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STABILITY AND INSTABILITY OF STANDING WAVES FOR THE GENERALIZED DAVEY–STEWARTSON SYSTEM

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Abstract. We study the stability and instability properties of standing waves for the equation $iu_t + \Delta u + a|u|^{p-1}u + E_1(|u|^2)u = 0$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, which derives from the generalized Davey-Stewartson system in the elliptic-elliptic case. We show that if $n = 2$ and $a(p-3) < 0$, then the standing waves generated by the set of minimizers for the associated variational problem are stable. We also show that if $n = 3$, $a > 0$ and $1 + 4/3 < p < 5$ or $a < 0$ and $1 < p < 3$, then the standing waves are strongly unstable. We employ the concentration compactness principle due to Lions and the compactness lemma due to Lieb to solve the associated minimization problem.

1. Introduction and main results. In the present paper, we consider the stability and instability of standing waves for the nonlinear Schrödinger equation

$$iu_t + \Delta u + a|u|^{p-1}u + E_1(|u|^2)u = 0, \quad t \geq 0,\ x \in \mathbb{R}^n, \tag{1.1}$$

where $a \in \mathbb{R}$, $p > 1$, $n \geq 2$ and E_1 is the singular integral operator with symbol $\sigma_1(\xi) = \xi_1^2/|\xi|^2$, $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$.

Equation (1.1) has its origin in fluid mechanics, where for $n = 2$ and $p = 3$, it describes the evolution of weakly nonlinear water waves that travel predominantly in one direction. More precisely, (1.1) is an n-dimensional extension of the generalized Davey–Stewartson system in the elliptic-elliptic case

$$\begin{cases} iu_t + \lambda u_{xx} + u_{yy} + a|u|^{p-1}u - uv_x = 0, \\ v_{xx} + \mu v_{yy} = -(|u|^2)_x, \end{cases}$$

where $\lambda, \mu > 0$ (see [8], [10], [6] and [7]).

By a standing wave, we mean a solution of (1.1) with the form

$$u(t,x) = e^{i\omega t}\varphi_\omega(x),$$

where $\omega > 0$ and φ_ω is a ground state of the stationary problem

$$\begin{cases} -\Delta\psi + \omega\psi - a|\psi|^{p-1}\psi - E_1(|\psi|^2)\psi = 0, \quad x \in \mathbb{R}^n, \\ \psi \in H^1(\mathbb{R}^n), \quad \psi \not\equiv 0. \end{cases} \tag{1.2_ω}$$

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We shall state the definition of ground state for (1.2_ω).

Definition 1.1. For $a \in \mathbb{R}$, $1 < p < 2^* - 1$ and $n = 2$ or 3, where $2^* = 2n/(n-2)$ if $n \geq 3$, otherwise $2^* = +\infty$, we define the ground states of (1.2_ω) as follows:

$$S_\omega(v) = \frac{1}{2}|\nabla v|_2^2 + \frac{\omega}{2}|v|_2^2 - \frac{a}{p+1}|v|_{p+1}^{p+1} - \frac{1}{4}\int |v|^2 E_1(|v|^2)dx,$$

$$\mathcal{X}_\omega = \text{the set of the solutions for } (1.2_\omega) = \{\psi \in H^1(\mathbb{R}^n) : S'_\omega(\psi) = 0,\ \psi \neq 0\},$$

$$\mathcal{G}_\omega = \text{the set of the ground states for } (1.2_\omega) = \{\varphi \in \mathcal{X}_\omega : S_\omega(\varphi) \leq S_\omega(\psi) \text{ for all } \psi \in \mathcal{X}_\omega\}.$$

For the existence of ground states for (1.2_ω), Cipolatti [6] showed the following result. For $a \in \mathbb{R}$, $1 < p < 2^* - 1$ and $n = 2$ or 3, we put

$$\omega^* = \begin{cases} +\infty, & \text{if } 1 < p < 3 \text{ and } a \in \mathbb{R},\ p = 3 \text{ and } a > -1 \text{ or } p > 3 \text{ and } a \geq 0, \\ \frac{p-3}{p-1}\left(\frac{-2}{a(p-1)}\right)^{2/(p-3)}, & \text{if } p > 3 \text{ and } a < 0. \end{cases}$$

Then, $\mathcal{G}_\omega$ is not empty for any $\omega \in (0, \omega^*)$.

Definition 1.2. For $\Omega \subset H^1(\mathbb{R}^n)$, we shall say that the set Ω is stable, if for any $\varepsilon > 0$, there exists $\delta > 0$ with the following property: if $u_0 \in H^1(\mathbb{R}^n)$ and the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies $\inf_{\varphi \in \Omega} \|u_0 - \varphi\|_{H^1} < \delta$, then

$$\sup_{0 \leq t < \infty} \inf_{\varphi \in \Omega} \|u(t) - \varphi\|_{H^1} < \varepsilon.$$

Otherwise, Ω is said to be unstable. Moreover, for $\varphi_\omega \in \mathcal{G}_\omega$, we shall say that the standing wave $u_\omega(t) = e^{i\omega t}\varphi_\omega$ is stable if $\mathcal{G}_\omega$ is stable, and that u_ω is unstable if $\mathcal{O}_\omega$ is unstable, where $\mathcal{O}_\omega = \{e^{i\theta}\tau_y\varphi_\omega : \theta \in \mathbb{R}, y \in \mathbb{R}^n\}$, $\tau_y v(x) = v(x-y)$. Furthermore, we shall say that u_ω is strongly unstable if for any $\varepsilon > 0$, there exists $u_0 \in H^1(\mathbb{R}^n)$ such that $\|u_0 - \varphi_\omega\|_{H^1} < \varepsilon$ and the solution $u(t)$ of (1.1) with $u(0) = u_0$ blows up in a finite time.

It follows from the invariances of (1.2_ω) that $\mathcal{O}_\omega \subset \mathcal{G}_\omega$. However, it is an open problem, to our knowledge, as to whether $\mathcal{G}_\omega = \mathcal{O}_\omega$ or not.

The unique local existence of H^1 solution to the Cauchy problem of (1.1) is already established. Let $a \in \mathbb{R}$, $1 < p < 2^* - 1$ and $n = 2$ or 3. Then, for any $u_0 \in H^1(\mathbb{R}^n)$ there exists $T^* = T^*(u_0) > 0$ and a unique solution $u(\cdot) \in C([0, T^*); H^1(\mathbb{R}^n))$ of (1.1) with $u(0) = u_0$ such that $T^* = +\infty$ or else, $T^* < +\infty$ and $\lim_{t \to T^*} |\nabla u(t)|_2 = +\infty$. Furthermore, $u(t)$ satisfies

$$|u(t)|_2 = |u_0|_2, \tag{1.3}$$

$$\mathcal{E}(u(t)) = \mathcal{E}(u_0) \tag{1.4}$$

for all $t \in [0, T^*)$, where

$$\mathcal{E}(v) = \frac{1}{2}|\nabla v|_2^2 - \frac{a}{p+1}|v|_{p+1}^{p+1} - \frac{1}{4}\int |v|^2 E_1(|v|^2)dx.$$

For details, see Theorem 4.3.1 in [3] at page 65 and [10].

Recently, many authors have been studying the problem of stability and instability of standing waves for nonlinear Schrödinger equations (see, e.g., [1, 4, 5, 7, 11, 17, 18, 19, 20, 22, 23]). In particular, for the nonlinear Schrödinger equation with single power nonlinearity

$$iu_t + \Delta u + |u|^{p-1}u = 0, \quad t \geq 0,\ x \in \mathbb{R}^n, \tag{1.5}$$

where $1 < p < 2^* - 1$ and $n \geq 1$, it is well known that if $p < 1 + 4/n$, then the standing wave u_ω is stable for any $\omega \in (0, \infty)$ (see Cazenave and Lions [5]), and if $p \geq 1 + 4/n$, then u_ω is strongly unstable for any $\omega \in (0, \infty)$ (see Berestycki and Cazenave [1] for $p \neq 1 + 4/n$, and Weinstein [22] for $p = 1 + 4/n$). Moreover, in this case, it is known that $\mathcal{G}_\omega = \mathcal{O}_\omega$ for any $\omega \in (0, \infty)$ (see Kwong [12]).

In [7], Cipolatti proved that if $a(p-3) \geq 0$ and $n = 2$ or 3, then the standing wave u_ω is unstable for any $\omega \in (0, \infty)$, and that if $n = 2$, $p = 3$ and $a > -1$, then u_ω is strongly unstable for any $\omega \in (0, \infty)$. After that, the author [17] proved that if $a > 0$, $p \geq 1 + 4/n$ and $n = 2$ or 3, then u_ω is unstable for any $\omega \in (0, \infty)$, and that if $n = 3$, $a > 0$ and $p < 1 + 4/3$, then there exists a positive constant $\omega_0 = \omega_0(a, p)$ such that u_ω is unstable for any $\omega \in (\omega_0, \infty)$. Moreover, under the following assumptions (H.1, 2), the author [18] showed that if $a > 0$, $p < 1 + 4/n$ and $n = 2$ or 3, then there exists a sequence (ω_k) such that $\omega_k > 0$, $\omega_k \to 0$ and u_{ω_k} is stable: (H.1) there is a choice $\varphi_\omega \in \mathcal{G}_\omega$ such that $\omega \mapsto \varphi_\omega$ is a C^1 mapping from the interval $(0, \omega^*)$ into $H^1(\mathbb{R}^n)$, and (H.2) if $n = 2$, then $|\varphi|_2 = |\varphi_\omega|_2$ holds for any $\varphi \in \mathcal{G}_\omega$.

Our main results in this paper are the following:

Theorem 1.1. *Assume that $n = 2$. For $\mu > 0$, we define that*

$$I_\mu = \inf\{\mathcal{E}(v) : v \in H^1(\mathbb{R}^2),\ |v|_2^2 = \mu\}, \tag{1.6$_\mu$}$$

$$\Sigma_\mu = \textit{the set of the minimizers for } (1.6_\mu) = \{\phi \in H^1(\mathbb{R}^2) : I_\mu = \mathcal{E}(\phi), |\phi|_2^2 = \mu\},$$

$$\mu_0 = \inf\{|v|_2^2 : v \in H^1(\mathbb{R}^2),\ v \neq 0,\ \mathcal{E}_0(v) \leq 0\},\ \textit{where} \tag{1.7}$$

$$\mathcal{E}_0(v) = \frac{1}{2}|\nabla v|_2^2 - \frac{1}{4}\int |v|^2 E_1(|v|^2)dx.$$

(1) *Let $a > 0$ and $1 < p < 3$. Then, for any $\mu \in (0, \mu_0)$, the set Σ_μ is not empty, and it is stable.*
(2) *Let $a < 0$ and $p > 3$. Then for any $\mu \in (\mu_0, \infty)$, the set Σ_μ is not empty, and it is stable.*

Theorem 1.2. *Assume that $n = 3$. If $a > 0$ and $1 + 4/3 < p < 5$ or $a < 0$ and $1 < p < 3$, then the standing wave $u_\omega(t) = e^{i\omega t}\varphi_\omega$, $\varphi_\omega \in \mathcal{G}_\omega$, is strongly unstable for any $\omega \in (0, \infty)$.*

Remark 1.1. The type of stability described in Theorem 1.1 is the same as in the papers due to Cazenave and Lions [5], and Cazenave and Esteban [4]. For any $\phi \in \Sigma_\mu$, there exists $\omega = \omega(\mu) > 0$ such that $\phi \in \mathcal{X}_\omega$, and $e^{i\theta}\phi$ is a positive function on $\mathbb{R}^n$ for some $\theta \in \mathbb{R}$ (see [5; Theorem II.1]). However, we do not know any relation between μ and $\omega(\mu)$. Therefore, we do not know whether $\mathcal{G}_\omega$ is stable or not in this case. On the other hand, we do not assume (H.1, 2) in Theorem 1.1, which was assumed in our previous paper [18]. Moreover, we note that when $n = 2$ and $a = 0$, the standing wave u_ω is strongly unstable for any $\omega \in (0, \infty)$ (see [7]). We use the concentration compactness principle due to Lions [14, 15], following [5] and [4], in the proof of Theorem 1.1.

Remark 1.2. From the Gagliardo–Nirenberg inequality, we have $\mu_0 > 0$. Indeed, if $v \in H^1(\mathbb{R}^2)$ satisfies $v \neq 0$ and $\mathcal{E}_0(v) \leq 0$, then we have

$$\int |v|^2 E_1(|v|^2)dx \leq |v|_4^4 \leq C|v|_2^2|\nabla v|_2^2 \leq \frac{C}{2}|v|_2^2 \int |v|^2 E_1(|v|^2)dx$$

for some $C > 0$. Hence, we have $\mu_0 \geq 2/C > 0$. Moreover, it is natural that the number μ_0 should appear in Theorem 1.1 (2), because when $n = 2$, $a < 0$ and $p > 3$, we can show that $|\psi|_2^2 > \mu_0$ holds for any $\psi \in \mathcal{X}_\omega$ and $\omega \in (0, \omega^*)$ (see Corollary 3.3 below).

Remark 1.3. Cipolatti [7] proved that if $n = 2$, $p = 3$ and $a > -1$, then u_ω is strongly unstable for any $\omega \in (0, \infty)$ by using the idea of Weinstein [22]. This is the so called critical case. On the other hand, in Theorem 1.2, we consider the super critical case, which was treated in the paper [1] by Berestycki and Cazenave for the case of local nonlinearity. They consider the associated minimization problem (see Proposition 4.1 below), and employ the radial rearrangement and radial compactness lemma due to Strauss [21] to obtain the suitable minimizer (see also Cazenave [3; Section 8.2]). Our proof of Theorem 1.2 is based on the idea of Berestycki and Cazenave [1]. However, since the equation (1.1) contains an anisotropic nonlinearity $E_1(|u|^2)u$, we use the compactness lemmas due to Fröhlich, Lieb and Loss [9], Lieb [13] and Brézis and Lieb [2] (see Lemmas 2.2, 2.3 and 2.4 below), following Nawa [16], instead of the radial compactness lemma.

Remark 1.4. It is interesting to compare Theorem 1.2 with the result by Cipolatti [7]. When $n = 2$ and $a(p-3) > 0$ or $n = 3$ and $a(p-3) \geq 0$, Cipolatti [7] constructed an unstable solution which exists globally in time, by using the Pohozaev multiplier $x \cdot \nabla\varphi_\omega$. On the other hand, the author [17] gave an unstable flow by using the scaling $\varphi_\omega^\lambda(x) = \lambda^{n/2}\varphi_\omega(\lambda x)$, when $a > 0$, $p \geq 1 + 4/n$ and $n = 2$ or 3. However, it was an open problem whether the unstable solutions of (1.1) given by [17] exist globally in time or blow up in a finite time. Theorem 1.2 gives an answer for the above problem in the case when $n = 3$, $a > 0$ and $1 + 4/3 < p < 5$ or $a < 0$ and $1 < p < 3$ (see Proof of Theorem 1.2 in Section 4).

Our plan in this paper is as follows. In Section 2, we give one proposition concerning (1.7) in Theorem 1.1. We give the proof of Theorems 1.1 and 1.2 in Sections 3 and 4, respectively.

In what follows, we omit the integral variables with respect to the spatial variable x, and we omit the integral region when it is the whole space $\mathbb{R}^n$. We denote the norms of $L^q(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ by $|\cdot|_q$ and $\|\cdot\|_{H^1}$, respectively. We put

$$\tau_y v(x) = v(x-y), \qquad v^\lambda(x) = \lambda^{n/2} v(\lambda x),\ \lambda > 0,$$
$$B_1(|v|^2) = \int |v|^2 E_1(|v|^2)dx = \int \sigma_1(\xi)|\mathcal{F}(|v|^2)|^2 d\xi,$$

where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^n$. We note that $|v^\lambda|_2 = |v|_2$ holds for any $\lambda > 0$.

2. Preliminaries. In this section, we consider (1.7) in Theorem 1.1.

Proposition 2.1. *Assume that $n = 2$. Let*

$$\mu_0 = \inf\big\{|v|_2^2 : v \in H^1(\mathbb{R}^2),\ v \neq 0,\ \mathcal{E}_0(v) \equiv \tfrac{1}{2}|\nabla v|_2^2 - \tfrac{1}{4}B_1(|v|^2) \leq 0\big\}, \tag{2.1}$$

$$J_0 = \inf\left\{J(v) \equiv |v|_2^2|\nabla v|_2^2/B_1(|v|^2) : v \in H^1(\mathbb{R}^2),\ v \neq 0\right\}. \tag{2.2}$$

Then, there exists a function $Q \in H^1(\mathbb{R}^2)$ such that $Q \not\equiv 0$, $|Q|_2^2 = \mu_0$ and $\mathcal{E}_0(Q) = 0$. Moreover, we have $J_0 = \mu_0/2$.

Following Nawa [16], we prove Proposition 2.1 by using the following basic lemmas, which will also be used in the proof of Theorem 1.2 in Section 4.

Lemma 2.2 (Fröhlich, Lieb and Loss [9]). *Let $1 < \alpha < \beta < \gamma$ and let $f(x)$ be a measurable function on $\mathbb{R}^n$ such that $|f|_\alpha \leq C_\alpha$, $|f|_\beta \geq C_\beta > 0$ and $|f|_\gamma \leq C_\gamma$ for some positive constants C_α, C_β and C_γ. Then we have meas$(\{x \in \mathbb{R}^n : |f(x)| > \eta\}) \geq C$ for some positive constants η and C depending on α, β, γ, C_α, C_β and C_γ, but not on f, where meas$(\cdot)$ is the Lebesgue measure on $\mathbb{R}^n$.*

Lemma 2.3 (Lieb [13]). *Let (f_j) be a bounded sequence of functions in $H^1(\mathbb{R}^n)$ such that meas$(\{x \in \mathbb{R}^n : |f_j(x)| > \eta\}) \geq C$ for some positive constants η and C. Then there exists a sequence $(y_j) \subset \mathbb{R}^n$ such that, for some subsequence (still denoted by the same letter), $\tau_{y_j} f_j \rightharpoonup f \not\equiv 0$ weakly in $H^1(\mathbb{R}^n)$.*

Lemma 2.4 (Brézis and Lieb [2]). *Let (f_j) be a bounded sequence in $L^q(\mathbb{R}^n)$ for $q \in (0,\infty)$. Suppose that $f_j \to f$ almost everywhere in $\mathbb{R}^n$. Then, we have*

$$|f_j|_q^q - |f_j - f|_q^q - |f|_q^q \to 0. \tag{2.3}$$

Remark 2.1. When $q = 2$, the assumption that $f_j \to f$ almost everywhere in $\mathbb{R}^n$ is not needed. That is, it is easily verified that if $f_j \rightharpoonup f$ weakly in $L^2(\mathbb{R}^n)$, then (2.3) holds with $q = 2$.

Proof of Proposition 2.1. In what follows, we shall often extract subsequences without explicitly mentioning this fact. Let (v_j) be a minimizing sequence for (2.1). Note that we do not know whether (v_j) is bounded in $H^1(\mathbb{R}^2)$ or not. So, we consider the following scaled functions:

$$Q_j(x) = \lambda_j v_j(\lambda_j x), \qquad \lambda_j = B_1(|v_j|^2)^{-1/2}.$$

Then, we have $|Q_j|_2^2 = |v_j|_2^2 \to \mu_0$, $B_1(|Q_j|^2) = 1$ and $\mathcal{E}_0(Q_j) = \lambda_j^2 \mathcal{E}_0(v_j) \le 0$. From Lemmas 2.2 and 2.3, we obtain a subsequence of (Q_j) (we still denote it by the same letter) such that

$$Q_j^1 \equiv \tau_{y_j} Q_j \rightharpoonup Q \not\equiv 0 \quad \text{ weakly in } H^1(\mathbb{R}^2)$$

for some $(y_j) \subset \mathbb{R}^2$. From the fact that $Q_j^1 \to Q$ almost everywhere in $\mathbb{R}^2$ and Egorov's theorem, we have $|Q_j^1|^2 \rightharpoonup |Q|^2$ weakly in $L^2(\mathbb{R}^2)$. Since the linear operator $v \mapsto \sigma_1(\xi)^{1/2}\mathcal{F}(v)$ is bounded on $L^2(\mathbb{R}^2)$, we also have $\sigma_1(\xi)^{1/2}\mathcal{F}(|Q_j^1|^2) \rightharpoonup \sigma_1(\xi)^{1/2}\mathcal{F}(|Q|^2)$ weakly in $L^2(\mathbb{R}^2)$. Therefore, from Lemma 2.4 and Remark 2.1, we obtain

$$\mathcal{E}_0(Q_j^1) - \mathcal{E}_0(Q_j^1 - Q) - \mathcal{E}_0(Q) \to 0, \tag{2.4}$$

$$|Q_j^1|_2^2 - |Q_j^1 - Q|_2^2 - |Q|_2^2 \to 0. \tag{2.5}$$

Now, we suppose that $\mathcal{E}_0(Q) > 0$. Then, it follows from (2.4) and the fact that $\mathcal{E}_0(Q_j^1) \le 0$ that $\mathcal{E}_0(Q_j^1 - Q) \le 0$ for sufficiently large j. Thus, by the definition of μ_0, we get $|Q_j^1 - Q|_2^2 \ge \mu_0$ for large j. Since $|Q_j^1|_2^2 \to \mu_0$, it follows from (2.5) that $|Q|_2^2 \le 0$, which is a contradiction. Hence, we obtain $\mathcal{E}_0(Q) \le 0$.

From the definition of μ_0, we have $\mu_0 \le |Q|_2^2 \le \liminf_{j\to\infty} |Q_j^1|_2^2 = \mu_0$. Hence, we obtain $|Q|_2^2 = \mu_0$. Here, we assume that $\mathcal{E}_0(Q) < 0$. Since $\mathcal{E}_0(\theta Q) = (\theta^2/2)|\nabla Q|_2^2 - (\theta^4/4)B_1(|Q|^2) > 0$ for sufficiently small $\theta > 0$, there exists $\theta_0 \in (0,1)$ such that $\mathcal{E}_0(\theta_0 Q) = 0$. Then, we get $|\theta_0 Q|_2^2 = \theta_0^2 |Q|_2^2 < |Q|_2^2 = \mu_0$, which contradicts the definition of μ_0. Hence, we have $\mathcal{E}_0(Q) = 0$.

Finally, let $(w_j) \subset H^1(\mathbb{R}^2)$ be a minimizing sequence for (2.2). We rescale w_j as follows:

$$W_j(x) = w_j(x/\nu_j), \qquad \nu_j = (2|\nabla w_j|_2^2 / B_1(|w_j|^2))^{1/2}.$$

Then, we get $J(W_j) = J(w_j)$ and $\mathcal{E}_0(W_j) = (1/2)|\nabla w_j|_2^2 - (\nu_j^2/4)B_1(|w_j|^2) = 0$, from which it follows that $|W_j|_2^2 \to 2J_0$. Therefore, by the definition of μ_0, we have $\mu_0 \le 2J_0$. On the other hand, by the definition of J_0, we have $J_0 \le J(Q) = |Q|_2^2/2 = \mu_0/2$. Hence, we obtain $J_0 = \mu_0/2$.

3. Proof of Theorem 1.1. We first consider the case when $n = 2$, $a > 0$ and $1 < p < 3$.

Lemma 3.1. *Assume that $n = 2$, $a > 0$ and $1 < p < 3$. Let $\mu \in (0, \mu_0)$. Then, any minimizing sequence for (1.6_μ) is relatively compact in $H^1(\mathbb{R}^2)$, up to a translation. In particular, Σ_μ is not empty.*

Proof. First, it follows from Proposition 2.1 that $I_\mu > -\infty$ for $\mu \in (0, \mu_0)$. In fact, for any $v \in H^1(\mathbb{R}^2)$ satisfying $|v|_2^2 = \mu < \mu_0$, we get

$$\mathcal{E}(v) \ge \frac{1}{2}\big(1 - \frac{\mu}{\mu_0}\big)|\nabla v|_2^2 - \frac{a}{p+1} C\mu |\nabla v|_2^{p-1}$$

for some $C > 0$. Thus, for $\mu \in (0, \mu_0)$, we have $I_\mu > -\infty$ and any minimizing sequence for (1.6_μ) is bounded in $H^1(\mathbb{R}^2)$. Moreover, for $v \in H^1(\mathbb{R}^2)$ satisfying $|v|_2^2 = \mu$, we get

$$\mathcal{E}(v^\lambda) = \frac{\lambda^2}{2}|\nabla v|_2^2 - \frac{a}{p+1}\lambda^{p-1}|v|_{p+1}^{p+1} - \frac{\lambda^2}{4}B_1(|v|^2) < 0$$

for sufficiently small $\lambda > 0$. Since $|v^\lambda|_2^2 = |v|_2^2 = \mu$, we have $I_\mu < 0$.

From Lemma II.1 in [14], if

$$I_{\theta\mu} < \theta I_\mu, \quad \mu \in (0, \mu_0),\ \theta \in (1, \mu_0/\mu), \tag{3.1}$$

then we have

$$I_\mu < I_\alpha + I_{\mu-\alpha}, \quad \mu \in (0, \mu_0),\ \alpha \in (0, \mu). \tag{3.2}$$

Moreover, from (3.2), we can prove Lemma 3.1 in the same way as the proof of Theorem I.2 in [15]. Therefore, it is enough to show (3.1).

From the fact that $I_\mu < 0$, there exists $\delta > 0$ such that

$$I_\mu = \inf\{\mathcal{E}(v) : v \in H^1(\mathbb{R}^2),\ |v|_2^2 = \mu,\ |\nabla v|_2^2 \geq \delta\}.$$

Then, we have

$$I_{\theta\mu} \leq \inf\{\mathcal{E}(v(\cdot/\sqrt{\theta})) : v \in H^1(\mathbb{R}^2),\ |v|_2^2 = \mu,\ |\nabla v|_2^2 \geq \delta\},$$

and

$$\mathcal{E}(v(\cdot/\sqrt{\theta})) = \frac{1}{2}|\nabla v|_2^2 - \frac{\theta a}{p+1}|v|_{p+1}^{p+1} - \frac{\theta}{4}B_1(|v|^2) \leq \theta\mathcal{E}(v) - \frac{1}{2}(\theta - 1)\delta.$$

Thus, we have

$$I_{\theta\mu} \leq \theta I_\mu - \frac{1}{2}(\theta - 1)\delta < \theta I_\mu,$$

which shows (3.1) and the proof of Lemma 3.1 is completed. □

Theorem 1.1 (1) is proved by the method of Cazenave and Lions [5], by using Lemma 3.1 (see also Cazenave and Esteban [4]).

Proof of Theorem 1.1 (1). We prove by contradiction that if Σ_μ is not stable, then there exists $\varepsilon_0 > 0$ and a sequence $(u_{0j}) \subset H^1(\mathbb{R}^2)$ such that

$$\inf_{\varphi \in \Sigma_\mu} \|u_{0j} - \varphi\|_{H^1} \to 0, \qquad \sup_{0 \leq t < \infty} \inf_{\varphi \in \Sigma_\mu} \|u_j(t) - \varphi\|_{H^1} \geq \varepsilon_0, \tag{3.3}$$

where $u_j(t)$ is a solution of (1.1) with $u_j(0) = u_{0j}$. By continuity in t, we can take $t_j > 0$ such that

$$\inf_{\varphi \in \Sigma_\mu} \|u_j(t_j) - \varphi\|_{H^1} = \varepsilon_0, \tag{3.4}$$

and the solution u_j existing at least in the time interval $[0, t_j]$. From (3.3) and the conservation laws (1.3) and (1.4), we have

$$|u_j(t_j)|_2^2 = |u_{0j}|_2^2 \to \mu, \tag{3.5}$$

$$\mathcal{E}(u_j(t_j)) = \mathcal{E}(u_{0j}) \to I_\mu. \tag{3.6}$$

From (3.5), (3.6) and Lemma 3.1, there exists a subsequence of $(u_j(t_j))$ (we still denote it by the same letter),

$$\tau_{y_j} u_j(t_j) \to \varphi_0 \quad \text{in} \quad H^1(\mathbb{R}^2)$$

for some $\varphi_0 \in \Sigma_\mu$ and some $(y_j) \subset \mathbb{R}^2$. This contradicts (3.4). This completes the proof of Theorem 1.1 (1). □

Next, we consider the case when $n = 2$, $a < 0$ and $p > 3$.

Lemma 3.2. *Assume that $n = 2$, $a < 0$ and $p > 3$. Let $\mu \in (\mu_0, \infty)$. Then, any minimizing sequence for (1.6_μ) is relatively compact in $H^1(\mathbb{R}^2)$, up to a translation. In particular, Σ_μ is not empty.*

Proof. First, for any $v \in H^1(\mathbb{R}^2)$ satisfying $|v|_2^2 = \mu$, there exists a positive constant C such that

$$\frac{1}{4} B_1(|v|^2) \le \frac{1}{4}|v|_4^4 \le \frac{|a|}{2(p+1)}|v|_{p+1}^{p+1} + C|v|_2^2,$$

which implies

$$\mathcal{E}(v) \ge \frac{1}{2}|\nabla v|_2^2 + \frac{|a|}{2(p+1)}|v|_{p+1}^{p+1} - C\mu.$$

Therefore, for any $\mu > 0$, we have $I_\mu > -\infty$ and any minimizing sequence for (1.6_μ) is bounded in $H^1(\mathbb{R}^2)$.

Next, we show that

$$I_\mu \ge 0, \quad \mu \in (0, \mu_0], \tag{3.7}$$

$$I_\mu < 0, \quad \mu \in (\mu_0, \infty). \tag{3.8}$$

In fact, for any $v \in H^1(\mathbb{R}^2)$ satisfying $|v|_2^2 = \mu \le \mu_0$, it follows from Proposition 2.1 that

$$\frac{1}{4} B_1(|v|^2) \le \frac{1}{2\mu_0}|v|_2^2 |\nabla v|_2^2 \le \frac{1}{2}|\nabla v|_2^2,$$

which implies $\mathcal{E}_0(v) \ge 0$. Thus, we have (3.7).

Next, for $\mu > \mu_0$, we put $w = \theta Q$ and $\theta = (\mu/\mu_0)^{1/2}$, where Q is given in Proposition 2.1. Then, we have $|w|_2^2 = \mu$ and

$$\mathcal{E}_0(w) = \frac{\theta^2}{2}|\nabla Q|_2^2 - \frac{\theta^4}{4} B_1(|Q|^2) = \frac{\theta^2}{4}(1 - \theta^2) B_1(|Q|^2) < 0.$$

Moreover, we have

$$\mathcal{E}(w^\lambda) = \lambda^2 \mathcal{E}_0(w) + \frac{|a|}{p+1}\lambda^{p-1}|w|_{p+1}^{p+1} < 0$$

for sufficiently small $\lambda > 0$. Since $|w^\lambda|_2^2 = |w|_2^2 = \mu$, we have (3.8). From (3.8), we can prove that

$$I_{\theta\mu} < \theta I_\mu, \quad \mu > \mu_0,\ \theta > 1 \tag{3.9}$$

similarly to (3.1) in the proof of Lemma 3.1. From (3.7), (3.8) and (3.9), we obtain

$$I_\mu < I_\alpha + I_{\mu-\alpha}, \quad \mu > \mu_0,\ \alpha \in (0,\mu). \tag{3.10}$$

Indeed, if for example $\alpha > \mu_0$ and $0 < \mu - \alpha \le \mu_0$, then we have

$$I_\mu = I_{(\mu/\alpha)\alpha} < \frac{\mu}{\alpha} I_\alpha < I_\alpha \le I_\alpha + I_{\mu-\alpha}.$$

Therefore, from (3.10), we can prove Lemma 3.2 in the same way as the proof of Theorem I.2 in [15]. □

Corollary 3.3. *Assume that* $n = 2$, $a < 0$ *and* $p > 3$. *Then,* $|\varphi_\omega|_2^2 > \mu_0$ *holds for any* $\varphi_\omega \in \mathcal{X}_\omega$ *and* $\omega \in (0, \omega^*)$.

Proof. Since $\psi \in \mathcal{X}_\omega$, we get

$$0 = \partial_\lambda S_\omega(\psi^\lambda)\big|_{\lambda=1} = |\nabla\psi|_2^2 + \frac{p-1}{p+1}|a||\psi|_{p+1}^{p+1} - \frac{1}{2}B_1(|\psi|^2).$$

Thus, we have

$$\mathcal{E}(\psi) = -\frac{p-3}{2(p+1)}|a||\psi|_{p+1}^{p+1} < 0,$$

which implies together with (3.7) that $|\psi|_2^2 > \mu_0$. □

Since we can prove Theorem 1.1(2) similarly to the proof of Theorem 1.1(1), we omit it.

4. Proof of Theorem 1.2. In this section we give the proof of Theorem 1.2. Since we fix the parameter ω, we drop the subscript ω. Thus, we write φ for φ_ω, S for S_ω, and so on. We put

$$P(v) = |\nabla v|_2^2 - \frac{3(p-1)}{2(p+1)} a|v|_{p+1}^{p+1} - \frac{3}{4}B_1(|v|^2). \tag{4.1}$$

The functional (4.1) is closely related to the pseudo-conformal conservation law (see (4.10) below). We also note that $P(v) = \partial_\lambda S(v^\lambda)\big|_{\lambda=1} = \partial_\lambda \mathcal{E}(v^\lambda)\big|_{\lambda=1}$.

We first prove a key proposition to obtain Theorem 1.2.

Proposition 4.1. *Assume the same condition as in Theorem* 1.2. *Then,* φ *is a ground state of* (1.2) *if and only if* $\varphi \in M$ *and* $m = S(\varphi)$, *where*

$$m = \inf\{S(v) : v \in M\}, \quad M = \{v \in H^1(\mathbb{R}^3) : v \ne 0,\ P(v) = 0\}. \tag{4.2}$$

We split the proof of Proposition 4.1 into the following three cases. Case 1: $a > 0$ and $1 + 4/3 < p \le 3$. Case 2: $a > 0$ and $3 < p < 5$. Case 3: $a < 0$ and $1 < p < 3$. In Case 1, we put

$$S^1(v) = S(v) - \frac{2}{3(p-1)}P(v) = \gamma_1|\nabla v|_2^2 + \frac{\omega}{2}|v|_2^2 + \gamma_2 B_1(|v|^2), \tag{4.3}$$

where $\gamma_1 = (3p-7)/(6(p-1)) > 0$ and $\gamma_2 = (3-p)/(4(p-1)) \geq 0$.

In Cases 2 and 3, we put

$$S^2(v) = S^3(v) = S(v) - \frac{1}{3}P(v) = \frac{1}{6}|\nabla v|_2^2 + \frac{\omega}{2}|v|_2^2 + \gamma_3 |v|_{p+1}^{p+1}, \tag{4.4}$$

where $\gamma_3 = (a(p-3))/(2(p+1)) > 0$.

Moreover, for $k = 1, 2, 3$, we set

$$m_k = \inf\{S^k(v) : v \in H^1(\mathbb{R}^3),\ v \neq 0,\ P(v) \leq 0\}. \tag{4.5}$$

Lemma 4.2. *Let $k = 1, 2, 3$. Then* (4.5) *is attained at some $w \in M$.*

Proof. We prove Case 1 only. The other cases can be proved analogously. Let (v_j) be a minimizing sequence for (4.5). Since $\gamma_1 > 0$ and $\gamma_2 \geq 0$, (v_j) is bounded in $H^1(\mathbb{R}^3)$. We claim that $\liminf_{j\to\infty} |v_j|_4^4 > 0$. In fact, suppose that $|v_j|_4^4 \to 0$. Then, we get $|v_j|_{p+1}^{p+1} \to 0$ and $B_1(|v_j|^2) \to 0$, which imply together with $P(v_j) \leq 0$ that $|\nabla v_j|_2^2 \to 0$. On the other hand, from the fact that $P(v_j) \leq 0$ and the Gagliardo–Nirenberg inequality, there exist positive constants C_1 and C_2 such that

$$1 \leq C_1 |v_j|_2^{(5-p)/2} |\nabla v_j|_2^{(3p-7)/2} + C_2 |v_j|_2 |\nabla v_j|_2.$$

Since (v_j) is bounded in $L^2(\mathbb{R}^3)$, this contradicts $|\nabla v_j|_2^2 \to 0$. Thus, we obtain

$$\liminf_{j\to\infty} |v_j|_4^4 > 0.$$

From Lemmas 2.2 and 2.3, we have a subsequence of (v_j) (we still denote it by the same letter) such that

$$w_j \equiv \tau_{y_j} v_j \rightharpoonup w \not\equiv 0 \quad \text{weakly in } H^1(\mathbb{R}^3)$$

for some $(y_j) \subset \mathbb{R}^3$. Moreover, similarly to the proof of Proposition 2.1, we have

$$w_j \to w \quad \text{a.e. on } \mathbb{R}^3,$$
$$\sigma_1(\xi)^{1/2}\mathcal{F}(|w_j|^2) \rightharpoonup \sigma_1(\xi)^{1/2}\mathcal{F}(|w|^2) \quad \text{weakly in } L^2(\mathbb{R}^3).$$

Therefore, from Lemma 2.4 and Remark 2.1, we obtain

$$P(w_j) - P(w_j - w) - P(w) \to 0, \tag{4.6}$$
$$S^1(w_j) - S^1(w_j - w) - S^1(w) \to 0. \tag{4.7}$$

Similarly to the proof of Proposition 2.1, from (4.6) and (4.7), we have $P(w) \leq 0$. Moreover, it follows from the definition of m_1 that

$$m_1 \leq S^1(w) \leq \liminf_{j\to\infty} S^1(w_j) = m_1.$$

Hence, we have $S^1(w) = m_1$.

Finally, we show that $P(w) = 0$. Suppose that $P(w) < 0$. Since

$$P(w^\lambda) = \lambda^2 |\nabla w|_2^2 - \frac{3(p-1)}{2(p+1)} a \lambda^{3(p-1)/2} |w|_{p+1}^{p+1} - \frac{3}{4}\lambda^3 B_1(|w|^2) > 0$$

for sufficiently small $\lambda > 0$, there exists $\lambda_0 \in (0,1)$ such that $P(w^{\lambda_0}) = 0$. Then, we have

$$S^1(w^{\lambda_0}) = \gamma_1 \lambda_0^2 |\nabla w|_2^2 + \frac{\omega}{2}|w|_2^2 + \gamma_2 \lambda_0^3 B_1(|w|^2) < S^1(w) = m_1,$$

which contradicts the definition of m_1. Hence, we have $P(w) = 0$. □

It follows from the fact that $m_k = m$ and Lemma 4.2 that there exists $w \in M$ such that $m = S(w)$.

Lemma 4.3. *If $w \in M$ satisfies $S(w) = m$, then we have $S'(w) = 0$.*

Proof. We prove Case 1 only. The other cases can be proved analogously. From $P(w) = 0$, we have

$$\begin{aligned}\langle P'(w), w\rangle &= 2|\nabla w|_2^2 - \frac{3(p-1)}{2} a |w|_{p+1}^{p+1} - 3B_1(|w|^2) \\ &= -\big((p-1)|\nabla w|_2^2 + \frac{3(3-p)}{4} B_1(|w|^2)\big) < 0.\end{aligned}$$

Thus, from our assumption, there exists a constant $\eta \in \mathbb{R}$ such that

$$S'(w) + \eta P'(w) = 0. \tag{4.8}$$

To conclude the proof, it is enough to show that $\eta = 0$.

We first claim that $2\eta + 1 > 0$. In fact, suppose that $2\eta + 1 \le 0$. Then, from (4.8) and $P(w) = 0$, we have

$$\begin{aligned}0 &= \langle S'(w) + \eta P'(w), w\rangle \\ &= (2\eta+1)|\nabla w|_2^2 + \omega |w|_2^2 - \big(\frac{3(p-1)}{2}\eta + 1\big) a |w|_{p+1}^{p+1} - (3\eta+1) B_1(|w|^2) \\ &= \frac{-3(p-1)^2\eta + p - 5}{3(p-1)} |\nabla w|_2^2 + \omega|w|_2^2 + \frac{3-p}{4(p-1)}\big(-3(p-1)\eta + 2\big) B_1(|w|^2) \\ &\ge \frac{(3p-7)(p+1)}{6(p-1)} |\nabla w|_2^2 + \omega |w|_2^2 + \frac{(3-p)(3p+1)}{8(p-1)} B_1(|w|^2) > 0.\end{aligned}$$

This is a contradiction. Thus, we have $2\eta+1 > 0$. Therefore, from (4.8) and [6; Theorem 2.4], we have

$$0 = \partial_\lambda (S + \eta P)(w^\lambda)\big|_{\lambda=1} = P(w) + \eta\big\{2|\nabla w|_2^2 - \frac{9(p-1)^2}{4(p+1)} a |w|_{p+1}^{p+1} - \frac{9}{4} B_1(|w|^2)\big\}. \tag{4.9}$$

Since $P(w)=0$, if $\eta\neq 0$, then it follows from (4.9) that

$$\begin{aligned}0&=2|\nabla w|_2^2-\frac{9(p-1)^2}{4(p+1)}a|w|_{p+1}^{p+1}-\frac{9}{4}B_1(|w|^2)\\&=-\Big\{\frac{(3p-7)}{2}|\nabla w|_2^2+\frac{9(3-p)}{8}B_1(|w|^2)\Big\}<0.\end{aligned}$$

This is a contradiction. Hence, we obtain that $\eta=0$. □

Since $P(\psi)=0$ for any $\psi\in\mathcal{X}$, Proposition 4.1 follows from Lemmas 4.2 and 4.3. Next, we give the proof of Theorem 1.2.

Lemma 4.4. *Let $\mathcal{A}=\{v\in H^1(\mathbb{R}^3):S(v)<m,\,P(v)<0\}$. If $u_0\in\mathcal{A}$ and $u(t)$ is the solution of* (1.1) *with $u(0)=u_0$, then we have $P(u(t))\le S(u_0)-m$ for any $t\in[0,T^*(u_0))$.*

Proof. From the conservation laws (1.3) and (1.4), we have $S(u(t))=S(u_0)<m$ for any $t\in[0,T^*(u_0))$. Moreover, from Proposition 4.1, we have $P(u(t))\neq 0$ for any $t\in[0,T^*(u_0))$. Since the mapping $t\mapsto P(u(t))$ is continuous, we have $P(u(t))<0$ for any $t\in[0,T^*(u_0))$. Thus, it follows from Lemma 4.2 that

$$m=m_k\le S^k(u(t))<S(u_0)-P(u(t))$$

for any $t\in[0,T^*(u_0))$, which shows Lemma 4.4. □

Lemma 4.5. *Let $\varphi\in\mathcal{G}$. Then, we have $\varphi^\lambda\in\mathcal{A}$ for any $\lambda>1$.*

Proof. We prove Case 1 only. The other cases can be proved analogously. Since $\partial_\lambda S(\varphi^\lambda)$

$$\begin{aligned}&=\lambda^{3(p-1)/2-1}\big(\lambda^{2-3(p-1)/2}|\nabla\varphi|_2^2-\frac{3(p-1)}{2(p+1)}a|\varphi|_{p+1}^{p+1}-\frac{3}{4}\lambda^{3-3(p-1)/2}B_1(|\varphi|^2)\big)\\&<\lambda^{3(p-1)/2-1}P(\varphi)=0,\end{aligned}$$

we have $S(\varphi^\lambda)<S(\varphi)=m$ for any $\lambda>1$. Moreover, we have $P(\varphi^\lambda)=\lambda\partial_\lambda S(\varphi^\lambda)<0$ for any $\lambda>1$. Hence, we have $\varphi^\lambda\in\mathcal{A}$ for any $\lambda>1$. □

Proof of Theorem 1.2. From Lemma 4.5, we have $\varphi^\lambda\in\mathcal{A}$ for any $\lambda>1$. Let $u_\lambda(t)$ be the solution of (1.1) with $u_\lambda(0)=\varphi^\lambda$. Since $\int|x|^2|\varphi^\lambda(x)|^2dx<\infty$ (see [6; Theorem 2.4]), we have the pseudo-conformal identity (see, e.g., [10]),

$$\frac{d^2}{dt^2}\int|x|^2|u_\lambda(t,x)|^2dx=8P(u_\lambda(t)),\quad t\in[0,T^*(\varphi^\lambda)).\tag{4.10}$$

Furthermore, from Lemma 4.4, we have that

$$P(u_\lambda(t))\le S(\varphi^\lambda)-m<0,\quad t\in[0,T^*(\varphi^\lambda)).\tag{4.11}$$

Therefore, it follows from (4.10) and (4.11) that $T^*(\varphi^\lambda) < \infty$. Since $\lim_{\lambda\to 1} \|\varphi^\lambda - \varphi\|_{H^1} = 0$, the proof is completed. $\square$

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