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A NONRESONANCE RESULT WITH RESPECT TO THE FUČIK SPECTRUM FOR A SECOND ORDER DIFFERENTIAL EQUATION

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1. Introduction. This paper deals with the existence of solutions to the following problem

$$\begin{cases} -u''(t) = f(t, u(t)), \quad t \in [0, 2\pi] \\ u(0) = u(2\pi); \quad u'(0) = u'(2\pi). \end{cases} \tag{1.1}$$

We are interested in cases where $f : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ is asymptotically linear and "asymmetric." By this we mean that the behaviour of $\frac{f(t,s)}{s}$ for s going to $+\infty$ may be different from what it is for s going to $-\infty$ ("jumping" nonlinearities).

Fučik and Dancer [1, 2], who first studied this problem, showed that the solvability of (1.1) strongly depends on the behaviour of $\lim_{s\to+\infty} \frac{f(t,s)}{s}$, $\lim_{s\to-\infty} \frac{f(t,s)}{s}$ with respect to some set $\Sigma \subset \mathbb{R}^2$. This set, the so-called Fučik spectrum, is the set of $(\alpha, \beta) \in \mathbb{R}^2$ for which the problem (1.1) with $f(t, s) = \alpha s_+ - \beta s_-$ has a nontrivial solution ($s_+ = \max\{s, 0\}$, $s_- = (-s)_+$). It is well known and it can be easily verified that Σ is composed of two lines $\mathbb{R} \times \{0\}$, $\{0\} \times \mathbb{R}$ and the curves C_N, $N \geq 1$

$$C_N = \big\{(\alpha, \beta) \in \mathbb{R}^+_* \times \mathbb{R}^+_*;\ \frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{N}\big\}. \tag{1.2}$$

Subsequently, various conditions have been introduced in order to guarantee nonresonance in problem 1.1 (cf. [3]–[6]). In particular, in [6], Gossez-Omari used the following assumptions:

$$\begin{cases} \text{There exist two points } (\alpha, \beta) \in C_N, (a, b) \in C_{N+1} \text{ for some} \\ N \geq 0 \text{ (here } C_0 = \{(0, 0)\}) \text{ such that} \\ \alpha \leq \lambda(t) = \liminf\limits_{s\to+\infty} \dfrac{f(t, s)}{s} \leq \limsup\limits_{s\to+\infty} \dfrac{f(t, s)}{s} = p(t) \leq a \\ \beta \leq \mu(t) = \liminf\limits_{s\to-\infty} \dfrac{f(t, s)}{s} \leq \limsup\limits_{s\to-\infty} \dfrac{f(t, s)}{s} = q(t) \leq b \end{cases} \tag{1.3}$$

and

$$\begin{cases} \text{the sets } \{t \in [0, 2\pi]; \lambda(t) > \alpha, \mu(t) > \beta\} \text{ and} \\ \{t \in [0, 2\pi]; p(t) < a, q(t) < b\} \quad \text{have positive measure.} \end{cases} \tag{1.4}$$

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The main argument used in [6] to prove the existence of solutions of (1.1) under conditions (1.3) and (1.4) was the degree theory of Leray-Shauder.

In a previous paper [7] we gave an existence theorem for problem (1.1) when $\frac{f(t,s)}{s}$ lies asymptotically in a rectangle between $(0,0)$ and the first curve C_1. We were able there to replace hypothesis (1.4) by a weaker condition bearing on the primitive of f, $F(t,s) = \int_0^s f(t,\rho)d\rho$. The method that we used in [7] was variational and our main tool was a variational characterization of the first curve C_1 (cf. Lemma 4.3 of [7]).

Recently, De Figueiredo and Ruf [8] have derived a minimax characterization of the full Fučik spectrum in the periodic case. Using this result, we present in this paper a variational approach to problem (1.1) in the case where $\frac{f(t,s)}{s}$ lies asymptotically in a rectangle R between two consecutive curves C_N and C_{N+1} of the Fučik spectrum.

For technical reasons, we impose one restriction, called condition (*), to the lowest point (α, β) of the rectangle R (as shown in Figure 1.1). Roughly speaking, condition (*) allows us, in a rather natural way, to split the surrounding space into two sets in which the functional is coercive/anticoercive (see Lemma 5.3 and Remark 5.4). We observe that condition (*) is not new in the litterature when dealing with variational methods. It appears for instance in [9, 10] and it is also used there to decompose the functional space.

It should be mentioned that our result improves the results of [6] when a "non-autonomous" nonlinearity f is considered. Indeed, in [6] Gossez-Omari are able to weaken hypotheses (1.3), (1.4) by using positive density conditions. These conditions can be interpreted in the autonomous case; i.e., when $f(t,s) = g(s) + h(t)$, in terms of the potential of g, $G(s) = \int_0^s g(\rho)d\rho$, and they coincide in that case with our conditions. They are however rather complicated in the general nonautonomous case. In the autonomous case the sharpest results are, to our knowledge, those by Habets-Omari-Zanolin in [11]. They weakened hypothesis (F) of our Theorem 2.1 by considering instead $h \in L^\infty$ and

$$\liminf_{u\to+\infty} \frac{2G(u)}{u^2} < a; \qquad \limsup_{u\to+\infty} \frac{2G(u)}{u^2} > \alpha,$$
$$\liminf_{u\to-\infty} \frac{2G(u)}{u^2} < b; \qquad \limsup_{u\to-\infty} \frac{2G(u)}{u^2} > \beta.$$

Their proof, which relies on a time-map technique, does not seem to be adaptable to the nonautonomous case that we consider.

The paper is organized as follows. In the next section we state the main result, Theorem 2.1. In Section 3, we explain the variational approach to problem (1.1). For later references we also include here the result of [8] mentioned above. In Section 4, we prove the Palais-Smale condition. Section 5 is devoted to the geometry of the functional and to complete the proof of Theorem 2.1.

2. Statement of the main result. Throughout this paper we denote by $\mathbb{N}$ the set of nonnegative integers, $0 \in \mathbb{N}$. Let $f : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the usual L^2-Caratheodory conditions and let $(\alpha,\beta)\in C_N$, $(a,b)\in C_{N+1}$ for some $N\geq 1$, two points belonging to two consecutive branches of the Fučik spectrum.

Let us suppose that the inequalities in (1.3) are satisfied with some uniformity in $t\in[0,2\pi]$; that is,

$$\begin{cases} \forall\epsilon>0 \text{ there exists } a_\epsilon\in L^2(0,2\pi) \text{ such that, for a.e. } t\in[0,2\pi],\\ (\alpha-\epsilon)s-a_\epsilon(t)\leq f(t,s)\leq(a+\epsilon)s+a_\epsilon(t) \text{ for } s\geq 0\\ (\beta-\epsilon)s+a_\epsilon(t)\geq f(t,s)\geq(b+\epsilon)s-a_\epsilon(t) \text{ for } s\leq 0. \end{cases} \tag{f}$$

Let us consider $F(t,s)=\int_0^s f(t,\rho)d\rho$ and the limits

$$\alpha(t)=\liminf_{s\to+\infty}\frac{2F(t,s)}{s^2},\qquad \beta(t)=\liminf_{s\to-\infty}\frac{2F(t,s)}{s^2},$$
$$a(t)=\limsup_{s\to+\infty}\frac{2F(t,s)}{s^2},\qquad b(t)=\limsup_{s\to-\infty}\frac{2F(t,s)}{s^2},$$

which clearly satisfy the following inequalities:

$$\alpha\leq\alpha(t)\leq a(t)\leq a,\qquad \beta\leq\beta(t)\leq b(t)\leq b\quad \text{a.e } t\in[0,2\pi]. \tag{2.1}$$

We will also assume that

$$\begin{aligned} &\text{meas}(\{t\in[0,2\pi];\alpha(t)>\alpha \text{ and } \beta(t)>\beta\})>0,\\ &\text{meas}(\{t\in[0,2\pi];a(t)<a \text{ and } b(t)<b\})>0. \end{aligned} \tag{F}$$

Let us denote by $\lambda_0<\lambda_1<\lambda_2<\dots$ the sequence of eigenvalues of $-u''$ with 2π-periodic boundary conditions; that is, $\lambda_i=i^2$. Observe that, according to our definition (1.2), we have $(\lambda_N,\lambda_N)\in C_N$ $\forall N\geq 1$.

Theorem 2.1. *Assume that f satisfies hypothesis (f) and (F) for some $(\alpha,\beta)\in C_N$, $(a,b)\in C_{N+1}$ with $N\geq 1$. Assume also that the following condition $(*)$ holds*

$$\lambda_{N-1}\leq\alpha\quad\textit{and}\quad\lambda_{N-1}\leq\beta. \tag{*}$$

Then there exists at least one solution $u\in H^{2,2}$ to problem (1.1).

Remark 2.2. (i) Theorem 2.1 was given in [7] in the case $(\alpha,\beta)=(0,0)$ and $(a,b)\in C_1$ (with a slight change in hypothesis (F) due to the fact that the eigenvalue associated to 0 does not change sign). (ii) The symmetric case $\alpha=\beta=\lambda_N$ and $a=b=\lambda_{N+1}$ in Theorem 2.1 was obtained by Costa-Oliveira in [12]. Actually, the result of [12] is stated for the P.D.E $-\Delta u=f(t,u(t))$ with Dirichlet boundary conditions. Their proof can be, however, easily adapted to problem (1.1). (iii) Our result also improves some results of [6] where strict conditions are imposed on $\lambda(t)$, $\mu(t)$, $p(t)$, $q(t)$ instead of $\alpha(t)$, $\beta(t)$, $a(t), b(t)$.

We show in the figure below some rectangles $R=[\alpha,a]\times[\beta,b]$ to which Theorem 2.1 applies.

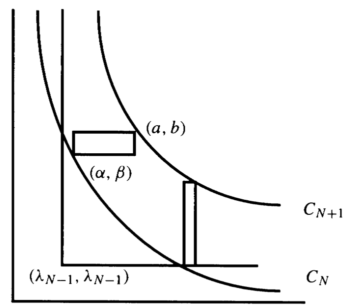

Figure 1.1.

3. The variational approach. We shall work in $H^1_{2\pi} = \{u \in H^1(0,2\pi); u(0) = u(2\pi)\}$. We denote by $\|\cdot\|$ the norm on $H^1_{2\pi}$; that is, $\|u\| = (\|u\|_2^2 + \|u'\|_2^2)^{1/2}$, where $\|u\|_2 = (\int_0^{2\pi} u(t)^2\,dt)^{1/2}$. The scalar product in $L^2(0,2\pi)$ will be denoted by $(\cdot,\cdot)$.

The functional associated with our problem is

$$\phi(u) = \frac{1}{2}\int_0^{2\pi} u'(t)^2\,dt - \int_0^{2\pi} F(t,u(t))\,dt. \tag{3.1}$$

According to hypothesis (f), f has linear growth and so ϕ is a well-defined C^1-functional on $H^1_{2\pi}$. The critical points of ϕ correspond to the weak solutions of (1.1).

We recall that a C^1-functional ϕ defined in a Banach space X satisfies the Palais-Smale condition ((P.S)) if any sequence (x_n) in X such that

(i) $(\phi(x_n))$ is bounded;

(ii) $\phi'(x_n) \to 0$ in X^*;

possesses a convergent subsequence.

Our Theorem 2.1 will follow from the next minimax principle (see [13], [14]) and from Corollary 5.6 of Section 5.

Theorem 3.1. ([13], [14]) *Let X be a Banach space and let $\phi : X \to \mathbb{R}$ be a C^1-functional satisfying the* (P.S) *condition. Let B be a compact metric space and $A \subset B$ a closed nonempty subset. Assume that there exists a map $h_0 \in C(A,X)$ such that*

$$\max_{z\in A}\phi(h_0(z)) < \max_{z\in B}\phi(h(z)) \tag{3.2}$$

for all $h \in \Lambda$, where $\Lambda = \{h \in C(B,X); h|_A = h_0\}$. Then the value

$$c = \inf_{h\in\Lambda}\max_{z\in B}\phi(h(z))$$

is a critical value of ϕ.

Remark 3.2. The value c is finite. Indeed, the Dugundji extension theorem secures the existence of at least one $h \in \Lambda$. Then

$$-\infty < \max_{z\in A}\phi(h_0(z)) \le c \le \max_{z\in B}\phi(h(z)) < \infty$$

since A and B are compact sets.

Remark 3.3. The result of Theorem 3.1 is still valid if one replaces X by a complete Finsler manifold of class $C^{1,1}$ and if $c < +\infty$. In that case, the (PS) condition has to be interpreted in terms of $d_M\phi$, the differential of ϕ on M. The key point in the proof of Theorem 3.1 is a deformation lemma which can also be derived for the manifold M (see for instance [15] for a general deformation lemma). We will use this extended version of Theorem 3.1 in the second part of Lemma 5.3.

In Section 5, we will define two sets A, B and a map $h_0 \in C(A, H^1_{2\pi})$ fulfilling the hypothesis of the minimax principle 3.1. The choice of these sets is mainly motivated by the results of [8] on a variational characterization of the Fučik spectrum. We recall briefly this characterization.

Let $\mu \in \mathbb{R}$ and $N \geq 1$. Let us denote by $(\lambda_N(\mu) + \mu, \lambda_N(\mu))$ the point of C_N lying on the line $y = x - \mu$. Observe that $-\mu, 0, \lambda_1(\mu), \lambda_2(\mu), \ldots$ is the sequence of eigenvalues of the following eigenvalue problem P_μ:

$$-u''(t) - \mu u_+(t) = \lambda u(t), \quad u(0) = u(2\pi);\ u'(0) = u'(2\pi). \tag{P_μ}$$

Let S^1 be the group $\mathbb{R}/2\pi\mathbb{Z}$ and denote by $T_\theta u$, $\theta \in S^1$ and $u \in H^1_{2\pi}$, the θ-translation of u; i.e., $T_\theta u = u(\theta + \cdot)$. (From here on we identify S^1 with $[0, 2\pi[$).

Let $A \subset H^1_{2\pi}$ be a translation invariant set. We associate with A a number $\gamma_0(A) \in \mathbb{N} \cup \{+\infty\}$, called the relative S^1-index of Benci ([16, 20]) which is defined as follows:

$\gamma_0(A) = \inf\{k \in \mathbb{N}; \exists h \in C_e(A, \mathbb{R} \times \mathbb{C}^k)$ such that $h(u) \neq 0\ \forall u \in A$ and $h(c) = (c, 0, 0, \ldots 0)\ \forall c \in \mathbb{R} \cap A\}$,

where

$C_e(A, \mathbb{R} \times \mathbb{C}^k) = \{h = (h_0, h_1, \ldots, h_k) : A \to \mathbb{R} \times \mathbb{C}^k$ continuous; $\exists (m_1, \ldots, m_k) \in (\mathbb{Z}^*)^k$ s.t. $h(T_\theta u) = (h_0(u), \ldots, e^{im_j\theta} h_j(u), \ldots) \forall u \in A\}$.

We define $\gamma_0(\phi) = 0$ and $\gamma_0(A) = +\infty$ if no k can be found in the previous definition. For more information about γ_0 see [16].

We are now ready to state the result of [8].

Theorem 3.4. ([8]) *Let $M = \{u \in H^1_{2\pi}; \int_0^{2\pi} u(t)^2\,dt = 1\}$. For any $\mu \in \mathbb{R}$ denote by $I_\mu(u) = \int_0^{2\pi} u'(t)^2\,dt - \mu \int_0^{2\pi} u_+(t)^2\,dt$ the functional associated with problem P_μ. Then*

$$\lambda_N(\mu) = \inf_{A \in \Gamma_N} \max_{u \in A} I_\mu(u) \quad \forall N \geq 1, \tag{3.3}$$

where $\Gamma_N = \{A \subset M; A$ is compact, invariant and $\gamma_0(A) \geq N\}$.

4. The Palais-Smale condition. We will use the following lemma. The reader is referred to [11] for its proof.

Lemma 4.1. [11] *Let $(\alpha, \beta) \in C_N$ and $(a, b) \in C_{N+1}$ for some $N \geq 1$. Let $m, n \in L^\infty(0, 2\pi)$ satisfy*

$$\alpha \leq m(t) \leq a, \quad \beta \leq n(t) \leq b \quad a.e. \tag{4.1}$$

Then any nontrivial solution of the equation

$$-u''(t) = m(t)u_+(t) - n(t)u_-(t), \quad u(0) = u(2\pi);\ u'(0) = u'(2\pi) \tag{4.2}$$

either has N positive bumps (in S^1) of length $\frac{\pi}{\sqrt{\alpha}}$ with $m(t) \equiv \alpha$ almost everywhere on each positive bump and N negative bumps of length $\frac{\pi}{\sqrt{\beta}}$ with $n(t) \equiv \beta$ almost everywhere on each negative bump or has $N+1$ positive bumps of length $\frac{\pi}{\sqrt{a}}$ with $m(t) \equiv a$ almost everywhere on each positive bump and $N+1$ negative bumps of length $\frac{\pi}{\sqrt{b}}$ with $n(t) \equiv b$ almost everywhere on each negative bump. If, in addition to (4.1), *the sets $\{t; m(t) > \alpha$ and $n(t) > \beta\}$ and $\{t; m(t) < a$ and $n(t) < b\}$ have positive measure, then* (4.2) *possesses only the trivial solution.*

Proposition 4.2. *Let $f : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ satisfy hypotheses (f) and (F) with $(\alpha, \beta) \in C_N$ and $(a, b) \in C_{N+1}$, $N \geq 1$. Then the functional ϕ satisfies the* (P.S) *condition.*

Proof. Let (u_n) be a sequence in $H^1_{2\pi}$ such that (i) There exists a constant $C > 0$ such that

$$|\phi(u_n)| \leq C \quad \forall n \in \mathbb{N}.$$

(ii) There exists a sequence $\epsilon_n \in \mathbb{R}^+$, $\lim_{n\to\infty} \epsilon_n = 0$ such that

$$|(u_n', v') - (f(\cdot, u_n(\cdot)), v)| \leq \epsilon_n \|v\| \quad \forall v \in H^1_{2\pi}.$$

We claim that (u_n) has a converging subsequence. To prove this, it is sufficient to show that the sequence (u_n) is bounded (see [13] Lemma 6.2). Suppose for the sake of contradiction that $\lim_{n\to+\infty} \|u_n\| = +\infty$ and denote $v_n = \frac{u_n}{\|u_n\|}$. Since $\|v_n\| = 1$, there exists some subsequence (still denoted v_n) and some $v_0 \in H^1_{2\pi}$ such that

$$\begin{aligned} v_n &\to v_0 \quad \text{(weakly) in } H^1_{2\pi} \\ v_n &\to v_0 \quad \text{(strongly) in } C^0([0, 2\pi]). \end{aligned}$$

By hypothesis (f) we have that $(\frac{f(\cdot, u_n(\cdot))}{\|u_n\|})$ is a bounded sequence in $L^2(0, 2\pi)$. Thus, up to a subsequence, $(\frac{f(\cdot, u_n(\cdot))}{\|u_n\|})$ converges weakly in $L^2(0, 2\pi)$ to some $f_0 \in L^2(0, 2\pi)$. It is easy to check that f_0 can be written in the following form:

$$f_0(t) = m(t) v_{0_+}(t) - n(t) v_{0_-}(t), \tag{4.3}$$

where $m(t), n(t) \in L^\infty(0, 2\pi)$ satisfy hypothesis (4.1) of Lemma 4.1. Dividing (ii) by $\|u_n\|^2$ we obtain, after passing to the limit,

$$(v_0', v') = (f_0, v) \quad \forall v \in H^1_{2\pi}. \tag{4.4}$$

By substituting (4.3) in (4.4) we finally obtain that v_0 satisfies equation (4.1) (cf. Lemma 4.1).

Let us prove that $v_0 \not\equiv 0$. We choose $v = \frac{v_n}{\|u_n\|}$ in (ii) and we write $\int_0^{2\pi} v_n'(t)^2\, dt = 1 - \int_0^{2\pi} v_n(t)^2\, dt$ to reach

$$\Big|1 - \int_0^{2\pi} v_n(t)^2\, dt - \int_0^{2\pi} \frac{f(t, u_n(t))}{\|u_n\|} v_n(t)\, dt\Big| \leq \frac{\epsilon_n}{\|u_n\|}.$$

After passing to the limit we find

$$1 - \|v_0\|_2^2 = (f_0, v_0). \tag{4.5}$$

By comparing (4.4) for $v = v_0$ with (4.5) we conclude that

$$\|v_0'\|_2^2 = 1 - \|v_0\|_2^2. \tag{4.6}$$

Hence $v_0 \not\equiv 0$.

Now, since v_0 is a nontrivial solution of equation (4.2) with $m, n \in L^\infty$ satisfying (4.1), we can apply the first part of Lemma 4.1. Suppose for instance that the first case occurs (one could proceed analogously for the second case). So we have

$$\begin{cases} m(t) \equiv \alpha & \text{in } \Omega_+ = \{t; v_0(t) > 0\} \\ n(t) \equiv \beta & \text{in } \Omega_- = \{t; v_0(t) < 0\} \end{cases} \tag{4.7}$$

and then, by (4.3), $f_0(t) = \alpha v_{0_+} - \beta v_{0_-}$. This gives in (4.5) the following equality:

$$1 - \|v_0\|_2^2 = \alpha \int_0^{2\pi} v_{0_+}(t)^2\,dt + \beta \int_0^{2\pi} v_{0_-}(t)^2\,dt. \tag{4.8}$$

On the other hand, by dividing (i) by $\|u_n\|^2$, we have

$$\Big|\frac{1}{2}(1 - \int_0^{2\pi} v_n(t)^2\,dt) - \int_0^{2\pi} \frac{F(t,u_n(t))}{\|u_n\|^2}\,dt\Big| \le \frac{C}{\|u_n\|^2}.$$

So, when n goes to $+\infty$ it becomes

$$1 - \|v_0\|_2^2 = \lim_{n\to\infty} \int_0^{2\pi} \frac{2F(t,u_n(t))}{\|u_n\|^2}\,dt,$$

and combining it with (4.8) we find

$$\alpha \int_0^{2\pi} v_{0_+}(t)^2\,dt + \beta \int_0^{2\pi} v_{0_-}(t)^2\,dt = \lim_{n\to\infty} \int_0^{2\pi} \frac{2F(t,u_n(t))}{\|u_n\|^2}\,dt. \tag{4.9}$$

We apply now Fatou's lemma and hypothesis (f) to the right-hand side of (4.9). We obtain

$$\begin{aligned} \alpha \int_0^{2\pi} v_{0_+}(t)^2 + \beta \int_0^{2\pi} v_0(t)^2 &\ge \int_0^{2\pi} \alpha(t) v_{0_+}(t)^2 + \int_0^{2\pi} \beta(t) v_{0_-}(t)^2 \\ &\ge \alpha \int_0^{2\pi} v_{0_+}(t)^2 + \beta \int_0^{2\pi} v_{0_-}(t)^2. \end{aligned}$$

Hence

$$\alpha \int_0^{2\pi} v_{0_+}(t)^2\,dt + \beta \int_0^{2\pi} v_{0_-}(t)^2\,dt = \int_0^{2\pi} \alpha(t) v_{0_+}(t)^2\,dt + \int_0^{2\pi} \beta(t) v_{0_-}(t)^2\,dt$$

which clearly contradicts hypothesis (F). This concludes the proof.

5. On the geometry of the functional. Let us denote by E_j, $j \in \mathbb{N}$, the proper subspace associated with the eigenvalue $\lambda_j = j^2$. That is,

$$E_j = \{r \sin j(\theta + \cdot); r \in \mathbb{R}^+, \theta \in S^1\} \text{ if } j \neq 0 \text{ and } E_0 = \mathbb{R}.$$

We define $F_k = \oplus_{j=0}^k E_j$ for $k \geq 1$.

Let $(\alpha, \beta) \in C_N$, $N \geq 1$, be a given point of the Fučik spectrum. We shall denote by F the set of solutions to the equation

$$\begin{cases} -u''(t) = \alpha u_+(t) - \beta u_-(t), & t \in]0, 2\pi[\\ u(0) = u(2\pi); \ u'(0) = u'(2\pi). \end{cases} \tag{5.1}$$

Thus, $F = \{r v_0(\theta + \cdot); r \in \mathbb{R}^+, \theta \in S^1\}$, where

$$v_0(t) = \begin{cases} \sqrt{\beta} \sin \sqrt{\alpha} t, & t \in [0, \pi/\sqrt{\alpha}] \\ \sqrt{\alpha} \sin \sqrt{\beta}(t - \frac{2\pi}{N}), & t \in [\frac{\pi}{\sqrt{\alpha}}, \frac{2\pi}{N}] \end{cases}$$

$$v_0(t) = v_0(t - \frac{2\pi k}{N}), \quad t \in [\frac{2\pi k}{N}, \frac{2\pi(k+1)}{N}], \quad k = 1, \dots, N-1.$$

For any $R > 0$ let us introduce the set $A_R = \{u \in F_{N-1} + F; \|u\|_2 = R\}$. Clearly, A_R is a compact, translation-invariant set. Let us take $\omega_0 \in H^1_{2\pi}$ to be a function such that $\omega_0 \notin F_{N-1} + F$ and $-\omega_0 \notin F_{N-1} + F$. Such a function ω_0 obviously exists because $(F_{N-1} + F) \cup (F_{N-1} - F) \neq H^1_{2\pi}$. We define the set

$$B_R = \{u \in F_{N-1} + F + \mathbb{R}^+ \omega_0; \|u\|_2 = R\}.$$

Notice that B_R is a compact set containing A_R. Observe also that, in the symmetric case $\alpha = \beta = \lambda_N$, the set A_R is a sphere of F_N of radius R and for $\omega_0 \in E_{N+1}$, B_R is a hemisphere of F_{N+1} having A_R as its border.

We will prove below that, for R large enough, the sets A_R, B_R and the map $h_0 = Id$ satisfy the hypothesis of Theorem 3.1. To this end we start by proving some basic facts about A_R and B_R.

Lemma 5.1. $F_{N-1} \cap F = \{0\}$.

Proof. Suppose first that $\alpha = \beta$. Thus $F = E_N$ and the claim of the lemma follows from the well-known property, $E_N \subset F_{N-1}^{\perp}$. Suppose now that $\alpha \neq \beta$. Observe that any nontrivial solution of equation (5.1) is of class C^2. On the other hand, all the functions in F_{N-1} are of class C^∞. Then $F_{N-1} \cap F = \{0\}$. □

From now on we denote the elements of E_j by e_j; those of F_{N-1} by e and those of F by v. We will also denote by $|A|$ the measure of a measurable set $A \subset [0, 2\pi]$.

Lemma 5.2. *For any $u \in F_{N-1} + F$, $u \not\equiv 0$, we have*

$$|\{t \in [0, 2\pi]; u(t) = 0\}| = 0. \tag{5.2}$$

Proof. We proceed by induction on the number of terms j in the sum $\oplus_{j=0}^{N-1} E_j + F$. Let $u \in e_0 + v$ with $e_0 \in E_0$ and $v \in F$. We can assume that $v \not\equiv 0$, $e_0 \not\equiv 0$; otherwise the statement of the lemma follows directly. It is also easy to check that u is a solution of the following equation:

$$-u''(t) = h(t),$$

where $h(t) = \alpha v_+(t) - \beta v_-(t)$. Since for any $\omega \in H^1_{2\pi}$, one has

$$\omega' \equiv 0 \quad \text{a.e. on} \quad \{t; \omega(t) = 0\}$$

(see [17] Chapter 2), it follows from the equation above that $\{t; u(t) = 0\} \subset \{t; h(t) = 0\}$. Since, moreover, $|\{t; h(t) = 0\}| = |\{t; v(t) = 0\}| = 0$ for any $v \in F$, we conclude that $|\{t; u(t) = 0\}| = 0$.

Suppose now by induction that (5.2) holds for any function of $F_k + F$ with $0 \le k < N - 1$ and take an arbitrary $u \in F_{k+1} + F$

$$u = e_0 + \ldots + e_k + e_{k+1} + v, \qquad e_j \in E_j,\ v \in F.$$

Then u is a solution of the equation

$$-u''(t) = \lambda_{k+1} u(t) + h(t),$$

where $h(t) = \sum_{j=0}^{k} (\lambda_j - \lambda_{k+1}) e_j + (\alpha - \lambda_{k+1}) v_+ - (\beta - \lambda_{k+1}) v_-$. By the result of [17] we obtain again

$$\{t; u(t) = 0\} \subset \{t; h(t) = 0\}.$$

Besides, $h(t) = 0$ if and only if $u_1(t) = 0$ (and $v(t) \ge 0$) or $u_2(t) = 0$ (and $v(t) \le 0$), where

$$u_1(t) = \sum_{j=0}^{k} (\lambda_j - \lambda_{k+1}) e_j(t) + (\alpha - \lambda_{k+1}) v(t),$$

$$u_2(t) = \sum_{j=0}^{k} (\lambda_j - \lambda_{k+1}) e_j(t) + (\beta - \lambda_{k+1}) v(t).$$

We now apply the hypothesis of induction to both u_1 and u_2 (changing, if necessary, u_1 to $-u_1$ and u_2 to $-u_2$ in order to be sure that the last term in the definition of u_1 and u_2 belongs to F). Thus

$$|\{t; u_1(t) = 0\}| = |\{t; u_2(t) = 0\}| = 0,$$

and hence

$$|\{t; u(t)=0\}| = |\{t; h(t)=0\}| = 0. \quad \square$$

We will need in the next lemma some estimates about the functional ϕ. It is easy to check (see [6], Remark 2.2) that hypotheses (f) and (F) imply the existence of functions $\gamma, \delta, c, d \in L^\infty(0,2\pi)$ such that

$$\alpha \le \gamma(t) \le c(t) \le a, \quad \beta \le \delta(t) \le d(t) \le b \qquad \text{a.e.,} \tag{5.3}$$

with

$$\begin{cases} \text{the sets } \{t; \gamma(t) > \alpha \text{ and } \delta(t) > \beta\},\ \{t; c(t) < a \text{ and } d(t) < b\} \\ \text{have positive measure} \end{cases} \tag{5.4}$$

and such that

$$\begin{cases} \forall \epsilon > 0 \ \ \exists b_\epsilon \in L^1(0,2\pi) \text{ such that} \\ \frac{1}{2}(\gamma(t)-\epsilon)s^2 - b_\epsilon(t) \le F(t,s) \le \frac{1}{2}(c(t)+\epsilon)s^2 + b_\epsilon(t) \text{ a.e. } t, \ \ \forall s \ge 0 \\ \frac{1}{2}(\delta(t)-\epsilon)s^2 - b_\epsilon(t) \le F(t,s) \le \frac{1}{2}(d(t)+\epsilon)s^2 + b_\epsilon(t) \text{ a.e. } t, \ \ \forall s \le 0. \end{cases} \tag{5.5}$$

Let us define the following functionals on $H^1_{2\pi}$:

$$\psi(u) = \int_0^{2\pi} u'(t)^2\,dt - \int_0^{2\pi} \gamma(t)u_+(t)^2\,dt - \int_0^{2\pi} \delta(t)u_-(t)^2\,dt$$
$$\chi(u) = \int_0^{2\pi} u'(t)^2\,dt - \int_0^{2\pi} c(t)u_+(t)^2\,dt - \int_0^{2\pi} d(t)u_-(t)^2\,dt.$$

Hence, by (5.5) we have that $\forall \epsilon > 0\ \exists b_\epsilon \in L^1(0,2\pi)$ such that

$$\chi(u) - \epsilon\|u\|_2^2 - 2\|b_\epsilon\|_1 \le 2\phi(u) \le \psi(u) + \epsilon\|u\|_2^2 + 2\|b_\epsilon\|_1$$

for all $u \in H^1_{2\pi}$. Here $\|b_\epsilon\|_1 = \int_0^{2\pi} |b_\epsilon(t)|\,dt$.

The following lemma will be crucial in the proof of Proposition 5.5.

Lemma 5.3. *Suppose that $(\alpha,\beta) \in C_N$ satisfies condition (*); i.e., $\alpha \ge \lambda_{N-1}$ and $\beta \ge \lambda_{N-1}$. Then* (i) $\max_{u\in A_1} \psi(u) < 0$. *Let $\Lambda = \{h \in C(B_1, M); h|_{A_1} = Id\}$. Then* (ii) $\inf_{h\in\Lambda} \max_{u\in B_1} \chi(h(u)) > 0$.

Proof. (i) From $\gamma(t) \ge \alpha$ almost everywhere and $\delta(t) \ge \beta$, we obtain

$$\psi(u) \le \int_0^{2\pi} u'(t)^2\,dt - \alpha\int_0^{2\pi} u_+(t)^2\,dt - \beta\int_0^{2\pi} u_-(t)^2\,dt \tag{5.6}$$

for all $u \in H^1_{2\pi}$.

Let $u \in A_1$ with $u = e+v$, $e \in F_{N-1}$, $v \in F$ and $\|u\| = 1$. Let us start by proving that $\psi(u) < 0$ when $e \equiv 0$. In this case $u = v$ with $v \not\equiv 0$. From (5.1) we deduce (after multiplying the equation by v and integrating) that

$$\int_0^{2\pi} v'(t)^2\,dt - \alpha\int_0^{2\pi} v_+(t)^2\,dt - \beta\int_0^{2\pi} v_-(t)^2\,dt = 0, \tag{5.7}$$

which implies, by (5.6), that $\psi(v) \leq 0$. Suppose now, for the sake of contradiction, that $\psi(v) = 0$. Then equality must hold in (5.6); that is,

$$\int \gamma(t) v_+(t)^2 \, dt + \int \delta(t) v_-(t)^2 \, dt = \alpha \int v_+(t)^2 \, dt + \beta \int v_-(t)^2 \, dt.$$

Then

$$\int (\gamma(t) - \alpha) v_+^2(t) \, dt = \int (\delta(t) - \beta) v_-^2(t) \, dt = 0$$

which is impossible by (5.4).

We have proved that $\psi(e+v) < 0$ when $e \equiv 0$. Suppose now that $e \not\equiv 0$. Let us introduce the map $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(s) = \int_0^{2\pi} (se+v)'^2 \, dt - \alpha \int_0^{2\pi} (se+v)_+^2 \, dt - \beta \int_0^{2\pi} (se+v)_-^2 \, dt.$$

We claim that

$$g(s) < 0 \quad \forall s \in \mathbb{R} \backslash \{0\}. \tag{5.8}$$

The proof of claim (5.8) will be a consequence of the three following facts:

(a) $g(0) = 0$.

This has already been proved in (5.7).

(b) $g'(0) = 0$.

The derivative of g is

$$g'(s) = 2\Big(\int_0^{2\pi} (se+v)' e' \, dt - \alpha \int_0^{2\pi} (se+v)_+ e \, dt + \beta \int_0^{2\pi} (se+v)_+ e \, dt \Big).$$

So, for $s = 0$ we have

$$g'(0) = 2\Big(\int_0^{2\pi} v' e' \, dt - \alpha \int_0^{2\pi} v_+ e \, dt + \beta \int_0^{2\pi} v_- e \, dt \Big).$$

Since v is a solution of (5.1), the right hand in the previous equality is zero. Thus $g'(0) = 0$.

(c) $g''(s) < 0 \ \forall s \in \mathbb{R}$.

Let us first justify briefly the existence of $g''(s)$. It is known (see for instance [18]) that the function $P(s) = \int_0^{2\pi} (se+v)_+ e \, dt$ is derivable at $s \in \mathbb{R}$ provided that

$$|\{t \in [0, \pi]; se(t) + v(t) = 0\}| = 0. \tag{5.9}$$

By Lemma 5.2, we have that (5.9) holds for all $s \in \mathbb{R}$ and so P is derivable in $\mathbb{R}$. The same conclusion is true for the negative part (observe that $\int_0^{2\pi} (se+v)_- e \, dt = \int_0^{2\pi} (-se-v)_+ e \, dt$).

Moreover, for all $s \in \mathbb{R}$ we have

$$P'(s) = \int_{\Omega_+^s} e^2 \, dt,$$

where $\Omega_+^s = \{t \in [0,2\pi]; se(t)+v(t) > 0\}$ (see [18]). Therefore

$$g''(s) = 2\Big(\int_0^{2\pi} e'^2\,dt - \alpha\int_{\Omega_+^s} e^2\,dt - \beta\int_{\Omega_-^s} e^2\,dt\Big) \tag{5.10}$$

with $\Omega_-^s = \{t \in [0,2\pi]; se(t)+v(t) < 0\}$.

Let us now prove that $g''(s) < 0$. Since $e \in F_{N-1}$ we have, on the one hand

$$\int_0^{2\pi} e'^2\,dt \le \lambda_{N-1}\int_0^{2\pi} e^2\,dt. \tag{5.11}$$

On the other hand, condition (*) implies that

$$\alpha\int_{\Omega_+^s} e^2\,dt + \beta\int_{\Omega_-^s} e^2\,dt \ge \lambda_{N-1}\int_0^{2\pi} e^2\,dt. \tag{5.12}$$

By substituing (5.11) and (5.12) in (5.10) we obtain that $g''(s) \le 0$. It remains to prove that $g''(s) = 0$ cannot occur. Suppose for the sake of contradiction that $g''(s) = 0$ for some $s \in \mathbb{R}$. Then the equality must hold in both (5.11) and (5.12). Observe that equality (5.11) yields $e \in E_{N-1}$ and that equality in (5.12) implies that

$$\underline{\text{either}}\ \ \alpha = \lambda_{N-1} \ \text{ and meas }\ \Omega_-^s = 0 \quad \underline{\text{or}}\ \ \beta = \lambda_{N-1} \ \text{ and meas }\ \Omega_+^s = 0. \tag{5.13}$$

Notice also that the alternative $(\alpha,\beta) = (\lambda_{N-1},\lambda_{N-1})$ never occurs because $(\lambda_{N-1}, \lambda_{N-1}) \notin C_N$.

We will reach a contradiction by showing that (5.13) can never happen for any $e \in E_{N-1}$. Suppose now for instance that $\alpha = \lambda_{N-1}$ and meas $\Omega_-^s = 0$; we would proceed analogously in the other case. We can assume that $N > 1$. Indeed, if $N = 1$ then $\lambda_{N-1} = \lambda_0 = 0$ and since any $(\alpha,\beta) \in C_1$ satisfies $\alpha, \beta > 0$, (5.13) fails.

By continuity of e, v we deduce from meas $\Omega_-^s = 0$ that

$$se(t) + v(t) \ge 0 \quad \forall t \in [0,2\pi].$$

Thus, any negative bump of $se(t)$ is contained in some positive bump of $v(t)$. Observe that $se(t)$ has exactly $N-1$ negative bumps, say $I_1,\dots,I_{N-1}$, and v has exactly N positive bumps $J_1,\dots,J_N$. We have, moreover, that $|I_k| = \frac{\pi}{\sqrt{\lambda_{N-1}}} = \frac{\pi}{\sqrt{\alpha}} = |J_\ell|\ \forall k = 1,\dots,N-1, \forall \ell = 1,\dots,N$. Therefore, each I_k corresponds exactly to one of the bumps J_ℓ.

We suppose, for simplicity, that $I_1 = J_1, \dots, I_{N-1} = J_{N-1}$. Hence, $J_N \subset \{t; se(t) > 0\}$. Since we also have that $\{t; v(t) < 0\} \subset \{t; se(t) > 0\}$, then

$$|J_N| + |\{t : v(t) < 0\}| \le |\{t : se(t) > 0\}|;$$

i.e., $\frac{\pi}{\sqrt{\alpha}} + \pi \le \pi$, which is absurd. This proves (c) and then the proof of claim (5.8) is completed.

If we now choose $s = 1$ in (5.8) it becomes

$$g(1) = \int_0^{2\pi} u'(t)^2\,dt - \alpha \int_0^{2\pi} u_+(t)^2\,dt - \beta \int_0^{2\pi} u_-(t)^2\,dt < 0.$$

This allows us to conclude that $\psi(u) < 0$ in (5.6).

(ii) According to hypothesis (f) we have

$$\chi(u) \geq \int_0^{2\pi} u'(t)^2\,dt - (a-b)\int_0^{2\pi} u_+^2(t)\,dt - b$$

for all $u \in M$. Therefore,

$$\inf \max_{\Lambda} \chi(u) \geq \inf \max_{\Lambda} I_\mu(u) - \lambda_{N+1}(\mu) \tag{5.14}$$

with $\mu = a - b$.

In order to apply the result of [8], i.e., Theorem 3.4, let us introduce the following translation-invariant set

$$B_1^* = \bigcup_{\theta \in S^1} T_\theta B_1.$$

We associate to each map $h \in \Lambda$ the map $h^* : B_1^* \to M$ defined by

$$h^*(T_\theta u) = T_\theta h(u) \quad \forall u \in B_1, \quad \forall \theta \in S^1.$$

Thus, $h^*(B_1^*)$ is a compact invariant subset of M.

Let us prove that $\gamma_0(h^*(B_1^*)) \geq N+1$. Suppose, on the contrary, that there exists some $0 \leq s < N+1$ and some $g \in C_e(h^*(B_1^*), \mathbb{R} \times \mathbb{C}^s \backslash \{(0,0)\})$. Then $g \circ h : B_1^* \to \mathbb{R} \times \mathbb{C}^s \backslash \{(0,0)\}$ is a continuous and equivariant map which leaves invariant $\mathbb{R} \cap B_1^*$. A closer look at the set B_1^* shows that this set is "equi-homeomorphic" to a sphere of $\mathbb{R} \times \mathbb{C}^{N+1}$. To be more precise, let $T^{N+1} = \{(c, z_1, \dots, z_N) \in \mathbb{R} \times \mathbb{C}^{N+1};$ $c^2 + \sum_{i=1}^{N+1} |z_j|^2 = \frac{1}{2\pi}\}$. Consider the map $J : T^{N+1} \to B_1^*$ defined by

$$\begin{aligned} &J(c, r_1 e^{i\theta_1}, \dots, r_{N+1} e^{i\theta_{N+1}}) \\ &= \frac{c \cdot \mathbf{1} + \sum_{j=0}^{N-1} r_j \sin j(\theta_j + \cdot) + r_N v_0(\theta_N + \cdot) + r_{N+1}\omega_0(\theta_{N+1} + \cdot)}{\|c \cdot \mathbf{1} + \sum_{j=0}^{N-1} r_j \sin j(\theta_j + \cdot) + r_N v_0(\theta_N + \cdot) + r_{N+1}\omega_0(\theta_{N+1} + \cdot)\|_2}. \end{aligned}$$

J is well defined because of the choice of ω_0 and Lemma 5.1. Moreover, J is an equivariant homeomorphism which leaves $T^{N+1} \cap \mathbb{R}$ invariant; i.e.,

$$J(c, 0, \dots, 0) = c \cdot \mathbf{1}$$

Then the composition $g \circ h \circ J : T^{N+1} \to \mathbb{R} \times \mathbb{C}^s \backslash \{(0,0)\}$ is a continuous equivariant map which leaves invariant the elements of $T^{N+1} \cap \mathbb{R}$. Since $S < N+1$, the existence of such a map contradicts the S^1-version of Borsuk-Ulam Theorem (see [19]). We conclude that

$$\gamma_0(h^*(B_1^*)) \geq N+1 \tag{5.15}$$

and so $h^*(B_1^*) \subset \Gamma_{N+1}$.

Besides, since I_μ is T_θ-invariant, i.e, $I_\mu(T_\theta u) = I_\mu(u)$ $\forall \theta \in S^1$, $\forall u \in H^1_{2\pi}$, then

$$\max_{u \in h(B_1)} I_\mu(u) = \max_{u \in h^*(B_1^*)} I_\mu(u). \tag{5.16}$$

It follows now from (5.14)–(5.16) and from Theorem 3.4 that

$$\inf \max_{\Lambda} \chi(u) \geq \inf \max_{\Gamma_{N+1}} I_\mu - \lambda_{N+1}(\mu) = 0.$$

Let us prove, by contradiction, that the strict inequality holds in the above inequality. Suppose that $\inf \max_{\Lambda} \chi(u) = 0$. Since $\chi(u) \leq \psi(u)$ $\forall u \in M$ (see (5.4)) we have, by the first part of this lemma, that

$$\max_{u \in A_1} \chi(u) \leq \max_{u \in A_1} \psi(u) < 0 = \inf \max_{\Lambda} \chi(u)$$

and, in particular,

$$\max_{u \in A_1} \chi(u) < \inf \max_{\Lambda} \chi(u). \tag{5.17}$$

We use now Remark 3.3 in Section 3. It is easily checked that the functional χ, restricted to M, satisfies the Palais-Smale condition. Since condition (5.17) implies condition (3.2) of Theorem 3.1, we have that $0 = \inf \max_\Lambda \chi(u)$ is a critical value of $\chi|_M$. Thus, there exists $u_0 \in M$ (in particular $u_0 \neq 0$) and some $\lambda \in \mathbb{R}$ such that

$$\chi(u_0) = 0, \quad \chi'(u_0)(v) = \lambda(u, v) \quad \forall v \in H^1_{2\pi}. \tag{5.18}$$

By homogeneity we have $\chi(u_0) = \frac{1}{2}\chi'(u_0)(u_0)$. Hence $\lambda = 0$ and the second equation in (5.17) becomes $\chi'(u_0) = 0$. Thus u_0 is a nontrivial solution of equation

$$-u''(t) = c(t)u_+(t) - d(t)u_-(t), \quad u(0) = u(2\pi); \quad u'(0) = u'(2\pi)$$

which contradicts Lemma 4.1 if one takes into account hypotheses (5.3) and (5.4). This proves part (ii) of the lemma.

Remark 5.4. Condition (*) has been used in (5.12) to prove the convexity of the map g yielding

$$\max_{u \in A_1} \Big(\int_0^{2\pi} u'(t)^2\, dt - \alpha \int_0^{2\pi} u_+(t)^2\, dt - \beta \int_0^{2\pi} u_-(t)^2\, dt \Big) = 0.$$

Then the nonresonance condition (5.4) implies that $\max_{u \in A_1} \psi(u) < 0$. Observe that the proof of part (ii) in Lemma 5.3 can be adapted, with minor changes in the proof, to any $A \subset \Gamma_N$ of the form $A = L(T^N)$, with $L \in C_e(T^N, M)$ satisfying $L(c, 0, 0, \dots, 0) = (c, 0, 0, \dots, 0)$ $\forall c \in \mathbb{R} \cap T^N$ and such that $\max_{u \in A} \psi(u) < 0$. One takes then $B = T^N \times \mathbb{R}^+$ and $\Lambda = \{h \in C(T^N \times \mathbb{R}^+, M); h|_{T^N} = L\}$. We do not know actually if such a set A exists for any $(\alpha, \beta) \in C_N$, $\gamma, \delta \in L^\infty(0, 2\pi)$ satisfying (5.3) and (5.4).

Proposition 5.5. *Assume that f satisfies hypotheses* (f) *and* (F) *with $(\alpha,\beta)\in C_N$ and $(a,b)\in C_{N+1}$ for $N\geq 1$. Assume also that $(\alpha,\beta)\in C_N$ satisfies conditions* (*) *and let ϕ be the functional defined in* (3.1). *We have*

(i)
$$\lim_{R\to+\infty}\max_{u\in A_R}\phi(u)=-\infty.$$

Let $\Lambda_R=\{h\in C(B_R,H^1_{2\pi});h|_{A_R}=Id\}$. Then for R large enough we have

(ii)
$$\max_{u\in A_R}\phi(u)<\max_{u\in B_R}\phi(h(u))\quad \textit{for all } h\in\Lambda_R.$$

As a consequence of the minimax principle 3.1 and of Proposition 5.5 we obtain

Corollary 5.6. *For R large enough, the value $c=\inf_{h\in\Lambda_R}\max_{u\in h(B_R)}\phi(u)$ is a critical value of ϕ.*

Proof of Proposition 5.5. We denote by $\xi=\max_{u\in A_1}\psi(u)$ and $\eta=\inf\max_\Lambda\chi$, where Λ is the family of maps defined in Lemma 5.3. Let us take $0<\epsilon<\min\{-\xi,\eta\}$ in (5.5). Thus, there exists $b_\epsilon\in L^1(0,2\pi)$ such that

$$\chi(u)-\epsilon\|u\|_2^2-2\|b_\epsilon\|_1\leq 2\phi(u)\leq\psi(u)+\epsilon\|u\|_2^2+2\|b_\epsilon\|_1\quad\text{for all } u\in H^1_{2\pi}.$$

(i) From the inequality $2\phi(u)\leq\psi(u)+\epsilon\|u\|_2^2+2\|b_\epsilon\|_1$ $\forall u\in H^1_{2\pi}$ we derived, for $\bar u=\frac{u}{\|u\|_2}$, the following estimate:

$$\max_{u\in A_R}2\phi(u)\leq\max_{\bar u\in A_1}R^2\psi(\bar u)+\epsilon R^2+2\|b_\epsilon\|_1=R^2(\xi+\epsilon)+2\|b_\epsilon\|_1.$$

Then
$$\lim\max_{u\in A_R}2\phi(u)=-\infty.$$

(ii) Let us fix an $R>0$ coming from part (i) such that

$$\max_{u\in A_R}\phi(u)<-\|b_\epsilon\|_1. \tag{5.19}$$

Let $h\in\Lambda_R$. We distinguish between two cases:
(a) $0\in h(B_R)$. Then

$$\max_{u\in B_R}\phi(h(u))\geq\phi(0)=0>-\|b_\epsilon\|_1>\max_{u\in A_R}\phi(u).$$

(b) $0\notin h(B_R)$. In this case we consider the map $\bar h:B_1\to M$ defined by $\bar h(\bar u)=\frac{h(R\bar u)}{\|h(R\bar u)\|_2}$. Thus, $\bar h\in\Lambda$. Therefore, $\max_{u\in\bar h(B_1)}\chi(u)\geq\eta$ and, using that $2\phi(v)\geq\chi(v)-\epsilon\|v\|_2^2-2\|b_\epsilon\|_1$ $\forall v\in H_{2\pi}$, we obtain

$$\max_{u\in B_R}\frac{2\phi(h(u))+2\|b_\epsilon\|_1}{\|h(u)\|_2^2}\geq\max_{u\in B_R}\chi(\frac{h(u)}{\|h(u)\|_2})-\epsilon\geq\eta-\epsilon>0.$$

Hence,

$$\max_{u \in B_R} \phi(h(u)) > -\|b_\epsilon\|_1$$

and, by (5.19), we conclude that

$$\max_{u \in B_R} \phi(h(u)) > \max_{u \in A_R} \phi(u).$$

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REFERENCES

[1] S. Fučik, *Boundary value problems with jumping nonlinearities*, Čas. Pǒst. Mat. 101 (1976), 69–87.

[2] E.N. Dancer, *On the Dirichlet problem for weakly nonlinear elliptic p.d.e.*, Proc. Royal Soc. Edinb. Sect. A, 76 (1977), 283–300.

[3] S. Invernizzi, *A note on nonuniform nonresonance for jumping nonlinearities*, Commen. Math. Univ. Carolinae, 27 (1986), 285–291.

[4] P. Drabek, S. Invernizzi, *On the periodic BVP for the forced Duffing equation with jumping nonlinearity*, Nonlinear Analysis, TMA, 10 (1986), 643–650.

[5] J. Mawhin, J. Ward, *Periodic solutions of some forced Liénard differential equations at resonance*, Arch. Math. , 41 (1983), 337–351.

[6] J.-P. Gossez, P.P. Omari, *Nonresonance with respect to the Fučik spectrum for periodic solutions of second order ode*, Nonlinear Analysis, TMA, 14, 1079–1104 (1990).

[7] M. Cuesta, J.-P. Gossez, *A variational approach to nonresonance with respect to the Fučik spectrum*, Nonlinear Analysis, TMA, 19 (1992), 487–500.

[8] D. De Figueiredo, B. Ruf, *On the periodic Fučik spectrum and a superlinear Sturm-Liouville equation*, Proc. Royal Soc. of Edinb., 123A (1993), 95–107.

[9] O. Kavian, *Quelques remarques sur le spectre demi-linéaire de certains opérateurs autoadjoints*, preprint.

[10] C. Maghalhaes, *Semilinear elliptic problem with crossing of multiple eigenvalues*, Comm. Part. Diff. Eq., 15 (1990), 1265–1292.

[11] P. Habets, P. Omari, F. Zanolin, *Nonresonance conditions on the potential with respect to the Fučik-spectrum for the periodic b.v.p.*, to appear.

[12] D. Costa, A. Oliveira, *Existence of solutions for a class of semilinear problems at double resonance*, Boll. Soc. Bras. Mat., 19 (1988), 21–37.

[13] D. De Figueiredo, *Lectures on Ekeland Variational principle and detours*, Lec. Notes College on Variational Problems in Analysis, Trieste (1988).

[14] H. Brezis, L. Nirenberg, *Remarks on finding critical points*, Comm. on Pure and Appl. Math., 44 (1991), 939–963.

[15] M. Struwe, "Variational Methods," Berlin, Springer, 1990.

[16] V. Benci, *A geometrical index for the group S^1 and some applications to the research of periodic solutions of ODE's*, Comm. Pure and Appl. Math., 34 (1981), 393–432.

[17] D. Kinderlehrer, G. Stampacchia, "An Introduction to Variational Inequalities and their Applications," Academic Press, 1980.

[18] S. Solimini, *Some remarks on the number of solutions of some nonlinear elliptic problems*, AIHP, Analyse Non Linéaire, 2 (1985), 143–156.

[19] L. Niremberg, *Comments on nonlinear problems*, Proc. S.A.F.A., Catania, Sep. 1981.

[20] H. Berestycki, J.M. Lasry, G. Mancini, B. Ruf, *Existence of multiple periodic orbits on star-shaped Hamitonian surfaces*, Comm. on Pure and Appl. Math., 38 (1985), 253–289.