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QUENCHING FOR A DIFFUSIVE EQUATION WITH A CONCENTRATED SINGULARITY

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Abstract. The diffusion equation with a concentrated singular reaction $v_t = v_{xx} + \epsilon\delta(x-a)f(v)$ $(\epsilon > 0, 0 < a < 1)$ is studied. Criteria for global existence and finite time quenching of the solution are established. The growth rate and estimate on quenching time are also given for a certain class of nonlinearities.

1. Introduction. In this paper, we consider the following semilinear heat equation:

$$v_t = v_{xx} + F(x,v), \qquad 0 < x < 1, \quad t > 0, \tag{1.1}$$

subject to the initial and boundary (Dirichlet- or Neumann-type) conditions,

$$v(x,t) = 0 \quad \text{or} \quad v_x(x,t) = 0, \qquad x = 0,1, \quad t > 0, \tag{1.2}$$

$$v(x,0) = 0, \qquad 0 \le x \le 1. \tag{1.3}$$

Here the nonlinear source has a strongly concentrated and localized spatial dependence of the form

$$F(x,v) = \epsilon\delta(x-a)f(v), \qquad 0 < a < 1, \tag{1.4}$$

where ϵ is a positive parameter and $\delta(x-a)$ is the Dirac delta distribution. The function $f(v)$ satisfies: $f(v) > 0$, $f'(v) > 0$, $f''(v) > 0$ for $v \ge 0$, and possesses a singularity at c $(c > 0)$; i.e., $\lim_{v\to c^-} f(v) = +\infty$. In the sequel, we shall use (PD) or (PN) to denote the Dirichlet or Neumann problem, respectively.

There are two reasons for considering this problem. On the one hand, it can be related via transformation to a certain class of physical problems of ignition that remain of wide interest. On the other hand, the problem itself is closely related to

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a popular model arising in the study of a polarization phenomenon. In either case, however, the situation of a highly localized forcing term has not been previously considered.

To elaborate, for certain nonlinearities such as $f(v) = (c - v)^{-\alpha}$ $(\alpha > 0)$, by means of the transformation $w = -\ln(c-v)$, the differential equation (1.1) becomes

$$w_t = w_{xx} - w_x^2 + \epsilon\delta(x-a)e^{(\alpha+1)w}. \tag{1.5}$$

Equation (1.5) can be treated as an approximation to the ignition of a combustible medium with damping, where either a heated wire or a pair of small electrodes supplies a large amount of energy to a very confined area.

Alternatively, problem (1.1)–(1.3) is related to a model that characterizes electric current transients in polarized ionic conductors as follows (in our notation):

$$\begin{aligned} &u_t = u_{xx} + \epsilon/(1-u), && 0 < x < 1, && t > 0,\\ &u(x,t) = 0, && x = 0, 1, && t > 0,\\ &u(x,0) = 0, && 0 \le x \le 1. \end{aligned} \tag{1.6}$$

In [3] the following results were proven for (1.6):

(a) If $\epsilon > 8$, there is a finite time T such that $\lim_{t\to T^-} u(\frac{1}{2}, t) = 1$;

(b) Whenever (a) holds, $\lim_{t\to T^-} \max_x u_t(x,t) = +\infty$.

Such a phenomenon is known as quenching; that is, the solution of the equation remains bounded, whereas its derivatives blow up at some finite moment. Since 1975, when the above results appeared, problem (1.6) and its various generalizations have been extensively studied (see [4, 5] and the literature cited therein). Unlike the spatially localized nonlinearity considered here, however, the quenching literature has only considered nonlinearities having either no explicit spatial dependence, i.e., $F = F(v)$, or there is a required spatial smoothness for $F = F(x, v)$. To the best of our knowledge, no one has done any problem of the type described here. Due to the presence of the delta function in the equation, certain conventional arguments may not apply. Therefore we want to undertake a study by converting problem (1.1)–(1.3) to an equivalent integral equation for v at the site of the concentrated source, i.e., $v(a,t)$.

For this purpose, we introduce an integral representation formula for (1.1)–(1.3) of the form

$$v(x,t) = \int_0^t \int_0^1 G(x,t;\xi,s)F(\xi, v(\xi,s))\,d\xi\,ds. \tag{1.7}$$

Here $G(x,t;\xi,s)$ is the Green's function for the linear heat equation with either Dirichlet or Neumann type boundary conditions. In particular, for (PD) $G = G_D$, where

$$G_D(x,t;\xi,s) = 2H(t-s)\sum_{n=1}^{\infty}\sin n\pi x \sin n\pi\xi \exp\bigl[-n^2\pi^2(t-s)\bigr] \tag{1.8}$$
$$= \frac{H(t-s)}{2\sqrt{\pi(t-s)}}\sum_{n=-\infty}^{\infty}\bigl\{\exp\bigl[-\frac{(x-\xi-2n)^2}{4(t-s)}\bigr] - \exp\bigl[-\frac{(x+\xi-2n)^2}{4(t-s)}\bigr]\bigr\},$$

while for (PN) $G = G_N$, where

$$\begin{aligned} G_N(x,t;\xi,s) &= 2H(t-s)\bigl\{1 + 2\sum_{n=1}^{\infty}\cos n\pi x \cos n\pi\xi \exp\bigl[-n^2\pi^2(t-s)\bigr]\bigr\} \\ &= \frac{H(t-s)}{2\sqrt{\pi(t-s)}}\sum_{n=-\infty}^{\infty}\bigl\{\exp\bigl[-\frac{(x-\xi-2n)^2}{4(t-s)}\bigr] + \exp\bigl[-\frac{(x+\xi-2n)^2}{4(t-s)}\bigr]\bigr\}. \end{aligned} \tag{1.9}$$

Each Green's function has been expressed in spectral as well as image representation (see [9]), since both forms will be used in the analysis to follow.

Substitution of the expression (1.4) for the source function in terms of the delta distribution into (1.7), and evaluation of (1.7) at $x = a$ yields

$$v(a,t) = \epsilon\int_0^t G(a,t;a,s) f(v(a,s))\,ds. \tag{1.10}$$

Thus, the initial-boundary value problem (1.1)–(1.3) has been reduced to a nonlinear Volterra equation. Accordingly, the kernel $G(a,t;a,s)$ is obtained from (1.8) for (PD) or from (1.9) for (PN). In what follows, we will focus on (1.10) to conduct an investigation on quenching solutions and growth rate at quenching. To be more specific, we show that for problem (PD) there exists an $\epsilon^* > 0$ such that if $\epsilon \le \epsilon^*$ every solution is global, while quenching occurs if $\epsilon > \epsilon^*$; for problem (PN) with any $\epsilon > 0$, the solution quenches in finite time. Furthermore, for $f(v) \sim (c-v)^{-\alpha}$ $(\alpha > 0)$ we establish the asymptotic behavior $c - v(a,t) \sim A(\hat{t}-t)^{\frac{1}{2(\alpha+1)}}$ as $t \to \hat{t}$, where $\hat{t}$ is the quenching time.

2. Criteria for quenching. We begin with the solvability of the solution of (1.10). To this end, it is convenient to express the integral equation in the form

$$u(t) = \mathbf{T}u(t) \equiv \epsilon\int_0^t k(t-s) f(u(s))\,ds, \qquad t \ge 0, \tag{2.1}$$

where $u(t) = v(a,t)$ and the form of the kernel $k(t-s)$ depends upon the choice of boundary conditions (1.2). For (PD), it follows from (1.8) that

$$k(t-s) = k_D(t-s) = 2\sum_{n=1}^{\infty}\sin^2 n\pi a \exp\bigl[-n^2\pi^2(t-s)\bigr] \tag{2.2}$$
$$= \frac{1}{2\sqrt{\pi(t-s)}}\sum_{n=-\infty}^{\infty}\bigl[\exp\bigl(-\frac{n^2}{t-s}\bigr) - \exp\bigl(-\frac{(a-n)^2}{t-s}\bigr)\bigr],$$

while for (PN), it follows from (1.9) that

$$k(t-s) = k_N(t-s) = 1 + 2\sum_{n=1}^{\infty} \cos^2 n\pi a \exp\bigl[-n^2\pi^2(t-s)\bigr] \qquad (2.3)$$
$$= \frac{1}{2\sqrt{\pi(t-s)}} \sum_{n=-\infty}^{\infty} \bigl[\exp\bigl(-\frac{n^2}{t-s}\bigr) + \exp\bigl(-\frac{(a-n)^2}{t-s}\bigr)\bigr].$$

Our analysis of (2.1) will utilize the techniques of [7, 8]. First, contraction mapping arguments will be employed to establish the local existence of a continuous solution. In view of the positivity of $k(t-s)$ and $f(u)$, any solution must be positive in the existence interval, and it must also be strictly increasing, since

$$u'(t) = \epsilon k(t) f(0) + \epsilon \int_0^t k(t-s) f'(u(s)) u'(s)\, ds > 0. \qquad (2.4)$$

Let $\Psi(u) = u/f(u)$, then $\Psi(0) = 0$, $\lim_{u\to c^-} \Psi(u) = 0$, and $\Psi(u) \in C^2(0,c)$. From $f''(u) > 0$ it follows that $\Psi(u)$ attains its unique maximum $\Psi(m)$ in $(0,c)$, and $\Psi(m) = 1/f'(m)$.

We now show that for any fixed μ $(0 < \mu < m)$ there exists a $t^* > 0$ such that (2.1) has a unique continuous solution on $[0, t^*)$ satisfying

$$0 \le u(t) \le \mu. \qquad (2.5)$$

Applying the contraction mapping principle, it is required that the integral operator $\mathbf{T}$ maps the space of continuous functions satisfying (2.5) into itself, so that

$$\mathbf{T}u(t) \le \epsilon f(\mu) K(t) \le \mu, \qquad 0 \le t < t^*. \qquad (2.6)$$

This inequality depends upon

$$K(t) \equiv \int_0^t k(t-s)\, ds, \qquad (2.7)$$

which is an increasing function in either case (2.2) or (2.3). To ensure the validity of (2.6), we then need $\epsilon K(t^*) \le \Psi(\mu)$, and it follows that

$$\epsilon f'(\mu) K(t) \le f'(\mu)\Psi(\mu) < f'(m)\Psi(m) = 1, \qquad 0 \le t < t^*. \qquad (2.8)$$

As a consequence, for arbitrary $u_1(t)$ and $u_2(t)$, which are continuous and satisfy (2.5), the contraction property of $\mathbf{T}$ follows from

$$\sup_{0\le t<t^*} |\mathbf{T}u_1(t) - \mathbf{T}u_2(t)| \le \epsilon f'(\mu) K(t) \sup_{0\le t<t^*} |u_1(t) - u_2(t)|. \qquad (2.9)$$

Clearly, the limiting value of t^* is determined by considering

$$\epsilon K(t^*) = \Psi(m). \tag{2.10}$$

This yields a lower bound on the extent of the interval $[0, t^*)$ over which there is a guaranteed unique solution $u(t)$ $(< c)$ of (2.1).

Next we examine the possibility of finite time quenching of the solution. Under the assumption that (2.1) has a continuous solution $u(t)$ $(< c)$ for $0 \le t \le \bar{t}$, we have that

$$u(t) \ge J(t) \equiv \epsilon \int_0^t k(\bar{t} - s) f(u(s))\, ds, \qquad 0 \le t \le \bar{t}, \tag{2.11}$$

since $k(\bar{t} - s) \le k(t - s)$ for either (2.2) or (2.3). It then follows that

$$J'(t) = \epsilon k(\bar{t} - t) f(u(t)) \ge \epsilon k(\bar{t} - t) f(J(t)). \tag{2.12}$$

Integrating (2.12) yields

$$\int_0^{J(\bar{t})} \frac{du}{f(u)} \ge \epsilon \int_0^{\bar{t}} k(\bar{t} - s)\, ds = \epsilon K(\bar{t}), \tag{2.13}$$

which implies

$$\kappa \equiv \int_0^c \frac{du}{f(u)} > \epsilon K(\bar{t}), \tag{2.14}$$

since $u < c$. Noting the fact that $\psi(u) \equiv 1/f(u) > 0$ on $[0, c)$ and $\lim_{u \to c^-} \psi(u) = 0$, one can see that the value of κ is finite. Thus, if there is a $t^{**} < \infty$ such that

$$\epsilon K(t^{**}) = \kappa \equiv \int_0^c \frac{du}{f(u)}, \tag{2.15}$$

then any continuous solution of (2.1) cannot be extended to the interval $[0, t^{**})$. Consequently, there is a $\hat{t}$ $(t^* < \hat{t} < t^{**})$ such that $u(t) \to c$ as $t \to \hat{t}$, where t^* and t^{**} are determined by (2.10) and (2.15), respectively. Further, $\lim_{t \to \hat{t}} u'(t) = +\infty$, since by (2.4)

$$u'(t) \ge \epsilon k(t) f(0) + \epsilon k(t) \int_0^t f'(u(s)) u'(s)\, ds = \epsilon k(t) f(u(t)).$$

Relationship (2.15) represents a balance between the diffusive nature of the kernel, as reflected by $K(t^{**})$, the constant κ, which depends upon the growth of the nonlinearity $f(u)$, and the parameter ϵ, hence it is the essential criterion for quenching.

For problem (PN), we use the spectral representation of the kernel in (2.3) to obtain

$$K_N(t) = \int_0^t k_N(t-s)\,ds = t + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos^2 n\pi a\big[1-\exp\big(-n^2\pi^2 t\big)\big]. \quad (2.16)$$

It is clear that $K_N(t) \to \infty$ as $t \to \infty$, and hence (2.15) will eventually be satisfied for any $\epsilon > 0$.

For problem (PD), we use the spectral representation of the kernel in (2.2) to obtain

$$\begin{aligned} K_D(t) = \int_0^t k_D(t-s)\,ds &= \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2 n\pi a\big[1-\exp\big(-n^2\pi^2 t\big)\big] \\ &= a(1-a) - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2 n\pi a\exp\big(-n^2\pi^2 t\big). \end{aligned} \quad (2.17)$$

In this case, if ϵ is so small that

$$\epsilon a(1-a) \le \Psi(m), \quad (2.18)$$

where $\Psi(m)$ is the maximum of $\Psi(u)$ on $[0,c]$, then (2.10) holds, and hence every solution of (2.1) exists globally over $[0,\infty)$.

On the other hand, if ϵ is sufficiently large such that

$$\epsilon a(1-a) > \kappa, \quad (2.19)$$

then quenching will always occur in finite time.

To indicate the effect of the parameter ϵ on quenching, we now treat the solution of (2.1) as a function of t depending upon the parameter ϵ and denote it $u(t,\epsilon)$. Let $\epsilon^* = \sup\{\epsilon \mid u(t,\epsilon) < c \text{ for } 0 \le t < \infty\}$. By virtue of (2.18) and (2.19), $0 < \epsilon^* < \infty$.

For any $\epsilon < \epsilon^*$, there is an $\overline{\epsilon}$ ($\epsilon < \overline{\epsilon} < \epsilon^*$) such that

$$\begin{aligned} u(t,\epsilon) &= u(t,\overline{\epsilon}) + (\epsilon-\overline{\epsilon})\int_0^t k_D(t-s)f(u(s,\epsilon))\,ds \\ &\quad + \overline{\epsilon}\int_0^t k_D(t-s)[f(u(s,\epsilon)) - f(u(s,\overline{\epsilon}))]\,ds \\ &\le u(t,\overline{\epsilon}) + (\epsilon-\overline{\epsilon})\int_0^t k_D(t-s)f(u(s,\epsilon))\,ds \\ &\le u(t,\overline{\epsilon}) + (\epsilon-\overline{\epsilon})f(0)\int_0^t k_D(t-s)\,ds = u(t,\overline{\epsilon}) - (\overline{\epsilon}-\epsilon)f(0)K_D(t), \end{aligned} \quad (2.20)$$

which implies that $\sup_{[0,\infty)} u(t,\epsilon) < c$. Letting $u_\infty(\epsilon) = \lim_{t\to\infty} u(t,\epsilon)$, we show

$$u_\infty(\epsilon) = \epsilon a(1-a) f(u_\infty(\epsilon)). \tag{2.21}$$

For any $\rho > 0$, since $\lim_{t\to\infty} e^{-\pi^2 t} = 0$ and $\lim_{t\to\infty} f(u(t,\epsilon)) = f(u_\infty(\epsilon))$, there is a $\tilde{t} > 0$ such that

$$e^{-\pi^2 t} < \rho/[2f(u_\infty(\epsilon))a(1-a)]$$

and

$$0 < f(u_\infty(\epsilon)) - f(u(t,\epsilon)) < \rho/2a(1-a),$$

whenever $t > \tilde{t}$. Thus, if $t > 2\tilde{t}$, making use of the spectral representation of $k_D(t-s)$ in (2.2), we find that

$$\begin{aligned}
0 &< a(1-a)f(u_\infty(\epsilon)) - \int_0^t k_D(t-s) f(u(s,\epsilon))\,ds \\
&= f(u_\infty(\epsilon))[a(1-a) - \int_0^t k_D(t-s)\,ds] \\
&\qquad + \int_0^t k_D(t-s)[f(u_\infty(\epsilon)) - f(u(s,\epsilon))]\,ds \\
&\le f(u_\infty(\epsilon)) \frac{2}{\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} \sin^2 n\pi a \exp(-n^2\pi^2 t) \qquad (2.22) \\
&\quad + \int_0^{\frac{t}{2}} k_D(t-s) f(u_\infty(\epsilon))\,ds + \int_{\frac{t}{2}}^t k_D(t-s)[f(u_\infty(\epsilon)) - f(u(s,\epsilon))]\,ds \\
&\le f(u_\infty(\epsilon)) \frac{2}{\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} \sin^2 n\pi a \exp(-\frac{1}{2} n^2\pi^2 t) \\
&\quad + [f(u_\infty(\epsilon)) - f(u(\frac{t}{2},\epsilon))] \int_0^t k_D(t-s)\,ds < \frac{\rho}{2} + \frac{\rho}{2},
\end{aligned}$$

which in conjunction with (2.1) leads to (2.21). Hence,

$$\epsilon a(1-a) = u_\infty(\epsilon)/f(u_\infty(\epsilon)) < \Psi(m), \tag{2.23}$$

and consequently,

$$\epsilon^* a(1-a) = \Psi(m). \tag{2.24}$$

Equation (2.24) gives the formula to compute ϵ^*. Moreover, it implies that $u(t,\epsilon^*)$ exists globally and tends to m as $t \to \infty$. If $\epsilon > \epsilon^*$, however, $u(t,\epsilon)$ quenches in finite time by the definition of ϵ^*.

In summary, we have the following results.

Theorem. (I) *For the integral equation* (2.1) *with* $k = k_D$*, there is a critical value* ϵ^* *such that*

(a) *If* $\epsilon \leq \epsilon^*$*, every solution is global and uniformly bounded away from* c*;*

(b) *If* $\epsilon > \epsilon^*$*, quenching occurs in finite time.*

(II) *For the equation* (2.1) *with* $k = k_N$*, the solution quenches in finite time for every* $\epsilon > 0$.

It is worth noting that such results, when related to (PD) or (PN), are well consistent with those in [2, 6].

3. Growth rate at quenching. In this section, we are limiting ourselves to the situation where quenching does occur in finite time; that is,

$$u(t) \to c, \qquad \text{as } t \to \hat{t} < \infty. \tag{3.1}$$

Based on this assumption, we will develop a self-consistent asymptotic analysis of (2.1). This asymptotic technique will describe the quenching behavior of the solution, although it will not be sufficient to determine the actual quenching time $\hat{t}$ exactly. The class of nonlinearities under consideration arises in various applications and has the asymptotic growth property

$$f(u) \sim (c-u)^{-\alpha}, \qquad \text{as} \;\; u \to c, \tag{3.2}$$

with $\alpha > 0$. Our arguments parallel those of [7]. Due to the singularity possessed by $f(u)$, however, several notable differences appear at the technical level, and hence the relevant discussion will be presented in detail.

Under the assumption that (3.1) holds, we introduce the transformation,

$$\eta = (\hat{t} - t)^{-1} - \eta_0, \quad \eta_0 = (\hat{t})^{-1}, \quad w(\eta) = c - u(t). \tag{3.3}$$

Thus, the quenching behavior (3.1) becomes

$$w(\eta) \to 0, \qquad \text{as} \;\; \eta \to \infty. \tag{3.4}$$

The transformation (3.3) converts (2.1) to the form

$$c - w(\eta) = \epsilon \int_0^\eta k\{(\eta - \zeta)[(\zeta + \eta_0)(\eta + \eta_0)]^{-1}\} \varphi(\zeta)\, d\zeta, \qquad \eta \geq 0, \tag{3.5}$$

where

$$\varphi(\eta) = (\eta + \eta_0)^{-2} f(c - w(\eta)). \tag{3.6}$$

The advantage of this transformation is that certain techniques developed in [1] can be utilized for the asymptotic evaluation, as $\eta \to \infty$, of integrals like that in (3.5).

Following the methods of [1], let $\zeta = \eta\tau$, so that (3.5) becomes

$$c - w(\eta) = \eta I(\eta), \tag{3.7}$$

where

$$I(\eta) = \epsilon \int_0^1 k\{\eta(1-\tau)[(\eta\tau + \eta_0)(\eta + \eta_0)]^{-1}\}\varphi(\eta\tau)\, d\tau. \tag{3.8}$$

By means of the image representation of the kernels in (2.2) and (2.3), in a similar manner as that of [7], we can show that

$$I(\eta) \sim \frac{1}{2}\int_0^1 \frac{\Phi(\eta\tau)}{\sqrt{\pi(1-\tau)}}\, d\tau, \qquad \eta \to \infty, \tag{3.9}$$

where

$$\Phi(\eta\tau) = (\eta\tau + \eta_0)^{1/2}\varphi(\eta\tau) = (\eta\tau + \eta_0)^{-3/2} f(c - w(\eta)). \tag{3.10}$$

$I(\eta)$ as given in (3.9) corresponds to (4.10.15) in [1] where

$$I(\eta) \sim \frac{1}{2}\eta^{-1/2} I^{1/2}[\Phi; \eta]$$

with $I^{1/2}$ being the Riemann fractional integral of order 1/2. The leading order behavior of $I(\eta)$ enables us to determine the growth of the solution near quenching for various nonlinearities f in (3.10). To investigate the asymptotic behavior of $I(\eta)$ as $\eta \to \infty$, the technique of [1] suggests employment of the Parseval formula for Mellin transforms to convert the integral in (3.7) to one defined in the complex plane. Thus, as $\eta \to \infty$, the asymptotic integral equation becomes

$$c - w(\eta) \sim \frac{\epsilon\eta}{4\pi i}\int_{b-i\infty}^{b+i\infty} \eta^{-z} \frac{\Gamma(1-z)}{\Gamma(\frac{3}{2} - z)} M[\Phi(\tau); z]\, d\tau, \qquad \eta \to \infty, \tag{3.11}$$

where the Mellin transform is defined by

$$M[v(\tau); z] \equiv \int_0^\infty \tau^{z-1} v(\tau)\, d\tau. \tag{3.12}$$

In order to proceed with the asymptotic analysis of (3.11), we need more information about $M[\Phi; z]$. To this end, we assume that

$$w(\eta) \sim A\eta^{-\beta}, \quad \text{as} \quad \eta \to \infty, \tag{3.13}$$

where A and β (> 0) are to be determined. From (3.2) and (3.10), we then have

$$\Phi(\eta) \sim A^{-\alpha}(\eta + \eta_0)^{-3/2}\eta^{\alpha\beta}, \qquad \text{as} \quad \eta \to \infty, \tag{3.14}$$

or equivalently,

$$\Phi(\eta) \sim A^{-\alpha}\eta^{\alpha\beta-3/2}. \tag{3.15}$$

Since $\Gamma(1-z)$ has a simple pole

$$\Gamma(1-z) \sim -\frac{1}{z-1}, \qquad \text{as} \quad z \to 1, \tag{3.16}$$

and $M[\Phi; z]$ has a simple pole

$$M[\Phi; z] \sim -\frac{A^{-\alpha}}{z-(3/2-\alpha\beta)}, \qquad \text{as} \quad z \to 3/2-\alpha\beta, \tag{3.17}$$

it is possible for either pole to be encountered first or they could be encountered simultaneously. Therefore we need to consider all three possibilities.

Case I. $3/2-\alpha\beta < 1$. The leading asymptotic contribution of the integral in (3.11) comes from the Mellin transform, and hence the vertical path of integration in the complex z-plane lies between the two poles. Equation (3.11) then takes the form

$$c - A\eta^{-\beta} \sim \frac{\epsilon A^{-\alpha}\Gamma(\alpha\beta-1/2)}{2\Gamma(\alpha\beta)}\eta^{\alpha\beta-1/2}, \qquad \text{as } \eta \to \infty. \tag{3.18}$$

Since $\alpha\beta - 1/2 > 0$, when matching the two sides of (3.18), we cannot obtain an asymptotic balance.

Case II. $3/2-\alpha\beta = 1$. To compute the asymptotic contribution from the integral in (3.11), the vertical path is displaced to the right. In this case the pole at $z = 1$ implied by (3.17) coalesces with that from (3.16) to give rise to a double pole at $z = 1$. Thus, (3.11) takes the form

$$c - A\eta^{-\beta} \sim \frac{\epsilon A^{-\alpha}}{2\Gamma(1/2)}, \qquad \text{as} \quad \eta \to \infty. \tag{3.19}$$

Because the right-hand side in (3.19) does not possess any minor term of the form $O(\eta^{-\gamma})$ $(\gamma > 0)$, we still cannot obtain an asymptotic balance.

Case III. $3/2-\alpha\beta > 1$. This time the leading asymptotic contribution comes from the Gamma function. A complete asymptotic analysis, beyond the leading order, can be carried out if the vertical path of integration is displaced to the right of the simple pole of $M[\Phi; z]$ as well. In this case, (3.11) is equivalent to

$$c - A\eta^{-\beta} \sim \frac{\epsilon}{2\sqrt{\pi}}M[\Phi; 1] + \frac{\epsilon A^{-\alpha}\Gamma(\alpha\beta-1/2)}{2\Gamma(\alpha\beta)}\eta^{\alpha\beta-1/2}, \qquad \text{as } \eta \to \infty. \tag{3.20}$$

Matching the minor terms on the two sides of (3.20) then yields

$$\beta = \frac{1}{2(\alpha+1)}, \tag{3.21}$$

and

$$A = \Big[\frac{\epsilon\Gamma(\alpha\beta+1/2)}{2(1/2-\alpha\beta)\Gamma(\alpha\beta)}\Big]^{\frac{1}{\alpha+1}} = \Big[\epsilon(\alpha+1)\Gamma\big(\frac{2\alpha+1}{2\alpha+2}\big)\Big/\Gamma\big(\frac{\alpha}{2\alpha+2}\big)\Big]^{\frac{1}{\alpha+1}}. \quad (3.22)$$

Thus we have found that when the nonlinear source function behaves like (3.2), then

$$c - u(t) \sim A(\hat{t}-t)^{\frac{1}{2(\alpha+1)}}, \qquad \text{as } t \to \hat{t}. \quad (3.23)$$

Finally, we present the estimate on quenching time $\hat{t}$ for $f(u) \sim (c-u)^{-\alpha}$. Matching the leading terms on the two sides of (3.20), we observe

$$c \sim \frac{\epsilon}{2\sqrt{\pi}} M[\Phi;1]. \quad (3.24)$$

In view of (3.12) and (3.14), we evaluate $M[\Phi;1]$ as follows:

$$\begin{aligned} M[\Phi;1] &\sim A^{-\alpha}\int_0^\infty (\eta+\eta_0)^{-3/2}\eta^{\alpha\beta}d\eta = A^{-\alpha}(\hat{t})^{-\alpha\beta}\int_0^{\hat{t}} t^{\alpha\beta}(\hat{t}-t)^{-1/2-\alpha\beta}dt \\ &= A^{-\alpha}(\hat{t})^{1/2-\alpha\beta}\int_0^1 \tau^{\alpha\beta}(1-\tau)^{-1/2-\alpha\beta}d\tau \quad (3.25) \\ &= A^{-\alpha}(\hat{t})^{1/2-\alpha\beta}\Gamma(1+\alpha\beta)\Gamma(1/2-\alpha\beta)/\Gamma(3/2). \end{aligned}$$

Substitution of (3.25) into (3.24) yields

$$(\hat{t})^{1/2-\alpha\beta} \sim \frac{2c\sqrt{\pi}A^\alpha\Gamma(3/2)}{\epsilon\Gamma(1+\alpha\beta)\Gamma(1/2-\alpha\beta)} = \frac{c\pi A^\alpha}{\epsilon\Gamma(1+\alpha\beta)\Gamma(1/2-\alpha\beta)}. \quad (3.26)$$

Then replacing A by (3.22), we obtain the following:

$$\hat{t} \sim \Big[\frac{2c\pi}{\alpha\Gamma\big(\frac{1}{2\alpha+2}\big)}\Big]^{2(\alpha+1)}\frac{(\alpha+1)^{4\alpha+2}\Gamma^{2\alpha}\big(\frac{2\alpha+1}{2\alpha+2}\big)}{\epsilon^2\Gamma^{4\alpha+2}\big(\frac{\alpha}{2\alpha+2}\big)}. \quad (3.27)$$

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