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SHARP SOBOLEV INTERPOLATION INEQUALITIES FOR THE STOKES OPERATOR

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Abstract. We prove Sobolev interpolation inequalities for the Stokes operator, giving sharp L^∞ estimates for solenoidal vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$. We use fundamental solutions of a generalized Stokes system in the proofs.

1. Statement of main results. Sobolev interpolation inequalities for the Stokes operator are useful in estimating the nonlinear terms in the Navier-Stokes equations and bounding the eigenvalues of the Stokes operator. See, for example, Heywood and Rannacher [2], Temam [5], Constantin and Foias [1]. In this paper, we derive some of them in $\mathbb{R}^2$ and $\mathbb{R}^3$, with sharp constants, and discuss the possibility that these results also hold in arbitrary two- or three-dimensional domains.

Notations. Let Ω denote an open set in $\mathbb{R}^n$, $n = 2, 3$. Let $\mathcal{C}_0^\infty(\Omega)$ denote the set of all smooth solenoidal vector fields with compact support in Ω. Let $\mathcal{L}^2(\Omega)$ and $\mathcal{H}_0^1(\Omega)$ respectively denote the L^2 and H^1 closure of $\mathcal{C}_0^\infty(\Omega)$. When $n = 3$, let $\hat{\mathcal{H}}_0^1(\Omega)$ denote the completion of $\mathcal{C}_0^\infty(\Omega)$ in the Dirichlet norm. The spaces $\mathcal{H}_0^1(\Omega)$ and $\hat{\mathcal{H}}_0^1(\Omega)$ consist of solenoidal vector functions that vanish on the boundary in a generalized sense, the latter being less restrictive in that its members need not necessarily be square integrable.

Let ∇ denote the gradient and Δ the Laplace operator. Following Ladyzhenskaya [4], we denote the Stokes operator by $\widetilde{\Delta}$. In this paper, $\widetilde{\Delta}$ is defined as $\Pi\Delta$, where Π denotes the orthogonal projection of $L^2(\Omega)$ onto the subspace $\mathcal{L}^2(\Omega)$. We note that functions orthogonal to $\mathcal{L}^2(\Omega)$ are in the form of ∇p for some $p \in H^1_{loc}(\Omega)$.

Let $\|\cdot\|$ and $\|\cdot\|_\infty$ respectively denote the L^2 and L^∞ norms over Ω. Let $(\cdot,\cdot)$ denote the L^2 inner product. Let K_0 denote the modified Bessel function of the third kind of order zero.

Our main results are the following.

Theorem 1. *If* $\mathbf{u} \in \mathcal{H}_0^1(\mathbb{R}^2)$ *and* $\widetilde{\Delta}\mathbf{u} \in \mathcal{L}^2(\mathbb{R}^2)$, *then*

$$\|\mathbf{u}\|_\infty \le \frac{1}{2\sqrt{\pi}} \left(\|\mathbf{u}\| \|\widetilde{\Delta}\mathbf{u}\| + \|\nabla\mathbf{u}\|^2 \right)^{1/2}. \tag{1}$$

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Equality holds if and only if

$$\mathbf{u}(x) = (\Delta - \nabla \mathrm{div}) \frac{\partial}{\partial \mu} \frac{K_0(\sqrt{\mu}|x - x_0|) + \ln |x - x_0|}{\mu} \mathbf{c}, \tag{2}$$

for arbitrary $x_0 \in \mathbb{R}^2$, $\mu > 0$ *and constant vector* $\mathbf{c}$.

Theorem 2. *If* $\mathbf{u} \in \hat{\mathcal{H}}_0^1(\mathbb{R}^3)$ *and* $\widetilde{\Delta}\mathbf{u} \in \mathcal{L}^2(\mathbb{R}^3)$, *then*

$$\|\mathbf{u}\|_\infty \le \frac{1}{\sqrt{3\pi}} \|\nabla \mathbf{u}\|^{1/2} \|\widetilde{\Delta}\mathbf{u}\|^{1/2}. \tag{3}$$

Equality holds if and only if

$$\mathbf{u}(x) = (\Delta - \nabla \mathrm{div}) \left(\frac{e^{-\sqrt{\mu}|x - x_0|} - 1}{\mu |x - x_0|} - \frac{|x - x_0|}{2} \right) \mathbf{c}, \tag{4}$$

for arbitrary $x_0 \in \mathbb{R}^3$, $\mu > 0$ *and constant vector* $\mathbf{c}$.

Corollary. *If* $n = 2$ *or* 3, $\mathbf{u} \in \mathcal{H}_0^1(\mathbb{R}^n)$ *and* $\widetilde{\Delta}\mathbf{u} \in \mathcal{L}^2(\mathbb{R}^n)$, *then*

$$\|\mathbf{u}\|_\infty \le \frac{1}{\sqrt{n\pi}} \|\mathbf{u}\|^{1-n/4} \|\widetilde{\Delta}\mathbf{u}\|^{n/4}. \tag{5}$$

The Theorems are proved by using integral representations of $\mathbf{u}$ by the fundamental solutions of a generalized Stokes system. We introduce these fundamental solutions and calculate some integrals of them in Section 2. Then we use the results to prove the Theorems and the Corollary in Section 3.

Arbitrary domains in plane and space are considered in Section 4. The Green's functions of the generalized Stokes system with Dirichlet boundary condition in an arbitrary open set Ω are constructed from the fundamental solutions. We make a simple conjecture that the integrals of the Green's functions are not larger than those corresponding to the fundamental solutions found in Section 2. If the conjecture is proven, then the inequalities (1), (2) and (3) above hold for Ω, by the same proofs of Section 3.

Analogous inequalities for the Laplacian have already been proven for arbitrary domains (Xie [7]), by using the maximum principle to estimate the Green's functions. The inequalities are used to establish existence and regularity theorems for the vector Burgers equation, which is very similar to the Navier-Stokes equations, in nonsmooth domains (Heywood and Xie [3]).

2. Fundamental solutions of a generalized Stokes system. Let $g_\mu(x, x_0)$ be the fundamental solutions of the Helmholtz equation in $\mathbb{R}^n$,

$$(-\Delta + \mu) g_\mu = \delta(x - x_0),$$

where $x_0 \in \mathbb{R}^n$ and $\mu \geq 0$ are fixed, and δ is the Dirac distribution. It is well known that

$$g_0 = -\frac{1}{2\pi} \ln |x - x_0| \qquad (n = 2,\ \mu = 0), \tag{6}$$

$$g_\mu = \frac{1}{2\pi} K_0(\sqrt{\mu}|x - x_0|) \quad (n = 2,\ \mu > 0), \tag{7}$$

$$g_\mu = \frac{e^{-\sqrt{\mu}|x-x_0|}}{4\pi|x - x_0|} \qquad (n = 3,\ \mu \geq 0). \tag{8}$$

When $\mu > 0$ we have

$$\|g_\mu\|^2 = \frac{1}{4\pi\mu} \qquad (n = 2), \tag{9}$$

$$\|g_\mu\|^2 = \frac{1}{8\pi\sqrt{\mu}} \quad (n = 3). \tag{10}$$

The proof of (9) can be found in Xie [7]. Equation (10) is easily obtained from (8) by integration.

Let ϕ_μ satisfy

$$\Delta\phi_\mu = g_\mu. \tag{11}$$

For $\mu > 0$, we have

$$\phi_\mu = \frac{g_\mu - g_0}{\mu}. \tag{12}$$

For $\mu = 0$ and $n = 3$, we have

$$\phi_0 = \frac{|x - x_0|}{8\pi}. \tag{13}$$

Let $1 \leq k \leq n$ and let $\mathbf{e}^k$ be the unit vector in the x_k direction. Let

$$\mathbf{U}_\mu = \Delta\phi_\mu \mathbf{e}^k - \nabla\frac{\partial\phi_\mu}{\partial x_k}, \tag{14}$$

$$P_\mu = (-\Delta + \mu)\frac{\partial\phi_\mu}{\partial x_k}. \tag{15}$$

It is easy to verify that they are the fundamental solutions of the generalized Stokes system

$$(-\Delta + \mu)\mathbf{U}_\mu + \nabla P_\mu = \delta(x - x_0)\mathbf{e}^k, \tag{16}$$

$$\text{div } \mathbf{U}_\mu = 0. \tag{17}$$

As an example we display the formulas in the case $n = 3$. Let $\mathbf{r}$ denote the vector with components $r_k = x_k - x_{0k}$. Then

$$\mathbf{U}_\mu = \frac{1}{4\pi r}\left(a(\sqrt{\mu}r)\mathbf{e}^k + b(\sqrt{\mu}r)\frac{r_k\mathbf{r}}{r^2}\right), \quad P_\mu = -\frac{r_k}{4\pi r^3},$$

where

$$a(s) = \frac{1 + s + s^2 - e^s}{s^2 e^s} \quad (s > 0), \quad a(0) = \frac{1}{2},$$
$$b(s) = \frac{3e^s - 3 - 3s - s^2}{s^2 e^s} \quad (s > 0), \quad b(0) = \frac{1}{2}.$$

Lemma. *For $\mu > 0$, we have*

$$\|\mathbf{U}_\mu\|^2 = \frac{1}{8\pi\mu} \quad (n = 2), \tag{18}$$

$$\|\mathbf{U}_\mu\|^2 = \frac{1}{12\pi\sqrt{\mu}} \quad (n = 3), \tag{19}$$

$$\|\nabla(\mathbf{U}_\mu - \mathbf{U}_0)\|^2 = \frac{\sqrt{\mu}}{12\pi} \quad (n = 3). \tag{20}$$

Proof. From (14) we have

$$\begin{aligned}
\|\mathbf{U}_\mu\|^2 &= \left(\Delta\phi_\mu \mathbf{e}^k - \nabla\frac{\partial\phi_\mu}{\partial x_k},\ \Delta\phi_\mu \mathbf{e}^k - \nabla\frac{\partial\phi_\mu}{\partial x_k}\right) \\
&= \|\Delta\phi_\mu\|^2 - 2\left(\Delta\phi_\mu,\ \frac{\partial^2\phi_\mu}{\partial x_k^2}\right) + \left(\nabla\frac{\partial\phi_\mu}{\partial x_k},\ \nabla\frac{\partial\phi_\mu}{\partial x_k}\right) \\
&= \|\Delta\phi_\mu\|^2 - \left(\Delta\phi_\mu,\ \frac{\partial^2\phi_\mu}{\partial x_k^2}\right).
\end{aligned}$$

Summing with respect to k and using (11), we obtain

$$\sum_{k=1}^{n} \|\mathbf{U}_\mu\|^2 = n\|\Delta\phi_\mu\|^2 - (\Delta\phi_\mu,\ \Delta\phi_\mu) = (n-1)\|g_\mu\|^2.$$

The right side is independent of k. Therefore, by symmetry, we have

$$\|\mathbf{U}_\mu\|^2 = \frac{n-1}{n}\|g_\mu\|^2.$$

Hence we obtain (18) and (19) respectively from (9) and (10).

When $n = 3$, we have, by a similar argument,

$$\|\nabla(\mathbf{U}_\mu - \mathbf{U}_0)\|^2 = \frac{2}{3}\|\nabla(g_\mu - g_0)\|^2.$$

We obtain (20) since

$$\|\nabla(g_\mu - g_0)\|^2 = -(g_\mu - g_0, \Delta(g_\mu - g_0)) = (g_0 - g_\mu, \mu g_\mu) = \frac{\sqrt{\mu}}{8\pi},$$

where the last step follows from (8) by integration. This completes the proof of the lemma.

3. Proof of the main results.

Proof of Theorem 1. By a well known Sobolev imbeding theorem, $\mathbf{u}$ is equivalent to a continuous function on Ω, so it makes sense to consider its pointwise value at any $x_0 \in \mathbb{R}^2$. It is easy to prove the representation formula,

$$u_k(x_0) = \left(\mathbf{U}_\mu,\ (-\widetilde{\Delta} + \mu)\mathbf{u}\right),$$

where $\mu > 0$ is arbitrary. By the Schwarz inequality, (18) and integration by parts, we have

$$u_k^2(x_0) \le \|\mathbf{U}_\mu\|^2\ \|\widetilde{\Delta}\mathbf{u} - \mu\mathbf{u}\|^2 = \frac{1}{8\pi\mu}\left(\|\widetilde{\Delta}\mathbf{u}\|^2 + 2\mu\|\nabla\mathbf{u}\|^2 + \mu^2\|\mathbf{u}\|^2\right).$$

Minimizing the bound, we obtain

$$u_k^2(x_0) \le \frac{1}{4\pi}\left(\|\mathbf{u}\|\|\widetilde{\Delta}\mathbf{u}\| + \|\nabla\mathbf{u}\|^2\right). \tag{21}$$

Since x_0 is arbitrary and any direction can be taken as the x_k direction, Inequality (1) is obtained.

The Schwarz inequality achieves equality if and only if

$$(-\widetilde{\Delta} + \mu)\mathbf{u} = c\mathbf{U}_\mu, \tag{22}$$

for some constant c. Since $\mathbf{u}$ is supposed to be solenoidal, it satisfies the system

$$\begin{aligned} (-\Delta + \mu)\mathbf{u} + \nabla p &= c\mathbf{U}_\mu, \\ \text{div } \mathbf{u} &= 0, \end{aligned}$$

for some p. But differentiating (16) and (17) with respect to μ yields

$$\begin{aligned} (-\Delta + \mu)\frac{\partial \mathbf{U}_\mu}{\partial \mu} + \nabla\frac{\partial P_\mu}{\partial \mu} &= -\mathbf{U}_\mu, \\ \text{div } \frac{\partial \mathbf{U}_\mu}{\partial \mu} &= 0. \end{aligned}$$

Therefore, by the uniqueness of the solution to the system (which is easy to prove), we have

$$\mathbf{u} = -c\frac{\partial \mathbf{U}_\mu}{\partial \mu}. \tag{23}$$

Let $\mathbf{u}$ be given as in (23) with an arbitrary $\mu > 0$. Then $u_k^2(x_0)$ is not less than the minimized bound in (21). Hence it must be equal to that bound. In view of the formulas (14), (12), (6) and (7), such $\mathbf{u}$ are those given in (2).

Proof of Theorem 2. The proof is similar to that of Theorem 1. We have the representation formula

$$u_k(x_0) = \left(\mathbf{U}_0,\ -\widetilde{\Delta}\mathbf{u}\right).$$

Integrating by parts, we have

$$u_k(x_0) = \left(\mathbf{U}_\mu - \mathbf{U}_0,\ \widetilde{\Delta}\mathbf{u}\right) - \left(\mathbf{U}_\mu,\ \widetilde{\Delta}\mathbf{u}\right) = -\left(\nabla(\mathbf{U}_\mu - \mathbf{U}_0),\ \nabla\mathbf{u}\right) - \left(\mathbf{U}_\mu,\ \widetilde{\Delta}\mathbf{u}\right).$$

By the Schwarz inequality, (19) and (20), we obtain

$$\begin{aligned} |u_k(x_0)| &\le \|\nabla(\mathbf{U}_\mu - \mathbf{U}_0)\|\|\nabla\mathbf{u}\| + \|\mathbf{U}_\mu\|\|\widetilde{\Delta}\mathbf{u}\| \\ &= \frac{1}{2\sqrt{3\pi}}\left(\mu^{1/4}\|\nabla\mathbf{u}\| + \mu^{-1/4}\|\widetilde{\Delta}\mathbf{u}\|\right). \end{aligned}$$

Minimizing the bound, we obtain

$$|u_k(x_0)| \le \frac{1}{\sqrt{3\pi}}\|\nabla\mathbf{u}\|^{1/2}\|\widetilde{\Delta}\mathbf{u}\|^{1/2}.$$

This yields (3).

The Schwarz inequality for the first integral achieves equality if and only if

$$\mathbf{u} = c(\mathbf{U}_\mu - \mathbf{U}_0) \tag{24}$$

for some constant c. If $\mathbf{u}$ is given by (24) with an arbitrary $\mu > 0$, then we have

$$\widetilde{\Delta}\mathbf{u} = c\mu\mathbf{U}_\mu,$$

so that the Schwarz inequality for the second integral also becomes an equality. Therefore the equality in (3) must hold for $\mathbf{u}$. Such functions are those given by (4), in view of the formulas (14), (12), (13) and (8).

Proof of the Corollary. The Corollary follows from the Theorems by using integration by parts and the Schwarz inequality,

$$\|\nabla\mathbf{u}\|^2 = -(\mathbf{u}, \widetilde{\Delta}\mathbf{u}) \le \|\mathbf{u}\|\|\widetilde{\Delta}\mathbf{u}\|.$$

4. Arbitrary domains. In an arbitrary open set Ω in $\mathbb{R}^n$, the formulas (14) and (15) of the fundamental solutions provide a simple way to construct Green's functions of the generalized Stokes system with Dirichlet boundary condition.

Given any $x_0 \in \Omega$, let $\theta \in C_0^\infty(\Omega)$ be such that $\theta = 1$ in a neighborhood of x_0. Let $\mathbf{V}$, Q be the solution of

$$\begin{aligned} (-\Delta + \mu)\mathbf{V} + \nabla Q = (-\Delta + \mu)\Delta\left((1-\theta)\phi_\mu\right)\mathbf{e}^k \quad \text{in } \Omega, \\ \mathbf{V} \in \mathcal{H}_0^1(\Omega) \quad \text{if } \mu > 0, \\ \mathbf{V} \in \hat{\mathcal{H}}_0^1(\Omega) \quad \text{if } \mu = 0 \text{ and } n = 3. \end{aligned}$$

It is easy to prove that the solution exists uniquely. Then one can verify that

$$\begin{aligned} \mathbf{U}_{\mu,\Omega} &= \Delta(\theta\phi_\mu)\mathbf{e}^k - \nabla\frac{\partial}{\partial x_k}(\theta\phi_\mu) + \mathbf{V}, \\ P_{\mu,\Omega} &= (-\Delta + \mu)\frac{\partial}{\partial x_k}(\theta\phi_\mu) + Q, \end{aligned}$$

are the Dirichlet Green's functions on Ω.

Conjecture. *For $\mathbf{U}_{\mu,\Omega}$, the equalities* (18), (19) *and* (20) *become inequalities with "$\leq$" signs.*

Clearly, if the conjecture is proven, then the inequalities (1), (3) and (5) hold for Ω, by the same proofs of the previous section.

At last, we mention that there is another approach to proving Theorem 2 for arbitrary domains that does not require the conjectured inequality form of (20), see Xie [6].

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