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ON THE EXISTENCE OF HOMOCLINIC ORBITS FOR A SECOND-ORDER HAMILTONIAN SYSTEM*

Addolorata Salvatore
Dipartimento di Matematica, via E. Orabona 4, 70125 Bari, Italy

(Submitted by: Jean Mawhin)

Abstract. In this paper we look for homoclinic solutions of the system

$$\ddot{q} - a(t) \mid q \mid^{p-2} q + W_q(t,q) = 0$$

where $p > 2$, $a(t) \to +\infty$ as $\mid q \mid \to +\infty$ and $W(t,\cdot)$ is even and quadratic or superquadratic at infinity and at the origin. Using a compact embedding between suitable weighted Sobolev spaces, we prove the existence of infinitely many homoclinic solutions of the problem.

1. Introduction and statement of the results. The goal of this paper is to prove the existence of infinitely many homoclinic orbits emanating from 0 of the second-order Hamiltonian system

$$\ddot{q} + V_q(t,q) = 0, \tag{1.1}$$

where $t \in \mathbb{R}$, $q \in \mathbb{R}^N$ and V_q denotes the gradient of V with respect to q.

Those solutions satisfy the conditions

$$\lim_{|t|\to+\infty} q(t) = \lim_{|t|\to+\infty} \dot{q}(t) = 0. \tag{1.2}$$

Recently, variational methods have been used to study this problem. Namely, homoclinic orbits have been found as critical points of the functional

$$f(q) = \frac{1}{2}\int_{\mathbb{R}} \mid \dot{q}(t) \mid^2 \, dt - \int_{\mathbb{R}} V(t,q(t))\, dt$$

defined on the Sobolev space $H^1(\mathbb{R},\mathbb{R}^N)$ equipped with the usual norm

$$\| q \|_{H^1} = \Big(\int_{\mathbb{R}} (\mid \dot{q} \mid^2 + \mid q \mid^2)\, dt\Big)^{1/2}.$$

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Since $H^1(\mathbb{R}, \mathbb{R}^N)$ is not compactly embedded in $L^2(\mathbb{R}, \mathbb{R}^N)$, the functional f in general does not satisfy the well-known Palais-Smale (briefly (P.S.)) condition, thus we cannot directly apply the classical results of the critical points theory.

More recently Omana and Willem in [6] have shown that, under particular assumptions on the potential $V(t,q)$, a compact embedding between suitable weighted Sobolev spaces permits one to verify the (P.S.) condition for the action functional f. Then, by using a symmetric mountain pass theorem, they prove the existence of infinitely many homoclinic orbits emanating from 0 of the system

$$\ddot{q} - a(t)q + W_q(t,q) = 0 \tag{1.3}$$

where $a(t) \to +\infty$ as $\mid t \mid \to +\infty$ and $W(t,q)$, besides other technical assumptions, satisfies the growth conditions $W(t,q)/ \mid q \mid^2 \to +\infty$ (respectively 0) as $\mid q \mid \to +\infty$ (respectively $\mid q \mid \to 0$).

Now we deal with the system

$$\ddot{q} - a(t) \mid q \mid^{p-2} q + W_q(t,q) = 0 \tag{1.4}$$

where $p > 2$ and

(V_1) $a(t)$ is a continuous, positive function on $\mathbb{R}$ such that for all $t \in \mathbb{R}$

$$a(t) \geq \gamma \mid t \mid^{\alpha} \quad \text{with} \quad \alpha > \frac{p-2}{2}, \quad \gamma > 0.$$

By adapting to system (1.4) some tools developed by Benci and Fortunato in the study of semilinear elliptic equations on $\mathbb{R}^N$ (see [4]), we can prove that the Palais-Smale condition still holds; then a direct use of the variational methods permits us to find homoclinic solutions of system (1.4) in two different cases.

First, we assume that $W(t,q)$ is a quadratic function; that is,

(V_2) there exists $\lambda > 0$ such that

$$W(t,q) = \frac{\lambda}{2} \mid q \mid^2 \qquad \text{for all} \ \ (t,q) \in \mathbb{R} \times \mathbb{R}^N.$$

Then system (1.4) becomes

$$\ddot{q} - a(t) \mid q \mid^{p-2} q + \lambda q = 0. \tag{1.5}$$

In this case the geometry of the mountain pass is destroyed and the functional f is bounded from below; nevertheless, just exploiting the lack of compactness, it is possible to get the existence of infinitely many homoclinic orbits of (1.5).

Indeed, the following theorem holds:

Theorem 1.1. *Assume that* (V_1) *holds. Then for any* $\lambda > 0$ *system* (1.5) *has infinitely many distinct pairs of homoclinic orbits* $\{q_n(t), -q_n(t)\}$, $n \in \mathbb{N}$, *such that*

$$\| q_n \|_X \to 0 \qquad for \;\; n \to +\infty \tag{1.6}$$

(*see Section* 2 *for the definition of the space* X). *Thus every* $\lambda > 0$ *is a bifurcation point for the problem* (1.5).

We point out that condition (1.6) implies that only the behavior of V near to 0 is involved; then Theorem 1.1 can be extended as follows:

Theorem 1.2. *Let* $V \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ *satisfying*
(V_2') *there exist* $r, \lambda > 0$ *and* $p > 2$ *such that*

$$V(t,q) = -\frac{a(t)}{p} \mid q \mid^p + \frac{\lambda}{2} \mid q \mid^2 \quad if \;\; \mid q \mid \leq r \;\; and \;\; t \in \mathbb{R}$$

where $a(t)$ *satisfies* (V_1). *Then system* (1.1) *has infinitely many distinct pairs of homoclinic orbits such that* $\| \; q_n \|_X \to 0$ *as* $n \to +\infty$.

Remark 1. Many authors have studied the existence of nontrivial homoclinic orbits emanating from 0 when the potential V has a local maximum at q=0. Now 0 is not a local maximum for the function $V(t,q)$. However 0 is an "asymptotically" local maximum; that is, $V(t,0) = 0$ and $\lim_{t\to+\infty} V(t,q) < 0$ for $\mid q \mid$ small.

Analogous results have been stated in [4] for a semilinear elliptic problem on $\mathbb{R}^N$.

Let us consider now the case where $W(t,q)$ has a superquadratic growth at infinity and at the origin. Indeed, we assume that

(W_1) $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and there exists a constant $\mu > p$ such that

$$0 < \mu W(t,q) \leq q W_q(t,q) \quad \text{for all} \;\; q \in \mathbb{R}^N - \{0\} \text{ and } t \in \mathbb{R};$$

(W_2) $W_q(t,q) = o(\mid q \mid^{p-1})$ as $\mid q \mid \to 0$ uniformly in $t \in \mathbb{R}$;

(W_3) there exists $W^* \in C(\mathbb{R}^N, \mathbb{R})$ such that

$$\mid W(t,q) \mid + \mid W_q(t,q) \mid \leq \mid W^*(q) \mid \quad \text{for every} \;\; q \in \mathbb{R}^N \text{ and } t \in \mathbb{R}.$$

Under those assumptions, f satisfies the geometrical conditions of the mountain pass theorem; then the following theorem can be stated:

Theorem 1.3. *Let us assume that* $(V_1), (W_1) - (W_3)$ *hold. Then there exists a homoclinic orbit* q *of* (1.4) *emanating from* 0 *such that*

$$0 < \int_{\mathbb{R}} \big[\frac{1}{2} \mid \dot{q} \mid^2 + \frac{a(t)}{p} \mid q \mid^p - W(t,q)\big] \, dt < +\infty.$$

Moreover, if $W(t,q)$ *is even in* q, *i.e.,*

(W_4) $W(t,q) = W(t,-q)$ *for all* $q \in \mathbb{R}^N$ *and* $t \in \mathbb{R}$,

then there exists an unbounded sequence in X *of homoclinic orbits of* (1.4) *emanating from* 0.

Remark 2. Assumptions (W_1)–(W_2) imply

$$W(t,q) = o(\mid q \mid^p) \quad \text{as} \quad \mid q \mid \to 0 \text{ uniformly in } t \in \mathbb{R}; \tag{1.7}$$

$$\text{for any } t \in \mathbb{R} \text{ there is } b(t) > 0 \text{ such that } W(t,q) \geq b(t) \mid q \mid^\mu \text{ for } \mid q \mid \geq 1. \tag{1.8}$$

Obviously by (1.7) it follows that the function $V(t,q) = -\frac{a(t)}{p} \mid q \mid^p + W(t,q)$ has a local maximum at $q = 0$ for any $t \in \mathbb{R}$.

If $p = 2$, Theorem 1.3 can be proved by replacing (V_1) with the more general assumption "$a(t) \to +\infty$ as $\mid t \mid \to +\infty$" (see also [6]).

2. Variational formulation and compact embeddings. If γ is a positive, continuous function on $\mathbb{R}$ and $s > 1$, let

$$L^s_\gamma = L^s(\mathbb{R}, \mathbb{R}^N; \gamma) = \big\{ q \in L^1_{loc}(\mathbb{R}, \mathbb{R}^N) \mid \int_{\mathbb{R}} \gamma(t) \mid q(t) \mid^s dt < +\infty \big\}.$$

L^s_γ equipped with the norm

$$\| q \|_{s,\gamma} = \Big(\int_{\mathbb{R}} \gamma(t) |q(t)|^s \, dt \Big)^{1/s} \tag{2.1}$$

is a reflexive Banach space. Moreover we set for any t, $1 \leq t \leq \infty$,

$$L^t = L^t(\mathbb{R}, \mathbb{R}^N), \qquad H^1 = H^1(\mathbb{R}, \mathbb{R}^N)$$

with the usual norms

$$\|q\|_\infty = \max_{t \in \mathbb{R}} \mid q(t) \mid, \quad \|q\|_t = \Big(\int_{\mathbb{R}} \mid q(t) \mid^t \, dt \Big)^{1/t},$$

$$\|q\|_{H^1} = \Big(\int_{\mathbb{R}} (\mid q \mid^2 + \mid \dot{q} \mid^2) \, dt \Big)^{1/2}.$$

Let $X = H^1 \cap L^p_a$ where $a(t)$ is the function introduced in (V_1). Then X with its standard norm $\| \cdot \|_X$ is a reflexive Banach space.

In order to find homoclinic orbits of the system (1.4), we look for critical points of the functional

$$f(q) = \int_{\mathbb{R}} \big[\frac{1}{2} \mid \dot{q} \mid^2 + \frac{a(t)}{p} \mid q \mid^p - W(t,q) \big] \, dt \qquad q \in X. \tag{2.2}$$

Indeed, if W satisfies (V_1)–(V_2) or (V_1), (W_1)–(W_3) the following lemma holds:

Lemma 2.1. *The functional f is continuously Fréchet-differentiable on X and any critical point of f is a homoclinic orbit of system* (1.4).

Proof. Obviously $f : X \to \mathbb{R}$. By Theorem 2.1 of [3] we can deduce that the map

$$q \to a(t) \mid q \mid^{p-2} q$$

is continuous from L^p_a in the dual space $L^{p'}_{a^{-p'/p}}$ where $p' = \dfrac{p}{p-1}$. As the embeddings $X \subset H^1 \subset L^\beta$ for all $\beta \geq 2$ are continuous, it follows that $f \in C^1(X, \mathbb{R})$ and any critical point of f is a classical solution of (1.4) with $q(\pm\infty) = 0 = \dot{q}(\pm\infty)$ (see e.g. [8]). □

The following lemma is essentially contained in [4].

Lemma 2.2. *If $a(t)$ satisfies assumption (V_1), then*

$$\text{the embedding } L^p_a \subset L^2 \text{ is continuous.} \tag{2.3}$$

Moreover, there exists a Hilbert space Z such that

$$\text{the embeddings } L^p_a \subset Z \subset L^2 \text{ are continuous} \tag{2.4}$$

$$\text{the embedding } H^1 \cap Z \subset L^2 \text{ is compact.} \tag{2.5}$$

Proof. Let $\beta = \frac{p}{p-2}$ and $\beta' = \frac{p}{2}$. It results that

$$\begin{aligned}\| q \|_2^2 = \int_{\mathbb{R}} a^{-1/\beta'} a^{1/\beta'} q^2 \, dt &\leq \Big(\int_{\mathbb{R}} (a^{-1/\beta'})^\beta \, dt\Big)^{1/\beta} \Big(\int_{\mathbb{R}} (a^{1/\beta'} q^2)^{\beta'} \, dt\Big)^{1/\beta'} \\ &= a_1 \Big(\int_{\mathbb{R}} a q^p \, dt\Big)^{2/p} = a_1 \| q \|^2_{p,a},\end{aligned}$$

where by (V_1), $a_1 = \int_{\mathbb{R}} a^{-2/(p-2)} \, dt < +\infty$. Then (2.3) follows.

In order to prove (2.4), let us point out that by (V_1) there exists a positive continuous function ρ on $\mathbb{R}$ such that $\rho(t) \to +\infty$ as $\mid t \mid +\infty$ and

$$a_2 = \Big(\int_{\mathbb{R}} \rho^\beta a^{-\beta/\beta'} \, dt\Big)^{1/\beta} < +\infty.$$

As

$$\begin{aligned}\| q \|^2_{2,\rho} = \int_{\mathbb{R}} \rho q^2 \, dt &= \int_{\mathbb{R}} \rho a^{-1/\beta'} a^{1/\beta'} q^2 \, dt \\ &\leq a_2 \Big(\int_{\mathbb{R}} a \mid q \mid^{2\beta'} \, dt\Big)^{1/\beta'} = a_2 \| q \|^2_{p,a},\end{aligned}$$

(2.4) holds taking $Z = L^2_\rho$.

Finally, as $H^1 \cap Z$ is the weighted Sobolev space $\Gamma^1(\mathbb{R}, \rho, 1)$, (2.5) follows by Theorem 2.7 of [3]. □

Remark 3. By Lemma 2.2 we deduce that $H^1 \cap L^p_a$ is compactly embedded in L^2.

If $p = 2$, by the proof of (2.4) we deduce that $H^1 \cap L^2_a$ is compactly embedded in L^2 under the weaker assumption "$a(t) \to +\infty$ as $\mid t \mid \to +\infty$" (see also [6]).

We are now able to prove the following crucial lemma:

Lemma 2.3. *The functional f satisfies the Palais-Smale condition.*

Proof. Let $\{q_n\} \subset X$ such that

$$\{f(q_n)\}_n \text{ is bounded} \tag{2.6}$$

and

$$f'(q_n) \to 0 \text{ as } n \to +\infty. \tag{2.7}$$

First, let us consider the case where $W(t,q) = \lambda \mid q \mid^2$. By (2.3) and (2.6) there exist a_3 and a_4 such that

$$a_3 \geq \frac{1}{2} \parallel \dot{q}_n \parallel_2^2 - \frac{\lambda}{2} \parallel q_n \parallel_2^2 + a_4 \parallel q_n \parallel_2^p$$

and therefore $\{\parallel q_n \parallel_2\}_n$ and $\{\parallel \dot{q}_n \parallel_2^2 + \frac{1}{p}\int_{\mathbb{R}} a(t) \mid q_n \mid^p dt\}_n$ are bounded, so $\{q_n\}_n$ is bounded in X.

By Lemma 2.2 there exists a subsequence, still denoted by $\{q_n\}_n$, such that

$$q_n \to q \qquad \text{in } \ L^2 \text{ as } n \to +\infty. \tag{2.8}$$

By (2.7) and (2.8) it follows that the sequence

$$\begin{aligned} &\langle f'(q_n) - f'(q), q_n - q\rangle \\ &= \parallel \dot{q}_n - \dot{q} \parallel_2^2 - \lambda \parallel q_n - q \parallel_2^2 + \int_{\mathbb{R}} a(t)\big[\mid q_n \mid^{p-2} q_n - \mid q \mid^{p-2} q\big](q_n - q)\, dt \end{aligned} \tag{2.9}$$

tends to 0 as $n \to +\infty$. Now it is easy to see that there exists $a_5 > 0$ such that

$$(\mid x \mid^{p-2} x - \mid y \mid^{p-2} y)(x-y) \geq a_5 \mid x - y \mid^p \qquad \forall x, y \in \mathbb{R}^N \tag{2.10}$$

and therefore by (2.9), (2.8) and (2.10) $q_n \to q$ in X.

Let us prove now the (P.S.) condition when the function W is superquadratic. Assume that (W_1)–(W_3) hold. By (2.6), (2.7), and (W_1) there exists $a_6 > 0$ such that

$$\begin{aligned} a_6 + \parallel q_n \parallel_2 &\geq f(q_n) - \frac{1}{\mu} f'(q_n) q_n \\ &\geq (\frac{1}{2} - \frac{1}{\mu}) \parallel \dot{q}_n \parallel_2^2 + (\frac{1}{p} - \frac{1}{\mu}) \int_{\mathbb{R}} a(t) \mid q_n \mid^p \ dt. \end{aligned}$$

Then if $p < \mu$, the sequence $\{\parallel q_n \parallel_X\}$ is bounded. Passing to a subsequence, $q_n \rightharpoonup q$ weakly in X. By (2.4) of Lemma 2.2 it follows that $q_n \to q$ in L^2 and therefore, arguing as in Lemma 2 of [6], we obtain

$$W_q(t, q_n) \to W_q(t,q) \qquad \text{in } \ L^2(\mathbb{R}). \tag{2.11}$$

As the sequence

$$\begin{aligned}\langle f'(q_n) - f'(q), q_n - q\rangle \\ &= \| \dot{q}_n - \dot{q} \|_2^2 + \int_{\mathbb{R}} a(t)\big[\mid q_n \mid^{p-2} q_n - \mid q \mid^{p-2} q\big](q_n - q)\, dt \\ &\quad - \int_{\mathbb{R}} \big[W_q(t, q_n) - W_q(t, q)\big](q_n - q)\, dt \end{aligned} \tag{2.12}$$

tends to 0 as $n \to +\infty$, by (2.12), (2.11) and (2.10) we can conclude that $q_n \to q$ in X. □

In order to find nontrivial solutions of problem (1.1), we will use the "genus" properties, so we recall the following definitions and results (see [5] and [7]).

Let E be a Banach space and $g \in C^1(E, \mathbb{R})$. We set

$$\Sigma = \big\{A \subset E - \{0\} : A \text{ is closed in } E \text{ and symmetric with respect to } 0\big\};$$

$$K_c = \big\{x \in E : g(x) = c,\ g'(x) = 0\big\} \qquad c \in \mathbb{R};$$
$$g^c = \big\{x \in E : g(x) \le c\big\}.$$

Definition 2.4. Given A$\in \Sigma$, $\gamma(A)$ denotes the genus of A, that is, the smallest integer n such that there exists an odd, continuous map of A into $\mathbb{R}^n - \{0\}$.

Theorem 2.5. *Let g be an even C^1 functional on E which satisfies the* (P.S.) *condition. If $n \in \mathbb{N}$, $n > 0$, let*

$$\Sigma_n = \big\{A \in \Sigma : \gamma(A) \ge n\big\}$$
$$c_n = \inf_{A \in \Sigma_n} \sup_{x \in A} g(x). \tag{2.13}$$

If $\Sigma_n \ne \emptyset$ and $c_n \in \mathbb{R}$, then c_n is a critical value of g. Moreover if $c = c_n = c_{n+1} = \cdots = c_{n+r} \in \mathbb{R}$, for some $n, r \in \mathbb{N}$, $c \ne g(0)$, then $\gamma(K_c) \ge r + 1$.

Remark 4. Let us recall that if $\gamma(K_c) \ge 2$, then K_c has an infinite number of distinct pairs of points.

In particular if g is unbounded from below and from above, the following theorem holds (see [1]):

Theorem 2.6 (Symmetric mountain pass theorem)**.** *Let g be an even C^1 functional on E which satisfies the* (P.S.) *condition. Let us assume that $g(0) = 0$ and*

i) *there are constants ρ and $\alpha > 0$ such that $g(x) \ge \alpha$ for all $x \in \partial B_\rho$;*
ii) *for each finite dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X})$ such that $g \le 0$ on $\tilde{X}/B_R$.*

Then g has infinitely many critical points.

3. Proof of the Theorems.

Proof of Theorem 1.1. The proof closely follows that of Theorem 1.1 of [4]; however, for completeness, we give now an outline of the arguments used. By (2.4) and Lemma 2.3, the action functional

$$f(q) = \frac{1}{2} \parallel \dot{q} \parallel_2^2 - \frac{\lambda}{2} \parallel q \parallel_2^2 + \frac{1}{p} \parallel q \parallel_{p,a}^p$$

is bounded from below and satisfies the (P.S.) condition. Now we prove that

$$\text{for any } n \in \mathbb{N} \text{ there exists } \varepsilon > 0 \text{ such that } \gamma(f^{-\varepsilon}) \geq n. \tag{3.1}$$

Taking $\delta \in \big(0, \lambda\big)$, we set

$$E = \big\{ q \in \mathcal{S}(\mathbb{R}) : \operatorname{supp} \tilde{q} \subset \big(-\sqrt{\lambda - \delta}, \sqrt{\lambda - \delta}\big) \big\},$$

where $\tilde{q}$ is the Fourier transform of q and $\mathcal{S}(\mathbb{R})$ is the space of the C^∞ functions rapidly decreasing at infinity with their derivatives. It results

$$\int_{\mathbb{R}} \mid \dot{q}(t) \mid^2 \, dt = \int_{\mathbb{R}} t^2 \mid \tilde{q} \mid^2 \, dt \leq (\lambda - \delta) \parallel q \parallel_2^2 \qquad \text{for all } q \in E. \tag{3.2}$$

As E is an infinite-dimensional vector space, we can consider for any $n \in \mathbb{N}$ a n-dimensional vector space $V_n \subset E$. Let $q \in V_n$, $\parallel q \parallel_X = 1$ and $\mu > 0$, then by (3.2)

$$f(\mu q) \leq -\frac{\mu^2 \delta}{2} \parallel q \parallel_2^2 + \frac{\mu^p}{p} \parallel q \parallel_{p,a}^p \leq -a_7 \mu^2 \delta + a_8 \mu^p, \tag{3.3}$$

where

$$a_7 = \inf \big\{ \frac{\parallel q \parallel_2^2}{2} : q \in V_n, \parallel q \parallel_X = 1 \big\},$$
$$a_8 = \sup \big\{ \frac{\parallel q \parallel_{p,a}^p}{p} : q \in V_n, \parallel q \parallel_X = 1 \big\}.$$

By (3.3) it follows that

$$\text{there exist } \varepsilon, \bar{\mu} > 0 \text{ such that } f(\bar{\mu} q) < -\varepsilon \quad \text{for any } q \in V_n, \ \parallel q \parallel_X = 1;$$

then

$$S_{\bar{\mu}} \cap V_n \subset f^{-\varepsilon}$$

where

$$S_{\bar\mu} = \{q \in X : \| q \|_X = \bar\mu\}.$$

By some properties of the genus (see 3°) of Proposition 7.5 and Proposition 7.7 of [7]) we deduce that

$$\gamma(f^{-\varepsilon}) \geq \gamma(S_{\bar\mu} \cap V_n) = n,$$

so the proof of (3.1) is achieved. Then if we set

$$c_n = \inf_{A \in \Sigma_n} \sup_{q \in A} f(q),$$

it follows that $c_n \leq -\varepsilon < 0$; that is, for any $n \in \mathbb{N}$, c_n is a real negative number. By Theorem 2.5 we deduce that f has infinitely many critical points q_n with critical value $c_n = f(q_n)$.

Moreover, we prove that

$$c_n \to 0 \qquad \text{as} \;\; n \to +\infty. \tag{3.4}$$

Let Z be the Hilbert space satisfying (2.4)–(2.5). Given $\delta > 0$, let $\mathcal{L}_\delta$ be the unique self-adjoint operator in L^2 with domain $D(\mathcal{L}_\delta) \subset Z \cap H^1$ such that

$$(\mathcal{L}_\delta q, v)_2 = (\dot q, \dot v)_2 + \delta(u, v)_Z,$$

where $(\cdot,\cdot)_2$ and $(\cdot,\cdot)_Z$ denote the inner products in L^2 and in Z. Since $\mathcal{L}_\delta$ has positive and discrete spectrum, let $\{\lambda_j(\delta)\}_j$ and $\{H_j(\delta)\}_j$ be the eigenvalues and the corresponding eigenspaces of $\mathcal{L}_\delta$. Taking

$$H_\delta = \begin{cases} \{0\} & \text{if } \sigma(\mathcal{L}_\delta) \cap [0,\lambda) = \emptyset \\ \bigoplus_{j=1}^{h} H_j & \text{if } \{\lambda_1(\delta), \dots, \lambda_h(\delta)\} = \sigma(\mathcal{L}_\delta) \cap [0,\lambda) \neq \emptyset, \end{cases}$$

by (2.4) it results that for any $q \in X \cap H_\delta^\perp$,

$$f(q) \geq \frac{\lambda}{2} \| q \|_2^2 - \frac{\delta}{2} \| q \|_Z^2 + \frac{1}{p} \| q \|_{p,a}^p - \frac{\lambda}{2} \| q \|_2^2 \geq -\frac{\delta}{2} \| q \|_Z^2 + a_9 \| q \|_Z^p$$

where $H_\delta^\perp$ is the orthogonal complement of H_δ in L^2. Therefore

$$\forall \varepsilon > 0 \; \exists \; \delta = \delta(\varepsilon) > 0 \text{ such that } f(q) > -\varepsilon \qquad \forall q \in X \cap H_\delta^\perp. \tag{3.5}$$

Now, if we set $k = k(\varepsilon) = \dim H_\delta \cap X$, by the intersection property of the genus (see Proposition 7.8 of [7]) we deduce that

$$A \cap (H_\delta^\perp \cap X) \neq \emptyset \qquad \forall A \in \Sigma_{k+1}. \tag{3.6}$$

By (3.5) and (3.6) we obtain that

$$\forall n \in \mathbb{N},\ n \geq k+1: \ c_n = \inf_{A \in \Sigma_n} \sup_{q \in A} f(q) \geq -\varepsilon. \tag{3.7}$$

As we have already proved that the c_n's are negative, (3.7) implies that (3.4) holds. Finally, as $f(q_n) \to 0$ and $f'(q_n) = 0$, we easily conclude that

$$\| q_n \|_{\mathrm{X}} \to 0. \tag{3.8}$$

Proof of Theorem 1.2. Let us assume now that V satisfies (V_1)–$(V_2^{'})$. Then for $| q | > r$, we can modify the potential $V(q)$ in such a way the new function $\hat{V}$ satisfies the more restrictive assumption (V_2). By Theorem 1.1 there exists a sequence of homoclinic solutions q_n of the system

$$\ddot{q}(t) + \hat{V}_q(t, q) = 0.$$

As (3.8) still holds, it follows that

$$\max_{t \in \mathbb{R}} | q_n(t) | \to 0 \qquad \text{as} \ \ n \to +\infty;$$

that is, for n large enough, $| q_n(t) | \leq r$ for any $t \in \mathbb{R}$ (where r is the constant of the assumption $(V_2'))$ and therefore q_n is a solution of (1.1).

Proof of Theorem 1.3. We shall prove that f satisfies the assumptions of the mountain pass theorem (see [1] and [7]). Indeed $f(0) = 0$ and by Lemma 2.3 f satisfies the (P.S.) condition.

Moreover, as $X \subset L^\infty$ and (1.7) holds, for any $\varepsilon > 0$ there exists $\rho > 0$ such that for any $q \in X$, $\| q \|_{\mathrm{X}} \leq \rho$,

$$V(q(t)) \leq \varepsilon \mid q(t) \mid^p \qquad \text{for all } t \in \mathbb{R}.$$

So, if $\| q \|_X = \rho$, we have

$$f(q) \geq \frac{1}{2} \| \dot{q} \|_2^2 + \frac{1}{p} \| q \|_{p,a}^p - \varepsilon \| q \|_p^p \ \ \geq \frac{1}{2} \| \dot{q} \|_2^2 + (\frac{1}{p} - \varepsilon a_{10}) \| q \|_{p,a}^p .$$

If we choose ε small enough, we can find a constant $\alpha > 0$, α depending on ρ, such that

$$f(q) \geq \alpha \qquad \text{for any } q \in X \text{ with } \ \| q \|_X = \rho.$$

Let $q_1 \in X$ such that $| q_1(t) | \geq 1$ on an open and nonempty interval $\mathrm{I} \subset \mathbb{R}$. For any $\sigma \geq 1$ by (1.8) we have

$$\begin{aligned} f(\sigma q_1) &\leq \frac{\sigma^2}{2} \| \dot{q}_1 \|_2^2 + \frac{\sigma^p}{p} \| q_1 \|_{p,a}^p - \int_I W(t, \sigma q_1)\, dt \\ &\leq \frac{\sigma^2}{2} \| \dot{q}_1 \|_2^2 + \frac{\sigma^p}{p} \| q_1 \|_{p,a}^p - \sigma^\mu \int_I b(t) \mid q_1 \mid^\mu \ dt. \end{aligned}$$

Since $\mu > p$, there is a σ large enough such that $\| \sigma q_1 \|_X > \rho$ and

$$f(\sigma q_1) \leq f(0) = 0.$$

Then by the mountain pass theorem there exists a nontrivial critical point of f.

Let us assume now W even. In order to prove a multiplicity result, we shall prove that for any finite-dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X})$ such that $f \leq 0$ on $\tilde{X}/B_R$. Let $\tilde{X}$ be as above. Taking $m = \inf_{\|q\|_\infty = 2} \int_{|q(t)|>1} b(t) \mid q \mid^\mu \, dt$, arguing as in [6] by (1.8) it follows that $m > 0$ and

$$\int_{\mathbb{R}} W(t,q)\, dt \geq \frac{m}{2\mu} \| q \|_\infty^\mu \qquad \text{for all } q \in \tilde{X}.$$

Therefore, as $\tilde{X}$ is finite dimensional, there exist a_{11} and a_{12} (depending on $\tilde{X}$) such that

$$f(q) \leq a_{11} \| q \|_\infty^2 + a_{12} \| q \|_\infty^p - \frac{m}{2\mu} \| q \|_\infty^\mu \qquad \text{for all } q \in \tilde{X}.$$

As $\mu > p$, there exists R (depending on $\tilde{X}$) such that

$$f(q) \leq 0 \qquad \text{for all } q \in \tilde{X}/B_R.$$

The symmetric mountain pass theorem implies that f has an unbounded sequence of critical values $c_n = f(q_n)$. Obviously for any $n \in \mathbb{N}$

$$\int_{\mathbb{R}} W_q(t,q_n)q_n\, dt = \| \dot{q}_n \|_2^2 + \| q_n \|_{p,a}^p \tag{3.9}$$

and

$$\begin{aligned} c_n &= f(q_n) - \tfrac{1}{2} f'(q_n)q_n \\ &= (\frac{1}{p} - \frac{1}{2}) \| q_n \|_{p,a}^p + \int_{\mathbb{R}} \left[\tfrac{1}{2} W_q(t,q_n)q_n - W(t,q_n) \right] dt \\ &\leq \tfrac{1}{2} \int_{\mathbb{R}} W_q(t,q_n)q_n\, dt. \end{aligned} \tag{3.10}$$

Since $c_n \to +\infty$ as $n \to +\infty$, by (3.9) and (3.10) it follows that $\{q_n\}_n$ is unbounded in X.

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