Differential and Integral Equations Volume 9, Number 2, March 1996, pp. 409 – 420

STABILIZATION OF TWO-DIMENSIONAL LINEAR SYSTEMS BY TIME-DELAY FEEDBACK CONTROLS

Bo Hu[1]
Department of Information, Shanghai University of Finance and Economics, Shanghai 200433, P.R. China

(Submitted by: Kenneth L. Cooke)

Abstract. A continuous-time, linear, time-invariant control system with a fixed time delay in the feedback loop is considered. We investigate the problem of feedback stabilization. Some easy-to-check criteria to the two-dimensional systems are presented.

1. Introduction. We consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1.1}$$

where A is an $(n \times n)$ - matrix, B is an $(n \times m)$ - matrix, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $t \in \mathbb{R}$. For given $\tau > 0$, we want to find an $(m \times n)$ - matrix K such that the feedback control

$$u(t) = Kx(t - \tau) \tag{1.2}$$

stabilizes (1.1), namely, the solution of

$$\dot{x}(t) = Ax(t) + BKx(t - \tau) \tag{1.3}$$

for any initial condition is asymptotically stable. This problem was initiated and discussed by J. Yong in [9] and [10].

Definition 1.1. Let $\tau > 0$ and system (1.1) (denoted by $[A, B]$) be given. System $[A, B]$ is said to be τ - stabilizable if there exists a matrix K such that (1.3) (denoted by $[A, BK, \tau]$) is asymptotically stable. System $[A, B]$ is said to be uniformly τ-stabilizable if there exists a matrix K such that $[A, BK, \tau']$ is asymptotically stable for all $\tau' \in [0, \tau]$.

In this paper, we will mainly investigate the problem of uniform τ- stabilizability. For simplicity, we only discuss completely controllable, single-input systems. By [8], without loss of generality, we may assume $[A, b]$ has the canonical form

$$A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -p_0 & -p_1 & \cdots & -p_{n-2} & -p_{n-1} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{1.4}$$

Received in revised form September 1993.
[1]This research was supported in part by the Chinese Education Commission Science Foundation.
AMS Subject Classifications: 93D15, 34K20, 93C05.

Then, from the known results on retarded functional differential equations (see [3]), we know that the problem of τ - stabilizability of $[\dot{A}, b]$ is equivalent to the problem of finding a vector

$$K = (-q_0, -q_1, \cdots, -q_{n-1})^T \in \mathbb{R}^n,$$

such that all the roots of the transcendental polynomial

$$\det(zI - A - bK^T e^{-\tau z}) = z^n + p_{n-1}z^{n-1} + \cdots + p_0 + (q_{n-1}z^{n-1} + \cdots + q_0)e^{-\tau z}$$

have negative real parts. Here $\det(zI - A) = z^n + p_{n-1}z^{n-1} + \cdots + p_0$.

For $n \geq 0$, we let

$$P^n = \{z^n + p_{n-1}z^{n-1} + \cdots + p_0 : p_{n-1}, \cdots, p_0 \in \mathbb{R}\},$$

$$Q^{n-1} = \{q_{n-1}z^{n-1} + \cdots + q_0 : q_{n-1}, \cdots, q_0 \in \mathbb{R}\}, Q^{-1} \equiv 0.$$

Also, we denote

$$C^- = \{\lambda \in C : \operatorname{Re} \lambda < 0\}.$$

We introduce the following definitions.

Definition 1.2. An entire function $H(z)$ is said to be stable if

$$N(H(z)) \equiv \{\lambda \in C : H(\lambda) = 0\} \subset C^-.$$

Definition 1.3. Let $\tau > 0, p(z) \in P^n$; $p(z)$ is said to be τ - stabilizable if there exists a $q(z) \in Q^{n-1}$ such that $p(z) + q(z)e^{-\tau z}$ is stable; $p(z)$ is said to be uniformly τ- stabilizable if there exists a $q(z) \in Q^{n-1}$ such that $p(z) + q(z)e^{-\tau' z}$ is stable for all $\tau' \in [0, \tau]$.

Now we claim that the (uniform) τ - stabilizability of $[A, b]$ is equivalent to the (uniform) τ - stabilizability of $p(z) = \det(zI - A)$. Hence, in the following sections, we state our results in the terms of polynomials.

The following results were proved in [9] and [10].

Proposition 1.4. *If $p(z)$ is (uniformly) τ - stabilizable, so is $zp(z)$.*

Proposition 1.5. *For $p(z) = z - a (a > 0), p(z)$ is uniformly τ - stabilizable if and only if $\tau < \frac{1}{a}$.*

Proposition 1.6. *If $p(z) = z^n - az^{n-1}$ $(a \geq 0,\ n \geq 2)$, then for any given $\tau > 0$, $p(z)$ is uniformly τ - stabilizable.*

We define

$$\begin{aligned} Y(p, q) &= \{y > 0 : |p(iy)| = |q(iy)|\}, \qquad (1.5)\\ \phi(y) &= \arg(p(iy)), \quad \text{for} \quad p(iy) \neq 0,\\ \psi(y) &= \arg(q(iy)), \quad \text{for} \quad q(iy) \neq 0. \end{aligned}$$

For definiteness, we let $\phi(y) = \phi(y+0)$, $\psi(y) = \psi(y+0)$, for all y such that $p(iy) = 0$ or $q(iy) = 0$.

When $Y(p,q) \neq \emptyset$, we let

$$Q_0^{n-1}(p) = \{q \in Q^{n-1} : N(p(z) + q(z)) \subset C^-\}. \tag{1.6}$$

Also, we define

$$\theta(p,q) = \begin{cases} \min_{y \in Y(p,q)} \{\frac{1}{y}(\psi(y) + \pi - \phi(y) - 2\pi[\frac{\psi(y)+\pi-\phi(y)}{2\pi}])\}, \\ \qquad\qquad\qquad \text{when } Y(p,q) \neq \emptyset, \\ \infty, \qquad\qquad\quad \text{when } Y(p,q) = \emptyset, \end{cases} \tag{1.7}$$

where $[x]$ is the greatest integer less than or equal to x.

The next lemma, which was proved in [10], is the main tool for us to use in the following sections.

Lemma 1.7. *Let $p(z) \in P^n, q(z) \in Q_0^{n-1}(p)$. Then, $p(z) + q(z)e^{-\tau' z}$ is stable for all $\tau' \in [0, \tau]$ if and only if*

$$\tau < \theta(p,q). \tag{1.8}$$

Thus, for $p(z) \in P^n, p(z)$ is uniformly τ- stabilizable if and only if

$$\tau < \sup_{q \in Q_0^{n-1}(p)} \theta(p,q). \tag{1.9}$$

We see that the right hand side of inequality (1.9) is very difficult to calculate, in general. In this paper , for polynomials of degree two, we carry out some computations with some technical analysis to obtain some easy-checking criteria. Our results recover some of [10].

Also, we get some easy-to-check criteria for the problem of τ-stabilizability for polynomials of degree two, which will be submitted later.

For the properties of some transcendental polynomials, we refer our readers to [1], [2], [4–7].

2. Main results. In this section, we will discuss all cases for polynomials of degree two (correspondingly, for systems, we use the terminology two-dimensional). Criteria for the uniform τ-stabilizability of these polynomials are presented, most of which are best possible.

First, we have the following.

Theorem 2.1. *Let $p(z) = z^2 + p_1 z - p_0$, $p_0 > 0$; then $p(z)$ is uniformly τ - stabilizable if and only if*

$$\tau < \frac{(p_1^2 + 2p_0)^{\frac{1}{2}} + p_1}{p_0}. \tag{2.1}$$

Proof will be given in Section 3.

Remark 2.2. In the above theorem, $p(z)$ has two real roots with different signs. So the system is not stable if the control is zero. If we let $p(z) = (z-r_1)(z+r_2), r_1 > 0$, $r_2 > 0$, then we have $p_1 = r_2 - r_1$, $p_0 = r_1 r_2$, and $\Lambda^* \equiv \frac{\sqrt{p_1^2+2p_0}+p_1}{p_0} = \frac{\sqrt{r_1^2+r_2^2}+r_2-r_1}{r_1 r_2}$. It is easy to verify that $\frac{1}{r_1} < \Lambda^* < \frac{2}{r_1}$. Thus, comparing with Proposition 1.5, noticing that $(z-r_1)$ is uniformly τ - stabilizable if and only $\tau < \frac{1}{r_1}$, $(z-r_1)(z+r_2)$ is uniformly τ - stabilizable if and only if $\tau < \Lambda^*$, and $\Lambda^* > \frac{1}{r_1}$; we can see that, when the degree of the polynomials (or correspondingly, dimension of the systems) increases from 1 to 2, the existence of the stable part $(z+r_2)$ will relax the condition for uniform τ-stabilizability. But the condition is still confined by the nonstable part $(z - r_1)$ (since $\frac{1}{r_1} < \Lambda^* < \frac{2}{r_1}$). Noting that $\Lambda^* \to \frac{2}{r_1}$ as $r_2 \to \infty$, and $\Lambda^* \to \frac{1}{r_1}$ as $r_2 \to 0$, and also that for $r_2 = 0$, from Proposition 1.6, $z^2 - r_1 z$ is uniformly τ - stabilizable for any given $\tau > 0$. Thus, for polynomials of degree two, just taking $p(z) = z^2 + p_1 z - p_0$ ($p_0 \geq 0$) into account, we see that the necessary and sufficient condition of uniform τ - stability is not globally continuous in concern with its coefficients. This feature is worthy of mentioning because it seems quite different from our intuition. For it is a known result that the delay, for which a transcendental polynomial is stable, always depends continuously on the latter's coefficients.

Next, we begin to discuss the case when $p(z) = z^2 - p_1 z + p_0$, $p_1 \geq 0$, $p_0 > 0$. In the present case, $p(z)$ has a pair of conjugate roots or two real positive roots (we treat $p_1 = 0$ as a special case, which will be studied separately.).

For $q_0 \in (-p_0, 0]$, $q_1 \in (p_1, \infty)$, we set

$$y_{I^+} = \left(\frac{2p_0 + q_1^2 - p_1^2 + ((2p_0 + q_0^2 - p_1^2)^2 + 4(q_0^2 - p_0^2))^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}} \tag{2.2}$$

and

$$\tau_{I^+}(q_0, q_1) = \frac{1}{y_{I^+}}\left(\pi - \tan^{-1}\frac{p_1 y_{I^+}}{y_{I^+}^2 - p_0} - \tan^{-1}\frac{q_1 y_{I^+}}{|q_0|}\right). \tag{2.3}$$

We have the following Theorem 2.2, the proof of which will be stated in Section 4.

Theorem 2.2. *Let $p(z) = z^2 - p_1 z + p_0$, $p_1 \geq 0$, $p_0 > 0$. Then, $p(z)$ is uniformly τ - stabilizable if and only if*

$$\tau < \sup_{q_1 \in (p_1, \infty), q_0 \in (-p_0, 0)} \tau_{I^+}(q_0, q_1), \tag{2.4}$$

where $\tau_{I^+}(q_0, q_1)$ is given by (2.2)–(2.3).

Now we can use the above theorem to get some useful results.

Corollary 2.3. *Let $p(z) = z^2 - p_1 z + p_0$, $p_1 > 0$, $p_0 > 0$. Suppose $p(z)$ is uniformly τ - stabilizable. Then*

$$\tau < \min\{\frac{1}{p_1}, \frac{\pi}{\sqrt{p_0}}\}. \tag{2.5}$$

Proof. From Theorem 2.2, noting that $y_{I^+} > \sqrt{p}_0$, we have $\tau < \frac{\pi}{\sqrt{p_0}}$. Also, since

$$\tau_{I^+}(q_0, q_1) = \frac{1}{y_{I^+}}(\tan^{-1}\frac{|q_0|}{q_1 y_{I^+}} + \tan^{-1}\frac{y_{I^+}^2 - p_0}{p_1 y_{I^+}^2})$$
$$< \frac{|q_0|}{q_1 y_{I^+}^2} + \frac{y_{I^+}^2 - p_0}{p_1 {y_{I^+}}^2} < \frac{p_0}{p_1 {y_{I^+}}^2} + \frac{{y_{I^+}}^2 - p_0}{p_1 {y_{I^+}}^2} = \frac{1}{p_1},$$

Corollary 2.3 then follows.

Remark 2.4. In [9], Yong proposed the following conjecture: completely controllable system $[A, B]$ is uniformly τ - stabilizable if and only if $\tau \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} < 1$, where $\sigma(A)$ is the spectrum of A. Then, in [10], he gave a counterexample denying the necessity of the above conjecture. What we want to point out is that the sufficiency of the conjecture is not true either. For an example, considering $p(z) = z^2 - 2\sqrt{p}_0 z + p_0$, $p_0 > 0$, we have two roots $\lambda_1 = \lambda_2 = \sqrt{p}_0$. From Corollary 2.3, we know the necessity for $p(z)$ to be uniformly τ - stabilizable is $2\sqrt{p}_0\tau < 1$. Thus, when the sufficiency of the conjecture is satisfied, namely, $\sqrt{p}_0\tau < 1$, the condition to the necessity may fail to hold. Hence, our assertion follows.

Let

$$f(x) = \frac{2p_1(x^2 + p_0)}{p_1^2 x^2 + (x^2 - p_0)^2}, \tag{2.6}$$

$$g(x) = \frac{p_1 x(x^2 + p_0)}{p_1^2 x^2 + (x^2 - p_0^2)^2} + \tan^{-1}\frac{p_1 x}{x^2 - p_0} - \frac{\pi}{2}. \tag{2.7}$$

We have the following.

Corollary 2.5. *Let $p(z) = z^2 - p_1 z + p_0$, $p_1 > 0$, $p_0 > 0$. Suppose $p(z)$ is uniformly τ - stabilizable. Then*

$$\tau < f(x^*), \tag{2.8}$$

where x^* is the unique real positive root of $g(x)$ on $(\sqrt{p_0}, \infty)$.

Proof. From (2.2)–(2.4), since $y_{I^+} > \sqrt{p_0 + |q_0|}$, we know

$$\tau_{I^+}(q_0, q_1) < \frac{1}{y_{\,I^+}}(\pi - 2\tan^{-1}\frac{p_1 y_{I^+}}{y_{I^+}^2 - p_0}). \tag{2.9}$$

Let

$$h(x) = \frac{2}{x}(\frac{\pi}{2} - \tan^{-1}\frac{p_1 x}{x^2 - p_0}), x > \sqrt{p}_0. \tag{2.10}$$

We get

$$h'(x) = \frac{2}{x^2}(\frac{p_1 x(x^2 + p_0)}{p_1^2 x^2 + (x^2 - p_0)^2} + \tan^{-1}\frac{p_1 x}{x^2 - p_0} - \frac{\pi}{2}) = \frac{2g(x)}{x^2}.$$

Also, we have

$$g'(x) = -\frac{2p_1x^2}{(p_1^2x^2 + (x^2 - p_0)^2)^2}(x^4 + 2p_0x^2 + p_0p_1^2 - 3p_0^2).$$

Since for $x > \sqrt{p_0}$, $g'(x) < 0$ and $g(\sqrt{p_0}) > 0$, we know that there is exactly one root x^* of $g(x)$ on $(\sqrt{p_0}, \infty)$. Obviously, when $x \in (\sqrt{p_0}, x^*)$, $h'(x) > 0$; when $x \in (x^*, \infty)$, $h'(x) < 0$. Hence, x^* is the point at which $h(x)$ takes its maximum on $(\sqrt{p_0}, \infty)$. Moreover, since $g(x) = \frac{x}{2}(f(x) - h(x))$ and $g(x^*) = 0$, we get $h(x^*) = f(x^*)$. Hence, (2.8) follows from (2.9). □

Let

$$v(t) = \tan^{-1} t + \frac{3t}{1+t^2} - \frac{\pi}{2}, \quad t > 0. \tag{2.11}$$

Also, we denote $t^* \approx 0.6657$ the unique root of $v(t)$ in $(0, \infty)$. It is easy to prove that $v(t)$ has exactly one positive root on $(0, \infty)$. We omit the proof here. We have the following.

Corollary 2.6. *Let $p(z) = z^2 - p_1z + p_0$, $p_1 > 0$, $p_0 > 0$. If $\frac{\sqrt{2}p_1}{\sqrt{p_0}} \leq t^*$, then $p(z)$ is uniformly τ - stabilizable if and only if $\tau < f(x^*)$. Where $f(x)$ is given by* (2.6), *x^* is defined as in Corollary* 2.5 *and t^* is the unique root of $v(t)$, which is given by* (2.11).

Proof. It is enough to prove the sufficiency. Now, we have $h(\sqrt{p_0}) > 0$, $h(\sqrt{2p_0}) \leq 0$, $h(x)$ is given by (2.10). Hence, $x^* \in (\sqrt{p_0}, \sqrt{2p_0}]$. If we set

$$q_1 = \left(p_1^2 + 2\varepsilon\right)^{\frac{1}{2}} \quad (\varepsilon > 0), \quad q_0 = (x^*)^2 - p_0,$$

and let $\varepsilon \to 0$, we have $y_{I^+} \to x^*, \tau_{I^+}(q_0, q_1) \to f(x^*)$. Thus, our conclusion follows from Theorem 2.2 and Corollary 2.5. □

For $p(z) = z^2 - p_1z + p_0$ $(p_1 > 0, p_0 > 0)$, $\frac{\sqrt{2}p_1}{\sqrt{p_0}} \leq t^*$ is equivalent to the condition that all roots of $p(z) = 0$ satisfy $|\mathrm{Im}\, z| \geq \frac{2\sqrt{8-(t^*)^2}}{t^*}\mathrm{Re}\, z$. We can easily draw the region, in the complex plane, that the last inequality describes.

Corollary 2.7. *Let $p(z) = z^2 - p_1z + p_0$, $p_1 > 0$, $p_0 > 0$. Suppose z_1 and z_2 are two roots of $p(z)$. Let $\lambda = \max\{|z_1|, |z_2|\}$. Then, when $\lambda \to \infty$, the right hand side of* (2.4) *in Theorem* 2.2 *tends to zero.*

Proof. It is easy to verify that $\lambda \leq \max\{\sqrt{p_0}, p_1\}$. Hence, we get

$$\min\{\frac{1}{\sqrt{p_0}}, \frac{1}{p_1}\} \leq \frac{1}{\lambda}.$$

From the proof of Corollary 2.3, we have

$$\sup_{q_1 \in (p_1, \infty), q_0 \in (p_0, 0)} \tau_{I^+}(q_0, q_1) \leq \min\{\frac{1}{p_1}, \frac{\pi}{\sqrt{p_0}}\} \leq \frac{\pi}{\lambda}.$$

Thus,

$$\sup_{q_1 \in (p_1, \infty), q_0 \in (-p_0, 0)} \tau_{I^+}(q_0, q_1) \to 0 \quad \text{as} \quad \lambda \to \infty. \qquad \Box$$

For $p_1 = 0$, we have the following.

Corollary 2.8. *Let $p(z) = z^2 + p_0$, $p_0 > 0$. Then, $p(z)$ is uniformly τ - stabilizable if and only if $\tau < \frac{\pi}{\sqrt{p_0}}$.*

Proof. From (2.3), we have

$$\tau_{I^+} = \frac{1}{y_{I^+}}(\pi - \tan^{-1}\frac{q_1 y_{I^+}}{|q_0|}).$$

Let $q_1 = \varepsilon^2$, $q_0 = -\varepsilon(\varepsilon > 0)$. Let $\varepsilon \to 0$; we get the result of sufficiency from (2.4). The necessity is obvious. Thus, we have proved our conclusion. □

In this section, for systems which are completely controllable, single-input, linear, time-invariant and two-dimensional, we have investigated the problem of uniform τ-stabilizability in terms of polynomials. Now, let us summarize our results in the following.

Theorem 2.9. *Let $p(z) = z^2 + p_1 z + p_0$. Then, $p(z)$ is uniformly τ- stabilizable if and only if $\tau < \Lambda^*$, where Λ^* is listed in the following table corresponding to the various coefficients:*

Λ^*	$p_1 < 0$	$p_1 = 0$	$p_1 > 0$
$p_0 < 0$	$(\sqrt{p_1^2 + 2\|p_0\|} + p_1)/\|p_0\|$		
$p_0 = 0$	∞		
$p_0 > 0$	$*$	$\frac{\pi}{\sqrt{p_0}}$	

Results corresponding to the place denoted by (*) are not yet complete, some of which are included in Corollary 2.3, Corollary 2.5-2.7.

3. Proof of Theorem 2.1. In this section, we are going to prove Theorem 2.1. In order to do so, we first need the following.

Lemma 3.1. *For $x > 0$, $k \in \mathbb{R}$, the following inequality holds:*

$$\sqrt{k^2+2}x - \tan^{-1}(x\sqrt{x^2+k^2+2}) > kx - \tan^{-1}\frac{kx}{1+x^2}. \tag{3.1}$$

Proof. Noting that for $x = 0$, the expressions on both sides take the same value, so it is sufficient to prove their derivatives satisfy

$$\sqrt{k^2+2} - \frac{2x^2+k^2+2}{(1+x^2(x^2+k^2+2))\sqrt{x^2+k^2+2}} > k - \frac{k(1-x^2)}{k^2x^2+(1+x^2)^2}.$$

The above inequality is equivalent to

$$(\sqrt{k^2+2}-k)(k^2x^2+(1+x^2)^2) + k - kx^2 > \frac{x^2}{\sqrt{x^2+k^2+2}} + \sqrt{x^2+k^2+2}.$$

Square the expression on both sides; it is enough to prove

$$\begin{aligned}&(2\sqrt{k^2+2}(\sqrt{k^2+2}-k)+((2+k^2)(\sqrt{k^2+2}-k)-k)^2)x^4\\&+2\sqrt{k^2+2}((2+k^2)(\sqrt{k^2+2}-k)-k)x^2>\frac{x^4}{x^2+k^2+2}+3x^2.\end{aligned}\tag{3.2}$$

Cancel x^2 and multiply (x^2+k^2+2) on both sides, compare the coefficients of terms x^2, x^4 (using the formula $\sqrt{k^2+2}-k=\frac{2}{\sqrt{k^2+2}+k}$); we can then prove inequality (3.2). Thus, inequality (3.1) holds.

Proof of Theorem 2.1. Let $q(z)=q_1z+q_0$ be such that $p(z)+q(z)e^{-\tau' z}$ is stable for all $\tau'\in[0,\tau]$. Then, it is necessary that $q_1>-p_1$, $q_0>p_0$. Now, we consider the equation

$$|f(iy)|=|q(iy)|.\tag{3.3}$$

It is equivalent to

$$y^4+(2p_0+p_1^2-q_1^2)y^2+p_0^2-q_0^2=0.\tag{3.4}$$

We discuss two cases.

1^0. $p_1\le 0$. In this case, equation (3.4) has exactly one solution on $(0,\infty)$,

$$y_1=\left(\frac{q_1^2-p_1^2-2p_0+((q_1^2-p_1^2-2p_0)^2+4(q_0^2-p_0^2))^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}>\sqrt{q_0-p_0}.\tag{3.5}$$

From Lemma 1.7, $p(z)$ is uniformly τ- stabilizable in the present case if and only if

$$\tau<\sup_{q_1\in(-p_1,\infty),q_0\in(p_0,\infty)}\tau_1(q_0,q_1),\tag{3.6}$$

where

$$\tau_1(q_0,q_1)\equiv\frac{1}{y_1}(\tan^{-1}\frac{q_1y_1}{q_0}-\tan^{-1}\frac{|p_1|y_1}{p_0+y_1^2}).\tag{3.7}$$

2^0. $p_1>0$. In this case, equation (3.4) has only the solution

$$y_2=\left(\frac{q_1^2-p_1^2-2p_0+((q_1^2-p_1^2-2p_0)^2+4(q_0^2-p_0^2))^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}.\tag{3.8}$$

When $q_1\in(-p_1,0]$, $y_2<\sqrt{q_0-p_0}$. From Lemma 1.7, $p(z)$ is uniformly τ- stabilizable in the present case if and only if

$$\tau<\sup_{q_1\in(-p_1,0],q_0\in(p_0,\infty)}\tau_2(q_0,q_1),\tag{3.9}$$

where

$$\tau_2(q_0,q_1)\equiv\frac{1}{y_2}(\tan^{-1}\frac{p_1y_2}{p_0+y_2^2}-\tan^{-1}\frac{|q_1|y_2}{q_0}).\tag{3.10}$$

When $q_1 \in (0, \infty)$, similarly, $p(z)$ is uniformly τ- stabilizable if and only if

$$\tau < \sup_{q_1 \in (0,\infty), q_0 \in (p_0,\infty)} \tau_3(q_0, q_1), \tag{3.11}$$

where

$$\tau_3(q_0, q_1) \equiv \frac{1}{y_2}(\tan^{-1} \frac{p_1 y_2}{p_0 + y_2^2} + \tan^{-1} \frac{q_1 y_2}{q_0}). \tag{3.12}$$

Consider that for $q_1 \in (-p_1, 0)$, we have $-q_1 \in (0, p_1)$, and note that y_2 takes the same value at $\pm q_1$ (with q_0 fixed); we can easily prove

$$\sup_{q_1 \in (0,\infty), q_0 \in (p_0,\infty)} \tau_3(q_0, q_1) > \sup_{q_1 \in (-p_1, 0], q_0 \in (p_0,\infty)} \tau_2(q_0, q_1).$$

Thus, we claim that, in case 2^0, $p(z)$ is uniformly τ- stabilizable if and only if (3.11) holds.

Now, we are ready to prove Theorem 2.1. In case 1^0, let $q_1 = (p_1^2 + 2p_0 + \varepsilon)^{\frac{1}{2}}$. From (3.4), we have $q_0 = (y_1^4 - \varepsilon y_1^2 + p_0^2)^{\frac{1}{2}}$. Since $q_1 > -p_1$, $q_0 > p_0$, we get $-2p_0 < \varepsilon < y_1^2$. Consider that

$$\frac{q_1 y_1}{q_0} = \frac{y_1 (p_1^2 + 2p_0 + \varepsilon)^{\frac{1}{2}}}{(y_1^4 - \varepsilon y_1^2 + p_0^2)^{\frac{1}{2}}} < \frac{y_1 (p_1^2 + 2p_0 + y_1^2)^{\frac{1}{2}}}{p_0};$$

we have

$$\sup_{q_1 \in (-p_1,\infty), q_0 \in (p_0,\infty)} \tau_1(q_0, q_1) \le \sup_{y>0} \frac{1}{y}(\tan^{-1} \frac{y_1 (p_1^2 + 2p_0 + y^2)^{\frac{1}{2}}}{p_0} - \tan^{-1} \frac{|p_1| y}{p_0 + y^2}). \tag{3.13}$$

Let $k = \frac{p_1}{\sqrt{p_0}}$, $x = \frac{y}{\sqrt{p_0}}$. From Lemma 3.1, we know that (3.13) is equivalent to

$$\sup_{q_1 \in (-p_1,\infty), q_0 \in (p_0,\infty)} \tau_1(q_0, q_1) \le \frac{(p_1^2 + 2p_0)^{\frac{1}{2}} + p_1}{p_0}.$$

Also, substituting $q_1 = (p_1^2 + 2p_0)^{\frac{1}{2}}$, $q_0 = (p_0^2 + \varepsilon^4)^{\frac{1}{2}}$ into (3.6), and letting $\varepsilon \to 0$, we obtain

$$\sup_{q_1 \in (-p_1,\infty), q_0 \in (p_0,\infty)} \tau_1(q_0, q_1) \ge \frac{(p_1^2 + 2p_0)^{\frac{1}{2}} + p_1}{p_0}.$$

Thus, Theorem 2.1 is true in case 1^0. Similarly, we can prove Theorem 2.1 in case 2^0.

4. Proof of Theorem 2.2. In this section, we will prove Theorem 2.2 in detail.

Let $q(z) = q_1 z + q_0$ be such that $p(z) + q(z)e^{-\tau' z}$ is stable for all $\tau' \in [0, \tau]$. Then, it is necessary that $q_1 > p_1$, $q_0 > -p_0$. Equation (3.4) is now equivalent to

$$y^4 + (p_1^2 - 2p_0 - q_1^2) y^2 + p_0^2 - q_0^2 = 0. \tag{4.1}$$

The above equation may have a different number of positive roots when q_0 changes, so we have to discuss it in different cases.

(1) $q_0 \in (-p_0, 0]$. In this case, equation (4.1) has two positive roots,

$$y_{I^\pm} = \Big(\frac{2p_0 + q_1^2 - p_1^2 \pm ((2p_0 + q_1^2 - p_1^2)^2 + 4(q_0^2 - p_0^2))^{\frac{1}{2}}}{2}\Big)^{\frac{1}{2}}, \tag{4.2}$$

satisfying

$$y_{I^+} > \sqrt{p_0 + |q_0|}, \quad y_{I^-} < \sqrt{p_0 - |q_0|}.$$

From Lemma 1.7, we know that $p(z)$ is uniformly τ - stabilizable in the present case if and only if

$$\tau < \sup_{q_1 \in (p_1, \infty), q_0 \in (-p_0, 0]} \min\{\tau_{I^+}(q_0, q_1), \tau_{I^-}(q_0, q_1)\},$$

where

$$\begin{aligned} \tau_{I^+}(q_0, q_1) &\equiv \frac{1}{y_{\,I^+}}(\pi - \tan^{-1}\frac{p_1 y_{I^+}}{y_{I^+}^2 - p_0} - \tan^{-1}\frac{q_1 y_{I^+}}{|q_0|}), \\ \tau_{I^-}(q_0, q_1) &\equiv \frac{1}{y_{\,I^-}}(2\pi + \tan^{-1}\frac{p_1 y_{I^-}}{p_0 - y_{I^-}^2} - \tan^{-1}\frac{q_1 y_{I^-}}{|q_0|}). \end{aligned} \tag{4.3}$$

Obviously,

$$\tau_{I^-}(q_0, q_1) > \tau_{I^+}(q_0, q_1).$$

Thus, in case (1), $p(z)$ is uniformly τ - stabilizable if and only if

$$\tau < \sup_{q_1 \in (p_1, \infty), q_0 \in (-p_0, 0]} \tau_{I^+}(q_0, q_1). \tag{4.4}$$

(2) $q_0 \in (0, p_0)$. Similarly, we know that $p(z)$ is uniformly τ - stabilizable in the present case if and only if

$$\tau < \sup_{q_1 \in (p_1, \infty), q_0 \in (0, p_0)} \min\{\tau_{II^+}(q_0, q_1), \tau_{II^-}(q_0, q_1)\},$$

where

$$\begin{aligned} \tau_{II^+}(q_0, q_1) &\equiv \frac{1}{y_{II^+}}(\tan^{-1}\frac{q_1 y_{II^+}}{q_0} - \tan^{-1}\frac{p_1 y_{II^+}}{y_{II^+}^2 - p_0}), \\ \tau_{II^-}(q_0, q_1) &\equiv \frac{1}{y_{II^-}}(\pi + \tan^{-1}\frac{q_1 y_{II^-}}{q_0} + \tan^{-1}\frac{p_1 y_{II^-}}{p_0 - y_{II^-}^2}). \end{aligned} \tag{4.5}$$

Here, $y_{II^\pm}$ have the same expressions as (4.2).

It is easy to verify that

$$\tau_{II^-}(q_0, q_1) > \tau_{II^+}(q_0, q_1).$$

Thus, in case (2), $p(z)$ is uniformly τ - stabilizable if and only if

$$\tau < \sup_{q_1 \in (p_1, \infty), q_0 \in (0, p_0)} \tau_{II^+}(q_0, q_1). \tag{4.6}$$

(3) $q_0 \in [p_0, \infty)$. We know that in the present case, equation (4.1) has exactly one positive root,

$$y_{III} = \Big(\frac{2p_0 + q_1^2 - p_1^2 + ((2p_0 + q_1^2 - p_1^2)^2 + 4(q_0^2 - p_0^2))^{\frac{1}{2}}}{2}\Big)^{\frac{1}{2}}.$$

Let

$$\tau_{III}(q_0, q_1) \equiv \frac{1}{y_{III}}(\tan^{-1}\frac{q_1 y_{III}}{q_0} - \tan^{-1}\frac{p_1 y_{III}}{{y_{III}}^2 - p_0}). \tag{4.7}$$

Similarly, we obtain that $p(z)$ is uniformly τ - stabilizable if and only if

$$\tau < \sup_{q_1 \in (p_1, \infty), q_0 \in [p_0, \infty)} \tau_{III}(q_0, q_1).$$

Now, we are ready to prove the following results.

Proposition 4.1. *Let $p(z) = z^2 - p_1 z + p_0$, $p_1 \geq 0$, $p_0 > 0$. Then, $p(z)$ is uniformly τ - stabilizable if*

$$\tau < \frac{1}{\sqrt{2p_0}}(\pi - 2\tan^{-1}\frac{p_1\sqrt{2p_0}}{p_0}). \tag{4.8}$$

Proof. Let $q_0 = -p_0 + \varepsilon$, $q_1 = p_1 + \varepsilon (\varepsilon > 0)$. Let $\varepsilon \to 0$; then (4.8) follows from (4.2)–(4.4).

Lemma 4.2. $\tau_{III}(q_0, q_1) < \frac{1}{\sqrt{2p_0}}(\pi - 2\tan^{-1}\frac{p_1\sqrt{2p_0}}{p_0}) = \frac{\sqrt{2}}{\sqrt{p_0}}\tan^{-1}\frac{\sqrt{p_0}}{\sqrt{2}p_1}$.

Proof. Using the formula $\tan^{-1}\frac{1}{x} = \frac{\pi}{2} - \tan^{-1} x$ $(x > 0)$, from (4.7), we have

$$\begin{aligned} \tau_{III}(q_0, q_1) &= \frac{1}{y_{III}}(\tan^{-1}\frac{q_1 y_{III}}{q_0} - \tan^{-1}\frac{p_1 y_{III}}{y_{III}^2 - p_0}) \\ &= \frac{1}{y_{III}}(\tan^{-1}\frac{y_{III}^2 - p_0}{p_1 y_{III}} - \tan^{-1}\frac{q_0}{q_1 y_{III}}) \\ &< \frac{1}{y_{III}}\tan^{-1}\frac{{y_{III}}^2 - p_0}{p_1 y_{III}} < \frac{1}{y_{III}}\tan^{-1}\frac{y_{III}}{p_1}. \end{aligned} \tag{4. 9}$$

Noticing that $\frac{1}{x}\tan^{-1}\frac{x}{p_1}$ is a strictly decreasing function on $(0, \infty)$ and $y_{III} > \sqrt{2p_0} > (\frac{p_0}{2})^{\frac{1}{2}}$, from (4.9), we get

$$\tau_{III}(q_0, q_1) < (\sqrt{\tfrac{p_0}{2}})^{-1}\tan^{-1}\frac{\sqrt{p_0}}{\sqrt{2}p_1} = \sqrt{\tfrac{2}{p_0}}\tan^{-1}\frac{\sqrt{p_0}}{\sqrt{2}p_1}.$$

Now, we can prove Theorem 2.2.

Proof of Theorem 2.2. For $q_0 \in (0, p_0)$ in case (2), we get $-q_0 \in (-p_0, 0)$ in case (1). For the same q_1, it is obvious that $y_{I^+} = y_{II^+}$. Thus, we have $\tau_{I^+}(-q_0, q_1) > \tau_{II^+}(q_0, q_1)$ and also

$$\sup_{q_1 \in (p_1, \infty), q_0 \in (-p_0, 0]} \tau_{I^+}(q_0, q_1) \geq \sup_{q_1 \in (p_1, \infty), q_0 \in (0, p_1)} \tau_{II^+}(q_0, q_1).$$

Obviously,

$$\sup_{q_1 \in (p_1, \infty), q_0 \in (-p_0, 0]} \tau_{I^+}(q_0, q_1) = \sup_{q_1 \in (p_1, \infty), q_0 \in (-p_0, 0)} \tau_{I^+}(q_0, q_1).$$

Then, Theorem 2.2 follows from the proof of Proposition 4.1, Lemma 4.2 and (4.4).

Acknowledgment. This paper is a part of the author's M.S. thesis under the direction of Professor Jiongmin Yong. The author thanks Professor Yong for his enthusiastic and heuristic help.

REFERENCES

[1] R. Bellman and K.L. Cooke, "Differential Difference Equations," Academic Press, New York, London, 1963.

[2] R. Datko, *A procedure for determination of the exponential stability of certain differential difference equations,* Quart. Appl. Math., 36 (1978), 279–292.

[3] J.K. Hale, "Theory of Functional Differential Equations," Springer, New York, Berlin, 1977.

[4] J.K. Hale, E.F. Infante, and E.P. Tsen, *Stability in linear delay equations,* J. Math. Anal. Appl., 105 (1985), 533–555.

[5] J.M. Mahaffy, *A test for stability of linear differential delay equations,* Quart. Appl. Math., 40 (1982), 193–202.

[6] L.S. Pontryagin, *On the zeros of some elementary transcendental functions,* Amer. Math. Soc., Transl., Ser. 2.1 (1955), 95–110.

[7] A. Thowsen, *An analytic stability test for a class of time delay systems,* IEEE Trans. Automat. Contr., Vol AC–26 (1981), 735–736.

[8] W.M. Wonham, "Linear Multivariable Control: A Geometric Approach," 2nd Edition, Springer, New York, 1979.

[9] J. Yong, *Stabilization of linear systems by time delay feedback controls,* Quart. Appl. Math., 45 (1987), 371–388.

[10] J. Yong, *Stabilization of linear systems by time delay feedback controls II,* Quart. Appl. Math., 46 (1988), 593–603.