Differential and Integral Equations Volume 9, Number 2, March 1996, pp. 343 – 362

GENERATORS OF TRANSLATION SEMIGROUPS AND ASYMPTOTIC BEHAVIOR OF THE SHARPE-LOTKA MODEL

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(Submitted by: Glenn Webb)

Abstract. Our first aim in this paper is to give a characterization of generators for the class of translation semigroups which are associated with a symbol ϕ and can brought into the form $u(t) = \phi(u_t)$. As applications, we consider two linear models from biology, a cell-cycle model based on unequal division and the Sharpe-Lotka age-dependent model. Our second aim is to show the relation of this last model to this class of semigroups in order to give a result on its asymptotic behavior. We show that the semigroup associated with the Sharpe-Lotka model is equivalent to a translation semigroup and both semigroups are essentially compact. It is also shown that the generators of the two semigroups have the same spectrum. We give properties of this spectrum and of its spectral bound.

I. Introduction. Let F be a Banach space, E be the Banach space

$$E = L^p((-r,0),F) \quad \text{where } 1 \leq p < \infty \text{ and } 0 < r \leq \infty.$$

From the Crandall-Liggett theorem [3], and using the same techniques as those used by A. Grabosch in [4] (i.e., for the case $E = L^1((-\infty,0),F,e^{\eta s}ds)$, $\eta > 0$), we can show, under the assumption that ϕ is a Lipschitz continuous operator from E into F, that for $p = 1$ the operator A_ϕ with domain $D(A_\phi) = \{f \in W^{1,p}((-r,0),F) : f(0) = \phi f\}$, and $A_\phi f = f'$, $f \in D(A_\phi)$, generates the semigroup solution $(T_\phi(t))_{t\geq 0}$ of the equation

$$u(t) = \phi(u_t) \tag{1}$$

in the sense

$$T_\phi(t)f = \lim_{n\to\infty} (I - \frac{t}{n}A_\phi)^{-n} f. \tag{2}$$

(Here we use the standard notation [6]; $u_t(s) = u(t+s)$). Our aim is to establish the existence of the semigroup solution of (1) on E with $1 \leq p < \infty$ and to show that the generator of $(T_\phi(t))_{t\geq 0}$ is exactly A_ϕ with domain $D(A_\phi)$, that is,

$$\lim_{t\to 0^+} \Big(\frac{T_\phi(t)f - f}{t}\Big) = A_\phi f, \quad \text{for} \quad f \in D(A_\phi). \tag{3}$$

For this, we present two different methods based on determining the semigroup trajectories. These methods lead to the same semigroup which is strongly continuous

Received December 1993, in revised form April 1994.

and satisfies (3). We will show that one of these methods allows us to use a weak assumption on ϕ, that is, ϕ is not necessarily Lipschitz continuous. We notice that in this case, we can not use the theorem of Crandall-Liggett.

We also recall that the generator of a semigroup (introduced by (3)), is unique and satisfies also the relation (2). However, the implication (2) $\Longrightarrow$ (3) does not hold. Furthermore, until now, there is no result on the uniqueness of the operator which generates a semigroup in the sense (2), except of course, for the linear case. Indeed, in the linear case we have the equivalence between (2) and (3).

For application, we will discuss two linear examples. First the cell proliferation model [7], for which the Crandall-Liggett theorem can not be applied, because under appropriate assumptions on the parameter functions of the model, the operator ϕ is not Lipschitz continuous. Secondly, we will show the relation of the Sharpe-Lotka age-dependent population model to translation semigroups and give some results on the spectral properties for this model. More precisely, we prove that the semigroup solution of this model is equivalent to a translation semigroup which is essentially compact because it is associated with an operator ϕ, which is weakly compact. We also show that both generators of the two semigroups have the same spectrum. Additionally, we show that the spectral bound of this spectrum is a simple pole of the resolvent of the generator of the translation semigroup, coincides with its exponential growth constant and is a dominant eigenvalue of this generator. All these arguments allow us to give the result on the asymptotic behavior, first for the translation semigroup, and secondly for the "Sharpe-Lotka" semigroup.

We recall that the spectral properties, compactness and the asymptotic behavior of translation semigroups can be found in [1], where we have already given an application to the cell cycle model.

Example 1. The cell-cycle proliferation model (see [7], [2], [5] and [1] for a detailed discussion) leads to the integro-difference equation

$$m(t,x) = \int_A^B g(x,u)m(t-\theta(x),u)du \tag{4}$$

with $0 < A < B < \infty$, and $m(t,x)$ is the density function for the "mother" cells of size x which divide at time t. The hypotheses considered on this model are

- (H_θ) θ is continuous, $0 < \theta_1 \leq \theta(x) \leq \theta_2 < \infty$, for all $x \in [0,\infty[$;
- (H'_θ) θ has a derivative on $[A,B]$, $0 < A \leq a_1 < a_2 \leq B < \infty$, and $|\theta'(x)| > \theta'_0 > 0$, $A \leq x \leq B$, for some $\theta'_0 > 0$.

Example 2. The model of linear age-dependent population describes the evolution of the density of a population $m(a,t)$ with respect to age a at time t (we refer to [9] and [12] for more details on this model).

The density function m satisfies the so-called balance law

$$\frac{\partial}{\partial t}m(a,t) + \frac{\partial}{\partial a}m(a,t) = -\mu(a)m(a,t) \tag{5}$$

and birth law

$$m(0,t) = \int_0^\infty \beta(a)m(a,t)da, \tag{6}$$

where $\mu : [0,\infty) \longrightarrow [0,\infty)$ is the age-specific mortality modulus and $\beta : [0,\infty) \longrightarrow [0,\infty)$ is the age-specific fertility modulus.

Furthermore, μ and β satisfy the hypothesis

$(H_{\mu,\beta})$ β not identically 0, $\underline{\mu} \le \mu(a) \le \overline{\mu}$, $\beta(a) \le \overline{\beta}$, for $a \ge 0$, where $\underline{\mu} \ge 0$, $\overline{\mu} > 0$ and $\overline{\beta} > 0$ are constants.

II. Translation semigroups and first order differential operators. We consider the Banach space $E = L^p((-r,0),F)$, where $1 \le p < \infty$, $0 < r \le \infty$, and F is a Banach space with norm denoted $\|\cdot\|_F$. The norm used on E is given by

$$\|f\|_E = \Big(\int_{-r}^0 \|f(s)\|_F^p ds\Big)^{1/p}, \quad \text{for all} \quad f \in E.$$

Definition II.1. A strongly continuous semigroup $(T(t))_{t\ge 0}$ on the Banach space E is called a translation semigroup if for all $f \in E$ and $t \ge 0$, we have

$$(T(t)f)(s) = f(t+s), \quad \text{for} \quad t+s<0, \text{ a.e. } s \in (-r,0).$$

Our next result gives a characterization of the generator of a translation semigroup on the Banach space E in the sense of (3). However it does not exactly give the domain of this generator. This result will be generalized in Theorem II.6 where the domain will be exactly determined for the case of translation semigroups associated with an equation of the type (1).

Proposition II.2. *Let $(T(t))_{t\ge 0}$ be a translation semigroup on the Banach space E. Then the generator of $(T(t))_{t\ge 0}$ in the sense of* (3) *satisfies*

i) *$D(A) \subseteq W^{1,p}((-r,0),F)$ and $Af = f'$ for all $f \in D(A)$,*
ii) *the map $f \mapsto f(0)$ is continuous from $(D(A), \|\,\|_A)$ into F (where $\|f\|_A = \|f\| + \|Af\|$, for $f \in D(A)$).*

We notice that this result was established for the case of the Banach space

$$E = L^1((-\infty,0),F,e^{\eta s}ds), \quad \eta > 0,$$

by Grabosch ([4], Proposition 1.4) and in order to prove this result we will follow the same ideas as in [4]. The reasons which lead us to give this proof are the facts that here we take $1 \le p < \infty$ and we work on $(-r,0)$ with $0 < r \le \infty$, however in [4] Grabosch takes $r = \infty$, $p = 1$ and introduces the density $e^{\eta s}ds$.

Proof. Let $f \in D(A)$ and $s, s' \in (-r,0)$ such that $s < s'$. We have

$$f(s) = \lim_{h\to 0^+} \frac{1}{h}(\int_0^h [T(\tau)f](s)d\tau) = \lim_{h\to 0^+}(\frac{1}{h}\int_0^h f(\tau+s)d\tau).$$

Hence

$$\begin{aligned} f(s') - f(s) &= \lim_{h\to 0^+} \frac{1}{h}\Big(\int_{s'}^{s'+h} f(\tau)d\tau - \int_{s}^{s+h} f(\tau)d\tau\Big) \\ &= \lim_{h\to 0^+} \frac{1}{h}\Big(\int_{s+h}^{s'+h} f(\tau)d\tau - \int_{s}^{s'} f(\tau)d\tau\Big) = \lim_{h\to 0^+} \frac{1}{h}\Big(\int_{s}^{s'} [f(h+\tau) - f(\tau)]d\tau\Big) \\ &= \lim_{h\to 0^+} \frac{1}{h}\Big(\int_{s}^{s'} [(T(h)f)(\tau) - f(\tau)]d\tau\Big) = \int_{s}^{s'} (Af)(\tau)d\tau. \end{aligned}$$

Then

$$f(s') = f(s) + \int_{s}^{s'} (Af)(\tau)d\tau, \quad \text{a.e. } s, s' \in (-r, 0).$$

Thus $f \in W^{1,p}((-r,0),F)$, $f' = Af$ and $f(0)$ exists. Now for $a \in (0,r)$, we have

$$\begin{aligned} \|f(0)\|_F &= \|\frac{1}{a}\int_{-a}^{0} f(0)d\tau\|_F = \|\frac{1}{a}\int_{-a}^{0}[f(\tau) + \int_{\tau}^{0} f'(\nu)d\nu]d\tau\|_F \\ &\le \frac{1}{a}P(a)\Big(\int_{-a}^{0} \|f(\tau)\|_F^p d\tau\Big)^{1/p} + \frac{1}{a}\int_{-a}^{0} P(-\tau)\Big(\int_{\tau}^{0} \|f'(\nu)\|_F^p d\nu\Big)^{1/p} d\tau, \end{aligned}$$

where

$$P(x) = \begin{cases} 1 & \text{if } p = 1 \\ x^{1/q} & \text{if } p > 1, \end{cases}$$

with $\frac{1}{p} + \frac{1}{q} = 1$, and for all $x \in \mathbb{R}$. Hence

$$\|f(0)\|_F \le \frac{1}{a}P(a)\Big(\|f\|_E + \int_{-a}^{0} \|f'\|_E d\tau\Big) \le P(a)\Big(\frac{1}{a}\|f\|_E + \|f'\|_E\Big).$$

This implies that the map $f \mapsto f(0)$ is continuous from $(D(A), \| \, \|_A)$ into F. $\square$

From this proposition, we can deduce that the generators of translation semigroups on E in the sense of (3) are of the form $Af = f'$ with $D(A) \subseteq \{f \in W^{1,p}((-r,0),F) : f(0) = \phi f\}$, where ϕ is a continuous operator from $(D(A), \|\cdot\|_A)$ into F.

In some applications one can show that the operator ϕ has further properties, that is, ϕ has an extension to the whole space E. In the sequel, we will limit ourselves first to the case where ϕ is a Lipschitz continuous operator (not necessarily linear) from E into F, with Lipschitz constant denoted by $|\, \phi \,|$. With ϕ we associate the operator A_ϕ on E defined by $A_\phi f = f'$ with domain

$$D(A_\phi) = \{f \in W^{1,p}((-r,0),F) : f(0) = \phi f\}.$$

Secondly we will treat the case where instead of this assumption a condition on an iterate of ϕ (which will be defined below) will be used. For the two cases we will consider two different methods, both based on determining the solution of a fixed point problem respectively (see Proposition 7 and its proof and also remark 10 for the difference between these methods). These problems are defined from the equation (1). We prove the existence of the solutions of these problems, which will lead to the semigroup solution, for which we will show that it is generated by the operator A_ϕ in the sense (3) with domain $D(A_\phi)$.

Before doing so we first want to give the equivalent result in conjunction with Grabosch's, concerning the conditions of Crandall-Liggett's theorem.

Proposition II.3. *Assume that ϕ is Lipschitz continuous from E into F. Then the operator A_ϕ satisfies*

i) *for $p = 1$, the operator $(A_\phi - \mid \phi \mid I)$ is dissipative,*
ii) *for $\lambda > \frac{|\phi|^p}{p}$, $R(\lambda I - A_\phi) = E$,*
iii) *the domain of A_ϕ is dense in the space E,*
iv) *for $p = 1$, A_ϕ is the generator of a strongly continuous semigroup $(T_\phi(t))_{t\geq 0}$ in the sense of (2) which satisfies*

$$\|T_\phi(t)f - T_\phi(t)g\|_E \leq e^{|\phi|t}\|f - g\|_E$$

for all $f, g \in E$ and $t \geq 0$.

Proof. From the Crandall-Liggett theorem [3], iv) is a direct consequence of i), ii), and iii). We notice that the proof of i) is exactly the same as in [4] and to prove ii) and iii) we will follow the same ideas as in [4].

Let $g \in E$ and $\lambda > \mid \phi \mid^p /p$. Under the assumption that $f \in D(A_\phi)$, the equation $\lambda f - A_\phi f = g$ gives $f(s) = [V_{\lambda,g} f](s)$, where $V_{\lambda,g}$ is the operator defined on E by

$$[V_{\lambda,g} f](s) = e^{\lambda s}\phi(f) + \int_s^0 e^{\lambda(s-\tau)} g(\tau) d\tau. \qquad (*)$$

For all $f, h \in E$, we have

$$\|V_{\lambda,g} f - V_{\lambda,g} h\|_E^p = \int_{-r}^0 e^{\lambda p s}\|\phi f - \phi h\|_F^p ds \leq (\lambda p)^{-1} \mid \phi \mid^p \|f - h\|_E^p.$$

This inequality and the fact that $\lambda > \mid \phi \mid^p /p$ imply that $V_{\lambda,g}$ is strictly contracting and therefore has a unique fixed point which we denote $f_{\lambda,g}^\phi$. We have $f_{\lambda,g}^\phi(0) = [V_{\lambda,g} f_{\lambda,g}^\phi](0) = \phi(f_{\lambda,g}^\phi)$ and from $(*)$ we deduce that $f_{\lambda,g}^\phi \in D(A_\phi)$ and $(\lambda I - A_\phi) f_{\lambda,g}^\phi = g$. Hence $R(\lambda I - A_\phi) = E$, for all $\lambda > \mid \phi \mid^p /p$.

Let us now show iii). We first notice that it is obvious to show that A_0 (i.e $\phi = 0$) is the generator of the strongly continuous linear semigroup $(T_0(t))_{t\geq 0}$ given by

$$T_0(t)f(s) = \begin{cases} f(t+s) & \text{if } t + s < 0 \\ 0 & \text{if } t + s \geq 0, \end{cases}$$

for all $f \in E$, $t \geq 0$ and almost every $s \in (-r, 0)$. From this fact we deduce that $D(A_0)$ is dense in E and we have $\lim_{\lambda \to \infty} \|f^0_{\lambda,g} - g\|_E = 0$. Now for almost every $s \in (-r, 0)$, we can write

$$g(s) - [f^{\phi}_{\lambda,g}](s) = -e^{\lambda s}[f^{\phi}_{\lambda,g}](0) + g(s) - [f^0_{\lambda,g}](s),$$

which implies that

$$\begin{aligned}\|g - f^{\phi}_{\lambda,g}\|_E &\leq (\lambda p)^{-1}\|\phi[f^{\phi}_{\lambda,g}]\|_F + \|g - f^0_{\lambda,g}\|_E \\ &\leq (\lambda p)^{-1}(\mid \phi \mid \|[f^{\phi}_{\lambda,g} - g\|_E + \|\phi(g)\|_F) + \|g - f^0_{\lambda,g}\|_E.\end{aligned}$$

Hence

$$\|g - f^{\phi}_{\lambda,g}\|_E \leq (1 - \frac{\mid \phi \mid}{\lambda p})^{-1}[(\lambda p)^{-1}\|\phi g\|_F + \|g - f^0_{\lambda,g}\|_E],$$

which gives that $\lim_{\lambda \to \infty} \|f^{\phi}_{\lambda,g} - g\|_E = 0$ and therefore the point ii) is satisfied because $f^{\phi}_{\lambda,g} \in D(A_{\phi})$ and g was arbitrary chosen in E. $\square$

Let us now introduce the two fixed-point problems mentioned above.

First problem. Taking $f \in E$, we consider the fixed point problem

$$\{ \text{ to find u} \in C([0,T], E) \text{ such that } K_f u = u\}, \tag{7}$$

where $0 < T < r$ and K_f is the operator defined on $C([0,T], E)$ by

$$[(K_f u)(t)](s) = (K_f u)(t, s) = \begin{cases} f(t+s) & \text{if } t+s < 0 \\ \phi(u(t+s)) & \text{if } t+s \geq 0, \end{cases} \tag{8}$$

for all $u \in C([0,T], E)$, $t \in [0,T]$ and almost every $s \in (-r, 0)$. The space $C([0,T], E)$ is endowed with the sup norm which we denote by $\| \cdot \|_C$.

Second problem. Let $0 < T < r$. We define the operator $\tilde{\phi}$ from the Banach space $E(T) = L^p((-r,T), F)$ into $L^p((0,T), F)$ by

$$(\tilde{\phi} h)(s) = \phi(h(s + \cdot)) = \phi(h_s), \tag{9}$$

for all $h \in E(T)$ and $s \in (0,T)$, where $h(s + \cdot)$ is the function of E defined by $(h(s + \cdot))(\tau) = h(s + \tau)$, almost every $\tau \in (-r, 0)$. For $f \in E$, we define the operator $\tilde{K}^t_f$ on the Banach space $E(t) := L^p((-r,t), F)$, $t \in (0,T)$, such that

$$(\tilde{K}^t_f u)(s) = \begin{cases} f(s) & \text{if } s < 0 \\ (\tilde{\phi} u)(s) & \text{if } 0 \leq s \leq t, \end{cases} \tag{10}$$

for all $u \in E(t)$ and $s \in (-r, t)$. And we consider the fixed point problem

$$\{\text{to find } u \in E(t) \text{ such that } \tilde{K}^t_f u = u\}, \tag{11}$$

where $E(t)$ is endowed with the usual norm which we denote $\| \cdot \|_t$.

Proposition II.4. *Under the assumption that ϕ is Lipschitz continuous from E into F, the operators K_f and $\tilde{K}^t_f$, $f \in E$, $t \in (0,T)$, are well defined. If furthermore we suppose $T <| \phi |^{-p}$, then each one of the fixed point problems (7) and (11) associated with K_f and $\tilde{K}^t_f$, respectively, has a unique solution.*

Proof. 1) It is clear that for $t \in (0,T)$, $\tilde{K}^t_g u$ is an element of the Banach space $E(t)$. Now let $f \in E$ and $u \in C([0,T],E)$. For all $t_1, t_2 \in [0,T]$, $t_1 < t_2$, we have

$$\|K_f u(t_2) - K_f u(t_1)\|^p_E = \int_{-r}^0 \|K_f u(t_2,s) - K_f u(t_1,s)\|^p_F ds$$
$$= \int_{-r}^{-t_2} \|f(t_2+s) - f(t_1+s)\|^p_F ds + \int_{-t_2}^{-t_1} \|\phi(u(t_2+s)) - f(t_1+s)\|^p_F ds$$
$$+ \int_{-t_1}^0 \|\phi(u(t_2+s)) - \phi(u(t_1+s))\|^p_F ds.$$

From this last expression and the fact that ϕ is Lipschitz continuous, we deduce that $\|K_f u(t_2) - K_f u(t_1)\|_E \to 0$ when $|t_2 - t_1| \to 0$, which shows that $K_f u \in C([0,T],E)$.

2) Taking $u_1, u_2 \in C((0,T),E)$ and $v, w \in E(t)$, we have

$$\|K_f u_2 - K_f u_1\|_C = \sup_{t\in[0,T]} (\|K_f u_2(t) - K_f u_1(t)\|_E)$$
$$= \sup_{t\in[0,T]} \{\int_{-t}^0 \|\phi(u_2(t+s)) - \phi(u_1(t+s))\|^p_F ds\}^{1/p}$$
$$\leq \sup_{t\in[0,T]} t^{1/p} \mid \phi \mid \|u_2 - u_1\|_C \leq T^{1/p} \mid \phi \mid \|u_2 - u_1\|_C.$$

Also, we have

$$\|\tilde{K}^t_f v - \tilde{K}^t_f w\|^p_t = \int_{-r}^t \|\tilde{K}^t_f v(s) - \tilde{K}^t_f w(s)\|^p_F ds = \int_0^t \|(\tilde{\phi} v)(s) - (\tilde{\phi} w)(s)\|^p_F ds$$
$$\leq| \phi |^p \int_0^t \int_{-r}^0 \|v_s(\tau) - w_s(\tau)\|^p_F d\tau ds \leq| \phi |^p \int_0^t \int_{-r}^0 \|v(s+\tau) - w(s+\tau)\|^p_F d\tau ds$$
$$\leq| \phi |^p \int_0^t \int_{-r+s}^s \|v(\tau) - w(\tau)\|^p_F d\tau ds \leq| \phi |^p \, t\|v - w\|^p_t. \tag{12}$$

Hence, if $T^{1/p} \mid \phi \mid< 1$, the operators K_f, and $\tilde{K}^t_g$, $f \in E$, $t \in (0,T)$, are contractions and therefore each one has a unique fixed point in $C((0,T),E)$ or $E(t)$, respectively.

The semigroup $(T_\phi(t))_{t\geq 0}$: Let $f \in E$, $T <| \phi |^{-p}$ and ϕ be Lipschitz continuous from E into F.

First method. Consider the problem (7) with the initial function f and denote $u^f_{(1)}$ its unique solution and $u^f_{(2)}$ its unique solution when it is considered with the initial function $u^f_{(1)}(T)$. Therefore, by successive iterations, we obtain a sequence $(u^f_{(n)})_{n\geq 1}$

such that for all $n \in \mathbb{N}$, $n \neq 0$, $u^f_{(n+1)}$ is the solution of (7) considered with the initial function $u^f_{(n)}(T)$. In this case we give the family $(T_\phi(t))_{t\geq 0}$ by

$$T_\phi(t)f = u^f_{(n)}(t-(n-1)T), \tag{13}$$

for all $n \geq 1$ and $t \in [(n-1)T, nT]$.

Second method. Considering now the second problem (11) we denote u^f the fixed point of the operator $\tilde{K}^T_f$. The family $(T_\phi(t))_{t\geq 0}$ is given by

$$T_\phi(t)f = \begin{cases} u^f_t, & \text{if } t \in [0,T] \\ u^{[T_\phi(nT)f]}_{t-nT}, & \text{if } t \in [nT,(n+1)T], \end{cases} \tag{14}$$

where $u^{[T_\phi(nT)f]}$, $n \in \mathbb{N}$, $n \neq 0$, is the unique fixed point of the operator $\tilde{K}^{t-nT}_{[T_\phi(nT)f]}$. Let us remark that in (14) we used Hale notation while in (13) we have not used it.

Theorem II.5. *Assume that ϕ is Lipschitz continuous from E into F. Then the family $(T_\phi(t))_{t\geq 0}$ given by either* (13) *or* (14) *defines a strongly continuous semigroup which is of translation and satisfies*

$$T_\phi(t)f(s) = \begin{cases} f(t+s) & \text{if } t+s<0 \\ \phi(T_\phi(t+s)f) & \text{if } t+s \geq 0, \end{cases} \tag{15}$$

$$\|T_\phi(t)f - T_\phi(t)g\|_E \leq e^{(|\phi|^p/p)t}\|f-g\|_E, \tag{16}$$

for all $f, g \in E$, $t \geq 0$ and almost every $s \in (-r, 0)$.

Proof. 1) By (10) it is obvious that $(T_\phi(t))_{t\geq 0}$ given by (14) satisfies (15). Proceeding by induction, we will show that the family defined by (13) also satisfies (15), for all $t \in [(n-1)T, nT]$, $n \geq 1$. We have for all $t \in [0,T]$ (i.e., $n = 1$), $f \in E$ and almost every $s \in (-r, 0)$,

$$T_\phi(t)f(s) = u^f_{(1)}(t,s) = (K_f u^f_{(1)})(t,s) = \begin{cases} f(t+s) & \text{if } t+s<0 \\ \phi(u^f_{(1)}(t+s)) & \text{if } t+s \geq 0. \end{cases}$$

So the equality (15) follows from the fact that $T_\phi(t)f = u^f_{(1)}(t)$, for $t \in [0,T]$.

Now we assume that for $n \geq 1$ and $t \in [(n-1)T, nT]$, $T_\phi(t)$ satisfies the property (15). Taking $t \in [nT, (n+1)T]$ and $f \in E$, from $T_\phi(t)f = u^f_{(n+1)}(t-nT)$, we have

$$\begin{aligned} T_\phi(t)f(s) &= \begin{cases} [u^f_{(n)}(T)](t-nT+s) & \text{if } t-nT+s<0 \\ \phi[u^f_{(n+1)}(t-nT+s)] & \text{if } t-nT+s \geq 0, \end{cases} \\ &= \begin{cases} [u^f_{(n)}(T)](t-nT+s) & \text{if } t-nT+s<0 \\ \phi[T_\phi(t+s)f] & \text{if } t-nT+s \geq 0. \end{cases} \end{aligned}$$

But $u^f_{(n)}(T) = u^f_{(n)}(nT-(n-1)T) = T_\phi(nT)f$, hence

$$T_\phi(t)f(s) = \begin{cases} [T_\phi(nT)f](t-nT+s) & \text{if } t-nT+s<0 \\ \phi[T_\phi(t+s)] & \text{if } t-nT+s \geq 0, \end{cases}$$
$$= \begin{cases} \begin{cases} f(t+s) & \text{if } t+s<0 \\ \phi[T_\phi(t+s)f] & \text{if } t+s\geq 0 \end{cases} & \text{if } t-nT+s<0 \\ \phi[T_\phi(t+s)f] & \text{if } t-nT+s\geq 0, \end{cases}$$

which implies that $T_\phi(t)$ satisfies (15).

2) Using the property (15), we obtain for $f, g \in E$ and $t>0$,

$$\|T_\phi(t)f - T_\phi(t)g\|^p_E$$
$$= \int_{-r}^{-t} \|f(t+s)-g(t+s)\|^p_F ds + \int_{-t}^{0} \|\phi[T_\phi(t+s)f] - \phi[T_\phi(t+s)g]\|^p_F ds.$$

Taking $\tau = t+s$, we obtain

$$\|T_\phi(t)f - T_\phi(t)g\|^p_E = \int_{-r+t}^{0} \|f(\tau)-g(\tau)\|^p_F d\tau + \int_{-t}^{0} \|\phi[T_\phi(\tau)f] - \phi[T_\phi(\tau)g]\|^p_F d\tau$$
$$\leq \|f-g\|^p_E + |\ \phi\ |^p \int_0^t \|T_\phi(\tau)f - T_\phi(\tau)g\|^p_E d\tau.$$

Now (16) is obtained by applying Gronwall's lemma.

3) The semigroup property: we have to show that $T_\phi(t+t') = T_\phi(t)T_\phi(t')$ for all $(t,t') \in \mathbb{R}^2_+$. For $t' \geq 0$ fixed let $t>t'$ and denote $G(t) = T_\phi(t-t')T_\phi(t')$. For all $f \in E$ and almost every $s \in (-r, 0)$, we have

$$[G(t)f](s) = \begin{cases} [T_\phi(t')f](t-t'+s) & \text{if } t-t'+s<0 \\ \phi[T_\phi(t-t'+s)T_\phi(t')f] & \text{if } t-t'+s\geq 0, \end{cases}$$
$$= \begin{cases} f(t+s) & \text{if } t+s<0 \text{ and } t-t'+s<0 \\ \phi[T_\phi(t+s)f] & \text{if } t+s\geq 0 \text{ and } t-t'+s<0 \\ \phi[G(t+s)f] & \text{if } t+s-t'\geq 0, \end{cases}$$
$$= \begin{cases} [T_\phi(t)f](s) & \text{if } t-t'+s<0 \\ \phi[G(t+s)f] & \text{if } t-t'+s\geq 0. \end{cases}$$

Let H be the operator on $C([0,T],E)$ given by

$$(Hu)(t,s) = \begin{cases} [T_\phi(t)f](s) & \text{if } t-t'+s<0 \\ \phi[u(t+s)] & \text{if } t-t'+s\geq 0. \end{cases}$$

Then, using the same techniques as in the proof of Proposition II.4, one can say first that H has a unique fixed point u which satisfies $u(t) = G(t)f = T_\phi(t)f$, for $t \in [0,T]$ with $T <|\ \phi\ |^{-p}$, and secondly that $G(t)f = T_\phi(t)f$, for all $t \geq 0$. Therefore, for all $t \geq t' \geq 0$, $T_\phi(t) = T_\phi(t-t')T_\phi(t')$, that is $T_\phi(t+t') = T_\phi(t)T_\phi(t')$.

For $t' \geq t$, we consider the operator $G(t') = T_\phi(t)T_\phi(t'-t)$ and we show as above that $G(t') = T_\phi(t')$ and so $T_\phi(t+t') = G(t+t') = T_\phi(t)T_\phi(t')$.

Theorem II.6. *Assume that ϕ is Lipschitz continuous from E into F; then the infinitesimal generator (in the sense (3)) of the strongly continuous semigroup $(T_\phi(t))_{t\geq 0}$ given by either (13) or (14) is the operator A_ϕ with domain $D(A_\phi)$.*

Proof. Let H be the subspace of functions of the Banach space $C^1([-r,0],F^*)$ which have compact support in $]-r,0[$. For all $f\in E$ and all $h\in H$, we have

$$\lim_{t\to 0^+}\langle\frac{T_\phi(t)f-f}{t},h\rangle=\lim_{t\to 0^+}\int_{-r}^{-t}\frac{f(t+s)-f(s)}{t}h(s)ds + \lim_{t\to 0^+}\int_{-t}^{0}\frac{\phi[T_\phi(t+s)f]-f(s)}{t}h(s)ds.$$

The last limit is equal to zero since the support of h is a closed subset of $]-r,0[$. Therefore

$$\begin{aligned}\lim_{t\to 0^+}\langle\frac{T_\phi(t)f-f}{t},h\rangle &=\lim_{t\to 0^+}\Big(\int_{-r}^{-t}\frac{f(t+s)h(s)}{t}ds-\int_{-r}^{-t}\frac{f(s)h(s)}{t}ds\Big)\\ &=\lim_{t\to 0^+}\Big(\int_{-r+t}^{0}\frac{f(\tau)h(\tau-t)}{t}d\tau-\int_{-r}^{-t}\frac{f(s)h(s)}{t}ds\Big)\\ &=\lim_{t\to 0^+}\Big(\int_{-r}^{0}\frac{f(\tau)h(\tau-t)}{t}d\tau-\int_{-r}^{0}\frac{f(\tau)h(\tau)}{t}d\tau\Big)\\ &\quad-\lim_{t\to 0^+}\Big(\int_{-r}^{-r+t}\frac{f(\tau)h(\tau-t)}{t}d\tau-\int_{-t}^{0}\frac{f(\tau)h(\tau)}{t}d\tau\Big).\end{aligned}$$

If we take t small enough, then the two last terms are equal to zero since the support of the function h is contained in $]-r,0[$. So

$$\lim_{t\to 0^+}\langle\frac{T_\phi(t)f-f}{t},h\rangle=\lim_{t\to 0^+}\int_{-r}^{0}\frac{f(\tau)[h(\tau-t)-h(\tau)]}{t}d\tau=\langle f,\frac{\partial}{\partial\tau}h\rangle.$$

From this we deduce that each element f in $D(A_\phi)$ satisfies the equality (3).

Now if f belongs to the domain of the generator of $(T_\phi(t))_{t\geq 0}$, then there exists $g\in E$ such that

$$\lim_{t\to 0^+}\|\frac{T_\phi(t)f-f}{t}-g\|_E=0,$$

and we have $\langle g,h\rangle=\langle f,\frac{\partial}{\partial\tau}h\rangle$, for all $h\in H$. But

$$\begin{aligned}\langle g,h\rangle&=\int_{-r}^{0}g(s)h(s)ds=\int_{-r}^{0}g(s)(\int_{0}^{s}h'(u)du)ds\\ &=-\int_{-r}^{0}g(s)(\int_{s}^{0}h'(u)du)ds=-\int_{-r}^{0}(\int_{-r}^{u}g(s)ds)h'(u)du.\end{aligned}$$

So, taking $G(u) = \int_{-r}^{u} g(s)ds$, we obtain

$$-\int_{-r}^{0} G(u)h'(u)du = \int_{-r}^{0} f(u)h'(u)du, \quad \text{for all } h \in H.$$

This implies that $f(u) = -G(u) + C$, where C is a constant, for all $u \in [-r, 0]$. Hence $f \in W^{1,p}(-r,0), F)$, and $f(0) = \phi f$ (since $T_\phi(t)f$, $t \geq 0$, is also in the domain of the generator of $(T_\phi(t))_{t\geq 0}$, and we have $(T_\phi(0)f)(0) = \phi(T_\phi(0)f))$, so we obtain that $f \in D(A_\phi)$, and

$$\lim_{t\to 0^+}\left(\frac{T_\phi(t)f - f}{t}\right) = A_\phi f.$$

Proposition II.7. *Let ϕ be an operator from E into F (not necessarily linear). Assume there exists $T \in]0, r]$, such that the operator $\tilde{\phi}$ is Lipschitz continuous from the Banach space $E(T) = L^p((-r,T), F)$ into $L^p((0,T), F)$, with Lipschitz constant $|\tilde{\phi}| < 1$. Then, the operator A_ϕ is the generator of a strongly continuous semigroup $(T_\phi(t))_{t\geq 0}$ which satisfies* (15) *and for all $t \geq 0$, we have*

$$\|T_\phi(t)f - T_\phi(t)g\|_E^p \leq \left(\frac{1+|\tilde{\phi}|^p}{1-|\tilde{\phi}|^p}\right)\|f-g\|_E^p, \quad f, g \in E. \tag{17}$$

Proof. To prove this result, we consider the second method above, i.e. the fixed point problem associated with the operator $\tilde{K}_f^T$, $f \in E$. We notice that the first method can not be applied here. We also notice that the proof of the fact that A_ϕ is the generator of $(T_\phi(t))_{t\geq 0}$ is the same as the one of Theorem II.6.

Using (12), we can say that $\tilde{K}_f^T$ has a unique fixed point u^f and we can define the family $(T_\phi(t))_{t\geq 0}$ by the formula (14), which satisfies clearly (15). And we have, for $t \in [0, T]$,

$$\begin{aligned}
&\|T_\phi(t)f - T_\phi(t)g\|_E^p = \int_{-r}^{-t} \|f(t+s) - g(t+s)\|_F^p ds + \int_{-t}^{0} \|\phi(u_{t+s}^f) - \phi(u_{t+s}^g)\|_F^p ds \\
&\leq \|f-g\|_E^p + \int_{-t}^{0} \|\phi(u_{t+s}^f) - \phi(u_{t+s}^g)\|_F^p ds \\
&\leq \|f-g\|_E^p + \int_{0}^{t} \|(\tilde{\phi}u^f)(\tau) - (\tilde{\phi}u^g)(\tau)\|_F^p d\tau \leq \|f-g\|_E^p + |\tilde{\phi}|^p \, \|u^f - u^g\|_{E(T)}^p \\
&\leq \|f-g\|_E^p + |\tilde{\phi}|^p \left(\int_{-r}^{p} \|f(s) - g(s)\|_F^p ds + \int_{0}^{T} \|u^f(s) - u^g(s)\|_F^p ds\right) \\
&\leq (1 + |\tilde{\phi}|^p)\|f-g\|_E^p + |\tilde{\phi}|^p \int_{-T}^{0} \|T_\phi(T)f(s) - T_\phi(T)g(s)\|_F^p ds.
\end{aligned}$$

Hence, from the assumption on $\tilde{\phi}$, we obtain

$$\max_{0\leq t\leq T} \|T_\phi(t)f - T_\phi(t)g\|_E^p \leq \left(\frac{1+|\tilde{\phi}|^p}{1-|\tilde{\phi}|^p}\right)\|f-g\|_E^p.$$

Using the same arguments, we have from (14), for all $n \geq 1$ and $t \in [nT, (n+1)T]$,

$$\max_{nT \leq t \leq (n+1)T} \|T_\phi(t)f - T_\phi(t)g\|_E^p \leq \left(\frac{1+ \mid \tilde{\phi} \mid^p}{1- \mid \tilde{\phi} \mid^p}\right) \|T_\phi(nT)f - T_\phi(nT)g\|_E^p,$$

which yields (17), if we proceed by induction. □

From the results obtained above, we can deduce the next Corollary which was also proved for the case $p = 1$ and ϕ Lipschitz continuous by R. Villella-Bressan [14] and A. Grabosch [4].

Corollary II.8. *Let ϕ be an operator from E into F (not necessarily linear) and assume one of the following hypotheses:*

(i) *ϕ is Lipschitz continuous from E into F;*

(ii) *there exists $T \in]0, r]$, such that the operator $\tilde{\phi}$ is Lipschitz continuous from the Banach space $E(T) = L^p((-r, T), F)$ into $L^p((0, T), F)$, with Lipschitz constant $\mid \tilde{\phi} \mid < 1$.*

Then the problem

$$\begin{cases} \text{find } u \in L^p_{loc}((-r, +\infty), F) \cap C([0, +\infty), F) \text{ such that } u(t) = \phi(u_t), \ t \geq 0 \\ u_0 = f \end{cases} \tag{18}$$

with $f \in E$, has a unique solution, which satisfies

$$u(t) = \begin{cases} f(t) & a.e. \ t \in (-r, 0) \\ \phi(T_\phi(t)f) & if \ t \geq 0, \end{cases} \tag{19}$$

where $(T_\phi(t))_{t \geq 0}$ is the strongly continuous semigroup of translation associated with the operator ϕ. Furthermore, if $f \in D(A_\phi)$ then $u(t) = [T_\phi(t)f](0)$, $t \geq 0$.

Remark II.9. We notice that all the results obtained above are still valid if, instead of the space E, we consider the Banach space $\bar{E} = L^p((-r, 0), F, e^{\eta s}ds)$, $\eta > 0$, $0 < r \leq \infty$ with the norm

$$\|f\|_{\bar{E}} = \left(\int_{-r}^0 \|f(s)\|_F^p e^{\eta s} ds\right)^{1/p}, \quad \text{for all } f \in \bar{E}.$$

III. Applications.

III.1 The cell-cycle model ([7], [1], [2] and [5]). The equation (4) can be written as $m(t) = \phi(m_t)$, where ϕ satisfies

$$\phi f(x) = \int_A^B g(x, u) f(-\theta(x), u) du, \tag{20}$$

for almost every $x \in (A, B)$ and $f \in E = L^p((-\theta_2, 0), F)$, where $F = L^p(A, B)$ and $1 \leq p < \infty$. Next we assume that the hypothesis (H_θ) is satisfied and that $g \in L^p((A, B), L^q(A, B))$ where $1 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. We denote

$$\|g\| = \left[\int_A^B \|g(x, \cdot)\|_{L^q}^p dx\right]^{\frac{1}{p}} = \left[\int_A^B \left(\int_A^B \mid g(x, u) \mid^q du\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}}$$

Proposition III.1.1. *Let $\|g\| < 1$ [resp. (H'_θ) and the condition $\|g(x,\cdot)\|_{L^q(A,B)} < C$, almost every $x \in (A,B)$, with $0 < C < \infty$] be satisfied. Then the operator A_ϕ associated with ϕ with its domain $D(A_\phi)$ is the generator of a strongly continuous semigroup $(T_\phi(t))_{t\geq 0)}$ on E in the sense* (3), *which is of translation, satisfies* (15) *and also* (17) (*resp.* (16)) *and is the semigroup solution of the equation* (4) (*in the sense of* (19)).

Proof. Let $f \in E$ and suppose first that (H'_θ) holds and $\|g(x,\cdot)\|_{L^q(A,B)} < C$, for almost every $x \in (A,B)$, with $0 < C < \infty$. Using the results obtained above, it is sufficient to show that ϕ is bounded from E into F. Taking $\tau = -\theta(x)$ in (20), we obtain

$$\|\phi f\|_F^p = \int_{-\theta(B)}^{-\theta(A)} \mid \int_A^B g(\theta^{-1}(-\tau),u)f(\tau,u)du \mid^p [\theta'(\theta^{-1}(-\tau)]^{-1}d\tau$$
$$\leq (\theta'_0)^{-1}\int_{-\theta_2}^{-\theta_1}[\int_A^B \mid g(\theta^{-1}(\tau),u) \mid^q du]^{\frac{p}{q}}[\int_A^B \mid f(\tau,u) \mid^p dud\tau \leq C^p(\theta'_0)^{-1}\|f\|_E^p.$$

Hence ϕ is a bounded operator from E into F.

Suppose now the condition $\|g\| < 1$ is satisfied. Let $T \leq \theta_1$ and consider the operator $\tilde{\phi}$ from the Banach space $E(T) = L^p((-\theta_2, T), F)$ into $L^p((0,T), F)$ associated with ϕ which is given by

$$[\tilde{\phi}f(s)](x) = [\phi(f_s)](x) = \int_A^B g(x,u)f(s-\theta(x),u)du \tag{21}$$

for all $f \in E(T)$ and almost every $s \in (0,T)$. We have

$$\|\tilde{\phi}f\|^p = \int_0^T \int_A^B \mid \int_A^B g(x,u)f(s-\theta(x),u)du \mid^p dxds$$
$$= \int_A^B \int_{-\theta(x)}^{T-\theta(x)} \mid \int_A^B g(x,u)f(s,u)du \mid^p dsdx$$
$$\leq \int_A^B [\int_A^B \mid g(x,u) \mid^q du]^{\frac{p}{q}} dx \int_{-\theta_2}^0 \int_A^B \mid f(s,x) \mid^p dxds \leq \|g\|^p\|f\|_{E(T)}^p.$$

Therefore, $\tilde{\phi}$ is Lipschitz continuous from $E(T)$ into $L^p((0,T), F)$ with Lipschitz constant $\mid \tilde{\phi} \mid < 1$. Hence, the result can be concluded from Proposition II.7.

Remark III.1.2. In the case where the condition (H'_θ) is not satisfied we can show, without taking $\|g\|_{L^q} < 1$, that we have the same result as in Proposition III.1.1. This is also due to the second method. Indeed, if we take $0 < T \leq \theta_1$, then the fixed point u^f of the operator $\tilde{K}_f^t$, $f \in E$, $t \in [0,T]$, in the Banach space $E(t) = L^p((-\theta_2, t), F)$, is u^f given by

$$u^f(s) = \begin{cases} f(s) & \text{if } s < 0 \\ (\tilde{\phi}f)(s) & \text{if } s \geq 0 \end{cases}$$

because we have from (21), for all $u \in E(t)$,

$$(\tilde{\phi}u)(s) = (\tilde{\phi}u_{/(-\theta_2,0)})(s), \quad \text{a.e. } s \in (0,t). \tag{22}$$

And the semigroup $(T_\phi(t))_{t\geq 0}$ on E is also given by the relation (14) and we show easily that it is strongly continuous, by using the same idea as in the proof of Proposition II.7. So, an important condition which yields this result is the fact that $\tilde{\phi}$ is bounded here and satisfies (22).

III.2. The Sharpe-Lotka model ([9], [10]). Our purpose here is to show that this model can be connected to a translation semigroup and that its asymptotic behavior can be determined by the study of properties on this translation semigroup, such as spectral properties, compactness and irreducibility.

In the following, we assume that the Hypothesis $(H_{\mu,\beta})$ is satisfied. Let ϕ (resp. χ) be the operator defined, respectively, from the Banach space $\bar{E} = L^1((0,+\infty),\mathbb{R})$ (resp. $E := L^1((0,\infty),\mathbb{R}, e^{\bar{\mu}s}ds)$) into $\mathbb{R}$ (resp. E) by

$$\phi f = \int_0^\infty \beta(s) e^{-\int_0^s \mu(\tau)d\tau} f(s)ds,$$

$$(\chi f)(s) = f(s)e^{\int_0^s \mu(\tau)d\tau}, \quad \text{a.e. } s \in (0,+\infty). \tag{23}$$

The norms which we consider on $\bar{E}$ and E are denoted respectively by $\|\cdot\|_{\bar{E}}$ and $\|\cdot\|_E$ and are given respectively by

$$\|f\|_{\bar{E}} = \int_0^\infty |f(s)|\, ds, \quad f \in \bar{E}, \qquad \|f\|_E = \int_0^\infty |f(s)|e^{\bar{\mu}s}\, ds, \quad f \in E.$$

Let A_ϕ be the first order derivative operator associated with ϕ, that is

$$D(A_\phi) = \{f \in W^{1,1}((0,+\infty),\mathbb{R}) : f(0) = \phi f\} \tag{24}$$

and $A_\phi f = -f'$, for all $f \in D(A_\phi)$. Notice that here we consider functions on $[0,+\infty)$ instead of $(-\infty, 0]$.

The operator ϕ being bounded linear from $\bar{E}$ into $\mathbb{R}$. We can use Theorems II.5 and II.6 to conclude the following result.

Corollary III.2.1. *The operator A_ϕ is the generator of a strongly continuous semigroup of translation $(T_\phi(t))_{t\geq 0}$ on $\bar{E}$ which satisfies* (16) *and*

$$T_\phi(t)f(s) = \begin{cases} f(s-t) & \text{if } t-s<0 \\ \phi(T_\phi(t-s)f) & \text{if } t-s \geq 0, \end{cases} \tag{25}$$

for all $f \in \bar{E}$, and almost every $s \in (0,+\infty)$.

Next, we will show that the semigroup solution of the Sharpe-Lotka model can be connected to the semigroup $(T_\phi(t))_{t\geq 0}$, by using the operator χ and we will relate its generator to the generator A_ϕ of $(T_\phi(t))_{t\geq 0}$, in order to conclude the result on the asymptotic behavior.

For this, we denote by $\sigma(A_\phi)$ the spectrum of A_ϕ, $R(\lambda, A_\phi)$ its resolvent and $s(A) = \sup\{Re(\lambda),\, \lambda \in \sigma(A_\phi)\}$ its spectral bound. And we introduce the family of operators $(\bar{\phi}_\lambda)_{\lambda\in C}$ defined on $\mathbb{R}$ and given by

$$\overline{\phi}_\lambda x = \phi(e^{-\lambda\cdot} \otimes x) = \int_0^\infty e^{-\lambda s} k(s) x ds, \quad x \in \mathbb{R},$$

where k is the function defined on $\mathbb{R}^+$ by

$$k(s) = \beta(s) e^{-\int_0^s \mu(\tau)d\tau}.$$

By $\sigma(\bar{\phi}_\lambda)$ we denote the spectrum of the operator $(\bar{\phi}_\lambda)$, $\lambda \in \mathbb{C}$, and by $r(\bar{\phi}_\lambda)$ its spectral radius, that is

$$r(\bar{\phi}_\lambda) = \sup\{\mid \lambda \mid : \lambda \in \sigma(\bar{\phi}_\lambda)\}.$$

Proposition III.2.2. *Let $(G(t))_{t\geq 0}$ be the family of operators on E given by*

$$G(t)f = \chi^{-1}(T_\phi(t)\chi f), \quad \textit{for all} \;\; f \in E. \tag{26}$$

Then we have

i) *$(G(t))_{t\geq 0}$ is a strongly continuous semigroup on E and satisfies*

$$G(t)f(a) = \begin{cases} f(a-t)e^{-\int_{a-t}^a \mu(\tau)d\tau} & \textit{if } a-t>0 \\ \phi\circ\chi(G(t-a)f)e^{-\int_0^a \mu(\tau)d\tau} & \textit{if } a-t\leq 0. \end{cases} \tag{27}$$

The semigroup $(G(t))_{t\geq 0}$ is the semigroup solution of equation (5).

ii) *Let A be the generator of the semigroup $(G(t))_{t\geq 0}$. Then $f \in D(A) \Longleftrightarrow h \in D(A_\phi)$ where $h = \chi(f)$, and we have $Af = \chi^{-1}(A_\phi h)$, that is,*

$$D(A) = \{f \in W^{1,1}((0,\infty), \mathbb{R}, e^{\bar{\mu}s}ds) : f(0) = \textstyle\int_0^\infty \beta(s)f(s)ds\}$$
$$Af(s) = -f'(s) - \mu(s)f(s), \; \textit{for all } s \in (0,\infty), f \in D(A).$$

iii) *$\sigma(A) = \sigma(A_\phi)$ and for all $\lambda \in \mathbb{C}$, such that $\operatorname{Re}\lambda > 0$, we have $\lambda \in \sigma(A)$ if and only if $\int_0^\infty e^{-\lambda s}k(s)\,ds = 1$ and in this case we have*

$$dim\,[Ker(A-\lambda I)] = dim\,[Ker(\overline{\phi}_\lambda - I)].$$

Proof. The points i) and ii) are obviously deduced from the properties on the semigroup $(T_\phi(t))_{t\geq 0}$ and its generator A_ϕ.

Let us now show iii). For $\lambda \in C$, $g \in E$, we set up the equation

$$Af - \lambda f = g, \quad f \in D(A). \tag{28}$$

Let $h = \chi f$. We have

$$h'(\theta) = (f'(\theta) + \mu(\theta)f(\theta))e^{\int_0^\theta \mu(\tau)d\tau} = -(Af)(\theta)e^{\int_0^\theta \mu(\tau)d\tau}.$$

Hence the equation (28) is equivalent to

$$A_\phi h(\theta) - \lambda h(\theta) = g(\theta)e^{\int_0^\theta \mu(\tau)d\tau} = (\chi g)(\theta).$$

Therefore we obtain $\lambda \in \sigma(A) \Longleftrightarrow \lambda \in \sigma(A_\phi)$. And, because $\overline{\phi}_\lambda$ is compact on $\mathbb{R}$, iii) is a direct consequence from [1, Lemma 5 and its proof] and the fact that

$$1 \in \sigma(\overline{\phi}_\lambda) \Longleftrightarrow \int_0^{+\infty} e^{-\lambda s}k(s)ds = 1.$$

Definition III.2.3. ([15]) Let X be a Banach lattice and X^* be its dual space.

(i) A positive operator T on X is said to be irreducible if, for all $x \in X$, $x > 0$, and for all $x^* \in X^*$, $x^* > 0$, there exists $n \in \mathbb{N}$ such that $\langle T^n x, x^*\rangle > 0$.

(ii) A strongly continuous semigroup $(T(t))_{t\geq 0}$ of positive operators on X, with generator denoted by A, is said to be irreducible if one of the following (equivalent) conditions are satisfied:

  (a) for all $x \in X$ and all $x^* \in X^*$, with $x > 0$ and $x^* > 0$, there exists $t_0 \geq 0$ such that $\langle T(t_0)x, x^*\rangle > 0$;

  (b) there exists $\lambda > s(A)$ such that $R(\lambda, A)x$ is quasi-interior point of X, for $x > 0$ (i.e., for all $x^* \in X^*$, $x^* > 0$, we have $\langle R(\lambda, A)x, x^*\rangle > 0$);

  (c) the operator $R(\lambda, A)$ is irreducible for one $\lambda > s(A)$.

Proposition III.2.4. *Assume that $(H_{\mu,\beta})$ is satisfied. Then we have*

(i) *The operator ϕ is weakly compact.*

(ii) *The equation $r(\overline{\phi}_\lambda) = 1$ has a unique real solution λ_0, and if $\lambda_0 > 0$ then λ_0 is a simple pole of the resolvent of A_ϕ, and the spectral bound $s(A_\phi) = \lambda_0$.*

(iii) *If $\lambda_0 > 0$, then we have $\bar{E} = Ker(A_\phi - \lambda_0 I) \oplus Range(A_\phi - \lambda_0 I)$ and $Ker(A_\phi - \lambda_0 I) = e^{\lambda_0 \cdot} \otimes Ker(I - \overline{\phi}_{\lambda_0}) = \{Ce^{\lambda_0 \cdot} \otimes x_{\lambda_0} : C \in \mathbb{R}\}$, where $x_{\lambda_0} > 0$ is an eigenvector of $\overline{\phi}_{\lambda_0}$ associated with the eigenvalue λ_0.*

Proof. From (22) and from [8, Theorem II.5.10], we can say that the operator ϕ is weakly compact on E, because it is defined by a bounded kernel.

Let us now prove (ii). The map $\lambda \mapsto r(\overline{\phi}_\lambda) = \int_0^{+\infty} e^{-\lambda s}k(s)ds$ is continuous and decreasing on $\mathbb{R}$, and we have $r(\overline{\phi}_\lambda) \to 0$ when $\lambda \to +\infty$, $r(\overline{\phi}_\lambda) \to +\infty$ when $\lambda \to -\infty$. There exists a unique $\lambda_0 \in \mathbb{R}$ such that $r(\overline{\phi}_{\lambda_0}) = 1$. Furthermore if $\lambda_0 > 0$, we have $\lambda \in \sigma(A_\phi) \Rightarrow Re(\lambda) < \lambda_0$, which implies that $\lambda_0 = s(A_\phi)$.

Now, since $\overline{\phi}_\lambda$ is irreducible and compact, then if $\lambda_0 > 0$, we can use the same arguments as in the proof of Proposition 9 in [1] to show that λ_0 is a simple pole of A_ϕ and therefore of A.

The last point (iii) is a direct consequence of the results obtained above and [1, Lemma 5 and its proof]. □

Denote $\omega(T(t))$ the growth constant of a semigroup $(T(t))_{t\geq 0}$ on a Banach space, that is

$$\omega(T(t)) = \inf\{\omega : \|T(t)\| \leq Me^{\omega t},\ M \geq 0\}.$$

Definition III.2.5 ([4]). A semigroup $(T(t))_{t\geq 0}$ on a Banach space E is said to be essentially compact (resp. essentially weakly compact) if there exist a semigroup $(H(t))_{t\geq 0}$ on E and a family of compact (resp. weakly compact) operators $(K(t))_{t\geq 0}$, such that, for all $t \geq 0$, we have $T(t) = H(t) + K(t)$ and $\omega(H(t)) < \omega(T(t))$.

Theorem III.2.6. *Let $(H_{\mu,\beta})$ be satisfied. Furthermore, we assume that the function k is absolutely continuous with the density $\frac{dk}{ds} \in L^\infty((0,+\infty),\mathbb{R})$ and $\int_0^{+\infty} k(s)ds > 1$. Then we have*

i) *The semigroup $(T_\phi(t))_{t\geq 0}$ is essentially compact.*
ii) *The spectral bound $\lambda_0 = s(A_\phi)$ of A_ϕ is a simple pole of the resolvent of A_ϕ, is a dominant eigenvalue of A_ϕ and coincides with $\omega(T_\phi(t))$.*
iii) *There exists a positive projection P of rank 1 such that the semigroup $(G(t))_{t\geq 0}$ solution of the Sharpe-Lotka model satisfies*

$$\|e^{-\lambda_0 t}G(t)f - Pf\|_{\bar{E}} \leq Me^{-\delta t}\|f\|_E, \quad f \in E,$$

for suitable constants $\delta > 0$, $M \geq 1$ and all $t \geq 0$.

The projection P is given by

$$Pf(a) = \frac{e^{-\lambda_0 a}\pi(a,0)\int_0^{+\infty}\beta(b)e^{-\lambda_0 b}[\int_0^b e^{\lambda_0 c}\pi(b,c)f(c)dc]db}{\int_0^{+\infty}\beta(b)be^{-\lambda_0 b}\pi(b,0)db} \tag{29}$$

for all $f \in E$ and $a \geq 0$, where

$$\pi(a,b) = e^{-\int_b^a \mu(\tau)d\tau}, \qquad 0 \leq b \leq a < \infty.$$

Remark III.2.7. i) The conditions used in Theorem III.2.6 are realistic hypotheses for the Sharpe-Lotka model. ii) We also notice that $(G(t))_{t\geq 0}$ is a C_0-semigroup on $(E, \|\cdot\|)$.

For the proof of Theorem III.2.6, we need the following lemma.

Lemma III.2.8. ([1]) *We consider the operator $\psi : E_1 = L^1((-r,0),\mathbb{R}) \to \mathbb{R}$, $0 < r \leq \infty$, given by*

$$\psi f = \int_{-r}^0 \xi(s)f(s)ds, \quad \text{for all } f \in E_1,$$

where $\xi \in L^\infty((-r,0),\mathbb{R})$. Then a necessary and sufficient condition for $\psi \circ A_\psi$ to have a continuous extension to E_1 is that η be absolutely continuous with respect to the Lebesgue measure, with the density $\frac{d\xi}{ds} \in L^q((-r,0),\mathbb{R})$ and $\xi(-r) = 0$.

For the proof of this lemma, we refer to the results of Proposition 5 and Lemma 3 in [1], for which we notice that, from their proofs, they are still valid for the case $r = +\infty$.

Proof of Theorem III.2.6. We will follow the same ideas as in [1] and [4] to prove this theorem. We can write $T_\phi(t) = H(t) + K(t)$, $t \geq 0$, where

$$(H(t)f)(s) = \begin{cases} f(s-t) & \text{if } t-s<0 \\ 0 & \text{if } t-s \geq 0 \end{cases} \tag{30}$$

$$(K(t)f)(s) = \begin{cases} 0 & \text{if } t-s<0 \\ \phi(T_\phi(t-s)f) & \text{if } t-s \geq 0 \end{cases} \tag{31}$$

From the hypothesis $(H_{\mu,\beta})$, we have $k(+\infty) = 0$. So, using Lemma III.2.8, we obtain that $\phi \circ A_\phi$ has a continuous extension to the space $\bar{E}$. So, we can use the same techniques as in [1, Proposition 7], to show that $K(t)$, $t \geq 0$, is compact. Furthermore, we obtain from the assumption $\int_0^{+\infty} k(s)ds > 1$ that $\lambda_0 > 0$ and from (29) that $\omega(H(t)) = 0$. Finally we conclude that $(T_\phi(t))_{t\geq 0}$ is essentially weakly compact.

The essential weak compactness of $(T_\phi(t))_{t\geq 0}$ on $\bar{E}$ yields that $(T_\phi(t))_{t\geq 0}$ is also essentially compact (see [1, Remark 3]). Therefore, we obtain from [11], $s(A_\phi) = \omega(T_\phi(t))$ and $s(A_\phi)$ is a dominant eigenvalue of A_ϕ.

Let us now prove the third point iii) of the theorem. We can write $\bar{E} = Ker(A_\phi - \lambda_0 I) \oplus Range(A_\phi - \lambda_0 I)$ and $Ker(A_\phi - \lambda_0 I) = e^{-\lambda_0 \cdot} \otimes Ker(I - \overline{\phi}_{\lambda_0})$. Let $x_{\lambda_0} > 0$ be an eigenvector of $\overline{\phi}_{\lambda_0}$ associated to the eigenvalue λ_0 and let $f \in \bar{E}$. Taking the projection of f onto $Ker(A_\phi - \lambda_0 I)$ and $Range(A_\phi - \lambda_0 I)$, we can write

$$f = \gamma(f)e^{-\lambda_0 \cdot} \otimes x_{\lambda_0} + \rho,$$

so that $T_\phi(t)f = \gamma(f)e^{\lambda_0(t-\cdot)}x_{\lambda_0} + (T_\phi(t))(\rho)$. Hence, from the fact that $K(t)$ is compact and $\omega(H(t)) = 0 < \lambda_0$, we have

$$T_\phi(t)f = \gamma(f)e^{\lambda_0(t-\cdot)}x_{\lambda_0} + \circ(e^{\lambda_0 t}). \tag{32}$$

So, there exits $M \geq 1$ and $\delta > 0$ such that

$$\|e^{-\lambda_0 t}T_\phi(t)f - \bar{P}f\|_{\bar{E}} \leq Me^{-\delta t}\|f\|_E, \quad f \in \bar{E},$$

where $\bar{P}f = \gamma(f)e^{-\lambda_0 \cdot}x_{\lambda_0}$. And from [1, Lemma 7], we obtain

$$\gamma(f) = \frac{\langle x^*_{\lambda_0}, \phi(\int_0^\theta e^{\lambda_0(-\theta+s)}f(s)ds)\rangle}{\langle x^*_{\lambda_0}, \phi(\theta e^{-\lambda_0\theta} \otimes x_{\lambda_0})\rangle}, \tag{33}$$

where $x^*_{\lambda_0} > 0$ is an eigenvector of $\bar{\phi}^*_{\lambda_0}$ such that $x^*_{\lambda_0}x_{\lambda_0} = 1$. The result on the asymptotic behavior for the semigroup $(G(t))_{t\geq 0}$ is a direct consequence of (32) and the relation (26). Indeed, if we take $Pf = \chi^{-1}(\bar{P}(\chi f))$, we have

$$\begin{aligned} \|e^{-\lambda_0 t}G(t)f - Pf\|_{\bar{E}} &= \|\chi[e^{-\lambda_0 t}G(t)f - Pf]\|_{\bar{E}} \\ &\leq \|e^{-\lambda_0 t}(T_\phi(t)\chi f) - [\gamma(\chi f)e^{-\lambda_0 \cdot}x_{\lambda_0}]\|_{\bar{E}} \leq Me^{-\delta t}\|f\|_E. \end{aligned}$$

And from (33), we obtain

$$\gamma(\chi f) = x_{\lambda_0}^{-1} \frac{\int_0^{+\infty} \beta(b) e^{-\lambda_0 b} [\int_0^b e^{\lambda_0 c} \pi(b,c) f(c) dc] db}{\int_0^{+\infty} \beta(b) b e^{-\lambda_0 b} \pi(b,0) db},$$

which implies (29) by taking $x_{\lambda_0} = 1$. $\square$

As a consequence of Theorem III.2.6, we have the following result in which we take

$$\bar{E} := L^1((0,+\infty), \mathbb{R}, e^{-\eta s} ds) \quad \text{and} \quad E := L^1((0,+\infty), \mathbb{R}, e^{(\bar{\mu}-\eta)s} ds)$$

with the respective norms

$$\|f\|_{\bar{E}} = \int_0^\infty |f(s)| e^{-\eta s} ds, \quad f \in \bar{E} \quad \text{and} \quad \|f\|_E = \int_0^\infty |f(s)| e^{(\bar{\mu}-\eta)s} ds, \quad f \in E,$$

where η is a constant such that $0 \leq \eta \bar{\mu}$.

Corollary III.2.9. *Let $(H_{\mu,\beta})$ be satisfied. Assume that $\int_0^\infty e^{\eta s} k(s)\, ds > 1$. Assume furthermore that the function k is absolutely continuous and satisfies $\frac{dk}{ds} \in L^\infty((0,\infty), \mathbb{R})$. Then ϕ given by (22) defines a bounded linear operator on $\bar{E}$, is associated with a translation semigroup $(T_\phi(t))_{t\geq 0}$ on $\bar{E}$ and the semigroup solution $(G(t))_{t\geq 0}$ of the Sharpe-Lotka model on E is given by (26). Furthermore there exists a positive projection P of rank 1 such that*

$$\|e^{-\lambda_0 t} G(t) f - Pf\|_{\bar{E}} \leq M e^{-\delta t} \|f\|_E, \quad f \in E,$$

for suitable constants $\delta > 0$, $M \geq 1$ and all $t \geq 0$, where λ_0 is the spectral bound of the generator A of $(G(t))_{t\geq 0}$. The projection P is given by (29).

Proof. The only new argument here, is the fact that

$$\omega(H(t)) = -\eta \, < \, \lambda_0 = s(A_\phi).$$

All the other results have the same proof as above.

III.3 Conclusion. For the cell proliferation model, the methods given in this paper allow us to use weak assumptions on the parameter functions of this model, to work in a Banach space of type L^P with p any value in $[1,\infty)$, and therefore to generalize the results obtained by Arino and Kimmel in [2]. We also recall that the study of the asymptotic behavior of such a model has been largely simplified and generalized by the use of properties of the class of translation semigroups which we have considered (see [1] for more details). This is also the case for our second example concerning the Sharpe-Lotka model which was also studied by Webb in [10]. Webb proved the same result on the asymptotic behavior for this model by using the notion of essential spectrum. In this work we first wanted to show how the study of this model can be related to our class of semigroups and secondly to simplify this study by using properties on this class. Indeed we showed in this paper that this study is simply reduced to the study of properties of the operator ϕ associated to this model and the family $(\bar{\phi}_\lambda)_\lambda$ defined from ϕ. This has also allowed us to give new results on this model, that its semigroup solution is essentially compact, and the spectral bound of the generator of this semigroup coincides with its growth constant. We also notice that the irreducibility of the semigroup $(T_\phi(t))_{t\geq 0}$ is not necessary to conclude this last result (see the proof of Proposition III.2.4 and [1] for more details). However, if we assume that the function β satisfies $\beta(s) > 0$, almost every $s \in (0, \infty)$, we obtain that the operator ϕ is irreducible and from [4, Proposition 3.7], that $(T_\phi(t))_{t\geq 0}$ is irreducible.

Acknowledgment. I would like to thank the referee for his helpful comments and suggestions on the presentation of this paper.

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