfying*

$$\lambda c^p|Q|^{1-\frac{p}{p^*}} < 1, \tag{12}$$

there exist constants $\rho_\circ = \rho_\circ(\lambda, p, n, S, c, |Q|) > 0$ and $M = M(\lambda, p, n, S, c, |Q|) > 0$ such that if $\|f\|_{p'} \le M$, then $F(u) \ge 0$ for all $\|u\|_{W^{1,p}} = \rho_\circ$.

(ii) If $\lambda \leq 0$, then there exists $\rho = \rho_\circ(p,n,S,c,|Q|) > 0$ and $M = M(p,n,S,c,|Q|) > 0$ such that if $\|f\|_{p'} \leq M$, then $F(u) \geq 0$ for all $\|u\|_{W^{1,p}} = \rho_\circ$.

Proof. As in the proof of Lemma 2 we obtain

$$\begin{aligned} F(u) &\geq t\big\{t^{p-1}\big[\frac{1}{p} - \frac{1}{p^*}S^{-p^*}t^{p^*-p} - \frac{\lambda}{p}c^p|Q|^{1-\frac{p}{p^*}}\big] - \|f\|_{p'}c|Q|^{\frac{1}{p}-\frac{1}{p^*}}\big\} \\ &= t\big(t^{p-1}h_1(t) - \|f\|_{p'}c|Q|^{\frac{1}{p}-\frac{1}{p^*}}\big). \end{aligned}$$

Since $\lambda \geq 0$ satisfies (12) we choose $\rho_\circ > 0$ such that $h_1(\rho_\circ) > 0$ and the first claim easily follows. The second claim follows from the estimate

$$F(u) \geq t\big\{t^{p-1}\big[\frac{1}{p} - \frac{1}{p^*}S^{-p^*}t^{p^*-p}\big] - \|f\|_{p'}c|Q|^{\frac{1}{p}-\frac{1}{p^*}}\big\}.$$

Lemma 4 $(1 < q < p < p^*)$**.** *(i) For a given $\lambda > 0$ there exist constants $\rho_\circ = \rho_\circ(p,n,S)$, $\delta = \delta(\lambda,p,q,n,S) > 0$ and $M = M(\lambda,p,q,n,S,|Q|) > 0$ such that if $\|f\|_{p'} \leq M$ and $|Q| \leq \delta$, then $F(u) \geq 0$ for all $\|u\|_{W^{1,p}} = \rho_\circ$.*
(ii) If $\lambda \leq 0$, then there exist constants $\rho_\circ = \rho_\circ(p,n,S)$ and $M = M(p,n,S,|Q|) > 0$ such that if $\|f\|_{p'} \leq M$, then $F(u) \geq 0$ for all $\|u\|_{W^{1,p}} = \rho_\circ$.

Proof. Letting $t^p = \int_Q |Du|^p\,dx$ we see that inequality (11) takes the form

$$\begin{aligned} F(u) &\geq t\big\{\big[\frac{t^{p-1}}{p} - \frac{1}{p^*}S^{-p^*}t^{p^*-1} - \frac{\lambda}{q}c^q|Q|^{1-\frac{q}{p^*}}t^{q-1}\big] - \|f\|_{p'}c^p|Q|^{\frac{1}{p}-\frac{1}{p^*}}\big\} \\ &= t\big\{\big[h_2(t) - \frac{\lambda}{q}c^q|Q|^{1-\frac{q}{p^*}}t^{q-1}\big] - \|f\|_{p'}c^p|Q|^{\frac{1}{p}-\frac{1}{p^*}}\big\}. \end{aligned}$$

Since $p^* > p$, we can find $\rho_\circ = \rho_\circ(p,n,S) > 0$, so that $h_2(\rho_\circ) > 0$. Then we choose $\delta = \delta(\lambda,p,q,n,S) > 0$ so that

$$h_2(\rho_\circ) - \frac{\lambda}{q}c^q|Q|^{1-\frac{q}{p^*}}\rho_\circ^{q-1} > 0$$

for $|Q| \leq \delta$. It is now clear that we can find $M = M(\lambda,p,q,n,S,|Q|) > 0$ such that if $\|f\|_{p'} \leq M$ and $|Q| \leq \delta$, then $F(u) \geq 0$ for all $\|u\|_{W^{1,p}} = \rho_\circ$. The second claim follows from the inequality

$$F(u) \geq t\big\{h_2(t) - \|f\|_{p'}c^p|Q|^{\frac{1}{p}-\frac{1}{p^*}}\big\}.$$

Inspection of the proof of Lemma 4 shows that the following version of this lemma holds.

Lemma 5 $(1 < q < p < p^*)$**.** *There exist constants*

$$\rho_\circ = \rho_\circ(p,n,S), \quad \lambda_\circ = \lambda_\circ(p,q,n,S,c,|Q|), \quad M = M(\lambda_\circ,p,q,n,|Q|)$$

such that if $0 \leq \lambda \leq \lambda_\circ$, $\|f\|_{p'} \leq M$, then $F(u) \geq 0$ for all $\|u\|_{W^{1,p}} = \rho_\circ$.

Since $p' > p^*$ we have $L^{p'}(Q) \subset L^{p^{*\prime}}(Q)$. From now on we always assume that $f \in L^{p'}(Q)$.

We are now in a position to establish the first existence result. For ease of notation we set

$$M_1 = \frac{p^{*\prime}\,(p')^{p^{*\prime}}\,(p^*)^{\frac{p^{*\prime}}{p^*}}\,S^{\frac{n}{p}}}{n^{\frac{p^{*\prime}}{p^*}+1}}.$$

In Theorem 1 below we apply Lemma 1. In the case $\lambda < 0$, due to (6), we may assume that $\frac{1}{n}S^{\frac{n}{p}} - K_3 > 0$ by taking M and $|Q|$ smaller if necessary.

Theorem 1 ($1 < p < q < p^*$)**.** *For a given $\lambda \geq 0$ let M and $\rho_\circ$ be constants determined by Lemma 2. Suppose that $f \not\equiv 0$ and that*

$$\max\big(\|f\|_{p'}, \|f\|_{p^{*\prime}}^{p^{*\prime}}\big) < \min\big(M, M_1\big).$$

Then there exists a function $u_\circ \in \overset{\circ}{W}{}^{1,p}(Q)$ such that

$$F(u_\circ) = \min_{\|u\|_{W^{1,p}} < \rho_\circ} F(u) < 0$$

and $u_\circ$ is a solution of problem (1), (2)*. For $\lambda < 0$ the assertion remains true for $|Q|$ and M such that $\frac{1}{n}S^{\frac{n}{p}} - K_3 > 0$.*

Proof. First, we observe that $\|f\|_{p^{*\prime}}^{p^{*\prime}} < M_1$ implies that

$$\frac{1}{n}S^{\frac{n}{p}} - K_1 > 0. \tag{13}$$

Since $f \not\equiv 0$, we can find $\phi \in \overset{\circ}{W}{}^{1,p}(Q)$ such that

$$\int_Q f(x)\phi(x)\,dx > 0$$

and consequently,

$$F(s\phi) = \frac{s^p}{p}\int_Q |\nabla\phi|^p\,dx - \frac{s^{p^*}}{p^*}\int_Q |\phi|^{p^*}\,dx - \frac{\lambda s^q}{q}\int_Q |\phi|^q\,dx - s\int_Q f\phi\,dx < 0$$

for $s > 0$ sufficiently small and hence

$$\inf_{\|u\|_{W^{1,p}} < \rho_\circ} F(u) < 0.$$

Inspection of the proof of Lemma 2 shows that for a given $\epsilon > 0$ there exists $\delta > 0$ such that $F(u) \geq -\epsilon$ for every $u \in \overset{\circ}{W}{}^{1,p}(Q)$ satisfying $\rho_\circ - \delta \leq \|u\|_{W^{1,p}} \leq \rho_\circ$. This observation implies that

$$F(u) \geq \frac{1}{2}\inf_{\|u\|_{W^{1,p}} \leq \rho_\circ} F(u) \tag{14}$$

for all u satisfying $\rho_\circ' \leq \|u\|_{W^{1,p}} < \rho_\circ$ for some $0 < \rho_\circ' < \rho_\circ$. Let $\{u_j\} \subset \overset{\circ}{W}{}^{1,p}(Q)$ be a minimizing sequence for $\inf_{\|u\|_{W^{1,p}} \leq \rho_\circ} F(u)$. By virtue of (14) we may assume that $\{u_j\} \subset \{\|u\|_{W^{1,p}} < \rho_\circ'\}$. Let $\overline{B(0,\rho_\circ)} = \{u \in \overset{\circ}{W}{}^{1,p}(Q) : \|u\|_{W^{1,p}} \leq \rho_\circ\}$ be equipped with a metric dist $(u_1,u_2) = \|u_1 - u_2\|_{W^{1,p}}$ for u_1, u_2 in $\overline{B(0,\rho_\circ)}$. It is clear that $\overline{B(0,\rho_\circ)}$ with dist (u_1,u_2) is a complete metric space. According to Ekeland's variational principle [8] we may assume that every u_j is a minimizer for

$$\inf\{F(u) + \delta_j\|u_j - u\|_{W^{1,p}} : u \in \overline{B(0,\rho_\circ)}$$

for some $\delta_j > 0$ with $\delta_j \to 0$ as $j \to \infty$. This implies that

$$-\Delta_p u_j - \lambda|u_j|^q - |u_j|^{p^*} - f \to 0$$

in $W^{-1,p'}(Q)$ as $j \to \infty$. We now apply Proposition 1(i),(v), with $C \leq 0$, which is possible because of (13). Consequently, $\{u_j\}$ has a subsequence converging to $u_\circ$ in $\overset{\circ}{W}{}^{1,p}(Q)$ and

$$F(u_\circ) = \inf\{F(u) : u \in \overline{B(0,\rho_\circ)}\}.$$

Similarly using Proposition 1(i),(iv),(v) and Lemma 3 we can prove

Theorem 2 ($1 < p = q < p^*$). *Suppose that $\lambda \in \mathbb{R}$ (if $\lambda > 0$ we additionally assume that* (12) *holds) and let M and $\rho_\circ$ be the constants determined by Lemma* 3. *Moreover, suppose that $f \not\equiv 0$ and that*

$$\max\left(\|f\|_{p'}, \|f\|_{p^{*\prime}}^{p^{*\prime}}\right) < \min\left(M, M_1\right).$$

Then there exists $u_\circ \in \overset{\circ}{W}{}^{1,p}(Q)$ such that

$$F(u_\circ) = \min_{\|u\|_{W^{1,p}} < \rho_\circ} F(u) < 0$$

and $u_\circ$ is a solution of problem (1), (2).

In the case $1 < q < p < p^*$ we can obtain the existence result for every $\lambda > 0$ with $|Q|$ small (Lemma 4(i)) or for a given domain Q with $\lambda > 0$ small (Lemma 5). Also, if $\lambda > 0$, we can drop the assumption $f \not\equiv 0$ because F admits a negative term $-\int_Q |u|^q \, dx$.

For a given $\lambda > 0$ let $\rho_\circ$, δ and M be constants determined by Lemma 4. By virtue of (6) we may assume, taking M and δ smaller if necessary, that

$$\frac{1}{n} S^{\frac{n}{p}} - K_2 > 0.$$

Theorem 3 ($1 < q < p < p^*$). (i) *For a given $\lambda > 0$ let M and δ be constants determined by Lemma* 4(i). *Suppose that $\|f\|_{p'} \leq M$ and $|Q| \leq \delta$. Then there exists a function $u_\circ \in \overset{\circ}{W}{}^{1,p}(Q)$ such that*

$$F(u_\circ) = \inf_{\|u\|_{W^{1,p}} < \rho_\circ} F(u) < 0 \tag{15}$$

and $u_\circ$ is a solution of problem (1), (2).
(ii) *For a given $\lambda \leq 0$ let $\rho_\circ$ and M be constants determined by Lemma* 4(ii). *Suppose that $f \not\equiv 0$ and $\|f\|_{p'} \leq M$. Then there exists a function $u_\circ \in \overset{\circ}{W}{}^{1,p}(Q)$ such that* (15) *holds and $u_\circ$ is a solution of* (1), (2).

Proof. We only point out that we can find $v \not\equiv 0$ such that $\int_Q fv \, dx \geq 0$ and since $q < p$ we must have

$$F(sv) = \frac{s^p}{p} \int_Q |Dv|^2 \, dx - \frac{s^{p^*}}{p^*} \int_Q |v|^{2^*} \, dx - \frac{\lambda}{q} s^q \int_Q |v|^q \, dx - s \int_Q fv \, dx < 0$$

for $s > 0$ sufficiently small. $\square$

Finally, let $\lambda_\circ$, $\rho_\circ$ and M be the constants from Lemma 5. Taking into account (7) we may always assume that

$$\frac{1}{n} S^{\frac{n}{p}} - K_2 > 0.$$

This can be achieved by taking $\lambda_\circ$ and M smaller if necessary.

Theorem 4 $(1 < q < p < p^*)$**.** *Let* $\lambda_\circ$, $\rho_\circ$ *and* M *be constants determined by Lemma* 5. *Suppose that* $\|f\|_{p'} \leq M$ *and that* $0 < \lambda < \lambda_\circ$. *Then there exists* $u_\circ \in \overset{\circ}{W}{}^{1,p}(Q)$ *such that*

$$F(u_\circ) = \inf_{\|u\|_{W^{1,p}} < \rho_\circ} F(u) < 0$$

and $u_\circ$ *is a solution of* (1), (2).

4. Existence of a second solution. To obtain the existence of a second solution we apply the mountain-pass theorem without the Palais-Smale condition [4].

Theorem 5. *Let* X *be a Banach space and let* $F \in C^1(X, \mathbb{R})$. *We suppose that there exist constants* $r > 0$ *and* $R > 0$ *such that*

(*i*) $F(u) > r$ *for all* $\|u\| = R$,
(*ii*) $F(0) = 0$ *and* $F(w_\circ) < r$ *for some* $w_\circ$ *with* $\|w_\circ\| > R$.

Let us set

$$\Gamma = \{g \in C([0,1]; X) : g(0) = 0,\, g(1) = w_\circ\}$$

and

$$r \leq d = \inf_{g \in \Gamma} \max_{t \in [0,1]} F(g(t)).$$

Then there exists a sequence $\{u_j\} \subset X$ *such that* $F(u_j) \to d$ *and* $F'(u_j) \to 0$ *in* X^*, *where* X^* *is a dual space of* X.

We commence with the case $1 < p < q < p^*$. It follows from the proof of Lemma 2 that

$$F(u) \geq r > 0 \text{ for } \|u\|_{W^{1,p}} = \rho_\circ$$

for some $\rho_\circ > 0$, by taking M smaller if necessary.

Theorem 6 $(1 < p < q < p^*)$**.** *For a given* $\lambda > 0$ *let* M *and* $\rho_\circ$ *be the constants determined by Lemma* 2. *Suppose that* $f \not\equiv 0$, $\max\big(\|f\|_{p'}, \|f\|_{p^{*\prime}}^{p^{*\prime}}\big) < \min(M, M_1)$. *Then there exists* $\bar\lambda > 0$ *such that for* $\lambda \geq \bar\lambda$ *problem* $(1), (2)$ *admits at least two distinct solutions.*

Proof. We have to show the existence of a solution w which is distinct from a solution $u_\circ$ from Theorem 1. We choose $v_\circ \in \overset{\circ}{W}{}^{1,p}(Q)$ such that $\|v_\circ\|_{p^*} = 1$ and $\int_Q f(x) v_\circ\, dx < 0$. We see that $\lim_{t\to\infty} F(tv_\circ) = -\infty$ and $F(t_\lambda v_\circ) = \sup_{t \geq 0} F(tv_\circ) > 0$ for some $t_\lambda > 0$ which satisfies

$$\begin{aligned} 0 &= t_\lambda^{p-1} \int_Q |\nabla v_\circ|^p\, dx - t_\lambda^{p^*-1} \int_Q |v_\circ|^{p^*}\, dx - \lambda t_\lambda^{q-1} \int_Q |v_\circ|^q\, dx - \int_Q f v_\circ\, dx \\ &= t_\lambda^{q-1} \Big[t_\lambda^{p-q} \int_Q |\nabla v_\circ|^p\, dx - t_\lambda^{p^*-q} - \lambda \int_Q |v_\circ|^q\, dx \Big] - \int_Q f v_\circ\, dx. \end{aligned}$$

Since $t_\lambda^{p^*-q} + \lambda \int_Q |v_\circ|^q\, dx \to \infty$ as $\lambda \to \infty$, we deduce that $t_\lambda \to 0$ as $\lambda \to \infty$. Consequently,

$$\lim_{\lambda\to\infty} \sup_{t \geq 0} F(tv_\circ) = 0$$

and there exists $\bar{\lambda} > 0$ such that

$$\sup_{t\geq 0} F(tv_\circ) < \frac{1}{n} S^{\frac{n}{p}} - K_1$$

for $\lambda \geq \bar{\lambda}$, since by (13) $\frac{1}{n} S^{\frac{n}{p}} - K_1 > 0$. If we now take $w_\circ = tv_\circ$ with t sufficiently large we get $F(w_\circ) < 0$ and hence

$$r \leq d = \inf_{g\in\Gamma} \max_{t\in[0,1]} F(g(t)) \leq \sup_{s\geq 0} F(sv_\circ) < \frac{1}{n} S^{\frac{n}{p}} - K_1,$$

where $\Gamma = \{g \in C([0,1], \overset{\circ}{W}{}^{1,p}(Q)) : g(0) = 0, g(1) = w_\circ\}$. We can now apply Theorem 5 to obtain the existence of a sequence $\{u_j\} \subset \overset{\circ}{W}{}^{1,p}(Q)$ such that $F(u_j) \to d$ and $F'(u_j) \to 0$ in $W^{-1,p'}(Q)$ as $j \to 0$. According to Proposition 1(i), $\{u_j\}$ has a subsequence converging in $\overset{\circ}{W}{}^{1,p}(Q)$ to w with $F(w) = d > 0$. The solution w is distinct from $u_\circ$ because $F(u_\circ) < 0$. □

In the case $1 < q \leq p < p^*$, $|Q|$ depends on λ (see Lemmas 4 and 5) and it is not clear whether, arguing as in Theorem 6, for a given $|Q|$ we can find $\bar{\lambda}$, so that for $\lambda \geq \bar{\lambda}$, $\sup_{t\geq 0} F(tv_\circ)$ is sufficiently small. However, one can suspect that for a fixed $\lambda \in \mathbb{R}$, $\sup_{t\geq 0} F(tv_\circ) \to 0$ as $|Q| \to 0$. We now show that this, in general, is false. Indeed, if $v_\circ \in \overset{\circ}{W}{}^{1,p}(Q)$ is such that $\|v_\circ\|_{p^*} = 1$, then by the Sobolev inequality $S \leq \|\nabla v_\circ\|_p$ and by the Hölder inequality $\|v_\circ\|_q \to 0$ as $|Q| \to 0$ for all $q < p^*$. Let us assume that $\int_Q f v_\circ \, dx \leq 0$ and $p = q$. By the previous remark we can assume that $\int_Q |\nabla v_\circ|^p \, dx - \lambda \int_Q |v_\circ|^p \, dx > 0$. Then

$$F(tv_\circ) \geq F_1(tv_\circ) \equiv \frac{t^p}{p} \int_Q |\nabla v_\circ|^p \, dx - \lambda \frac{t^p}{p} \int_Q |v_\circ|^p \, dx - \frac{t^{p^*}}{p^*}$$

and there exists $t_1 = t_1(Q) > 0$ such that

$$F_1(t_1 v_\circ) = \max_{t\geq 0} F_1(tv_\circ)$$

with

$$\frac{d}{dt} F_1(tv_\circ) \mid_{t=t_1} = t_1^{p-1} \int_Q |\nabla v_\circ|^p \, dx - \lambda t_1^{p-1} \int_Q |v_\circ|^p \, dx - t_1^{p^*-1} = 0.$$

Hence, we have

$$t_1 = \Big(\int_Q |\nabla v_\circ|^p \, dx - \lambda \int_Q |v_\circ|^p \, dx \Big)^{\frac{1}{p^*-p}}$$

and a straightforward calculation gives

$$F_1(t_1 v_\circ) = \big(\frac{1}{p} - \frac{1}{p^*}\big) \Big(\int_Q |\nabla v_\circ|^p \, dx - \lambda \int_Q |v_\circ|^p \, dx \Big)^{\frac{p^*}{p^*-p}}.$$

Since $\int_Q |\nabla v_\circ|^p \, dx \geq S$ and $\int_Q |v_\circ|^p \, dx \to 0$ as $|Q| \to 0$, we see that $F_1(t_1 v_\circ) \not\to 0$ and also $F(t_1 v_\circ) \not\to 0$. Similar calculations can be carried out in the case $q < p < p^*$.

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