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PERIODIC PARABOLIC EQUATIONS ON $\mathbb{R}^N$ AND APPLICATIONS

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Abstract. Periodic parabolic reaction-diffusion equations on $\mathbb{R}^N$ are considered. Using the method of sub- and supersolutions time-periodic solutions are obtained, which are stable with respect to the L_∞-norm. Particular attention is devoted to Fisher's equation from population genetics.

1. Introduction. Reaction-diffusion equations are one of the types of parabolic equations which have important applications to natural sciences such as biomathematics or chemical combustion theory. Within this class of equations, periodic equations seem to be of particular interest since they can take into account seasonal fluctuations occurring in the phenomena they are modelling. For the mathematical treatment of such equations on bounded domains we mainly refer to the recent monograph, [19]. In contrast to the bounded domain case very little is known about periodic-parabolic problems on unbounded domains. In [22] an abstract framework is developed to treat such reaction-diffusion equations on $\mathbb{R}^N$.

It should be remarked that the boundedness of the domain implies that all the maps involved in the functional analytic formulation of the problem are compact. Moreover, the spaces one uses in this case are often such that the positive cone has nonempty interior. It is exactly these two properties which fail to hold when one deals with reaction-diffusion equations on all of $\mathbb{R}^N$ having the following type:

$$\begin{cases} \partial_t u - \Delta u = f(x,t,u) & \text{on } \mathbb{R}^N \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where we assume $f\colon \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to be sufficiently smooth as well as $f(\cdot,\cdot,0) \equiv 0$. Since we are interested in the periodic version of (1.1), the reaction term f is assumed to be T-periodic in the time variable t with fixed period $T > 0$. As initial values we consider u_0 belonging to the space $C_0(\mathbb{R}^N)$ or $\mathrm{BUC}(\mathbb{R}^N)$.

Our main concern is to derive results on the existence of stable positive T-periodic solutions of (1.1) under suitable assumptions on the nonlinearity f. Here, stability is to be understood with respect to the L_∞-norm. Our results can be basically obtained by sub- and supersolution techniques. Although the abstract theory developed for problems on bounded domains cannot be directly applied to (1.1), the results obtained for such problems may serve as guidelines for the corresponding results on $\mathbb{R}^N$.

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In this treatise we focus our attention on some special classes of nonlinearities, which comprise a periodic version of Fisher's equation from population genetics. We shall now present one of our main results.

Consider the periodic parabolic initial value problem

$$\begin{cases} \partial_t u - \Delta u = m(x,t)h(u) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

where m is a smooth T-periodic function, which is negatively bounded away from zero at infinity, and $h \in C^2(\mathbb{R})$ is a concave function satisfying

$$h(0) = h(1) = 0 \qquad \text{and} \qquad h'(0) > 0.$$

We emphasize that h need not be strictly concave.

Equation (1.2) can be interpreted as the periodic Fisher equation from population genetics with indefinite weight function m (cf. [19] and the references therein). Since the function u in (1.2) represents a relative density (cf. Section 3), only solutions with values in $[0,1]$ are of interest. Of course, the zero solution $u \equiv 0$ solves (1.2).

We shall prove the following result (cf. Theorem 3.3):

Theorem 1.1. (i) *If the trivial solution $u \equiv 0$ is linearly stable, then there is no nontrivial T-periodic solution of* (1.2) *and $u \equiv 0$ is globally asymptotically L_∞-stable with respect to initial data in $\{u_0 \in \mathrm{BUC}(\mathbb{R}^N) : 0 \le u_0 \ll 1\}$.*

(ii) *If the trivial solution $u \equiv 0$ is linearly unstable, then there exists a unique nontrivial T-periodic solution u^*, which is globally asymptotically L_∞-stable with respect to initial data in $\{u_0 \in \mathrm{BUC}(\mathbb{R}^N) : 0 < u_0 \ll 1\}$.*

(iii) *Let the trivial solution $u \equiv 0$ be neutrally stable. If h is not linear in some interval $[0, s_0]$ with $s_0 > 0$, then there is no nontrivial T-periodic solution of* (1.2) *and $u \equiv 0$ is globally asymptotically L_∞-stable with respect to initial data in $\{u_0 \in \mathrm{BUC}(\mathbb{R}^N) : 0 \le u_0 \ll 1\}$. If h is linear on such an interval $[0, s_0]$, there exists a one-parameter family $\mathcal{A} := \{\varepsilon\phi : 0 \le \varepsilon \le s_0\}$ of L_∞-stable T-periodic solutions of* (1.1).

This theorem generalizes Theorem 6.1 in [22], where strict concavity of h is assumed. Moreover, Theorem 1.1 can be considered as an extension of a result by P. Hess in [18], where he studies this type of equation on bounded domains subject to the various boundary conditions. Furthermore, we point out that as a special case we obtain the corresponding result for the associated autonomous problem, by letting the period T tend to zero.

2. The periodic parabolic equation on $\mathbb{R}^N$.

2.1. The initial value problem. Consider the following semilinear nonautonomous reaction-diffusion problem on $\mathbb{R}^N$ ($N \ge 1$):

$$\begin{cases} \partial_t u - \Delta u = f(x,t,u) & \text{on } \mathbb{R}^N \times (0,\infty], \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N, \end{cases} \tag{2.1}$$

for initial values $u_0 \in \mathrm{BUC}(\mathbb{R}^N)$; i.e., u_0 is bounded and uniformly continuous on $\mathbb{R}^N$. Throughout we assume that f is a sufficiently smooth (Hölder continuous) function being periodic in t with given period $T > 0$, and $f(\cdot,\cdot,0) \equiv 0$ on $\mathbb{R}^N \times \mathbb{R}$.

We are interested in existence and stablity properties of *T-periodic solutions*; i.e., global solutions of (2.1) which are T-periodic in time.

Let $\underline{u}$ and $\overline{u}$ be sub- and supersolutions, respectively, of (2.1) satisfying

$$\begin{aligned} &\underline{u}(\cdot,0) \le \underline{u}(\cdot,T), \quad \overline{u}(\cdot,0) \ge \overline{u}(\cdot,T) \quad \text{on } \mathbb{R}^N, \\ &\underline{u}_0 := \underline{u}(\cdot,0) \le \overline{u}(\cdot,0) =: \overline{u}_0 \qquad\quad \text{on } \mathbb{R}^N. \end{aligned} \tag{2.2}$$

For precise definitions of sub- and supersolutions we refer to [22] or [26]. Set

$$V := [\underline{u}_0, \overline{u}_0]_{\mathrm{BUC}(\mathbb{R}^N)} := \{u_0 \in \mathrm{BUC}(\mathbb{R}^N) \colon \underline{u}_0 \le u_0 \le \overline{u}_0\}.$$

Let X_0 be either $C_0(\mathbb{R}^N)$ or $\mathrm{BUC}(\mathbb{R}^N)$. Due to the results in [22] and [26] on classical solvability and existence of global solutions of (2.1) and from the comparison principle (cf. Theorem 1.7 in [22]) it follows that for initial values $u_0 \in V \cap X_0$ the solution $u(\cdot;u_0)$ of (2.1) exists for all times; i.e., $t^+(u_0) = +\infty$, and satisfies

$$\underline{u}_0 \le \underline{u}(\cdot,T) \le u(T;u_0) \le \overline{u}(\cdot,T) \le \overline{u}_0.$$

Thus, defining the *time-T-map* S by

$$S \colon V \cap X_0 \to X_0\,, \quad u_0 \mapsto u(T;u_0),$$

we immediately obtain that S is continuous and strictly order preserving, and fixed points of S are in one-to-one correspondence with the T-periodic solutions of (2.1) in $V \cap X_0$.

Remark 2.1. Here, the positive cone X_0^+ consists of all pointwise positive functions in X_0. Moreover, we have the following:

(i) The positive cone of $\mathrm{BUC}(\mathbb{R}^N)$ has nonempty interior. It consists of all positive functions which are bounded away from zero.

(ii) The above is not true for $C_0(\mathbb{R}^N)$. The positive cone of $C_0(\mathbb{R}^N)$ has empty interior and the quasi-interior points of $C_0(\mathbb{R}^N)^+$ are exactly the functions which are everywhere positive.

(iii) Let $w_1, w_2 \in X_0$. If $w_1(x) < w_2(x)$ holds pointwise on $\mathbb{R}^N$, then we shall write $w_1 <_p w_2$. We point out that, while $w_1 \ll w_2$ is equivalent to $w_1 <_p w_2$ in $C_0(\mathbb{R}^N)$, this is not true in $\mathrm{BUC}(\mathbb{R}^N)$.

In the following we want to establish conditions under which (asymptotically) stable positive T-periodic solutions of (2.1) exist. In particular, we are interested in the stability with respect to the L_∞-norm. First, we prove existence of order-stable T-periodic solutions (cf. Proposition 2.7), and then, we show that in some cases order stability implies L_∞-stability (cf. Proposition 2.12). The relevant definitions concerning stability may be found in [21], [11] and [26].

2.2. Existence of order-stable fixed points. A fixed point v of S in V is said to be *order stable from below* if there exists a strictly increasing sequence $(v_n)_{n\in\mathbb{N}}$ of (not necessarily strict) subsolutions of S in V converging to v from below. The fixed point v is called *strongly order stable from below* if the subsolutions are strict. (*Strong*) *order stability from above* is defined analogously by replacing increasing by decreasing and subsolution by supersolution. A (*strongly*) *order-stable* fixed point has the corresponding property from above and below.

As above we assume that there exists an order-related pair $\underline{u}_0 \leq \overline{u}_0$ of sub- and supersolutions of S. From the results in [22] and [26] we obtain by the monotone iteration scheme minimal and maximal fixed points of S in V by

$$u_{\min} := \lim_{n\to\infty} S^n(\underline{u}_0) \qquad \text{and} \qquad u_{\max} := \lim_{n\to\infty} S^n(\overline{u}_0).$$

Throughout this section we assume that both fixed points $u_{\min}$ and $u_{\max}$ belong to $C_0(\mathbb{R}^N)$. This means that every fixed point of S in V belongs to $[u_{\min}, u_{\max}]_{C_0(\mathbb{R}^N)}$.

The next result is crucial for our proof of the existence of order-stable fixed points of S in V.

Proposition 2.2. *Assume that there exist positive constants R_0 and δ_0 such that*

$$f_u(x,t,u) \leq 0 \qquad \text{for } |x| \geq R_0,\ |u| \leq \delta_0 \text{ and } t \in [0,T]. \tag{2.3}$$

Let $v_1 < v_2$ be fixed points of S in V. Then precisely one of the three alternatives holds:

(i) *There exists a further fixed point of S between v_1 and v_2.*

(ii) *There exists an entire orbit $(y_n)_{n\in\mathbb{Z}}$ consisting of strict subsolutions of S, connecting v_1 and v_2; i.e., $y_n \to v_1$ as $n \to -\infty$, $y_n \to v_2$ as $n \to +\infty$, and there is no (strict) supersolution between v_1 and v_2.*

(iii) *There exists an entire orbit $(x_n)_{n\in\mathbb{Z}}$ consisting of strict supersolutions of S, connecting v_2 and v_1; i.e., $x_n \to v_2$ as $n \to -\infty$, $x_n \to v_1$ as $n \to +\infty$, and there is no (strict) subsolution between v_1 and v_2.*

Remark 2.3. Proposition 2.2 is a more general version of Proposition 4.3 in [22] and goes back to a result due to E. N. Dancer and P. Hess in [8]. The proof of Proposition 4.3 in [22] is based on techniques developed by E. N. Dancer in [6] and [7], which depend heavily on the spectral properties of the period map S. The advantage of our approach here is twofold: First, our assumptions are weaker than those in [22] and second, using comparison principle arguments we were able to simplify the proof of the above result to some extent. Note that it is not possible to adapt the arguments in [22] to our situation.

One of the essential ingredients of the proof of Proposition 2.2 is contained in the following lemmas, where we use an idea similar to that in Lemma 3.1 in [27].

Lemma 2.4. *Assume* (2.3) *and let $\underline{u}_0 \in C_0(\mathbb{R}^N)$ be a subsolution of S and $\overline{u}_0 \in C_0(\mathbb{R}^N)$ be a supersolution of S satisfying*

$$\underline{u}(x,t) := u(x,t;\underline{u}_0) \leq u(x,t;\overline{u}_0) =: \overline{u}(x,t)$$

for $|x| \le R_0$ *and* $t \in [0,T]$ *and*

$$|\overline{u}(x,t) - \underline{u}(x,t)| \le \delta_0 \qquad \text{for } |x| \ge R_0 \text{ and } t \in [0,T].$$

Then $\underline{u} \le \overline{u}$ *on* $\mathbb{R}^N \times [0,T]$.

Proof. Putting $w := \underline{u} - \overline{u}$ we claim that $w \le 0$ on $\mathbb{R}^N \times [0,T]$. From our assumption we immediately conclude that $w(\cdot,0) \le w(\cdot,T)$ and

$$\partial_t w - \Delta w = f(x,t,\underline{u}) - f(x,t,\overline{u}) = f_u(x,t,\xi)w \tag{2.4}$$

on $\mathbb{R}^N \times [0,T]$ for an appropriate function ξ, which satisfies $|\xi(x,t)| \le \delta_0$ for $|x| \ge R_0$ and $t \in [0,T]$.

If $w(\cdot,0) \le 0$ on $\mathbb{R}^N$, the maximum principle immediately yields the assertion. Suppose for the sake of contradiction that $w(\cdot,0) \not\le 0$. Since $w(\cdot,0)$ belongs to $C_0(\mathbb{R}^N)$, there exists $x_0 \in \mathbb{R}^N$ such that

$$M := w(x_0,0) = \max\nolimits_{x\in\mathbb{R}^N} w(x,0) > 0.$$

Moreover, it is clear that $|x_0| > R_0$. Setting $\tilde{w} := w - M$ we have

$$\begin{cases} \partial_t \tilde{w} - \Delta\tilde{w} - f_u(x,t,\xi)\tilde{w} \le 0 & \text{for } |x| > R_0,\ t \in (0,T], \\ \tilde{w} < 0 & \text{for } |x| = R_0,\ t \in (0,T], \\ \tilde{w}(\cdot,0) \le 0 & \text{for } |x| \ge R_0. \end{cases}$$

From the maximum principle we conclude $w(x_0,T) < M = w(x_0,0)$. This gives the required contradiction. □

Lemma 2.5. *Assume* (2.3) *and let* $v_1 < v_2$ *be fixed points of* S *in* V. *Suppose there exists an entire orbit* $(y_n)_{n\in\mathbb{Z}}$ *consisting of strict subsolutions of* S, *connecting* v_1 *and* v_2; *i.e.,* $y_n \to v_1$ *as* $n \to -\infty$ *and* $y_n \to v_2$ *as* $n \to +\infty$. *Then there is no supersolution of* S *between* v_1 *and* v_2.

Proof. Assume for the sake of contradiction that there exists a supersolution $\overline{w}_0$ of S between v_1 and v_2. Without loss of generality we may assume that

$$v_2(x,t) - v_1(x,t) \le \delta_0$$

for $x \in \mathbb{R}^N$ satisfying $|x| \ge R_0$ and $t \in [0,T]$. Since there are strict subsolutions y_n arbitrarily close to v_1 there exists an $n_0 \in \mathbb{Z}$ such that

$$u(x,t;y_{n_0}) \le u(x,t;\overline{w}_0) \qquad \text{for } |x| \le R_0 \text{ and } t \in [0,T].$$

Thus, Lemma 2.4 immediately implies $y_{n_0} \le \overline{w}_0$. But this is a contradiction, since $S^k(y_{n_0}) = y_{k+n_0}$ tends to v_2 as $k \to +\infty$. □

Remark 2.6. Inspecting the proof of Lemma 2.5 we see that even more is true. Call a fixed point v of S in $\tilde{V} := [u_{\min}, u_{\max}]_{C_0(\mathbb{R}^N)}$ *order repelling from above* or *from below*, whenever there exist arbitrarily close strict subsolutions in $\tilde{V}$ lying above v or arbitrarily close supersolutions in $\tilde{V}$ lying below v. Then a fixed point v of S in $\tilde{V}$ cannot be order stable and order repelling at the same time. This means that it is not possible that a fixed point v of S in V has both strict subsolutions and strict supersolutions in $\tilde{V}$ on the same side arbitrarily close to it. While this is trivial on bounded domains, this is far from being evident in our situation.

Proof of Proposition 2.2. By the results in [8] or [19] it suffices to show that in case (ii) there are no supersolutions of S in $(v_1, v_2)_{C_0(\mathbb{R}^N)}$. This is by no means obvious, since it may be the case that there exist a supersolution $\overline{w}$ and strict subsolutions arbitrarily close to v_1, which are not order related to $\overline{w}$. But due to Lemma 2.5 this cannot occur. This proves the claim. □

If $S^n(\underline{u}_0)$ or $S^n(\overline{u}_0)$ do not converge uniformly on $\mathbb{R}^N$ to $u_{\min}$ or $u_{\max}$, respectively, it is not clear whether these fixed points are order stable from below or from above. Nevertheless, we get the following result on existence of order-stable fixed points.

Proposition 2.7. *Let* (2.3) *be satisfied. If there exist strict sub- and supersolutions* $\underline{u}_n$ *and* $\overline{u}_n$ *of* S *such that*

$$\underline{u}_n < u_{\min} \leq u_{\max} < \overline{u}_n \quad \textit{and} \quad \limsup_{|x|\to\infty}\big(\overline{u}_n(x) - \underline{u}_n(x)\big) < \frac{1}{n}$$

for $n \in \mathbb{N} \setminus \{0\}$*, then there always exists an order-stable fixed point of* S *in* V*. More precisely, there exists a strongly order-stable fixed point or a nontrivial totally ordered continuum* $\mathcal{F} := \{u_1 \leq u \leq u_2 : S(u) = u\}$ *of order-stable fixed points of* S *in* V*, where* u_1 *and* u_2 *are fixed points of* S*, which are strongly order stable from below and from above, respectively.*

Proof. Putting

$$\underline{w}_n := \sup\{\underline{u}_n, S^n(\underline{u}_0)\} \qquad \text{and} \qquad \overline{w}_n := \inf\{\overline{u}_n, S^n(\overline{u}_0)\}$$

for $n \in \mathbb{N}$ we obtain strict sub- and supersolutions of S converging to $u_{\min}$ and $u_{\max}$, respectively, uniformly on $\mathbb{R}^N$ as $n \to \infty$ (cf. Theorems 2.3 and 2.4 in [22]). Hence, $u_{\min}$ is strongly order stable from below and $u_{\max}$ is strongly order stable from above and both fixed points lie in $C_0(\mathbb{R}^N)$. Thus, thanks to Proposition 2.2, we may argue as in the proof of Corollary 4.7 in [22] to obtain the assertion. □

Remark 2.8. It should be remarked that Proposition 2.2 as well as Proposition 2.7 rely essentially on the fact that S maps order intervals in $C_0(\mathbb{R}^N)$ into relatively compact sets in $C_0(\mathbb{R}^N)$.

2.3. Periodic parabolic eigenvalue problems on all of $\mathbb{R}^N$.

Periodic parabolic eigenvalue problems on bounded domains have been widely studied (cf. [4], [19]). In [13], [10] and [12] D. Daners and P. Koch Medina have investigated such problems on all of $\mathbb{R}^N$. In the following we shall make use of their work.

Let v be a T-periodic solution of (2.1) in $C_0(\mathbb{R}^N)$. We consider the following eigenvalue problem:

$$\begin{cases} \partial_t\phi - \Delta\phi - f_u(x,t,v)\phi = \lambda\phi & \text{on } \mathbb{R}^N \times \mathbb{R}, \\ \phi(\cdot,t) \in C_0(\mathbb{R}^N) & \text{for } t \in \mathbb{R}, \\ \phi \text{ is } T\text{-periodic.} \end{cases} \tag{2.5}$$

Denote by S_v the period map associated to (2.5) and by $r(S_v)$ the spectral radius of S_v.

Proposition 2.9. *Let v be a T-periodic solution of* (2.1) *in* $C_0(\mathbb{R}^N)$ *and suppose* (2.3).

(i) *If $r(S_v) > 1$, then there exist a unique principal eigenvalue $\lambda_1(v)$ and a corresponding positive eigenfunction ϕ of* (2.5)*. Moreover, $\lambda_1(v)$ is given by $\lambda_1(v) = -\frac{1}{T}\log r(S_v)$.*

(ii) *Assertion* (i) *remains true, if $r(S_v) = 1$, provided there exists a constant $c > 0$ such that $f_u(x,t,u) \le -c$ for $|x| \ge R_0$, $t \in \mathbb{R}$ and $|u| \le \delta_0$.*

Proof. The assertion can be directly obtained from the results in [13] by using an approximation argument. □

On the other hand, from the results in [19] on periodic-parabolic eigenvalue problems on bounded domains it is well known that for each $R > 0$ there exist the principal eigenvalue $\lambda_R(v)$ and a positive eigenfunction $\phi_{R,v}$ of the periodic parabolic eigenvalue problem on the bounded domain $\overline{\mathbb{B}}(0,R)$:

$$\begin{cases} \partial_t\phi - \Delta\phi - f_u(x,t,v)\phi = \lambda\phi & \text{in } \mathbb{B}(0,R) \times \mathbb{R}, \\ \phi = 0 & \text{on } \partial\mathbb{B}(0,R) \times \mathbb{R}, \\ \phi \text{ is } T\text{-periodic.} \end{cases} \tag{2.6}$$

As in Lemma 6.6 in [13] we conclude that $R \mapsto \lambda_R(v)$ is strictly decreasing and bounded for $R \to +\infty$, provided f satisfies (2.3). Thus, we may define

$$\lambda_\infty(v) := \lim_{R\to\infty} \lambda_R(v). \tag{2.7}$$

Moreover, we mention that estimates for the principal eigenvalue $\lambda_R(v)$ from above and from below are given in [19]. Such estimates are useful, whenever one is interested in sufficient conditions for $\lambda_\infty(v) < 0$.

It is natural to ask, whether there is any relationship between $\lambda_\infty(v)$, the *asymptotic principal eigenvalue*, and $\lambda_1(v)$, the principal eigenvalue of the eigenvalue problem on $\mathbb{R}^N$. A partial answer is given by the following proposition.

Proposition 2.10. *Let v be a T-periodic solution of* (2.1) *in* $C_0(\mathbb{R}^N)$*. If* (2.3) *holds, then the following assertions are true.*

(i) (*Daners, Koch*) $\lambda_\infty(v) \ge -\frac{1}{T}\log r(S_v)$.

(ii) *Assume $r(S_v) < 1$ or $r(S_v) = 1$, and suppose in the latter case that there exists a constant $c > 0$ such that $f_u(x,t,u) \le -c$ for $|x| \ge R_0$, $t \in \mathbb{R}$ and $|u| \le \delta_0$. Then $\lambda_\infty(v) = \lambda_1(v) = -\frac{1}{T}\log r(S_v)$.*

Proof. The proof of (i) may be found in [13].

The proof of assertion (ii) is split up in two parts. Let $\lambda_1(v) < 0$ be the principal eigenvalue and ϕ be a corresponding positive eigenfunction of (2.5). Supposing $\lambda_1(v) < \lambda_\infty(v)$ we shall derive a contradiction. Let $\lambda < 0$ be fixed in $\big(\lambda_1(v), \lambda_\infty(v)\big)$. Put

$$\delta := \frac{\min\{\phi(x,t) : |x| \le R \ \text{ and } \ t \in [0,T]\}\,\big(\lambda - \lambda_1(v)\big)}{1 + \max\{|f_u\big(x,t,v(x,t)\big)| : |x| \le R \ \text{ and } \ t \in [0,T]\}} > 0,$$

where $R \ge R_0$ is chosen such that $|v(x,t)| \le \delta_0$ for $|x| \ge R$ and $t \in [0,T]$. Defining functions

$$\psi_\varepsilon := \varepsilon(\phi - \delta) \qquad \text{for } \varepsilon > 0$$

we get

$$\partial_t \psi_\varepsilon - \Delta\psi_\varepsilon - \big(f_u(x,t,v) + \lambda\big)\psi_\varepsilon = \varepsilon\phi\big(\lambda_1(v) - \lambda + \delta\frac{\lambda + f_u(x,t,v)}{\phi(x,t)}\big) \le 0$$

on $\mathbb{R}^N \times \mathbb{R}$. Thus, putting $\phi_\varepsilon := \sup\{\psi_\varepsilon, 0\}$ we obtain arbitrarily small positive T-subsolutions of the restricted problem

$$\begin{cases} \partial_t u - \Delta u = \big(f_u(x,t,v) + \lambda\big)u & \text{in } \mathbb{B}(0,R_1) \times (0,\infty), \\ u = 0 & \text{on } \partial\mathbb{B}(0,R_1) \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{B}(0,R_1) \end{cases} \tag{2.8}$$

for sufficiently large $R_1 > 0$. This implies that the zero solution of (2.8) is not stable; i.e., $\lambda_{R_1}(v) - \lambda \le 0$, which gives the desired contradiction.

A similar argument can be used if $\lambda_1(v) = 0$. One only has to observe that $\lambda \in \big(\lambda_1(v), \lambda_\infty(v)\big)$ can be chosen so small that $f_u\big(x,t,v(x,t)\big) \le -\lambda$ holds for $|x| \ge R$ and $t \in [0,T]$. Hence, the proof is complete. $\square$

Remark 2.11. Proposition 2.10 yields a characterization of the instability of the trivial solution $u \equiv 0$ of (2.1): The zero solution of (2.1) is linearly unstable if and only if it is linearly unstable on some sufficiently large bounded domain. Moreover, we point out that assertion (ii) of Proposition 2.10 remained open in [13].

2.4. L_∞-stable T-periodic solutions. In the following we shall establish conditions under which order stability implies L_∞-stability of a T-periodic solution of (2.1).

A T-periodic solution v of (2.1) is said to be L_∞*-stable from above* (*from below*) if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $u_0 \in X_0$ satisfying $\|u_0 - v(0)\|_\infty < \delta$ and $u_0 \ge v(0)$ ($u_0 \le v(0)$) we have $t^+(u_0) = \infty$ and $\|u(t;u_0) - v(t)\|_\infty < \varepsilon$ for $t \ge 0$. Moreover, a T-periodic solution v of (2.1) is called (*strongly*) *order stable* if its initial value $v_0 := v(0)$ has this property as a fixed point of the period map S.

Now we are interested in the L_∞-stability of an arbitrary order-stable fixed point of S in $[u_{\min}, u_{\max}]_{C_0(\mathbb{R}^N)}$ with respect to V. Note that it is not always clear, whether order stability implies L_∞-stability.

However (cf. Proposition 4.12 in [22]), if $f_u(x,t,0) \le -c$ for large $|x|$ and some positive constant c, then each order-stable T-periodic solution of (2.1) in V is L_∞-stable with respect to V. Moreover, if the T-periodic solution is strongly order stable, then it is (locally) asymptotically L_∞-stable with respect to V.

Assuming only (2.3) we prove the following proposition:

Proposition 2.12. *Let* (2.3) *be satisfied and let* v *be a strongly order-stable* T*-periodic solution of* (2.1) *with* $v_0 := v(\cdot,0) \in V$*. Suppose there exist sequences* $(\underline{u}_n)_{n\in\mathbb{N}}$ *and* $(\overline{u}_n)_{n\in\mathbb{N}}$ *of strict sub- and supersolutions of* S *satisfying*

$$\underline{u}_n \ll u_{\min} \le v_0 \le u_{\max} \ll \overline{u}_n \qquad \text{in } \operatorname{BUC}(\mathbb{R}^N)$$

and

$$\limsup_{|x|\to\infty} \big(\overline{u}_n(x) - \underline{u}_n(x)\big) < \frac{1}{n} \qquad \text{for } n \in \mathbb{N}\setminus\{0\}.$$

Then v *is* L_∞*-stable.*

Proof. We only show that v is L_∞-stable from above. Let $\varepsilon > 0$ be given. The idea of the proof is to construct a strict supersolution $\overline{v}$ of S such that

$$v_0 \ll \overline{v} \le v_0 + \varepsilon \qquad \text{in } \operatorname{BUC}(\mathbb{R}^N).$$

From this the claim follows immediately.

In case $v_0 = u_{\max}$ the assertion follows directly from the proof of Proposition 2.7. Hence, consider the case $v_0 < u_{\max}$. Since v_0 is strongly order stable from above, there exist strict supersolutions v_k of S such that $v_k \searrow v_0$ uniformly on $\mathbb{R}^N$ as $k \to \infty$. Replacing v_k by $\tilde{v}_k := \inf\{v_k, u_{\max}\}$ we may assume without loss of generality that the strict supersolutions v_k of S belong to $C_0(\mathbb{R}^N)$.

We fix $k_0 \in \mathbb{N}$ such that

$$0 < v_{k_0}(x,t) - v(x,t) \le \min\big\{\delta_0, \frac{\varepsilon}{2}\big\} \qquad \text{on } \mathbb{R}^N \times [0,T],$$

where $v_{k_0}(x,t) := u(x,t;v_{k_0})$. Since v_{k_0} is a strict supersolution of S in V, there exists a fixed point $v_1 \ge v_0$ of S in V such that $S^n(v_{k_0}) \searrow v_1$ uniformly on $\mathbb{R}^N$ as $n \to \infty$. Choose $w_0 \in (v_1, \overline{u}_0]_{\operatorname{BUC}(\mathbb{R}^N)}$ satisfying

(i) $w_0 \gg v_1$ in $\operatorname{BUC}(\mathbb{R}^N)$,
(ii) $S(w_0) \le v_0 + \varepsilon$,
(iii) $u\big(x,t;S(w_0)\big) < u(x,t;v_{k_0})$ for $|x| \le R_0$ and $t \in [0,T]$.

Note that $S(w_0) \not\ge w_0$. Indeed, assuming for the sake of contradiction that $S(w_0) \ge w_0$ we would obtain by an iteration process a fixed point of S in $[w_0, \overline{u}_0]_{\operatorname{BUC}(\mathbb{R}^N)}$, which is an obvious contradiction of

$$\lim_{n\to\infty} S^n(\overline{u}_0) = u_{\max} \in C_0(\mathbb{R}^N).$$

Furthermore, using a similar argument we conclude that there are no subsolutions of S in $(v_1, v_{k_0}]_{C_0(\mathbb{R}^N)}$. Without loss of generality we assume that $\overline{u}_{2n} = \overline{u}_{2n+1}$ for all $n \in \mathbb{N}$. Defining the following sequence:

$$\begin{aligned} w_1 &:= \inf\{w_0, S(w_0), \overline{u}_0\}, \\ w_2 &:= \inf\{w_0, S(w_1), \overline{u}_1\}, \\ &\ \ \vdots \\ w_{n+1} &:= \inf\{w_0, S(w_n), \overline{u}_n\}. \end{aligned}$$

It is evident due to $S(w_0) \ngeq w_0$ that $w_{n+1} < w_n$ and therefore $S(w_{n+1}) <_p S(w_n)$ for $n \in \mathbb{N}$. As in the proof of Theorem 2.4 in [22] we conclude that $S(w_n) \searrow w$ uniformly on bounded subsets of $\mathbb{R}^N$ as $n \to \infty$. Since

$$S(w_{n+1})(x) - w(x) \leq \overline{u}_n(x) - v_0(x) \leq \frac{1}{n} \quad \text{for} \quad |x| \geq R_n$$

and some constant $R_n > 0$, the convergence even takes place in $\mathrm{BUC}(\mathbb{R}^N)$. Due to this fact and since $S(w_{n+1}) \leq S^2(w_n)$ we conclude that $w \leq S(w)$; i.e., w is a subsolution of S. Using (iii) and $w \leq S(w_0)$ we get

$$w(x,t) := u(x,t;w) < u(x,t;v_{k_0}) \quad \text{for} \quad |x| \leq R_0,\, t \in [0,T].$$

Moreover, it is not difficult to see that $v_1 \ll S(w_n)$ in $\mathrm{BUC}(\mathbb{R}^N)$ for $n \in \mathbb{N}$ and $w \in C_0(\mathbb{R}^N)$. Thus, Lemma 2.4 implies $v_1 \leq w \leq v_{k_0}$ and therefore $v_1 = w$. Since $v_1 \ll w_0$ in $\mathrm{BUC}(\mathbb{R}^N)$ and $S(w_n) \searrow v_1$ uniformly on $\mathbb{R}^N$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $S(w_{n_0}) \leq v_0 + \varepsilon$ as well as $S(w_{n_0}) \leq w_0$. Without loss of generality we assume that n_0 is an odd integer and therefore $\overline{u}_{n_0} = \overline{u}_{n_0-1}$. Hence, we have

$$w_{n_0+1} = \inf\{w_0, S(w_{n_0}), \overline{u}_{n_0}\} = \inf\{S(w_{n_0}), \overline{u}_{n_0-1}\}.$$

Since $S(w_{n_0}) \leq S(\overline{u}_{n_0-1}) \leq \overline{u}_{n_0-1}$, we conclude that

$$w_{n_0} > w_{n_0+1} = \inf\{S(w_{n_0}), \overline{u}_{n_0-1}\} = S(w_{n_0}).$$

Thus, $\overline{v} := S(w_{n_0})$ is a strict supersolution of S lying in the order interval $[v_0, v_0 + \varepsilon]_{\mathrm{BUC}(\mathbb{R}^N)}$. From the maximum principle it is obvious that $\overline{v} \gg v_0$ in $\mathrm{BUC}(\mathbb{R}^N)$. This proves that $\overline{v}$ has the desired properties. $\square$

3. Periodic Fisher's equation with indefinite weight. In this section we apply our results to the well-known Fisher equation from population genetics (cf. [15] or [16]). Consider a population of diploid individuals distributed, for example, on a planar habitat, and suppose that the gene at a specific locus in a specific chromosome pair occurs in two allelic forms denoted by a and A. Thus, the population is divided into three classes: aa, aA and AA. The individuals having the genotypes aa and AA are called *homozygotes* and the individuals with the mixed-type genotype aA are called *heterozygotes*.

Let $v(x,t)$ denote the relative density of the allele A at the point x of the habitat at time t. Under some additional assumptions D.G. Aronson and H.F. Weinberger have shown in [3] that the relative density v is close to the solution of a reaction-diffusion equation of the form

$$\partial_t u - \Delta u = F(u). \tag{3.1}$$

Here, we are interested in the periodic Fisher equation on $\mathbb{R}^N$ for $N \geq 1$ with indefinite weight function m:

$$\begin{cases} \partial_t u - \Delta u = m(x,t)F(u) & \text{on } \mathbb{R}^N \times (0,\infty), \\ \quad u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N, \end{cases} \tag{3.2}$$

where we assume that m is a sufficiently smooth (Hölder-continuous) T-periodic function satisfying

$$m(x,t) \leq -c|x|^{-\alpha} \qquad \text{for } |x| \geq R_0,\ t \in \mathbb{R} \tag{3.3}$$

and some given constants $R_0 > 0$ and $\alpha \in [0, 2)$.

Note that the weight function m may change sign, since assumption (3.3) only imposes that m is always negative outside some bounded region of $\mathbb{R}^N$. The function F represents the effect of natural selection (cf. [5] or [19]), and it satisfies $F(0) = F(1) = 0$. As in [2] we shall be concerned with the following three cases: *heterozygote intermediate*, *heterozygote inferior* and *heterozygote superior* case.

In the heterozygote intermediate case the viability of the heterozygote class aA lies between the viabilities of the homozygote classes aa and AA. If the weight m is positive in some subregions, then the allele A has some advantage over a in these subregions. On the other hand, a negative weight in some subregions means that a is more favourable than A.

In the other two cases the heterozygote is either more or less viable than both homozygotes. If m changes sign, then it switches from the heterozygote superior to the heterozygote inferior case or vice versa.

In the following we shall present our results for the periodic parabolic initial value problem (3.2) with initial values $u_0 \in [0, 1]_{\mathrm{BUC}(\mathbb{R}^N)}$. Our main interest shall be existence of T-periodic solutions, stability and asymptotic behaviour. Of course, for all cases $u \equiv 0$ and $u \equiv 1$ are trivial T-periodic solutions of (3.2).

3.1. Heterozygote intermediate. Let h be a sufficiently smooth function satisfying

$$\begin{aligned} &h(0) = h(1) = 0, \quad h(u) > 0 \text{ in } (0,1), \\ &h'(0) > 0, \quad \text{and} \quad h'(1) < 0. \end{aligned} \tag{3.4}$$

First we shall consider the initial value problem (3.2) assuming (3.3) and $F = h$. This corresponds to the heterozygote intermediate case of the periodic Fisher equation allowing an indefinite weight function m as described above.

Theorem 3.1. *Suppose* (3.3) *with* $F = h$ *satisfying* (3.4). *Then the following is true:*

(i) *The trivial solution* $u \equiv 1$ *is unstable. There exists a* T*-periodic solution* $u_{\max}$ *in* $[0, 1)_{\mathrm{BUC}(\mathbb{R}^N)} \cap C_0(\mathbb{R}^N)$ *of* (3.2), *which is* L_∞*-stable from above with respect to initial data in* $[0, 1]_{\mathrm{BUC}(\mathbb{R}^N)}$, *and every further* T*-periodic solution of* (3.2) *in* $[0, 1)_{\mathrm{BUC}(\mathbb{R}^N)}$ *lies between* 0 *and* $u_{\max}$.

(ii) *If* $\lambda_\infty(0) < 0$, *then the trivial solution* $u \equiv 0$ *is unstable. Moreover, there exists a nontrivial* L_∞*-stable* T*-periodic solution or a totally ordered continuum* $\mathcal{A} := \{u_1 \leq u \leq u_2 : u \text{ is a } T\text{-periodic solution of } (3.2)\}$ *of nontrivial* T*-periodic solutions of* (3.2), *where* u_1 *and* u_2 *are* T*-periodic solutions of* (3.2), *which are* L_∞*-stable from below and from above, respectively.*

Remark 3.2. Note that in general $u_{\max}$ is not asymptotically L_∞-stable from above (cf. Remark 19.2 in [26]). However, as before we readily obtain that $S^n(u_0)$ converges to $u_{\max}$ uniformly on bounded subsets of $\mathbb{R}^N$ as $n \to \infty$, whenever $u_0 \in [u_{\max}, 1)_{\mathrm{BUC}(\mathbb{R}^N)}$.

Proof of Theorem 3.1. The idea of the proof is to find suitable sub- and supersolutions such that Propositions 2.7 and 2.12 can be applied. For a detailed discussion of the construction of such sub- and supersolutions we mainly refer to [26]. However, the main idea is contained in [23] and [22].

In fact, thanks to Theorem 17.2 and Proposition 14.6 in [26] there exists a strict supersolution $\overline{u}_0$ of S such that all nontrivial fixed points of S in $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)}$ are contained in $[0,\overline{u}_0]_{\mathrm{BUC}(\mathbb{R}^N)}$. From Proposition 4.3 and Theorem 17.1 in [26] it follows that

$$u_{\max} := \lim_{n\to\infty} S^n(\overline{u}_0)$$

is a fixed point of S in $[0,1)_{\mathrm{BUC}(\mathbb{R}^N)} \cap C_0(\mathbb{R}^N)$, and every further nontrivial fixed point of S in $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)}$ lies in $[0,u_{\max}]_{C_0(\mathbb{R}^N)}$. By Remark 14.5 in [26] we obtain strict supersolutions of S, which are small at infinity. Hence, the L_∞-stability of $u_{\max}$ from above follows from Proposition 2.12. This proves assertion (i).

If $\lambda_\infty(0) < 0$, then there exist by Proposition 12.3 in [26] strict subsolutions of S satisfying the hypotheses of Proposition 2.7 . Hence, assertion (ii) follows by Proposition 2.12. □

Imposing some further conditions we get a more detailed picture of the situation. This is done by the next theorem.

Theorem 3.3. *In addition, we assume that h is concave in $[0,1]$ and that assumption (3.3) is fulfilled for $\alpha = 0$. Then the following alternative holds*:

(i) *If the trivial solution $u \equiv 0$ is linearly stable, then each solution of (3.2) with initial value $u_0 \in [0,1)_{\mathrm{BUC}(\mathbb{R}^N)}$ converges to 0 uniformly on bounded subsets of $\mathbb{R}^N$ as $t \to \infty$. Moreover, $u \equiv 0$ is globally asymptotically L_∞-stable with respect to initial data in $\{u_0 \in \mathrm{BUC}(\mathbb{R}^N) : 0 \le u_0 \ll 1\}$.*

(ii) *If the trivial solution $u \equiv 0$ is linearly unstable, then there exists a unique nontrivial T-periodic solution $u^* \in (0,1)_{\mathrm{BUC}(\mathbb{R}^N)}$ lying in $C_0(\mathbb{R}^N)$, and each solution of (3.2) having its initial value $u_0 \in (0,1)_{\mathrm{BUC}(\mathbb{R}^N)}$ converges to u^* uniformly on bounded subsets of $\mathbb{R}^N$ as $t \to \infty$. Moreover, u^* is globally asymptotically L_∞-stable with respect to initial data in $\{u_0 \in \mathrm{BUC}(\mathbb{R}^N) : 0 < u_0 \ll 1\}$.*

(iii) *Let the trivial solution $u \equiv 0$ be neutrally stable. If h is not linear in some interval $[0,s_0]$ with $s_0 > 0$, then each solution of (3.2) with initial value $u_0 \in [0,1)_{\mathrm{BUC}(\mathbb{R}^N)}$ converges to 0 uniformly on bounded subsets of $\mathbb{R}^N$ as $t \to \infty$. Moreover, $u \equiv 0$ is globally asymptotically L_∞-stable with respect to initial data in $\{u_0 \in \mathrm{BUC}(\mathbb{R}^N) : 0 \le u_0 \ll 1\}$. If, however, h is linear on such an interval $[0,s_0]$, there exists a one-parameter family $\mathcal{A} := \{\varepsilon\phi : 0 \le \varepsilon \le s_0\}$ of L_∞-stable T-periodic solutions of (3.2) in $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)} \cap C_0(\mathbb{R}^N)$, and there is no other nontrivial T-periodic solution in the order interval $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)}$. Moreover, each solution of (3.2) having its initial value $u_0 \in [0,1)_{\mathrm{BUC}(\mathbb{R}^N)}$ converges to the order interval $[0,s_0\phi]_{C_0(\mathbb{R}^N)}$ uniformly on bounded subsets of $\mathbb{R}^N$ as $t \to \infty$, and $[0,s_0\phi]_{C_0(\mathbb{R}^N)}$ is globally asymptotically stable with respect to initial data in $\{u_0 \in \mathrm{BUC}(\mathbb{R}^N) : 0 \le u_0 \ll 1\}$.*

Remark 3.4. This theorem parallels the results on bounded domains as proved by P. Hess in [18]. In contrast to our Theorem 3.3 he is able to give a complete description of the long-time behaviour of all solutions in the bounded domain case (cf. [18]). More precisely, he proves that each solution converges to a T-periodic solution as $t \to \infty$. This result is a consequence of a stabilization result (cf. Theorem 3.3 in [19]) for strongly monotone discrete-time dynamical systems. For a more detailed overview of such convergence results we refer to [20] and [9]. It should be noticed that, since our problem (3.2) does not generate a strongly monotone discrete-time dynamical system, neither a stabilization result (as in [1]) nor a generic convergence result (as in [25]) is available. The lack of such a result prevents us from inferring convergence of each solution of (3.2) to a periodic solution.

The proof of Theorem 3.3 is based on the following two lemmas.

Lemma 3.5. *Let u in $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)}$ be a nontrivial T-periodic solution of* (3.2)*. Then u is linearly stable or neutrally stable. It is neutrally stable if and only if*

$$|\nabla u|^2 h''(u) \equiv 0 \qquad \text{on } \mathbb{R}^N \times [0,T].$$

Proof. Since every nontrivial T-periodic solution of (3.2) lies in $C_0(\mathbb{R}^N)$, we may proceed as in the proof of Proposition 6.3 in [22]. □

Lemma 3.6. (i) *If the nontrivial T-periodic solution u of* (3.2) *in $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)}$ is neutrally stable, then h is linear on some interval $[0,s_0]$ with $s_0 \ge \|u\|_\infty$, and the trivial solution $u \equiv 0$ is neutrally stable.*

(ii) *Conversely, if $u \equiv 0$ is neutrally stable and h is linear on $[0,s_0]$, $s_0 > 0$ maximal, there exists a one-parameter family $\{\varepsilon\phi : 0 < \varepsilon \le s_0\}$ of nontrivial T-periodic solutions of* (3.2) *in $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)} \cap C_0(\mathbb{R}^N)$, and there are no other nontrivial T-periodic solutions of* (3.2) *in $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)} \cap C_0(\mathbb{R}^N)$.*

Proof. The proof of Lemma 3.6 is similar to that of Proposition 3.1 in [18], and shall be omitted. For a detailed proof we refer the reader to the proof of Lemma 19.6 in [26]. □

Proof of Theorem 3.3. (i) Let the trivial solution $u \equiv 0$ be linearly stable. Any nontrivial T-periodic solution v of (3.2) is linearly stable by Lemmas 3.5 and 3.6. By Theorem 6.1 in [19] there exists an unstable T-periodic solution between 0 and v, which is impossible. Thus there is no nontrivial T-periodic solution of (3.2) in $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)}$ in this case. Moreover, the stability property follows from Remark 3.2 and Theorem 2.6 in [22].

(ii) If the trivial solution $u \equiv 0$ is linearly unstable, any nontrivial T-periodic solution of (3.2) in $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)}$ is again linearly stable by Lemmas 3.5 and 3.6. The existence of a nontrivial T-periodic solution $u^* \in [0,1]_{\mathrm{BUC}(\mathbb{R}^N)} \cap C_0(\mathbb{R}^N)$ is guaranteed by the presence of small positive strict T-subsolutions (cf. Proposition 12.1 in [26] or Lemma 5.3 in [22]) and suitable strict T-supersolutions (cf. Proposition 14.3 in [26] or Lemma 5.2 in [22]). The uniqueness of u^* is a consequence of the above-quoted Theorem 6.1 in [19]. The stability property of u^* follows as in case (i).

(iii) Let the trivial solution $u \equiv 0$ be neutrally stable. If there is no nontrivial interval $[0, s_0]$, in which h is linear, any nontrivial T-periodic solution v of (3.2) in $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)}$ is linearly stable by Lemma 3.6. We embed (3.2) in the following parameter-dependent problem:

$$\begin{cases} \partial_t w - \Delta w = \lambda m(x,t) h(w) & \text{on } \mathbb{R}^N \times \mathbb{R}, \\ w \text{ is } T\text{-periodic} \end{cases} \tag{3.5}$$

for λ in a neighbourhood of 1. Since $r(S_v) < 1$, we may solve (3.5) by the implicit function theorem in a neighbourhood of $(1, v)$; in particular, for $\lambda^* < 1$ close to 1 we obtain a T-periodic solution v_{λ^*} of (3.5) near v satisfying $v_{\lambda^*} \not\leq 0$. This is possible, since v is everywhere positive. Putting now

$$\underline{u}_0 := \sup\{0, v_{\lambda^*}(\cdot, 0)\}$$

we obtain a strict T-subsolution for $(3.5)_{\lambda^*}$. By standard arguments we get a positive nontrivial T-periodic solution of $(3.5)_{\lambda^*}$, which is linearly stable. Since the zero solution of $(3.5)_{\lambda^*}$ is linearly stable thanks to the results in [13], there exists an unstable T-periodic solution of $(3.5)_{\lambda^*}$ between them, which is impossible by Lemma 3.6. Hence (3.2) admits no nontrivial T-periodic solution in $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)}$. The required stability properties are obtained as in case (i).

Finally, if h is linear on some interval $[0, s_0]$ with $s_0 > 0$, we have the family $\mathcal{A} := \{\varepsilon\phi : 0 \leq \varepsilon \leq s_0\}$ of T-periodic solutions of (3.2) in $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)} \cap C_0(\mathbb{R}^N)$ by Lemma 3.6 and there are no other T-periodic solutions in $[0,1)_{\mathrm{BUC}(\mathbb{R}^N)}$.

All T-periodic solutions are order stable and therefore by Proposition 4.12 in [22] L_∞-stable. The stability property of $[0, s_0\phi]_{C_0(\mathbb{R}^N)}$ follows as before. □

3.2. Heterozygote inferior. Next, we shall consider equation (3.2) for $F(u) = h(u)(a - u)$, where a is a sufficiently smooth T-periodic function with values in $(0,1)$. In terms of the Fisher equation this models the heterozygote inferior case for large $|x|$.

Theorem 3.7. *Let $a(x,t)$ be a smooth T-periodic function with values in $(0,1)$, and set $F(u) = h(u)(a-u)$. Suppose in addition to (3.3) and (3.4) that there exists a positive constant C such that $-C|x|^{-\alpha} \leq m(x,t)$ for $|x| \geq R_0$ and $t \in \mathbb{R}$. If there exist $R > 0$ and $\underline{a} \in (0,1)$ satisfying*

$$\int_0^{\underline{a}} h(s)(\underline{a} - s)\, ds > \int_{\underline{a}}^1 h(s)\Big(\frac{C}{c} s - \underline{a}\Big)\, ds$$

and

$$0 < \underline{a} \leq a(x,t) < 1 \qquad \text{on } \big(\mathbb{R}^N \setminus \mathbb{B}(0,R)\big) \times [0,T],$$

then the following is true:

(i) *There exists a T-periodic solution $u_{\max}$ in $[0,1)_{\mathrm{BUC}(\mathbb{R}^N)} \cap C_0(\mathbb{R}^N)$ for (3.2), which is L_∞-stable from above with respect to initial data in $[0,1)_{\mathrm{BUC}(\mathbb{R}^N)}$.*

(ii) *If $\lambda_\infty(0) < 0$, then the trivial solution $u \equiv 0$ of (3.2) is unstable. Moreover, there exists a nontrivial L_∞-stable T-periodic solution or a totally ordered continuum*

$\mathcal{A} := \{u_1 \le u \le u_2 : u$ *is a* T*-periodic solution of* $(3.2)\}$ *of nontrivial* T*-periodic solutions of* (3.2) *between* 0 *and* $u_{\max}$*, where* u_1 *and* u_2 *are* T*-periodic solutions of* (3.2)*, which are* L_∞*-stable from below and from above, respectively.*

Remark 3.8. (i) We point out that the T-periodic solution $u_{\max}$ in Theorem 3.7(i) need not be maximal in $[0,1)_{\mathrm{BUC}(\mathbb{R}^N)}$. However, we have $u_{\max} \ge u$ for all T-periodic solutions u of (3.2) in $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)}$ satisfying

$$\limsup_{|x|\to\infty} u(x,0) < \delta \in (0,\underline{a}),$$

provided

$$\int_\delta^{\underline{a}} h(s)(\underline{a}-s)\,ds > \int_{\underline{a}}^1 h(s)\big(\frac{C}{c}s-\underline{a}\big)\,ds.$$

(ii) An analogous result is true for T-periodic solutions of (3.2) near $u \equiv 1$ at infinity, provided there exists $\overline{a} \in (0,1)$ satisfying

$$\int_{\overline{a}}^0 h(s)(s-\overline{a})\,ds > \int_0^{\overline{a}} h(s)\big(\frac{C}{c}\overline{a}-s\big)\,ds$$

and

$$0 < a(x,t) \le \overline{a} < 1 \qquad \text{on } \big(\mathbb{R}^N \setminus \mathbb{B}(0,R)\big) \times [0,T]$$

for some $R > 0$.

(iii) Furthermore, depending on the function a various combinations of the situations in Theorem 3.7 and Remark 3.8(ii) are possible. The simplest one shall be discussed in Corollary 3.9.

Proof of Theorem 3.7. The assertion follows in a way similar to the proof of Theorem 3.1 by constructing suitable sub- and supersolutions. A detailed proof can be found in [26]. □

As a special case of Theorem 3.7 and Remark 3.8(ii) we consider the following one-dimensional T-periodic initial value problem (cf. [23]):

$$\begin{cases} \partial_t u - \partial_{xx}^2 u = u(1-u)\big(u-a(x,t)\big) & \text{on } \mathbb{R}\times(0,\infty), \\ \qquad u(\cdot,0) = u_0 & \text{on } \mathbb{R}. \end{cases} \tag{3.6}$$

We are interested in T-periodic solutions u of (3.6) lying in $[0,1]_{\mathrm{BUC}(\mathbb{R}^N)}$, which satisfy

$$\lim_{x\to+\infty} u(x,t) = 1 \qquad \text{and} \qquad \lim_{x\to-\infty} u(x,t) = 0$$

uniformly in $t \in [0,T]$. Usually such solutions are called *clines* (cf. [24], [15] or [16]). The result in question is the following:

Corollary 3.9. *Let a be given as in Theorem* 3.7 *and assume that*

$$\limsup_{x\to+\infty}\big[\max_{t\in[0,T]} a(x,t)\big] < \frac{1}{2} < \liminf_{x\to-\infty}\big[\min_{t\in[0,T]} a(x,t)\big].$$

(i) *Then there exists a* (*locally*) *asymptotically* L_∞*-stable* T*-periodic cline or a totally ordered continuum consisting of* L_∞*-stable* T*-periodic clines.*

(ii) *If, in addition,* $0 \ll a(\cdot,t) \ll 1$ *in* $\mathrm{BUC}(\mathbb{R}^N)$ *for* $t\in[0,T]$, *i.e., if the function a is bounded away from* 0 *and* 1, *then both trivial solutions are also* (*locally*) *asymptotically* L_∞*-stable with respect to* $[0,1]_{\mathrm{BUC}(\mathbb{R})}$. *Hence, there exist at least three* L_∞*-stable* T*-periodic solutions of* (3.6) *in this case.*

Proof. For a proof of Corollary 3.9 we refer to [26].

3.3. Heterozygote superior at infinity. Finally, we consider the case that the heterozygote is more viable than either of the homozygotes outside some bounded domain. This is accomplished by setting $F(u) = h(u)(u-a)$. The following theorem is proved in [26].

Theorem 3.10. *Suppose* (3.3) *and* (3.4). *Set* $F(u) = h(u)(u-a)$, *where* $a(x,t)$ *is a given smooth* T*-periodic function satisfying*

$$0 < \underline{a} \le a(x,t) \le \overline{a} < 1 \qquad \text{for } \ |x| \ge R \text{ and } t\in[0,T]$$

and some constants $\underline{a}$, $\overline{a}$ *and* $R > 0$. *Then both trivial solutions of* (3.2) *are unstable and there exist a minimal and a maximal nontrivial* T*-periodic solution* $u_{\min} \le u_{\max}$ *in* $(0,1)_{\mathrm{BUC}(\mathbb{R}^N)}$. *Every solution of* (3.2) *with initial value* $u_0 \in (0,1)_{\mathrm{BUC}(\mathbb{R}^N)}$ *converges uniformly on bounded subsets of* $\mathbb{R}^N$ *to the order interval* $[u_{\min}, u_{\max}]_{\mathrm{BUC}(\mathbb{R}^N)}$ *as* $t\to\infty$.

If in addition there exists a constant $a_\infty \in (0,1)$ *such that* $a(x,t)$ *tends to* a_∞ *as* $|x|\to\infty$ *uniformly in* $t\in[0,T]$, *then* $u_{\min}$ *and* $u_{\max}$ *are* L_∞*-stable from below and from above, respectively. Moreover, all nontrivial* T*-periodic solutions are contained in* $[u_{\min}, u_{\max}]_{\mathrm{BUC}(\mathbb{R}^N)}$ *and converge to* a_∞ *as* $|x|\to\infty$. *More precisely, there exists a nontrivial* L_∞*-stable* T*-periodic solution or a totally ordered continuum* $\mathcal{A} := \{u_1 \le u \le u_2 : u \text{ is a } T\text{-periodic solution of } (3.2)\}$ *of nontrivial* T*-periodic solutions of* (3.2) *between* $u_{\min}$ *and* $u_{\max}$, *where* u_1 *and* u_2 *are* T*-periodic solutions of* (3.2), *which are* L_∞*-stable from below and from above, respectively.*

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