Differential and Integral Equations Volume 9, Number 2, March 1996, pp. 239 – 246

A COUNTEREXAMPLE ON COMPETING SPECIES EQUATIONS

E.N. Dancer
School of Mathematics and Statistics, University of Sydney, N.S.W. 2006, Australia

(Submitted by: Reza Aftabizadeh)

Abstract. We construct an example showing that there need not be a strictly positive solution of a competing species system with diffusion and Dirichlet boundary conditions in a case unresolved by earlier work.

In this paper we use the calculations in [9] to show that for certain parameter values the competing species system

$$\begin{aligned} -\Delta u &= u(a-u-bv) \quad \text{in } \Omega \\ -\Delta v &= v(d-v-cu) \quad \text{in } \Omega \\ u = v &= 0 \qquad\qquad\quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

has no nontrivial positive solution. Here Ω is a bounded domain in R^m with smooth boundary and by a positive solution, we mean a solution (u,v) with $u(x)>0$ and $v(x)>0$ on all of Ω. In particular, we resolve a case left open in the author's earlier paper, [2]. We need to explain the case in more detail. Assume that $a,d>\lambda_1$ where λ_1 denotes the smallest eigenvalue of $-\Delta$ under Dirichlet boundary conditions. (As in [2], the cases where $a\le\lambda_1$ or $d\le\lambda_1$ are easy to resolve.) Now, as in [2], there is a unique nontrivial nonnegative solution $(\overline{u},0)$ of (1) with second component zero such that $\overline{u}(x)>0$ on Ω. Moreover, there exists $\overline{c}>0$ such that $(\overline{u},0)$ is stable (as a solution of the natural corresponding parabolic system) if $c>\overline{c}$ and is unstable if $c<\overline{c}$. Note that for later purposes it is convenient to use the notation in [9] rather than [2]. Similarly there is a nontrivial nonnegative solution of the form $(0,\overline{v})$ and a $\overline{b}>0$ such that $(0,\overline{v})$ is stable if $b>\overline{b}$ and is unstable if $b<\overline{b}$. In [2], we obtained a good understanding of for which c and b (1) has a positive solution. However, one case that was left open was the case where $c=\overline{c}, b=\overline{b}$. It is easy to give examples where $c=\overline{c}$ and $b=\overline{b}$ and where there is a positive solution. (We choose $a=d$ and look for solutions where u and v are multiples of $\overline{u}$.) The main purpose of this paper is to show that for many smooth domains Ω (including some convex ones in all dimensions except two) there exist $a,d>\lambda_1$ such that (1) has no positive solution for $c=\overline{c}, b=\overline{b}$. This result is particularly interesting since there are two published proofs of the contrary result in the literature ([11], [12]). The errors in those proofs are incorrect degree calculations as in Lemma 4.5 in [12]. Note that it is easier to give counterexamples if we allow more general nonlinearities (as in [3]) or if we allow

Received for publication January 1995.
AMS Subject Classifications: 35J65, 92D25.

the coefficients to be space dependent as in [10]. Our counterexamples are for the original model. In addition, we also construct (for some domains) examples where $a \neq d$, $c = \overline{c}$, $b = \overline{b}$ but there is a positive solution of (1). Thus it seems complicated to decide when (1) has a positive solution in the case where $c = \overline{c}$, $b = \overline{b}$.

The main counterexample is constructed by a bifurcation theory argument when a, d are near λ_1. It depends on knowing the bifurcation equation to fifth order. The calculations needed come from [9]. I should like to thank Dr. Furter for providing some unpublished calculations of his and for correcting some minor errors in the calculations in [9].

It is necessary to have a copy of [9] available in reading this paper.

1. The counterexamples. We look for positive solutions of (1) when a, d are near λ_1. It is well known and easy to prove that if (u, v) is a positive solution of (1), then $u \leq \overline{u}$. Thus $\|u\|_\infty$ is small if a is near λ_1. Hence we see that positive solutions of (1) are small if a, d are near λ_1. Thus any positive solution of (1) for a, d near λ_1 must come from the bifurcation equation for solutions near $(0, 0)$.

As in [9], the bifurcation equation for small solutions is the pair of equations

$$\begin{aligned} 0 = \lambda x &- Mx(x+by) - Nx(2x^2 + b(c+3)xy + b(b+1)y^2) \\ &- Kx(5x^3 + b(c^2+4c+10)x^2y + 3b[(b+1)(c+1)+b]xy^2 \\ &+ b(b^2+2b+2)y^3) - Lx(2\lambda x^2 + b(3\lambda + c\mu)xy + b(b\lambda+\mu)y^2) \\ &- xf_5^1(x,y,\lambda,\mu) + \text{terms of sixth order (at least) in } x, y, \lambda, \mu. \end{aligned} \tag{2}$$

The second equation is the same as the first except that the roles of λ and μ, x and y and c and b are all interchanged. (The fifth-order term is $-yf_5^2(x, y, \lambda, \mu)$.) Here $\lambda = a - \lambda_1$, $\mu = d - \lambda_1$, $x = (u, \phi)$, $y = (v, \phi)$, formulae for M, N, K, L appear in (3.6) in [9], ϕ is the positive eigenfunction of -Δ corresponding to λ_1 normalized so that $\|\phi\|_2 = 1$ and (,) denotes the usual scalar product on $L^2(\Omega)$. In fact, we will only need certain terms in f_5^1. Note that the higher-order terms in (2) must have x as a factor because $u = 0$ solves the first equation of (1) whatever a, d, b, c, v are and that the right-hand side must be real analytic in all variables (simultaneously) because (1) is (in appropriate spaces). Henceforth, by smooth, we will mean real analytic.

There is one other special property of these bifurcation equations. If $a = d = \lambda_1$ and if $bc = 1$, it is easy to check that (1) has solutions $u = b\alpha\phi$, $v = -\alpha\phi$ for all α. Note that these are not positive solutions. Thus, in this case there is a curve of solutions with $u + bv = 0$. Hence we see that if $a = d = \lambda_1$ and $b = c^{-1}, (x + by)$ must be a factor of (2) and also of the second bifurcation equation. Note it is this which partially (but not totally) explains the unusual degeneracy of the bifurcation equations noted in [9].

We will look at the case where $bc = 1 + \delta$ where δ is small and positive and c is fixed ($\delta = -\epsilon$ in the notation of [9]). It is proved in [9] that for each small negative δ there is a $\mu(\delta)$ of the form

$$(-C)^{-\frac{1}{2}}(\delta)^{\frac{1}{2}} + \epsilon h(\delta^{\frac{1}{2}}), \tag{3}$$

where h is smooth, and a λ which is a smooth function of μ of the form

$$\lambda = \lambda(\mu) = b\mu - c^{-2}(c-1)NM^{-2}\mu^2 + \text{ higher order terms in } \mu \tag{4}$$

such that the two semi-trivial solutions $(x,0),(0,y)$ of the pair of bifurcation equations (corresponding to $(\overline{u},0)$ and $(0,\overline{v})$) are both degenerate solutions of the pair of bifurcation equations (degenerate in the sense that the linearization is not invertible). Here, as in [9], $C = 2c^{-2}(c-1)^2M^{-4}(N^2 - KM) < 0$. By the Liapounov-Schmidt reduction, it follows that $(\overline{u},0)$ and $(0,\overline{v})$ are degenerate solutions of (1). Now, as in [1], $(\overline{u},0)$ is degenerate if and only if the equation

$$\begin{aligned} -\Delta k = &(d - c\overline{u})k \quad \text{in } \Omega \\ K = &0 \qquad \text{on } \partial\,\Omega \end{aligned}$$

has a nontrivial solution. Since d is close to λ_1 for the above choice of coefficients, a simple comparison argument shows that k must not change sign and hence it follows from [2] that $c = \overline{c}$. Similarly $b = \overline{b}$.

Hence we see that it suffices to show that if $bc = 1 + \delta$ with δ positive and small, $\mu = \mu(\delta)$, $\lambda = \lambda(\mu) = \lambda(\mu(\delta))$, the pair of bifurcation equations have no small solutions with $x, y > 0$. By dividing the first bifurcation equation by x, the second by $b^{-1}y$ and subtracting, we obtain the equation

$$\begin{aligned} &- c^{-2}(c-1)NM^{-2}\mu(\delta)^2 + h.o.t. \text{ (in } \mu(\delta)) \\ &= -\delta x + Nb^{-1}(b-1)(x+by)^2 + K(b-1)b^{-1}(x+by)^2[(3b+1)x + b(b+3)y] \\ &\quad + (\lambda - b\mu)L(x+by)^2 + f_5^1(x,y,\lambda(\delta),\mu(\delta)) - bf_5^2(x,y,\lambda(\delta),\mu(\delta)) \\ &\qquad + \text{ terms which are of order five at least (in } x,y,\lambda,\mu) \\ &\qquad + \text{terms of the form } \delta q(x,y,\lambda,\mu). \end{aligned} \tag{5}$$

Here $\lambda = \lambda(\delta)$, $\mu = \mu(\delta)$ and all terms in q are at least quadratic in x and y. Here we have used the formula for $\lambda(\mu)$ and that $bc = 1 + \delta$. Before considering this equation further, we need to consider (2) more carefully. Since $x, b, y > 0$, all the latter terms on the right-hand side of (2) are dominated by the first. Hence we see that $x + by \le K\lambda(\delta)$. Since $x, y \ge 0$ and by the formula for $\lambda(\delta)$, we see that we can write $x = X\delta^{\frac{1}{2}}, y = Y\delta^{\frac{1}{2}}$ where X and Y are bounded. Substituting this in (2) and by using the formula for $\lambda(\delta)$ obtaining by substituting (3) in (4), we obtain

$$\begin{aligned} A_1 + A_2\delta^{\frac{1}{2}} + h.o.t. = M(X+bY) + \delta^{\frac{1}{2}}N(X+bY)(2X + (b+1)Y) \\ + \delta K(X+bY)\widetilde{Q}(X,Y) + 0(\delta^{\frac{3}{2}}), \end{aligned} \tag{6}$$

where $A_1 = c^{-2}(c-1)NM^{-2}C^{-1}$. Note that our earlier general theory ensures that $X + bY$ is a factor of the cubic term. Everything here is a smooth function of $\delta^{\frac{1}{2}}, X, Y$. Hence we can use the implicit function theorem to solve for $X + bY$ as a smooth function of X and $\delta^{\frac{1}{2}}$ and find

$$X + bY = M^{-1}A_1 + \delta^{\frac{1}{2}}g_1(X) + \delta g_2(X) + 0(\delta^{\frac{3}{2}}). \tag{7}$$

(Note that there can be no higher order terms in $\delta^{\frac{1}{2}}$ and X except the first which are independent of δ because for $\delta = 0$ all the terms on the right-hand side of (6) except the first are zero.) By substituting (7) in (6), we see by some easy calculations that $g_1(X)$ is affine in X and $g_2(X)$ is quadratic in X. Note that by quadratic in X we mean at most quadratic in X. For future reference, note that it follows from (7) that if we expand $(X + bY)^2$ as a function of $\delta^{\frac{1}{2}}$ and X that the $\delta^{\frac{1}{2}}$ terms are affine in X. This is useful a little later.

We now substitute our formulae for x, y in (5). We see that

$$\begin{aligned} A_2\delta + A_3\delta^{\frac{3}{2}} + A_4\delta^2 + h.o.t. \\ = -\delta^{\frac{3}{2}}X + \delta N b^{-1}(b-1)(X+bY)^2 \\ + \delta^{\frac{3}{2}} K b^{-1}(b-1)(X+bY)^2[(3b+1)X + b(b+3)Y] \\ + \delta L(\lambda(\delta) - b\mu(\delta))(X+bY)^2 \\ + \delta^2(f_5^1(X,Y,0,0) - bf_5^2(X,Y,0,0)) + \delta^2 q^2(X,Y) + o(\delta^2). \end{aligned} \tag{8}$$

Here the term $q^2(X,Y)$ is the quadratic term in $q(X,Y,0,0)$. We can use (7) to eliminate Y and obtain an equation involving X and δ. We consider the terms in $\delta^1, \delta^{\frac{3}{2}}, \delta^2$ when this is done.

The δ terms give

$$A_2 = Nb^{-1}(b-1)(A_1 M^{-1})^2$$

and it is easy to check that this is an identity. (There are also general reasons below why it must be an identity.) For the $\delta^{\frac{3}{2}}$ terms, these will be of the form

$$A_3 = -X + C_1 + C_2, \tag{9}$$

where C_1 is the coefficient of the $\delta^{\frac{1}{2}}$ term in $Nb^{-1}(b-1)(X+bY)^2$ (when we eliminate Y) while C_2 is the constant term (in δ) in $Kb^{-1}(b-1)(X+bY)^2[(3b+1)X + b(b+3)Y]$ when we eliminate Y. Note that the fourth term does not contribute because $\lambda(\delta) - b\mu(\delta)$ is $0(\mu(\delta))^2 = 0(\delta)$. Since the $\delta^{\frac{1}{2}}$ terms in $(X + bY)^2$ are affine in X, C_1 is affine in X while C_2 is affine in X. Hence (9) is an equation affine in X. However it must have two solutions $(\overline{X}, 0), (0, \overline{Y})$ corresponding the two semi-trivial solutions and so it must vanish identically. That $(\overline{X}, 0)$ and $(0, \overline{Y})$ are solutions of (8) needs some explanation. $(\overline{u}, 0)$ and $(0, \overline{v})$ are degenerate solutions of (1) (because $b = \overline{b}$ and $c = \overline{c}$) and thus after the Liapounov-Schmidt reduction they must give degenerate solutions of the bifurcation equations. It then follows easily that they must still be solutions of the equations obtained when x and y are divided from the two equations respectively.

Now consider the δ^2 terms of (8). We first need to find the δ terms in $(X+bY)^2$. By (7) and since g_1 is affine in X and since g_2 is quadratic in X, one easily finds that the δ terms in $(X + bY)^2$ are quadratic in X. Thus the second term on the right-hand side of (8) gives a δ^2 term which is quadratic in X. (It is easy to see the first term gives no contribution.) For the third term, we need to find the $\delta^{\frac{1}{2}}$ terms in $(X+bY)^2[(3b+1)X+b(b+1)Y]$. By using (7) (up to $\delta^{\frac{1}{2}}$ terms) and noting that $g_1(X)$ is affine, one easily sees that we only obtain quadratic terms (in X) in the $\delta^{\frac{1}{2}}$ term here. (We write $(3b+1)X+b(b+1)Y=(b+1)(X+bY)+2bX$.) For the fourth term, we note that since $\lambda(\delta)-b\mu(\delta)\sim\delta$ we want the constant term (in δ) in $(X+bY)^2$ and this is constant. We can use a similar argument for the term $\delta^2 q^2(X,Y)$ (and obtain quadratic terms). The remaining term we must find is the constant term (in δ) in $f_5^1(X,Y,0,0)-bf_5^2(X,Y,0,0)$ when $bc=1$. By our earlier remarks each of the terms here will be $(X+bY)$ times a cubic term and hence if we look for terms independent of δ it can be at most cubic in X. (It is possible to show that the cubic term is zero.) Hence we see that δ^2 term in (8) is a cubic polynomial $Q(X)$ (possibly degenerate).

We use this to prove that if $R\neq 0$ where R is defined on page 112 of [9], then for small positive δ, (1) has no positive solution. In this case, we will prove below that Q has two simple roots corresponding to the zeros $(\overline{X},0)$ and $(0,\overline{Y})$ of (2) (and the other corresponding equation). Since Q is cubic it follows that Q has at most one other root τ and τ is simple. If τ exists, we see by the implicit function theorem that there must be a unique zero of (8) near τ for small positive δ which is nondegenerate and a unique zero of (8) near $(\overline{X},0)$ and $(0,\overline{Y})$. By reversing our previous arguments (i.e. our reductions), it follows that in this case, (1) has either $(0,0),(\overline{u},0)$ and $(0,\overline{v})$ as its only nonnegative solutions or it has these three solutions plus exactly one other positive solution (r,s) which is nondegenerate. (Note that I do not claim that the solution corresponding to τ is necessarily positive.) In the former case, we are finished. Consider the latter case. By Theorem 1 in [4], (r,s) will have index $\pm$ 1 in the natural cone $K=K_1\oplus K_1$ where K_1 is the natural in $C_o(\Omega)$ when it exists. Here the index is for the natural mapping A defined in [2]. On the other hand, it is proved in [2] that the sum of the indices of the positive solutions is 1 and $(0,0)$ has index 0. Hence the sum of the indices of $(\overline{u},0)$ and $(0,\overline{v})$ can only be 0 or 2. Hence we will have constructed our example if we prove the sum of the indices of $(\overline{u},0)$ and $(0,\overline{v})$ is 1.

We still have to justify two claims in the previous paragraph. Firstly, we will evaluate the indices of $(\overline{u},0)$ and $(0,\overline{v})$. We assume $R<0$ and $c>1$. The other cases are similar. It is proved in [9] (see Theorem 3.5, Figure 6 and Appendix A3) that as we change d near $\lambda_1+\mu(\delta)$ there is a bifurcation of positive solutions near $(0,\overline{v})$ but we only have positive solutions for $d<\lambda_1+\mu(\delta)$. Hence by homotopy invariance, the index of $(0,\overline{v})$ must stay unchanged as we increase d above $\lambda_1+\mu(\delta)$. If $d>\lambda_1+\mu(\delta)$, it is well known (cf. [2] or Figure 6 in [9]) that $(0,\overline{v})$ is unstable and thus has index 0 (by [4]). Hence the index $(0,\overline{v})$ is also zero in our case. Similarly, if we increase d above $\lambda_1+\mu(\delta)$ there are no positive solutions near $(\overline{u},0)$. Moreover, $(\overline{u},0)$ becomes stable and thus $(\overline{u},0)$ will have index 1 for $\mu>\lambda_1+\mu(\delta)$ and hence for $d=\lambda_1+\mu(\delta)$. This justifies our second claim of the previous paragraph.

To justify the first claim is more tedious. We use Appendix A3 of [9]. Let $(\widetilde{x},0)$ denote the solution of the bifurcation equations ((2) and the corresponding second equation) with second component zero. They show that the order of vanishing of (8) in x at $(\widetilde{x},0)$ has the same order of magnitude as that of $\mathrm{Jac}_1(\hat{\mu})$ where Jac_1 is the Jacobian defined in [9]. (It is basically the Jacobian of p and q at $(\widetilde{x},0)$

where the original bifurcation equations (2) and its analogue are written in the form $\lambda x - xp = 0$, $\mu y - yq = 0$.) Note that $\hat{\mu}(-\delta)$ in [9] is the same as $\mu(\delta)$ here. There are some slight changes due to rescalings. We discuss these in a moment. While $\mathrm{Jac}_1(\hat{\mu})$ can be calculated from the formula for $\mathrm{Jac}_1(\mu)$ in Appendix A3 in [9], it is easier to proceed indirectly. Firstly, by definition, $\mathrm{Jac}_1(\hat{\mu}_{1,0}(-\delta)) = 0$ where $\mu_{1,0}(-\delta)$ is defined in 3.3 of [9] and $\hat{\mu}_{1,0}(-\delta) \sim \delta^{\frac{1}{2}}$ for small δ. On the other hand we can differentiate the formula for $\mathrm{Jac}_1(\mu)$ in A3 of [9] and find that $\partial/\partial\mu\ \mathrm{Jac}_1(\mu) \sim \delta^{\frac{1}{2}}$ when $\mu \sim \delta^{\frac{1}{2}}$. (Note that $\mathrm{Jac}_1(\mu)$ is smooth in μ and so the differentiation can be justified.) Since $\hat{\mu}(-\delta) - \mu_{1,0}(-\delta) \sim R\delta << \delta^{\frac{1}{2}}$ it follows that

$$\mathrm{Jac}_1(\hat{\mu}(-\delta)) \sim \delta^{\frac{1}{2}}(\hat{\mu}(-\delta) - \mu_{1,0}(-\delta)) \sim R\delta^{\frac{3}{2}}.$$

In our derivation of (8), we have rescaled variables by $\delta^{\frac{1}{2}}$ and divided one equation by $\delta^{\frac{1}{2}}$. Thus our Jacobian is really $\sim R\delta^2$. (Note that all Liapounov-Schmidt-type reductions give equivalent answers.) Thus by our remarks above, the bifurcation equation (8) at $(\overline{X}, 0)$ has order of vanishing in $X \sim R\delta^2$. Hence we see that $Q'(\overline{X}) \neq 0$. Similarly $Q'(0) \neq 0$. This completes the construction of our example.

Remarks. The assumption that $R \neq 0$ is crucial. Otherwise our argument shows that the cubic Q has two degenerate roots and hence vanishes identically. Hence, if $R = 0$, we would need to use higher-order terms.

We do not know an example where $R = 0$. We suspect that it is not difficult to prove that $R \neq 0$ for generic domains Ω. Note that it is shown in [9] that $R \neq 0$ for an interval in R. (It is easy to prove that whether or not $R = 0$ is unchanged by translations or dilations of Ω.) R could also be explicitly calculated for higher-dimensional cubes. We briefly sketch a proof that there exist domains C^2 close to balls (and thus strictly convex domains) where $R \neq 0$ if $n \neq 2$. First note that the theory of domain variation (cf. [5] or [8]) implies that the condition $R \neq 0$ persists under very general small perturbations of Ω. Firstly, we see that $R \neq 0$ for a very thin annulus $\{x : 1 - \delta < \|x\| < 1\}$. To see this, first note that in this case all the functions appearing in the definition of R will be radically symmetric. Thus all the equations appearing in the definition of R are ordinary differential equations. Moreover, as in [7], if $\{r : 1 - \delta < r < 1\}$ is rescaled to the unit interval, the ordinary differential operator appearing in the definition of R for the thin annulus is close to that for the set $\Omega = [0, 1]$. Hence we see that (after the rescaling) R for the thin annulus must be close to that for the interval and hence is nonzero (because each of the terms in the definition of R is only changed slightly). We then modify the thin annulus by deleting $\{x : 1 - \delta < \|x\| < 1, \sum_{i=2}^n x_i^2 < \delta'\}$ where δ' is small (compared with δ). If $n \geq 3$, this is a domain perturbation of the type we discussed in earlier work (cp. [4] or [5]) and hence if δ' is small, R will be nonzero for this new domain D_1. Note that D_1 is contractible. If we modify this new domain by rounding of the corners, we obtain a smooth domain $\widetilde{D}$ with $R \neq 0$. It is easy but tedious to define a smooth map $H : \overline{B} \times [0, 1] \longrightarrow R^n$ (where B is the unit ball) such that $H(\cdot, 0) = I$, H_t is a diffeomorphism for each t and $H(B \times \{1\}) = \widetilde{D}$. By using standard smoothing results (cf. Narasimhan [13]) we can approximate H by a real analytic map at the expense of perturbing $\widetilde{D}$ slightly. (The new $\widetilde{D}$ will have $R \neq 0$.) Define $D_t = H(B \times \{t\})$ and let $R(t)$ be R for the domain D_t. Then $R(t)$ is real analytic in t. This is easy but tedious. One uses the ideas in Saut and Teman ([14]) to locally reduce to a fixed-domain problem depending real analytically on t and one also uses that the first eigenvalue and eigenfunction depend real analytically upon t because the first eigenvalue is simple (cp. Crandall and Rabinowitz, [1]). Since $R(t)$ is real analytic in t and $R(1) \neq 0$, there exist t's arbitrarily close to zero where $R(t) \neq 0$. This proves our claim. If $n = 2$, one can probably obtain a similar example starting from the square.

Remark. It appears unlikely that this method can be used to give examples where $(\overline{u}, 0)$ and $(0, \overline{v})$ are nondegenerate and unstable but (1) has a nonunique positive solution unless one calculates higher-order terms.

Lastly we show very briefly how domain variation methods give examples where $b = \overline{b}$, $c = \overline{c}$, $(\overline{u}, 0)$ and $(0, \overline{v})$ are isolated nonnegative solutions and there is a positive solution. This is valid if $n \geq 2$. This is a minor variant of arguments in Section 4 of [2]. We have changed the notation from [2] slightly by using b and c rather than c and e. Thus $\overline{b}$ and $\overline{c}$ are the analogues of $\overline{c}$ and $\overline{e}$ there. In Section 4 of [2] we constructed an example where $f_1(\overline{b}) < 0$ and $f_2(\overline{c}) < 0$. Here f_1 and f_2 are defined in [2]. By a direction of bifurcation argument (cp. near the end of the construction of our first counterexample here or page 843 of [2]) $(\overline{u}, 0)$ and $(0, \overline{v})$ are isolated as nonnegative solutions and each has index 0 (for the map A of [2] on the natural cone K in $C_o(\Omega) \oplus C_o(\Omega)$). A simple degree argument as in Section 1 of [2] then ensures that (1) has a positive solution (since $(0, 0)$ has index zero and the sum of the indices of the nonnegative solutions is 1). This completes the construction of this example.

With a little care, one can deduce that both our counterexamples persist for perturbations of a and d (but keeping $b = \overline{b}$ and $c = \overline{c}$). Thus both counterexamples occur for a relatively open set of parameters.

REFERENCES

[1] M. Crandall and P. Rabinowitz, *Bifurcation, perturbation of simple eigenvalues and linearized stability,* Archives Rat. Mech. Anal., 52 (1973), 161–180.

[2] E.N. Dancer, *The existence and uniqueness of positive solutions for competing species models with diffusion,* Trans. Amer. Math. Soc., 326 (1991), 829–859.

[3] E.N. Dancer, *Counterexamples to some conjectures on the number of solutions of nonlinear equations,* Math. Ann., 272 (1985), 421–440.

[4] E.N. Dancer, *On the indices of fixed points of mappings in cones and applications,* J. Math. Anal. Appl., 91 (1983), 131–151.

[5] E.N. Dancer, *The effect of domain shape on the number of positive solutions of certain nonlinear equations,* J. Diff Eqns., 74 (1988), 120–156.

[6] E.N. Dancer, *On positive solutions of some pairs of differential equations II,* J. Diff. Eqns., 66 (1985), 236–258.

[7] E.N. Dancer, *Global breaking of symmetry of positive solutions on two-dimensional annuli,* Diff. Integral Eqns., 5 (1992), 903–914.

[8] E.N. Dancer, *A note on an equation with critical exponent,* Bull. London Math. Soc., 20 (1988), 600–602.

[9] J. Eilbeck, J. Furter, and J. Lopez-Gomez, *Coexistence in a competition model with diffusion,* J. Diff Eqns., 107 (1994), 96–139.

[10] J. Furter and J. Lopez-Gomez, *On the existence and uniqueness of coexistent states for the Lotka-Volterra competition model with diffusion and spatial dependent coefficients,* preprint.

[11] Lige Li and R. Logan, *Positive solutions to general elliptic Lotka-Volterra models,* Diff. Integral Eqns., 4 (1991), 817–834.

[12] R. Logan and A. Ghoreishi, *Positive solutions of a class of a biological models in a heterogeneous environment,* Bull. Austral. Math. Soc., 44 (1991), 79–94.

[13] R. Narasimhan, “Analysis on Real and Complex Manifolds,” North Holland, Amsterdam, 1985.

[14] J. Saut and R. Temam, *Generic properties of nonlinear boundary value problems,* Comm. Partial Diff. Eqns., 4 (1979), 293–319.