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TRAJECTORIES OF DYNAMICAL SYSTEMS JOINING TWO GIVEN SUBMANIFOLDS

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Abstract. In this paper we study ordinary differential equations on a noncomplete Riemannian manifold. Using the Ljusternik-Schnirelmann theory, we prove the existence of infinitely many solutions joining two given submanifolds.

1. Introduction and statement of the results. Let $(\mathcal{M}, \langle\cdot,\cdot\rangle_R)$ be a Riemannian manifold, M_0, M_1 closed submanifolds of $\mathcal{M}$ and $V : \mathcal{M} \to \mathbb{R}$ a C^1 potential function. In this paper we look for curves $x : [0,1] \to \mathcal{M}$ satisfying the equation

$$D_t\dot{x}(t) = -\nabla_R V\left(x\left(t\right)\right) \tag{1.1}$$

and orthogonal to M_0 and M_1, that satisfy the boundary conditions

$$\begin{cases} x(0) \in M_0 & x(1) \in M_1 \\ \dot{x}(0) \in (T_{x(0)}(M_0))^{\perp} & \dot{x}(1) \in (T_{x(1)}(M_1))^{\perp}, \end{cases} \tag{1.2}$$

where $D_t\dot{x}(t)$ denotes the covariant derivative of $\dot{x}(t)$ along the direction of $x(t)$, $\nabla_R V(x(t))$ is the Riemannian gradient with respect to x of V in $x(t)$ and $T_x(\mathcal{M})$ (respectively $T_x(M_0)$ or $T_x(M_1)$) is the tangent space to $\mathcal{M}$ at x (resp. to M_0 or M_1 if x belongs to M_0 or M_1).

If M_0 and M_1 are reduced to a single point, say x_0 and x_1 respectively, then existence of solutions of (1.1) joining x_0 and x_1 has been studied by many authors (see for example [7], [9], [12]). If $V = 0$, the existence of geodesics on $\mathcal{M}$ starting orthogonal to M_0 and ending orthogonal to M_1 has been stated in [8] when $\mathcal{M}$ is complete and M_0 or M_1 is compact.

The aim of this paper is to extend those results in three directions; indeed we shall consider problem (1.1)–(1.2) when $V \neq 0$, $\mathcal{M}$ is not complete and M_0 and M_1 are not compact. We shall study problem (1.1)–(1.2) by variational methods. In fact it is known that its solutions are the critical points of the functional

$$f(x) = \tfrac{1}{2}\int_0^1 \langle\dot{x}(t), \dot{x}(t)\rangle_R dt - \int_0^1 V(x(t))dt \tag{1.3}$$

defined on a suitable Riemannian manifold (see section 1).

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First, we shall assume that

$$(M)\quad \begin{cases} \text{i)} & M_0 \text{ or } M_1 \text{ is compact;} \\ & \text{otherwise} \\ \text{ii)} & \text{there exists } x_0 \in \mathcal{M} \text{ such that for any } x \in M_0 \text{ and for any } y \in M_1, \\ & \lim d(x,y) = +\infty \text{ as } d(x,x_0) \to +\infty \text{ or } d(y,x_0) \to +\infty, \end{cases}$$

where $d(\cdot,\cdot)$ denotes the canonical distance on $\mathcal{M}$ induced by the Riemannian manifold.

We shall prove the following theorem.

Theorem 1.1. *Let* $(\mathcal{M}, \langle\cdot,\cdot\rangle_R)$ *be a finite dimensional, connected, complete Riemannian manifold of class* C^3 *and* M_0, M_1 *closed submanifolds of* $\mathcal{M}$ *satisfying condition* (M). *Let* $V : \mathcal{M} \to \mathbb{R}$ *be a* C^1 *function satisfying*

(V_1) *there exists* $c_0 \in \mathbb{R}$ *such that* $V(x) < c_0$ *for any* $x \in \mathcal{M}$.

Then problem (1.1)–(1.2) *has at least one solution.*

Moreover, if $\mathcal{M}$ *is not contractible in itself and* M_0 *and* M_1 *are contractible in* $\mathcal{M}$, *then there exist infinitely many nonconstant solutions* x_n *of* (1.1) *orthogonal to* M_0 *and* M_1 *such that* $f(x_n) \to +\infty$.

Remark 1.2. If we assume $\nabla V(x) \neq 0$ for any $x \in M_0 \cap M_1$, clearly the solution we found is nonconstant. We can avoid this assumption if (1.1)–(1.2) has infinitely many solutions x_n whose critical values positively diverge because the set $M_0 \cap M_1$ is compact by assumption (M) and then the function V is bounded on $M_0 \cap M_1$.

Theorem 1.1 can be extended to a noncomplete Riemannian manifold. For this, let us introduce the following definition (see [3], [4]).

Definition 1.3. Let $\mathcal{M}$ be a connected open subset of an n-dimensional Riemannian manifold $\mathcal{M}^*$ of class C^3 and $\partial\mathcal{M}$ its topological boundary. We say that $\mathcal{M}$ has a convex boundary if there exists $\phi \in C(M \cup \partial M, \mathbb{R}_+) \cap C^2(\mathcal{M}, \mathbb{R}_+)$ such that

i) $\phi(x) = 0$ if and only if $x \in \partial\mathcal{M}$;

ii) for any $\eta > 0$ the set $\{x \in \mathcal{M} : \phi(x) \geq \eta\}$ is complete (with respect to the Riemannian structure of $\mathcal{M}$);

iii) there exist some constants α, β, and $\delta > 0$ such that for any $x \in \mathcal{M}$, $\phi(x) < \delta$,

$$\alpha \leq \langle \nabla_R\phi(x), \nabla_R\phi(x)\rangle_R \leq \beta \tag{1.4}$$

and

$$H_R^\phi(x)[v,v] \leq \gamma\langle v,v\rangle_R\phi(x) \qquad \text{for any } v \in T_x(\mathcal{M}), \tag{1.5}$$

where $\nabla_R\phi(x)$ is the Riemannian gradient of ϕ at x and $H_R^\phi(x)[v,v]$ denotes the Riemannian Hessian of ϕ at x in the direction v.

Let us point out that by i) and iii) it follows that

$$\limsup_{x\to\bar{x}\in\partial\mathcal{M}} H_R^\phi(x)[v,v] \leq 0 \qquad \text{for any } v \in T_x(\mathcal{M}) \text{ with } \langle v,v\rangle \leq 1.$$

The convexity assumption in Definition 1.3 permits one to prove the existence of solutions of (1.1)–(1.2) even if the manifold $\mathcal{M}$ is not complete. Indeed, if $\mathcal{M}$ has a convex boundary and M_0 and M_1 do not satisfy assumption (M), Theorem 1.1 can be generalized as follows.

Theorem 1.4. *Let $(\mathcal{M}, \langle\cdot,\cdot\rangle_R)$ be a Riemannian manifold of class C^3 with convex boundary and M_0, M_1 closed submanifolds of $\mathcal{M}$. Let us assume that there exist a point $x_0 \in \mathcal{M}$ and a positive constant R such that*

($\mathcal{M}_1$) *there exist $U : \mathcal{M} \to \mathbb{R}_+$ of class C^2 and a positive constant ν such that for any $x \in \mathcal{M}$ with $d(x, x_0) \geq R$,*

$$H_R^U(x)[\xi,\xi] \geq \nu\langle\xi,\xi\rangle \quad \text{for any } \xi \in T_x(M);$$

($\mathcal{M}_2$) *$\nabla_R U(x) \in T_x(M_0)$ for any $x \in M_0$ with $d(x,x_0) \geq R$, $\nabla_R U(x) \in T_x(M_1)$ for any $x \in M_1$ with $d(x,x_0) \geq R$;*

($\mathcal{M}_3$) *there exist some positive constants ρ and μ such that $\phi(x) \geq \mu > 0$ for any $x \in \mathcal{M}$, $d(x,x_0) \geq \rho$, where ϕ is the function introduced in Definition* 1.3;

($\mathcal{M}_4$) *there exists a positive constant σ such that $d(x,y) \geq \sigma$ for any $x \in M_0$ with $d(x,x_0) \geq R$ and for any $y \in M_1$ with $d(y,x_0) \geq R$.*

Now let $V : \mathcal{M} \to \mathbb{R}$ be a C^1 function satisfying (V_1) *and*

(V_2) $\lim_{d(x,x_0)\to+\infty} V(x) = c_0$;

(V_3) $\langle\nabla_R\phi(x), \nabla_R V(x)\rangle_R \geq 0$ *for any $x \in \mathcal{M}$ with $\phi(x) < \delta$ (where δ is the constant introduced in* (iii) *of Definition* 1.3);

(V_4) $\limsup_{d(x,x_0)\to+\infty} \langle\nabla_R V(x), \nabla_R U(x)\rangle_R = 0$.

Then problem (1.1)–(1.2) *has at least one solution.*

Now, instead of ($\mathcal{M}_4$), *let us assume that $\mathcal{M}$ is not contractible in itself and M_0 and M_1 are contractible in $\mathcal{M}$; then there exist infinitely many nonconstant solutions x_n of* (1.1)–(1.2) *such that $f(x_n) \to +\infty$.*

Remark 1.5. If M_0 and M_1 are reduced to a single point, the existence of infinitely many solutions joining two given points of a manifold with convex boundary has been proved in [12].

Remark 1.6. In Theorem 1.4 the function U is introduced in order to give a hypothesis on $\mathcal{M}$ at infinity, so we can always assume U bounded on bounded sets. Moreover, without loss of generality, we suppose that the element x_0 introduced in Theorem 1.4 is the same which appears in condition (M)-(ii).

Examples of manifolds satisfying stronger assumptions than ($\mathcal{M}_1$) are given in [7] (see also Proposition 1.9 for a manifold satisfying ($\mathcal{M}_1$) and ($\mathcal{M}_2$).

Let us point out that assumptions ($\mathcal{M}_1$)–($\mathcal{M}_2$) replace assumption (M) while the convexity assumption introduced in Definition 1.3 permits one to overcome the lack of completeness of $\mathcal{M}$; in fact the following corollaries hold.

Corollary 1.7. *Let $(\mathcal{M}, \langle\cdot,\cdot\rangle_R)$ be a connected Riemannian manifold of class C^3 with convex boundary and M_0, M_1 closed submanifolds of $\mathcal{M}$ satisfying assumption* (M). *Let $V : \mathcal{M} \to \mathbb{R}$ be a C^1 function satisfying* (V_1)–(V_3). *Then problem* (1.1)–(1.2) *has at least one solution.*

Moreover, if $\mathcal{M}$ is not contractible in itself and M_0, M_1 are contractible in $\mathcal{M}$, then there exist infinitely many solutions x_n of (1.1) *orthogonal to M_0 and M_1 such that $f(x_n) \to +\infty$.*

Corollary 1.8. *Let $(\mathcal{M}, \langle\cdot,\cdot\rangle_R)$ be a connected finite dimensional, complete Riemannian manifold of class C^3 and M_0, M_1 closed submanifolds of $\mathcal{M}$. Let us assume that $(\mathcal{M}_1)$, $(\mathcal{M}_2)$ and $(\mathcal{M}_4)$ hold and let V satisfy (V_1), (V_2) and (V_4). Then the conclusion of Theorem* 1.4 *still holds.*

In particular the following result can be proved.

Proposition 1.9. *Let M_0 and M_1 be linear subspaces of $\mathbb{R}^n$ such that $M_0 \cap M_1 = \{0\}$ and let $V : \mathbb{R}^n \to \mathbb{R}$ be a potential function satisfying (V_1). Then there exists at least one solution of the equation $\ddot{x} + \nabla V(x) = 0$ orthogonal to M_0 and M_1.*

2. Variational formulation and abstract theorems. Let $(\mathcal{M}, \langle\cdot,\cdot\rangle_R)$ be a finite dimensional C^3 Riemannian manifold and let us denote by $d(\cdot,\cdot)$ the canonical distance induced by the Riemannian structure. Let

$$\begin{aligned} W^1 &= W^{1,2}([0,1],\mathcal{M}) \\ &= \{x : [0,1] \to \mathcal{M},\ x \text{ absolutely continuous}, \int_0^1 \langle \dot{x}, \dot{x}\rangle dt < +\infty\}. \end{aligned}$$

We will consider the path space of the curves $x(t)$ joining M_0 and M_1; that is,

$$\Omega^1 = \Omega^1(\mathcal{M}, M_0, M_1) = \{x \in W^{1,2}([0,1],\mathcal{M}) : x(0) \in M_0,\ x(1) \in M_1\}.$$

If $\mathcal{M}$ is complete and M_0, M_1 are closed in $\mathcal{M}$, then Ω^1 is a complete Riemannian manifold and its tangent space at $x \in \Omega^1$ is given by

$$\begin{aligned} T_x(\Omega^1) = \big\{\xi \in W^{1,2}([0,1], T\mathcal{M}) : \xi(t) \in T_{x(t)}(\mathcal{M}), \\ (\xi(0), \xi(1)) \in T_{(x(0),x(1))}(M_0 \times M_1)\big\}, \end{aligned}$$

where $T\mathcal{M}$ is the tangent bundle of $\mathcal{M}$ and $W^{1,2}([0,1], T\mathcal{M})$ is the set of the absolutely continuous curves $\xi : [0,1] \to T\mathcal{M}$ such that

$$\langle \xi, \xi\rangle_1 = \int_0^1 (\langle D_t\xi(t), D_t\xi(t)\rangle_R + \langle \xi(t), \xi(t)\rangle_R) dt < +\infty,$$

D_t being the covariant derivative with respect to the Riemannian structure.

Let us denote by $C([0,1],\mathcal{M})$ the space of the continuous curves in $\mathcal{M}$ endowed with the metric

$$d_\infty(x, x') = \sup_{0\le t\le 1} d(x(t), x'(t)).$$

On Ω^1 let us define the action functional f by

$$f(x) = \tfrac{1}{2}\int_0^1 \langle \dot{x}(t), \dot{x}(t)\rangle_R dt - \int_0^1 V(x(t))dt \qquad x \in \Omega^1.$$

It is easy to see (cf. [9]) that f is C^1 on Ω^1, moreover its critical points are the normal solutions of (1.1) joining M_0 and M_1; that is, x is a critical point of f on Ω^1 if and only if

$$D_t\dot{x}(t) = \nabla_R V(x(t))$$

and

$$\langle \dot{x}(0), \xi \rangle_R = 0 \;\; \forall \xi \in T_{x(0)}(M_0), \qquad \langle \dot{x}(1), \xi \rangle_R = 0 \;\; \forall \xi \in T_{x(1)}(M_1).$$

In order to prove the existence of critical points of f, we will use the well known Ljusternik-Schnirelmann theory, so we recall the following definitions and results (see e.g. [13], [14]).

Definition 2.1. Let X be a topological space. Given $A \subseteq X$, $cat_X(A)$ is the category of A in X; that is, the least number of closed and contractible subsets of X covering A. If it is not possible to cover A with a finite number of such sets, then $cat_X(A) = +\infty$.

We denote $catX = cat_X(X)$.

Definition 2.2. Let Λ be a Riemannian manifold and $g : \Lambda \to \mathbb{R}$ a given C^1 functional; g satisfies the Palais-Smale condition, briefly (P.S), if any $(x_n)_{n\in\mathbb{N}} \subset \Lambda$, such that $(g(x_n))_{n\in\mathbb{N}}$ is bounded and $g'(x_n) \to 0$ as $n \to +\infty$, has a subsequence which converges in Λ.

Moreover, let g^c be the sublevel of g relatively to c; that is,

$$g^c = \{x \in \Lambda : g(x) \leq c\} \qquad \text{for any } c \in \mathbb{R}.$$

Theorem 2.3 (Ljusternik-Schnirelmann)**.** *Let Λ be a Riemannian manifold and g a C^1 functional on Λ which satisfies the* (P.S) *condition. Let us assume that Λ is complete or every sublevel of g in Λ is complete. For $k \in \mathbb{N}$, $k > 0$, define*

$$\Gamma_k = \{A \subseteq \Lambda : cat_\Lambda(A) \geq k\}, \qquad c_k = \inf_{A\in\Gamma_k} \sup_{x\in A} g(x).$$

If $\Gamma_k \neq \emptyset$ and $c_k \in \mathbb{R}$, then c_k is a critical value of g.

Remark 2.4. Let Λ and g be as in Theorem 2.3. If g is bounded from below, then $cat_\Lambda(g^c) < +\infty$ for any $c \in \mathbb{R}$.

Theorem 2.5. *Let $\mathcal{M}$ be an open connected not contractible in itself subset of $\mathbb{R}^n$ and M_0, M_1 two closed nonempty subsets of $\mathcal{M}$ which are contractible in $\mathcal{M}$. Then $cat(\Omega^1) = +\infty$ and Ω^1 possesses compact subsets of arbitrary high category.*

Proof. (see [5] and [6]).

3. Proof of the theorems. In order to prove Theorem 1.1 we need the following result.

Lemma 3.1. *Under the assumptions of Theorem* 1.1, *the functional f satisfies the Palais-Smale condition.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a (P.S) sequence; that is,

$$(f(x_n))_{n\in\mathbb{N}} \;\; \text{is bounded} \tag{3.1}$$

and

$$f'(x_n) \to 0 \quad \text{as } n \to +\infty. \tag{3.2}$$

By (3.1) and (V_1) it follows that

$$\Big(\int_0^1 \langle \dot{x}_n(t), \dot{x}_n(t)\rangle_R dt\Big)_{n\in\mathbb{N}} \text{ is bounded.} \tag{3.3}$$

If M_0 or M_1 is compact, one of the two real sequences

$$(d(x_0, x_n(0))_{n\in\mathbb{N}}, \qquad (d(x_0, x_n(1))_{n\in\mathbb{N}}$$

is bounded too; then by (3.3), $(x_n)_{n\in\mathbb{N}}$ is bounded in Ω^1. If M_0 and M_1 are not compact, as

$$(d(x_n(0), x_n(1)) \leq \int_0^1 \langle \dot{x}_n(t), \dot{x}_n(t)\rangle_R dt,$$

it follows that $(d(x_n(0), x_n(1))_{n\in\mathbb{N}}$ is bounded and therefore by (M)-(ii) also in this case we can conclude that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in Ω^1.

Now, by using the well known Nash imbedding theorem (see [10]), the complete Riemannian manifold $\mathcal{M}$ can be seen as a closed subset of $\mathbb{R}^N$ for a suitable N, so there exists $x \in \Omega^1$ such that $x_n \rightharpoonup x$ weakly in $W^1([0,1], \mathbb{R}^N)$ and uniformly in $[0,1]$ up to a subsequence.

Then by Lemma 2.1 of [2], arguing in a standard way it is possible to show that $x \in \Omega^1$ and $x_n \to x$ in W^1.

Proof of Theorem 1.1. As f is bounded from below and satisfies (P.S) condition, by Theorem 2.3, it has at least $cat(\Omega^1)$ critical points. By Theorem 2.5 the existence of infinitely many solutions $(x_n)_{n\in\mathbb{N}}$ of (1.1)–(1.2) follows when $\mathcal{M}$ is not contractible in itself and M_0 and M_1 are contractible in $\mathcal{M}$.

Now if $f(x_n)_{n\in\mathbb{N}}$ is bounded, then taking $c \geq f(x_n)$ for any n, we have that $cat_{\Omega^1}(f^c) = cat(\Omega^1) = +\infty$ which contradicts remark 2.4.

Finally, as $f(x_n) \to +\infty$, the solutions found are nonconstant in t (see Remark 1.2). □

Let us consider now the case where assumption (M) does not hold and $\mathcal{M}$ is not complete. Then the action functional f does not satisfy (P.S) condition because there are (P.S) sequences which are not bounded in Ω^1 and (P.S) sequences which are bounded in Ω^1 but they converge to an element $x \in \partial\Omega^1$. These difficulties arise also in the study of periodic solutions of the equation (1.1) on a noncomplete Riemannian manifold; then, as in [11], we shall penalize the functional f in such a way the penalized functional satisfies (P.S) condition and its critical points are the critical points of f and therefore solutions of problem (1.1)–(1.2).

More precisely, for any $\varepsilon > 0$ and for any $x \in \Omega^1$ let

$$f_\varepsilon(x) = \tfrac{1}{2}\int_0^1 \langle \dot{x}, \dot{x}\rangle_R dt - \int_0^1 V(x)dt + \int_0^1 U_{1,\varepsilon}(x)dt + \int_0^1 U_{2,\varepsilon}(x)dt,$$

where $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$ are defined

$$U_{1,\varepsilon}(x) = \psi_{1,\varepsilon}\Big(\frac{1}{\phi^2(x)}\Big), \tag{3.4}$$

where $\psi_{1,\varepsilon} : \mathbb{R} \to \mathbb{R}$ is a C^2 function satisfying

$$\psi_{1,\varepsilon}(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1/\varepsilon \\ t & \text{if } t \ge 1 + 1/\varepsilon, \end{cases}$$

$\psi'_{1,\varepsilon}(t) > 0$ if $t > \frac{1}{\varepsilon}$ and $\psi_{1,\varepsilon}(t) \le \psi_{1,\varepsilon'}(t)$ for all $t \ge 0$ and $\varepsilon \le \varepsilon'$. Moreover, we set

$$U_{2,\varepsilon}(x) = \psi_{2,\varepsilon}(U(x)), \tag{3.5}$$

where U is the function introduced in assumption $(\mathcal{M}_1)$ and $\psi_{2,\varepsilon}$ is the positive increasing function defined by

$$\psi_{2,\varepsilon}(t) = \begin{cases} 0 & \text{if } t \le 1/\varepsilon \\ \sum\limits_{n=3}^{\infty} \frac{1}{n!}\nu^n (t - \frac{1}{\varepsilon})^n & \text{if } t > 1/\varepsilon, \end{cases}$$

where ν is the constant introduced in assumption $(\mathcal{M}_1)$.

We need the following technical lemmas.

Lemma 3.2. *Let U be a C^2 scalar field on $\mathcal{M}$ satisfying the assumption $(\mathcal{M}_1)$ for a fixed point $x_0 \in \mathcal{M}$. Then there exist some constants $c_1, c_2, c_3 > 0$ such that for any $x \in \mathcal{M}$,*

$$(\langle \nabla_R U(x), \nabla_R U(x) \rangle_R)^{\frac{1}{2}} \ge \nu d(x, x_0) - c_1, \tag{3.6}$$

$$U(x) \ge \frac{\nu}{2} d^2(x, x_0) - c_2 d(x, x_0) - c_3. \tag{3.7}$$

Proof. Cf. Lemma 2.2 of [3].

Remark 3.3. Taken U, R and ρ as in $(\mathcal{M}_1)$ and in $(\mathcal{M}_3)$, by Remark 1.6 we can choose $R > \rho$ and U_0 such that $U(x) \ge U_0 \Rightarrow d(x, x_0) \ge R+1$, where $U_0 \ge \sup\{U(x) : x \in \mathcal{M}, d(x, x_0) \le R + 1\}$.

Lemma 3.4. *Let $\mathcal{M}$ be a Riemannian manifold and there exists $x_0 \in \mathcal{M}$ such that $(\mathcal{M}_3)$ holds. Let $(x_n)_{n\in\mathbb{N}} \subset \Omega^1$ be such that*

1) *the sequence $(\int_0^1 \langle \dot{x}_n(t), \dot{x}_n(t) \rangle_R dt)_{n\in\mathbb{N}}$ is bounded;*
2) *there exists $(s_n)_{n\in\mathbb{N}} \subset [0, 1]$ such that $\lim_n \phi(x_n(s_n)) = 0$.*

Then (up to subsequences)

$$\lim_n \int_0^1 \frac{1}{\phi^2(x_n)} dt = +\infty. \tag{3.8}$$

Proof. Cf. Lemma 3.2 of [1]. □

The following lemma states a very important property of the functional f_ε.

Lemma 3.5. *Let $\mathcal{M}$ be a Riemannian manifold with convex boundary satisfying $(\mathcal{M}_1)-(\mathcal{M}_3)$. Given M, $\rho > 0$, there exists $\varepsilon_0 = \varepsilon_0(M, \rho)$ such that for any $\varepsilon \in]0, \varepsilon_0]$, any critical point $x_\varepsilon \in \Omega^1$ of the penalized functional f_ε satisfying*

$$-c_0 + \rho \leq f_\varepsilon(x_\varepsilon) \leq M \tag{3.9}$$

is a critical point of f.

Proof. In order to prove Lemma 3.5, it is sufficient to prove that there exist ε_1, $\varepsilon_2 > 0$ such that for any x_ε satisfying (3.9),

$$\sup_{t\in[0,1]} U(x_\varepsilon(t)) \leq \frac{1}{\varepsilon} \qquad \text{if } \ \varepsilon \leq \varepsilon_1 \tag{3.10}$$

and

$$\inf_{t\in[0,1]} \phi(x_\varepsilon(t)) \geq \sqrt{\varepsilon} \qquad \text{if } \ \varepsilon \leq \varepsilon_2. \tag{3.11}$$

In fact by (3.10) and (3.11) it follows that for $\varepsilon \leq \min\{\varepsilon_1, \varepsilon_2\}$,

$$U_{1,\varepsilon}(x_\varepsilon(t)) = U_{2,\varepsilon}(x_\varepsilon(t)) = 0 \quad \text{for any} \ \ t \in [0,1]$$

and therefore x_ε is a critical point of f.

Let us assume that (3.10) is not true, that is there exist $\varepsilon_n \to 0$ and $(x_n)_{n\in\mathbb{N}} \subset \Omega^1$ such that x_n is a critical point of $f_n = f_{\varepsilon_n}$ which satisfies (3.9) and

$$\sup_{t\in[0,1]} U(x_n(t)) > \frac{1}{\varepsilon_n}\,. \tag{3.12}$$

Let us prove that (3.12) implies

$$\sup\{d(x_n(t), x_0) : t \in [0,1],\ n \in \mathbb{N}\} = +\infty. \tag{3.13}$$

Namely, if (3.13) does not hold, there exists $\tilde{R} > 0$ such that $d(x_n(t), x_0) \leq \tilde{R}$ for any t, n, while if n is large enough by (3.12), Remark 3.3 and $(\mathcal{M}_3)$ it follows that $\phi(x_n(t_n)) \geq \mu$, so $x_n(t_n)) \in A_{\tilde{R},\mu}$, where

$$A_{\tilde{R},\mu} = \{x \in \mathcal{M} : d(x, x_0) \leq \tilde{R},\ \phi(x) \geq \mu\}.$$

Then U is bounded on the compact set $A_{\tilde{R},\mu}$, and this contradicts (3.12).

Obviously by (3.9) we have that

$$\Big(\int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle dt\Big)_{n\in\mathbb{N}} \qquad \text{is bounded;}$$

then, as (3.13) holds,

$$\inf\{d(x_n(t), x_0) : t \in [0,1]\} \to +\infty \quad \text{as} \ \ n \to +\infty. \tag{3.14}$$

It follows that, for n large, $U_{1,\varepsilon_n}(x_n(t)) = 0$ for any $t \in [0,1]$, thus x_n satisfies the equation

$$D_t\dot{x}_n = -\nabla_R V(x_n) + \psi'_{2,\varepsilon_n}(U(x_n))\nabla_R U(x_n). \tag{3.15}$$

For any $t \in [0,1]$, let $u_n(t) = U(x_n(t))$. By (3.15) if n is large enough,

$$\begin{aligned}\ddot{u}_n(t) = H_R^U(x_n(t))[\dot{x}_n(t),\dot{x}_n(t)] - \langle\nabla_R U(x_n(t)),\nabla_R V(x_n(t))\rangle_R\\ + \langle\nabla_R U(x_n(t)),\nabla_R U(x_n(t))\rangle_R \psi'_{2,\varepsilon_n}(U(x_n(t)).\end{aligned}$$

By (3.14) and $(\mathcal{M}_2)$, as $\dot{x}_n(0) \in (T_{x_n(0)}(M_0))^\perp$ and $\dot{x}_n(1) \in (T_{x_n(1)}(\mathcal{M}_1))^\perp$ it follows that

$$\begin{aligned}0 = \int_0^1 \ddot{u}_n(t)dt \geq \nu\int_0^1 \langle\dot{x}_n(t),\dot{x}_n(t)\rangle_R dt - \int_0^1 \langle\nabla_R U(x_n(t)),\nabla_R V(x_n(t))\rangle_R dt\\ + \int_0^1 \psi'_{2,\varepsilon_n}(U(x_n(t))\langle\nabla_R U(x_n(t)),\nabla_R U(x_n(t))\rangle_R dt.\end{aligned} \tag{3.16}$$

By the expression of f_n it follows that

$$\tfrac{1}{2}\int_0^1 \langle\dot{x}_n(t),\dot{x}_n(t)\rangle_R dt = f_n(x_n) + \int_0^1 V(x_n(t))dt - \int_0^1 \psi_{2,\varepsilon_n}(U(x_n(t)))dt. \tag{3.17}$$

By (3.14) and (3.6) it is easy to see that, for n large,

$$\langle\nabla_R U(x_n(t)),\nabla_R U(x_n(t))\rangle_R \geq 2 \qquad \text{for any} \quad t \in [0,1] \tag{3.18}$$

while by (V_4) and (3.14) it follows that

$$-\int_0^1 \langle\nabla_R U(x_n(t)),\nabla_R V(x_n(t))\rangle_R dt \geq o(1). \tag{3.19}$$

By (3.16), (3.19) and (3.9) we deduce

$$0 \geq 2\nu\big(-c_0 + \rho + \int_0^1 V(x_n(t))dt\big) + o(1) + 2\int_0^1 (\psi'_{2,\varepsilon_n}(U(x_n(t)) - \nu\psi_{2,\varepsilon_n}(U(x_n(t)))dt. \tag{3.20}$$

Finally, by the definition of $\psi_{2,\varepsilon}$, by (V_2) and by (3.20) we have the contradiction

$$0 \geq 2\nu\rho + o(1)$$

thus (3.10) holds.

We prove now (3.11). If it does not hold, there exist $\varepsilon_n \to 0$, $(t_n)_{n\in\mathbb{N}} \subset [0,1]$ and $(x_n)_{n\in\mathbb{N}} \subset \Omega^1$ such that x_n is a critical point of $f_n = f_{\varepsilon_n}$ satisfying (3.9) and

$$\phi(x_n(t_n)) < \sqrt{\varepsilon_n}, \tag{3.21}$$

where t_n is the minimum point of the map $v_n(t) = \phi(x_n(t))$.

By (3.10), $U_{2,\varepsilon_n}(x_n(t)) = 0$ for any $t \in [0,1]$ so x_n satisfies the equation

$$D_t \dot{x}_n = -\nabla_R V(x_n) - 2\psi'_{1,\varepsilon_n}(\frac{1}{\phi^2(x_n)})\frac{\nabla_R \phi(x_n)}{\phi^3(x_n)}. \tag{3.22}$$

By (3.9) and (V_1) it follows that the sequence

$$\Big(\int_0^1 \langle \dot{x}_n, \dot{x}_n\rangle_R dt\Big)_{n\in\mathbb{N}} \quad \text{is bounded,}$$

then by (3.21) and $(\mathcal{M}_3)$ also $(x_n)_{n\in\mathbb{N}}$ is bounded in Ω^1 and, passing to a subsequence, $x_n \rightharpoonup x$ weakly in W^1 and uniformly in $[0,1]$.

By (3.21), $x \in \partial\Omega^1$; that is, there exists $t_0 \in [0,1]$ such that $\phi(x(t_0)) = 0$. As the closure of the set $\{x_n(0), x_n(1)\big|_{n\in\mathbb{N}}\}$ is compact in $\mathcal{M}$, there exists $\bar{\delta} > 0$ such that $\phi(x_n(0)) \geq \bar{\delta}$ and $\phi(x_n(1)) \geq \bar{\delta}$ for any n. Obviously for n large

$$v_n(t_n) < \delta_1 = \min\{\delta/2, \bar{\delta}\},$$

where $\delta > 0$ is the constant introduced in (iii) of Definition 1.1. It follows that for n large the minimum point t_n of v_n belongs to the open interval $(0,1)$ and therefore

$$\dot{v}_n(t_n) = 0, \quad \ddot{v}_n(t_n) \geq 0.$$

Now for any $n \in \mathbb{N}$ let w_n be the first instant, after t_n, for which $v_n(w_n) = \delta_1$. Then, for n large,

$$v_n(t) \leq \delta_1 < \delta \qquad \text{for any} \quad t \in [t_n, w_n]. \tag{3.23}$$

By (3.22) it follows that

$$\begin{aligned} \ddot{v}_n(t) = {} & H_R^{\phi}(x_n(t))[\dot{x}_n(t), \dot{x}_n(t)] - \langle \nabla_R \phi(x_n(t)), \nabla_R V(x_n(t))\rangle_R \\ & - 2\psi'_{1,\varepsilon_n}(\frac{1}{\phi^2(x_n(t))})\langle \nabla_R \phi(x_n(t)), \nabla_R \phi(x_n(t))\rangle_R \frac{1}{\phi^3(x_n(t))}. \end{aligned} \tag{3.24}$$

As $\mathcal{M}$ has convex boundary and (V_3) holds, by (3.23) and (3.24) and by definition of $\psi_{1,\varepsilon}$, we deduce that for n large and for any $t \in [t_n, w_n]$,

$$\ddot{v}_n(t) \leq \gamma \langle \dot{x}_n(t), \dot{x}_n(t)\rangle_R v_n(t). \tag{3.25}$$

As $\dot{v}_n(t_n) = 0$ for any n, by (3.25) it follows that for n large and for any $t \in [t_n, w_n]$,

$$v_n(t) \leq v_n(t_n) + \gamma \int_{t_n}^t \langle \dot{x}_n(\tau), \dot{x}_n(\tau)\rangle_R v_n(\tau) d\tau.$$

By Gronwall's lemma we conclude that

$$v_n(t) \leq v_n(t_n) \exp\big[\gamma \int_{t_n}^t \langle \dot{x}_n(\tau), \dot{x}_n(\tau)\rangle_R d\tau\big]$$

and therefore, as $(\int_0^1 \langle \dot{x}_n(t), \dot{x}_n(t)\rangle_R dt)_{n\in\mathbb{N}}$ is bounded, there exists $M > 0$ such that for n large,

$$v_n(t) \le M v_n(t_n) \qquad \text{for any} \;\; t \in [t_n, w_n].$$

Taking $t = w_n$ we have the contradiction $\delta \le M v_n(t_n)$ thus the proof of (3.11) is achieved.

Remark 3.6. If $\mathcal{M}$ satisfies the additional assumption $(\mathcal{M}_4)$ and the weaker condition

$$f_\varepsilon(x_\varepsilon) \le M \quad \text{for all} \;\; \varepsilon \le \varepsilon_0$$

holds, then there exists $\varepsilon_n \to 0$ such that any critical point x_n of $f_n = f_{\varepsilon_n}$ is a critical point of f.

In fact if $f_n(x_n) \to -c_0$ and $\sup_{t\in[0,1]} U(x_n(t)) > \frac{1}{\varepsilon_n}$, then arguing as in the proof of the Lemma 3.5, we can prove that (3.14) still holds and therefore

$$\int_0^1 \langle \dot{x}_n(t), \dot{x}_n(t)\rangle dt \to 0$$

which contradicts $(\mathcal{M}_4)$.

Proof of Theorem 1.4. Arguing as in Lemma 2.4 of [11] it is possible to prove that for any $\varepsilon > 0$ the functional f_ε satisfies (P.S) condition and that its sublevels are complete.

As f_ε is bounded from below, it attains its minimum at a point $x_\varepsilon \in \Omega^1$. Obviously, $f_\varepsilon(x_\varepsilon) \le f_1(x_1)$ for all $\varepsilon \in [0,1]$. By Remark 3.6 it follows that the functional f has at least a critical point.

In order to give now the multiplicity result, let

$$c_{k,\varepsilon} = \inf_{A\in\Gamma_k} \sup_{x\in A} f_\varepsilon(x).$$

By Theorem 2.3, for any $\varepsilon > 0$ and for any $k \in \mathbb{N}$, $c_{k,\varepsilon}$ is a critical value of f_ε; that is, there exists x_ε critical point of f_ε such that $f_\varepsilon(x_\varepsilon) = c_{k,\varepsilon}$.

Now we will pass from the critical points of the penalized functionals to those of the "energy". Then for any $a \in \mathbb{R}$ we set

$$f_a = \{x \in \Omega^1 : f(x) \ge a\}, \qquad f_{\varepsilon,a} = \{x \in \Omega^1 : f_\varepsilon(x) \ge a\}.$$

Let us point out that the sublevels of f have finite category even if f does not satisfy (P.S) (see Lemma 2.5 of [11]) and therefore by Theorem 2.5 there exists $k_1 = k_1(a) \in \mathbb{N}$ such that

$$B \cap f_a \ne \emptyset \qquad \text{for any } B \subseteq \Omega^1 \text{ with } cat_{\Omega^1}(B) \ge k_1.$$

Since $f_a \subseteq f_{\varepsilon,a}$ for any $\varepsilon > 0$, it follows that

$$B \cap f_{\varepsilon,a} \ne \emptyset \qquad \text{for any } B \subseteq \Omega^1 \text{ with } cat_{\Omega^1}(B) \ge k_1$$

which implies

$$c_{k_1,\varepsilon} \geq a. \tag{3.26}$$

Let K be a compact set in Ω^1 such that $cat_{\Omega^1}(K) \geq k_1$. Taking $M = \max_{x\in K} f_1(x)$, as $f_\varepsilon \leq f_{\varepsilon'}$ if $\varepsilon \leq \varepsilon'$,

$$c_{k_1,\varepsilon} = f_\varepsilon(x_\varepsilon) \leq \sup f_\varepsilon(K) \leq M \qquad \text{for any } \varepsilon \in (0,1]. \tag{3.27}$$

Choosing a large enough, by (3.26), (3.27) and Lemma 3.5 there exists ε_0 small enough such that for any $\varepsilon \leq \varepsilon_0$, x_ε is a critical point of f. Since $f_\varepsilon(x_\varepsilon) \geq a$ and the potential V is bounded on the closed set $\mathcal{M}_0 \cap \mathcal{M}_1$, the conclusion of Theorem 1.4 follows.

Proof of Corollary 1.7. As the manifold $\mathcal{M}$ has a convex boundary and M_0, M_1 satisfy assumption (M), the functional f does not satisfy (P.S) condition because there are (P.S) sequences which converge to an element of $\partial\Omega^1$. Then we shall consider only the penalization term $U_{1,\varepsilon}$ defined in (3.5). Setting

$$f_\varepsilon(x) = \frac{1}{2}\int_0^1 \langle \dot{x}, \dot{x}\rangle_R dt - \int_0^1 V(x)dt + \int_0^1 U_{1,\varepsilon}(x)dt$$

it is easy to see that Lemma 3.5 still holds if we replace (3.9) by the weaker condition $f_\varepsilon(x_\varepsilon) \leq M$. Then the existence and the multiplicity of solutions follows arguing as in the proof of Theorem 1.4.

Proof of Corollary 1.8. It suffices to take the penalization term $U_{2,\varepsilon}$ and to simplify the arguments of Theorem 1.4.

Proof of Proposition 1.9. As M_0 and M_1 satisfy the condition (M)-(ii), the conclusion follows by Theorem 1.1.

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