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ON SMOOTH GLOBAL SOLUTIONS OF A KIRCHOFF TYPE EQUATION ON UNBOUNDED DOMAINS

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Abstract. We consider the quasilinear hyperbolic equation

$$u_{tt} - M\Big(\int_\Omega |\operatorname{grad} u|^2\,dx\Big)\,\Delta u = 0, \tag{1}$$

where $x \in \Omega = \mathbb{R}^n$, t denotes time and $M(s)$ is a smooth function satisfying $M(s) > 0$ for all $s \geq 0$. We prove that there are no non-trivial "breathers" for equation (1). Here, a "breather" means a time periodic solution which is "small" as $|x| \to +\infty$. We also present a simpler proof of the so-called Pohozaev's second conservation law for (1) solving the global Cauchy problem for "non-physical" nonlinearity M arising from this conservation law.

1. Introduction. In this paper we discuss the quasilinear hyperbolic equation

$$u_{tt} - M\Big(\int_\Omega |\operatorname{grad} u|^2\,dx\Big)\,\Delta u = 0, \tag{1.1}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\Omega = \mathbb{R}^n$ and M is a smooth function satisfying $M(s) > 0$ for all $s \geq 0$. It is perhaps surprising that some basic problems remain unsolved for equation (1.1) despite the large number of publications on the subject; see for example [7], [6], [1], [5] and the references therein. Suppose that functions u_0 and u_1 are given and belong to the Sobolev spaces $H^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ respectively. Consider equation (1.1) with initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{1.2}$$

Can we find a (unique) global solution u of (1.1) with initial conditions (1.2)? As far as we know, this problem remains unsolved although there are partial answers discussed in the above references. A reader can be tempted to search for periodic solutions in time of (1.1). Unfortunately, we prove in the next section that this task is impossible, as long as u is smooth and "well behaved" at infinity for each t. Our second remark concerning (1.1) has to do with the appearance of a "non-physical"

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function $M(s)$ for which (1.1) has global solutions with "non-analytical" initial data. Recently S.I. Pohozaev [8] studied equation (1.1) on bounded domains and obtained a new conservation law (which is the key for proving global existence) as long as $M(s)$ looks like $(as+b)^{-2}$ where a and b are positive numbers. It is known that, in most applications, the function $M(s)$ should be bigger than or equal to $as+b$ for all $s \geq 0$. This requirement fails in Pohozaev's paper [8]. In Section 3 we give a new proof of Pohozaev's conservation law for (1.1) and prove the existence (and uniqueness) of a global solution of (1.1) in $\Omega = \mathbb{R}^n$ when $M(s) = (as+b)^{-2}$.

We shall use standard notation: Δ denotes the Laplace operator, that is $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$. The gradient of $u = u(x,t)$ with respect to space variables will be denoted by grad u. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth field. We denote by $\operatorname{Div} F$ the divergence of F with respect to the space variables. The expression dS_y will always mean the surface measure with respect to the variable y. The dot $\bullet$ will always mean the usual inner product in $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$, $x \neq 0$; then the radial derivative of a real value function h is given by

$$h_r(x) = \frac{\partial h}{\partial r}(x) = \frac{x}{|x|} \bullet \operatorname{grad} h.$$

In Section 3 we will consider complex valued solutions of (1.1) and we shall use the standard notation, that is $\bar{u}(x,t)$ will mean the complex conjugate of u, $\hat{u}(\xi,t)$ will denote the Fourier transform of u in the spatial variable, etc.

2. Non-existence of periodic solutions in time. In this section we discuss the non-existence of time-periodic solutions of our model equation (1.1) as long as the solution remains "small" as $|x| \to +\infty$. Such solutions are sometimes called "breathers". Our method is based on multipliers and was used for related problems (with local nonlinearities) by W. Strauss [9] who extended an earlier result of J. Coron [2] (see also [3]) on the subject.

Theorem 1. *Let $M \in C^1([0,\infty);(0,\infty))$ and $u \in C^2(\mathbb{R}; H^2(\mathbb{R}^n))$ be a solution of equation (1.1) where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Suppose that u is periodic in t with period T, u has finite energy and*

$$\lim_{R\to+\infty} R \int_{|x|=R} [u_t^2 + |\operatorname{grad} u|^2] \, dS_x = 0;$$

then $u \equiv 0$.

Proof. Suppose $u \not\equiv 0$. Let

$$m(u) = x \cdot \operatorname{grad} u + \frac{n}{2} u = r u_r + \frac{n}{2} u,$$

where $r = |x|$. Direct calculation gives us the identity

$$u_t (m(u))_t = \frac{r}{2}(u_t^2)_r + \frac{n}{2} u_t^2 = \operatorname{div}\left(\frac{x}{2} u_t^2\right). \tag{2.1}$$

Using (2.1) and the divergence theorem we obtain

$$\int_{\mathbb{R}^n} u_t(m(u))_t dx = \int_{\mathbb{R}^n} \operatorname{div}\big(\frac{x}{2}u_t^2\big)\, dx = \frac{1}{2}\lim_{R\to\infty} R\int_{|x|=R} u_t^2 dS_x = 0. \tag{2.2}$$

From (2.2) it follows that

$$\frac{d}{dt}\int_{\mathbb{R}^n} u_t m(u) dx = \int_{\mathbb{R}^n} u_{tt} m(u)\, dx. \tag{2.3}$$

Using (2.3) and integrating in time from zero to T we obtain

$$\int_0^T \int_{\mathbb{R}^n} u_{tt} m(u)\, dx\, dt = \int_{\mathbb{R}^n} u_t m(u)\, dx\Big|_{t=T} - \int_{\mathbb{R}^n} u_t m(u)\, dx\Big|_{t=0} = 0 \tag{2.4}$$

because u is periodic in time with period T. Clearly we have

$$\frac{\partial}{\partial x_j} m(u) = \sum_{k=1}^{n} \big(x_k \frac{\partial^2 u}{\partial x_j \partial x_k} + \delta_k^j \frac{\partial u}{\partial x_k}\big) + \frac{n}{2}\frac{\partial u}{\partial x_j},$$

where $\delta_k^j = 0$ if $j \neq k$ and $\delta_j^j = 1$. Integration by parts and the above identity imply

$$\begin{aligned}\int_{\mathbb{R}^n} (-\Delta u) m(u)\, dx &= \int_{\mathbb{R}^n} \operatorname{grad} u \cdot \operatorname{grad} m(u)\, dx \\ &= \frac{n+2}{2}\int_{\mathbb{R}^n} |\operatorname{grad} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^n} r(|\operatorname{grad} u|^2)_r dx.\end{aligned} \tag{2.5}$$

The identity

$$\operatorname{div}\big(\frac{x}{2}|\operatorname{grad} u|^2\big) = \frac{1}{2} r(|\operatorname{grad} u|^2)_r + \frac{n}{2}|\operatorname{grad} u|^2$$

and the divergence theorem imply that

$$\begin{aligned}&\int_{\mathbb{R}^n} (-\Delta u) m(u)\, dx = \int_{\mathbb{R}^n} |\operatorname{grad} u|^2 dx + \int_{\mathbb{R}^n} \operatorname{div}\big(\frac{x}{2}|\operatorname{grad} u|^2\big)\, dx \\ &= \int_{\mathbb{R}^n} |\operatorname{grad} u|^2 dx + \frac{1}{2}\lim_{R\to\infty} R\int_{|x|=R} |\operatorname{grad} u|^2 dS_x = \int_{\mathbb{R}^n} |\operatorname{grad} u|^2 dx.\end{aligned} \tag{2.6}$$

Finally, let us multiply equation (1.1) by $m(u)$ and integrate over $\mathbb{R}^n \times [0,T]$ to obtain

$$\int_0^T \int_{\mathbb{R}^n} u_{tt} m(u)\, dx\, dt - \int_0^T M\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2 dy\Big) \int_{\mathbb{R}^n} \Delta u\, m(u)\, dx\, dt = 0.$$

Using (2.4) and (2.6) we conclude that

$$\int_0^T M\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2 dy\Big)\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2 dx\Big)\, dt = 0;$$

consequently, $\int_{\mathbb{R}^n} |\operatorname{grad} u|^2 dx = 0$ for all $0 \le t \le T$. By assumption u has finite energy, thus

$$\int_{\mathbb{R}^n} u_t^2 dx \le \int_{\mathbb{R}^n} [u_t^2 + \tilde{M}(0)|\operatorname{grad} u|^2]\, dx = \text{constant} < +\infty,$$

where $\tilde{M}(s) = \int_0^s M(\tau)\, d\tau$. Our above discussion shows that u is independent of x. This implies that $u_{tt} = 0$ which contradicts the fact that $\int_{\mathbb{R}^n} u_t^2 dx < +\infty$. This proves Theorem 1.

3. The Cauchy problem: Pohozaev's second conservation law. In this section we study the Cauchy problem associated with equation (1.1) in the non-periodic case. Recently, W.I. Pohozaev [8] obtained a new conservation law for equation (1.1) in the case of bounded domains. Our description below is different and simpler from his fairly general discussion given in [8]. Let us consider $\Omega = \mathbb{R}^n$ with $M(s) > 0$ and $M \in C^1$. Let $u = u(x,t)$ be a solution of

$$L[u] = u_{tt} - M\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2 dy\Big)\Delta u = 0. \tag{3.1}$$

Let $m_1(u)$ and $m_2(u)$ be convenient multipliers to be chosen later. Let us multiply (formally) (3.1) by Δu_t and integrate over $\mathbb{R}^n$. Next, we multiply (3.1) by Δu and integrate over $\mathbb{R}^n$. Adding these two identities we obtain

$$m_1(u) \int_{\mathbb{R}^n} \Delta u_t L[u]\, dx + m_2(u) \int_{\mathbb{R}^n} \Delta u L[u]\, dx = 0. \tag{3.2}$$

Substitution in (3.2) of $L[u]$ by its expression in (3.1) gives us, after straightforward simplifications,

$$\begin{aligned}
&\frac{d}{dt}\Big\{ -\frac{1}{2} m_1(u) \int_{\mathbb{R}^n} |\operatorname{grad} u_t|^2 dx - \frac{1}{2} m_1(u) M\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2 dx\Big) \int_{\mathbb{R}^n} |\Delta u|^2 dx \\
&\quad - m_2(u) \int_{\mathbb{R}^n} \operatorname{grad} u \cdot \operatorname{grad} u_t\, dx \Big\} + \frac{1}{2} \int_{\mathbb{R}^n} |\operatorname{grad} u_t|^2 dx \Big(\frac{d}{dt} m_1(u)\Big) \\
&\quad + \frac{1}{2} M\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2\, dx\Big) \int_{\mathbb{R}^n} |\Delta u|^2\, dx \Big(\frac{d}{dt} m_1(u)\Big) \\
&\quad + m_1(u) M'\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2\, dx\Big) \int_{\mathbb{R}^n} \operatorname{grad} u \cdot \operatorname{grad} u_t\, dx \int_{\mathbb{R}^n} |\Delta u|^2\, dx \\
&\quad + \int_{\mathbb{R}^n} \operatorname{grad} u \cdot \operatorname{grad} u_t\, dx \Big(\frac{d}{dt} m_2(u)\Big) + m_2(u) \int_{\mathbb{R}^n} |\operatorname{grad} u_t|^2\, dx \\
&\quad - m_2(u) M\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2\, dx\Big) \int_{\mathbb{R}^n} |\Delta u|^2\, dx = 0,
\end{aligned} \tag{3.3}$$

where $M'(s) = \frac{d}{ds} M(s)$. Let us choose m_1 and m_2 such that $\frac{d}{dt} m_1(u) = -2m_2(u)$. Substitution in (3.3) give us

$$\frac{d}{dt} A(u) + B(u) = 0, \tag{3.4}$$

where

$$A(u) = -\frac{1}{2}m_1(u)\int_{\mathbb{R}^n} |\operatorname{grad} u_t|^2\,dx$$
$$-\frac{1}{2}m_1(u)M\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2 dx\Big)\int_{\mathbb{R}^n} |\Delta u|^2 dx - m_2(u)\int_{\mathbb{R}^n} \operatorname{grad} u\cdot\operatorname{grad} u_t\,dx$$

and

$$\begin{aligned} B(u) = & -2m_2(u)M\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2\,dx\Big)\int_{\mathbb{R}^n} |\Delta u|^2\,dx \\ & + m_1(u)M'\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2\,dx\Big)\int_{\mathbb{R}^n} \operatorname{grad} u\cdot\operatorname{grad} u_t\,dx\int_{\mathbb{R}^n} |\Delta u|^2\,dx \\ & + \int_{\mathbb{R}^n} \operatorname{grad} u\cdot\operatorname{grad} u_t\,dx\big(\frac{d}{dt}m_2(u)\big). \end{aligned}$$

Choosing

$$m_1(u) = -\Big[a\int_{\mathbb{R}^n} |\operatorname{grad} u|^2\,dx + b\Big], \tag{3.5}$$

where $a > 0$, $b > 0$, we get $\frac{d}{dt}m_1(u) = -2a\int_{\mathbb{R}^n} \operatorname{grad} u\cdot\operatorname{grad} u_t\,dx$; that is m_2 should be given by

$$m_2(u) = a\int_{\mathbb{R}^n} \operatorname{grad} u\cdot\operatorname{grad} u_t\,dx. \tag{3.6}$$

Substitution of (3.5) and (3.6) into (3.4) give us

$$\begin{aligned} & \frac{d}{dt}\Big\{\frac{1}{2}\Big[a\int_{\mathbb{R}^n} |\operatorname{grad} u|^2\,dx + b\Big]\int_{\mathbb{R}^n} |\operatorname{grad} u_t|^2\,dx \\ & + \frac{1}{2}\Big[a\int_{\mathbb{R}^n} |\operatorname{grad} u|^2\,dx + b\Big]M\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2\,dx\Big)\int_{\mathbb{R}^n} |\Delta u|^2\,dx \\ & - \frac{a}{2}\Big(\int_{\mathbb{R}^n} \operatorname{grad} u\cdot\operatorname{grad} u_t\,dx\Big)^2\Big\} \\ & - \int_{\mathbb{R}^n} \operatorname{grad} u\cdot\operatorname{grad} u_t\,dx\int_{\mathbb{R}^n} |\Delta u|^2 dx\Big[2aM\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2 dx\Big) \\ & + \Big(a\int_{\mathbb{R}^n} |\operatorname{grad} u|^2 dx + b\Big)M'\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2 dx\Big)\Big] = 0. \end{aligned} \tag{3.7}$$

Next, we choose $M(s)$ such that it satisfies the ordinary differential equation

$$2aM(s) + (as+b)M'(s) = 0 \tag{3.8}$$

for all $s \geq 0$. The general solution of (3.8) is

$$M(s) = \frac{c}{(as+b)^2},$$

where c is a constant. It follows by (3.7) and the above choice of M that

$$\Big[a\int_{\mathbb{R}^n}|\mathrm{grad}\,u|^2dx+b\Big]\Big\{\int_{\mathbb{R}^n}|\mathrm{grad}\,u_t|^2dx+M\Big(\int_{\mathbb{R}^n}|\mathrm{grad}\,u|^2dx\Big)\int_{\mathbb{R}^n}|\Delta u|^2dx\Big\} \\ -\frac{a}{2}\Big(\int_{\mathbb{R}^n}\mathrm{grad}\,u\cdot\mathrm{grad}\,u_t\,dx\Big)^2=\text{Constant} \tag{3.9}$$

for all $t\geq 0$. This is the so–called Pohozaev's second conservation law associated with (1.1).

Observations. 1. Observe that $M(s)=(as+b)^{-2}$ does not have the "good" physical requirements that should be satisfied by the nonlinear term M.

2. Instead of (3.8), in Pohozaev's paper [8] is found that $M(s)$ should satisfy the nonlinear ordinary differential equation, $2MM''=3(M')^2$ whose general solution is again $M(s)=c(as+b)^{-2}$.

The nonlinearity $M(s)=(as+b)^{-2}$ is such that if a solution exists then it will satisfy a priori estimates in finite time, given in the next lemma.

Lemma 1. *Let $u_0\in H^2(\mathbb{R}^n)$, $u_1\in H^1(\mathbb{R}^n)$. Suppose that there exists a solution u of (3.1) with $u(x,0)=u_0(x)$, $u_t(x,0)=u_1(x)$ and $M(s)=(as+b)^{-2}$, $(a,b>0)$ such that $u\in C(0,T;H^2)$, $u_t\in C(0,T;H^1)$ for some $T>0$. Then u satisfies the a priori estimates*

a) $\displaystyle\int_{\mathbb{R}^n}u_t^2\,dx\leq C_1,$ b) $\displaystyle\int_{\mathbb{R}^n}|\mathrm{grad}\,u_t|^2\,dx\leq C_2,$

c) $\displaystyle\int_{\mathbb{R}^n}|\mathrm{grad}\,u|^2\,dx\leq C_3(T),$ d) $\displaystyle\int_{\mathbb{R}^n}|\Delta u|^2\,dx\leq C_4(T),$

for all $0\leq t\leq T$ where C_1 and C_2 are constants depending only on the initial data and $C_3(T)$ and $C_4(T)$ depend on T and the initial data.

Proof. Multiply equation (3.1) by u_t and integrate over $\mathbb{R}^n$ to obtain that

$$\int_{\mathbb{R}^n}u_t^2\,dx+\tilde{M}\Big(\int_{\mathbb{R}^n}|\mathrm{grad}\,u|^2\,dx\Big)=\text{Constant},$$

where $\tilde{M}(s)=\int_0^s M(\tau)\,d\tau$. This proves a). Gronwall's inequality implies that there exists a constant $\tilde{C}_1(T)>0$ such that $\int_{\mathbb{R}^n}u^2\,dx\leq\tilde{C}_1(T)$ for all $0\leq t\leq T$. Identity (3.9) implies b).

Now, Hölder's inequality and b) imply that

$$\frac{d}{dt}\int_{\mathbb{R}^n}|\mathrm{grad}\,u|^2\,dx\leq 2C_2^{1/2}\Big(\int_{\mathbb{R}^n}|\mathrm{grad}\,u|^2\,dx\Big)^{1/2}$$

which implies c). Finally d) follows using (3.9), b) and c). Now, let us consider equation (3.1) where $M(s)=(as+b)^{-2}$ with a and b positive constants. We shall try to identify a suitable space of functions where the initial data can be taken to obtain a global solution of (3.1). Let $v = v(y,t)$ be the Fourier transform of u in the spatial variable only, that is

$$u(x,t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} v(y,t) e^{ix\cdot y}\, dy. \tag{3.10}$$

Then, formally, if u solves (3.1) with $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$ then v should be the solution of

$$\begin{cases} v_{tt} + |y|^2 M\Big(\displaystyle\int_{\mathbb{R}^n} |y|^2 |v(y,t)|^2\, dy\Big) v(y,t) = 0, \\ v(y,0) = \hat{u}_0(y), \quad v_t(y,0) = \hat{u}_1(y), \end{cases} \tag{3.11}$$

where $\hat{u}_0$ and $\hat{u}_1$ denote the Fourier transform of u_0 and u_1 respectively. The idea is quite simple and was introduced in [6] for this class of problems: To solve (3.11) globally, we approximate it by a sequence of truncated problems and using a priori estimates similar to the ones obtained in Lemma 1, we are able to show that it contains a subsequence approaching to a function v which is the solution of (3.11). Finally we show that u given by (3.10) solves (3.1). Several technical items are similar to the ones given in [6], so we just give details on the ones which are new.

For each $R > 0$, let $v_R = v_R(y,t)$ be the solution of the problem

$$\begin{cases} \dfrac{\partial^2 v_R}{\partial t^2} + |y|^2 M\Big(\displaystyle\int_{|z|\le R} |z|^2 |v_R|^2\, dz\Big)\, v_R(y,t) = 0, \quad |y| \le R,\ t > 0, \\ v_R(y,0) = \hat{u}_0(y), \quad \dfrac{\partial v_R}{\partial t}(y,0) = \hat{u}_1(y), \end{cases} \tag{3.12}$$

and define $v_R(y,t) \equiv 0$, if $|y| > R$; here $\hat{u}_0$ and $\hat{u}_1$ denote the Fourier transform of u_0 and u_1 respectively.

From now on we shall assume that the initial data for problem (3.1) satisfy the hypothesis

(H) $$u_0 \in H^{\frac{n+9}{2}}(\mathbb{R}^n), \quad u_1 \in H^{\frac{n+7}{2}}(\mathbb{R}^n), \quad u_0, u_1 \in L^1(\mathbb{R}^n).$$

The standard sequence of successive approximations for (3.12) converges, thus we have

Lemma 2. *Let $M(s) = (as+b)^{-2}$ with a and b are positive constants. Assume* (H). *Then there exists $T_0 > 0$ such that problem* (3.12) *has a local solution $v_R = v_R(y,t)$.*

Proof. In order to simplify notation, let us call $v = v_R$. By Zorn's lemma we can assume that v exists in a maximal interval $[0, T_{\max})$. To show that $T_{\max} = +\infty$ it is sufficient to prove that there exists a constant $C = C(T_{\max}) > 0$ such that $|v| \le C$, $|v_t| \le C$ for all $0 \le t < T_{\max}$. This is done by considering the analog of operator (3.1),

$$\hat{L}_R[v] = v_{tt} + |y|^2 M\Big(\int_{|z|\le R} |z|^2 |v|^2 dz\Big) v = 0,$$

and defining

$$\hat{m}_1(v) = -a\Big[\int_{|z|\le R} |z|^2 |v|^2 dz + b\Big], \quad \hat{m}_2(v) = a\, Re \int_{|z|\le R} |z|^2 v \bar{v}_t dz.$$

Next, we proceed in the same way as we did from (3.2) to (3.9) in order to obtain a (truncated) Pohozaev's identity,

$$\Big[\int_{|z|\le R}|z|^2|v|^2dz+b\Big]\Big\{\int_{|z|\le R}|z|^2|v_t|^2dz+M\Big(\int_{|z|\le R}|z|^2|v|^2dz\Big)\int_{|z|\le R}|z|^4|v|^2dz\Big\}$$
$$-\frac{a}{2}\Big(Re\int_{|z|\le R}|z|^2v\bar{v}_tdz\Big)^2=\text{constant} \tag{3.13}$$

for all $0\le t<T_{\max}$. Now, identity (3.13) implies, in particular, that

$$\int_{|z|\le R}|z|^2|v_t|^2dz\le C \quad \text{and} \quad \int_{|z|\le R}|z|^2|v|^2dz\le C(T_{\max}). \tag{3.14}$$

Taking the complex conjugate of equation (3.12) and multiplying by v_t we get

$$v_t\bar{v}_{tt}+|y|^2M\Big(\int_{|z|\le R}|z|^2|v|^2dz\Big)\bar{v}v_t=0. \tag{3.15}$$

Multiplying equation (3.12) by $\bar{v}_t$ and adding to (3.15) we obtain that

$$\frac{d}{dt}|v_t|^2+|y|^2M\Big(\int_{|z|\le R}|z|^2|v|^2dz\Big)\frac{d}{dt}|v|^2=0 \tag{3.16}$$

for all $0\le t<T_{\max}$. Let $E_R(t)$ be given by

$$E_R(t)=|v_t|^2+|y|^2|v|^2M\Big(\int_{|z|\le R}|z|^2|v|^2dz\Big).$$

Using (3.16), Schwarz's inequality (3.14) and $M(s)=(as+b)^{-2}$ we obtain that

$$\begin{aligned}\frac{d}{dt}E_R(t)&\le\frac{4a}{b^3}|y|^2|v|^2\Big(\int_{|z|\le R}|z|^2|v|^2dz\Big)^{1/2}\Big(\int_{|z|\le R}|z|^2|v_t|^2dz\Big)^{1/2}\\&\le\frac{4a}{b^3}|y|^2|v|^2C(T_{\max})\le\frac{4a}{b^3}C(T_{\max})E_R(t)\Big(\int_{|z|\le R}|z|^2|v|^2dz+b\Big)^2\\&\le C(T_{\max},a,b)E_R(t).\end{aligned}$$

Hence, Gronwall's inequality implies that $E_R(t)$ is bounded for all $0\le t<T_{\max}$. It follows that $|v|$ and $|v_t|$ are bounded for all $0\le t<T_{\max}$. This proves Theorem 2.

Lemma 3. *Let $T>0$ be an arbitrary fixed number. Assume that* (H) *holds and let $v=v_R(y,t)$ be the global solution of* (3.12) *obtained in Theorem* 2. *Then, there exists a constant $C=C(T)>0$ such that*

1) $\displaystyle\int_{|z|\le R}|v_t|^2\,dz\le C,$ 2) $\displaystyle\int_{|z|\le R}|v|^2\,dz\le C(T),$

3) $\displaystyle\int_{|z|\le R}|z|^{n+7}|v_t|^2\,dz\le C,$ 4) $\displaystyle\int_{|z|\le R}|z|^{n+9}|v|^2\,dz\le C,$

for all $0 \le t \le T$.

Proof. Using identity (3.16) we can write

$$\frac{d}{dt}\Big[\int_{|z|\le R}|v_t|^2\,dz+\tilde{M}\Big(\int_{|z|\le R}|z|^2|v|^2\,dz\Big)\Big]=0, \tag{3.17}$$

where $\tilde{M}(s)=\int_0^s M(\tau)\,d\tau$. Part 1) follows from (3.17).

Item 2) follows from 1) and Gronwall's inequality. Multiply equation (3.12) by $|y|^{n+7}\bar{v}_t$ to get

$$|y|^{n+7}v_{tt}\bar{v}_t+M\Big(\int_{|z|\le R}|z|^2|v|^2\,dz\Big)\,|y|^{n+9}v\bar{v}_t=0. \tag{3.18}$$

Taking the complex conjugate of equation (3.12), multiplying it by $|y|^{n+7}v_t$, adding the result with (3.18) and integrating over $|z|\le R$ we obtain that

$$\begin{aligned}&\frac{d}{dt}\Big[\int_{|z|\le R}|z|^{n+7}|v_t|^2\,dz+M\Big(\int_{|z|\le R}|z|^2|v|^2\,dz\Big)\int_{|z|\le R}|z|^{n+9}|v|^2dz\Big]\\&\quad=2\int_{|z|\le R}|z|^{n+9}|v|^2dzM'\Big(\int_{|z|\le R}|z|^2|v|^2\,dz\Big)\,Re\int_{|z|\le R}|z|^2v\bar{v}_t\,dz.\end{aligned} \tag{3.19}$$

Now we use (3.14) and $M(s)=(as+b)^{-2}$ to get a bound for the left hand side of (3.19):

$$\le \text{Constant}\,M\Big(\int_{|z|\le R}|z|^2|v|^2\,dz\Big)\int_{|z|\le R}|z|^{n+9}|v|^2\,dz, \tag{3.20}$$

where the constant in (3.20) depends only on T, a, b and the initial data. From (3.19) and (3.20) we get item 3). Furthermore, the above discussion shows that there exists a constant $C=C(T)>0$ such that

$$M\Big(\int_{|z|\le R}|z|^2|v|^2\,dz\Big)\int_{|z|\le R}|z|^{n+9}|v|^2\,dz\le C(T). \tag{3.21}$$

Since $M(s)=(as+b)^{-2}$ and using (3.14), we deduce item 4) from (3.21).

Our next step consists in showing that there exists a subsequence of $\{v_R\}_{R>0}$ which converges to a solution of (3.11). Let $T>0$ be an arbitrary fixed number and let us define

$$h_R(t)=M\Big(\int_{|z|\le R}|z|^2|v_R|^2\,dz\Big),$$

where $v_R=v_R(y,t)$ is the global solution of (3.12) and $M(s)=(as+b)^{-2}$. Clearly

$$|h_R(t)|\le b^{-2}\quad\text{for all}\quad 0\le t\le T \tag{3.22}$$

and

$$|h_R'(t)|\le 4ab^{-3}C(T)\quad\text{for all}\quad 0\le t\le T$$

because of Schwarz's inequality and (3.14). Consequently, the family of functions $\{h_R(t)\colon\ R = 1, 2, \dots\}$ satisfies the hypothesis of the Arzela–Ascoli theorem in $0 \le t \le T$. Therefore, there exists a subsequence $\{h_{R_j}(T)\}$ such that h_{R_j} converges uniformly on $0 \le t \le T$ to a continuous function which we shall denote by $h_\infty(t)$. Let us denote by $w = w(y,t)$ the solution of the problem

$$w_{tt} + |y|^2\, h_\infty(t) w = 0, \quad w(y,0) = \hat{u}_0(y), \quad w_t(y,0) = \hat{u}_1(y). \tag{3.23}$$

Clearly (3.23) has a solution defined on $0 \le t \le T$. Since the initial data for (3.23) and for the equation (3.12) are the same it is not difficult to show that, for each $y \in \mathbb{R}^n$, $v_{R_j}(y,t) \to w(y,t)$ and $\frac{\partial v_{R_j}}{\partial t}(y,t) \to \frac{\partial w}{\partial t}(y,t)$ uniformly on $0 \le t \le T$.

Lemma 4. *Let $T > 0$ be an arbitrary fixed number. Assume that* (H) *holds and let $w = w(y,t)$ be the solution of* (3.23) *on $0 \le t \le T$; then*

a) $\int_{\mathbb{R}^n} |y|^2 |w(y,t)|^2\, dy < +\infty$ *and*

b) $h_\infty(t) = M\big(\int_{\mathbb{R}^n} |z|^2 |w|^2\, dz\big)$ *for all $0 \le t \le T$.*

Proof. Since v_{R_j} and w satisfy (3.12) and (3.23) respectively then we can write

$$v_{R_j}(y,t) = \hat{u}_0(y) + \hat{u}_1(y)t - \int_0^t |y|^2 (t-s) h_{R_j}(s) v_{R_j}(y,s)\, ds \tag{3.24}$$

and

$$w(y,t) = \hat{u}_0(y) + \hat{u}_1(y)t - \int_0^t |y|^2 (t-s) h_\infty(s) w(y,s)\, ds \tag{3.25}$$

for all $0 \le t \le T$. Subtraction of (3.24) from (3.25) gives us the estimate

$$\begin{aligned} |v_{R_j}(y,t) - w(y,t)| \le T^2 |y|^2 \sup_{0\le s\le T} |h_\infty(s) - h_{R_j}(s)| \sup_{0\le s\le T} |w(y,s)| \\ + T|y|^2 b^{-2} \int_0^t |v_{R_j}(y,s) - w(y,s)|\, ds, \end{aligned} \tag{3.26}$$

where we used (3.22). Gronwall's inequality implies that

$$|v_{R_j}(y,t) - w(y,t)| \le \big[T^2 |y|^2 \sup_{0\le s\le T} |h_\infty(s) - h_{R_j}(s)| \sup_{0\le s\le T} |w(y,s)|\big]\, e^{T^2|y|^2 b^{-2}}$$

for all $0 \le t \le T$. Let us fix a number $R_0 > 0$ and consider (3.27) for $|y| \le R_0$; then

$$|v_{R_j}(y,t) - w(y,t)| \le \big[T^2 R_0^2 \sup_{0\le s\le T} |h_\infty(s) - h_{R_j}(s)| \sup_{0\le s\le T} |w(y,s)|\big]\, e^{T^2 R_0^2 b^{-2}}.$$

Consequently,

$$|y|^2 v_{R_j}(y,t) \to |y|^2 w(y,t)$$

uniformly in $|y| \le R_0$ as $j \to +\infty$. This implies that

$$\lim_{j\to+\infty} \int_{|y|\le R_0} |y|^2 |v_{R_j}(y,t)|^2\, dy = \int_{|y|\le R_0} |y|^2 |w(y,t)|^2\, dy \tag{3.28}$$

for all $0 \le t \le T$. Also, from (3.14) we know that

$$\int_{|y|\le R_0} |y|^2 |v_{R_j}(y,t)|^2\, dy \le C(T)$$

for all $0 \le t \le T$ and j sufficiently large. Thus (3.28) implies that

$$\int_{|y|\le R_0} |y|^2 |w(y,t)|^2\, dy \le C(T) \quad \text{for all } \ 0 \le t \le T.$$

Letting $R_0 \to +\infty$ we conclude the proof of item a). Next, we estimate the difference $\big|h_{R_j}(t) - M\big(\int_{\mathbb{R}^n} |z|^2|w|^2\, dz\big)\big|$ using $M(s) = (as+b)^{-2}$ and item a):

$$\begin{aligned}\Big|h_{R_j}(t) - M\Big(\int_{\mathbb{R}^n} |z|^2|w|^2\, dz\Big)\Big| &\le \Big(\int_{\mathbb{R}^n} |z|^2 |v_{R_j}|^2 dz\Big) - M\Big(\int_{\mathbb{R}^n} |z|^2|w|^2 dz\Big)\Big| \\ &\le 2(aC(T)+b)b^{-4}\Big|\int_{\mathbb{R}^n} |z|^2(|v_{R_j} - |w|^2) dz\Big|. \end{aligned} \tag{3.29}$$

Let $\varepsilon > 0$ and split the integral on the right hand side of (3.29) into two pieces

$$\int_{\mathbb{R}^n} = \int_{|z|\le R_0} + \int_{|z|\ge R_0},$$

where we choose $R_0 > 0$ large enough so that

$$\int_{|z|\le R_0} |z|^2|w|^2\, dz < \varepsilon, \quad \int_{|z|\ge R_0} |z|^2 |v_{R_j}|^2\, dz < \varepsilon,$$

which is possible for large j because of our discussion in item a).

Since $v_{R_j} \to w$ uniformly in $|z| \le R_0$, there exists $j_0 > 0$ sufficiently large such that for all $j \ge j_0$ we have

$$\Big|\int_{|z|\le R_0} |z|^2(|v_{R_j} - |w|^2)\, dz\Big| \le \varepsilon.$$

Consequently, from (3.29) we conclude that

$$h_{R_j}(t) \to M\Big(\int_{\mathbb{R}^n} |z|^2|w|^2\, dz\Big).$$

Since

$$\Big|h_\infty(t) - M\Big(\int_{\mathbb{R}^n} |z|^2|w|^2\, dz\Big)\Big| \le |h_\infty(t) - h_{R_j}(t)| + \Big|h_{R_j}(t) - M\Big(\int_{\mathbb{R}^n} |z|^2|w|^2\, dz\Big)\Big| \tag{3.30}$$

and h_{R_j} converges uniformly to h_∞ then the right hand side of (3.30) approaches zero as $j \to +\infty$. This proves Lemma 4.

Lemma 5. *Let $T > 0$ be an arbitrary fixed number. Assume that* (H) *holds and let $w = w(y,t)$ be the solution of* (3.23) *on $0 \le t \le T$. Then there exists $C = C(T) > 0$ such that*

$$\text{a)} \quad \int_{\mathbb{R}^n} |w_t|^2\, dy \le C, \qquad \text{b)} \quad \int_{\mathbb{R}^n} |w(y,t)|^2\, dy \le C,$$

$$\text{c)} \quad \int_{\mathbb{R}^n} |y|^{n+7} |w_t(y,t)|^2\, dy \le C, \qquad \text{d)} \quad \int_{\mathbb{R}^n} |y|^{n+9} |w|^2\, dy \le C,$$

for all $a \le t \le T$.

Proof. Let $R_0 > 0$. We know $\frac{\partial v_{R_j}}{\partial t}(y,t) \to \frac{\partial w}{\partial t}(y,t)$ uniformly on $0 \le t \le T$. Consequently,

$$\int_{|y| \le R_0} \Big| \frac{\partial v_{R_j}}{\partial t} \Big|^2\, dy \to \int_{|y| \le R_0} \Big| \frac{\partial w}{\partial t} \Big|^2\, dy.$$

Now, part 1) of Lemma 3 implies that there exists a constant C such that

$$\int_{|y| \le R_0} \Big| \frac{\partial w}{\partial t} \Big|^2\, dy \le C;$$

letting $R_0 \to \infty$ we obtain item a). All remaining estimates are done in a similar way.

Theorem 3. *Let $T > 0$, $M(s) = (as + b)^{-2}$ with a, $b > 0$ and u_0, u_1 satisfying assumption* (H)*, then the Cauchy problem*

$$\begin{cases} u_{tt} - M\Big(\displaystyle\int_{\mathbb{R}^n} |\operatorname{grad} u|^2\, dx \Big) \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u(x,0) = u_0(x), \quad u_x(x,0) = u_1(x), & x \in \mathbb{R}^n \end{cases}$$

has a unique solution $u \in C^2(\mathbb{R}^n \times [0,T); \mathbb{C})$.

Proof. Define $u = u(x,t)$ for any $x \in \mathbb{R}^n$, $0 \le t \le T$ by

$$u(x,t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} w(y,t) e^{ix \cdot y}\, dy,$$

where w is the solution of (3.23) defined for $0 \le t \le T$. Observe that u is well defined because by Lemma 5 we have

$$|u(x,t)|^2 \le \int_{\mathbb{R}^n} (1 + |y|^{n+9}) |w|^2\, dy \int_{\mathbb{R}^n} (1 + |y|^{n+9})^{-1}\, dy < +\infty.$$

The continuity of w in time, Lemma 5 and the Lebesgue dominated convergence theorem imply that u is continuous (jointly) in x and t. Since

$$\frac{\partial}{\partial x_j} [w(y,t) e^{ix \cdot y}] = i y_j e^{ix \cdot y} w(y,t)$$

and

$$|iy_j e^{ix\cdot y} w(y,t)| \le |y|\,|w|$$

with

$$\int_{\mathbb{R}^n} |y|\,|w|\,dy < +\infty,$$

we deduce from Lemma 5 that $\frac{\partial u}{\partial x_j}$ exists and

$$\frac{\partial u}{\partial x_j} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} iy_j w(y,t) e^{ix\cdot y}\,dy.$$

It can be checked that $\frac{\partial u}{\partial x_j}$ is continuous. Similar discussion shows that $\frac{\partial^2 u}{\partial x_j \partial x_k}$, $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial t^2}$ exist, are continuous and

$$\frac{\partial^2 u(x,t)}{\partial x_j \partial x_k} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} -y_j y_k w(y,t) e^{ix\cdot y}\,dy, \tag{3.31}$$

$$\frac{\partial^2 u}{\partial t^2} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} -w_{tt}(y,t) e^{ix\cdot y}\,dy. \tag{3.32}$$

Observe that Plancherel's theorem implies that

$$M\Big(\int_{\mathbb{R}^n} |\operatorname{grad} u|^2\,dx\Big) = M\Big(\int_{\mathbb{R}^n} |y|^2|w|^2\,dy\Big). \tag{3.33}$$

Therefore, substitution of (3.31)–(3.32) and (3.33) together with (3.23) implies that u solves the Cauchy problem. Uniqueness follows by using Gronwall's inequality as in [6].

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