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CONDITIONS FOR A CENTRE IN A SIMPLE CLASS OF CUBIC SYSTEMS

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Abstract. We determine necessary and sufficient conditions for a planar system of the form

$$\frac{dx}{dt} = ax + by + x(\alpha x + \beta y + Ax^2 + Bxy + Cy^2)$$
$$\frac{dy}{dt} = cx + dy + y(\alpha x + \beta y + Ax^2 + Bxy + Cy^2)\,,$$

where $a, b, c, d, \alpha, \beta, A, B$ and C are real constants, to possess a centre at the origin. The approach for computing the necessity of the conditions (i.e., for the calculation of the focal values) is somewhat unconventional, since it relies on the properties of an integrating factor, rather than a first integral, which expedites the procedure. The subclass consisting of those systems which possess a centre at the origin is studied further. In particular, first integrals are given, and phase portraits are drawn of all topologically inequivalent nonlinear members.

1. Introduction. In this article, we shall be chiefly concerned with polynomial systems which are of the form

$$\begin{aligned}\frac{dx}{dt} &= -y + xf(x,y)\\ \frac{dy}{dt} &= x + yf(x,y)\,,\end{aligned} \tag{1.1}$$

where $f(x,y) = \alpha x + \beta y + Ax^2 + Bxy + Cy^2$, and α, β, A, B and C are (real) constants. It will be shown that the origin is a centre for system (1.1) if and only if

$$A + C = 0 \quad \text{and} \quad A\alpha^2 + B\alpha\beta + C\beta^2 = 0. \tag{1.2}$$

If $\alpha = \beta = 0$, i.e., if the system (1.1) has cubic homogeneous nonlinearities, then it is in fact possible to effect a rotation so that $A = C = 0$ and $B \geq 0$; if $B > 0$, a uniform rescaling of x and y may be employed to ensure that $B = 1$. In such a case, the origin is a centre if and only if the system (1.1) either is linear (i.e., $f(x,y) \equiv 0$) or may be brought to the form

$$\begin{aligned}\frac{dx}{dt} &= -y(1 - x^2)\\ \frac{dy}{dt} &= x(1 + y^2),\end{aligned} \tag{1.3}$$

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which, it may be observed, is invariant under the reflections $x' = \delta x$, $y' = \epsilon y$, $t' = \delta\epsilon t$, where $\delta^2 = \epsilon^2 = 1$. In particular, with $\delta = \epsilon = -1$, we see that (1.3) is a cubic system which is symmetric with respect to the centre, and so it is a member of the class which was studied by Rousseau and Schlomiuk ([22]). Moreover, with $\delta = -\epsilon = 1$ and $\delta = -\epsilon = -1$, it follows that (1.3) is a member of the subclass of Type II in the nomenclature of Rousseau and Schlomiuk ([22]).

On the other hand, if $(\alpha, \beta) \neq (0, 0)$, then, by means of a suitable rotation, we may in fact always arrange that $\beta = 0$, in which case $\alpha \neq 0$, and, by means of a uniform rescaling of x and y, we may then ensure that $\alpha = 1$. Consequently, using (1.2), we then find that the origin is a centre if and only if $A = C = 0$, i.e., if and only if system (1.1) may be brought to the form

$$\begin{aligned} \frac{dx}{dt} &= -y + x^2 + Bx^2y \\ \frac{dy}{dt} &= x + xy + Bxy^2, \end{aligned} \tag{1.4}$$

which is invariant under the reflection $x' = -x$, $y' = y$, $t' = -t$. It may be noted that in all cases, therefore, we have, without loss of generality, that $A = C = 0$. As discussed in Section 3, the special system (1.3) may be regarded as the limiting case of system (1.4) when $B \to \pm\infty$.

In fact, the centre of system (1.4) is rationally reversible (see, e.g., Żołądek, [26]), as may easily be seen by applying the transformation $x \to X = x^2$, $y \to Y = y$ and $t \to T$, where $\frac{dT}{dt} = x$, on either half-plane where $x > 0$ or $x < 0$. This results in the system

$$\begin{aligned} \frac{dX}{dT} &= 2(-Y + X + BXY) \\ \frac{dY}{dT} &= 1 + Y + BY^2, \end{aligned} \tag{1.5}$$

which is defined for *all* (real) X and Y, and which possesses no critical points for $X \geq 0$. This reduction also shows how a general integration of (1.4) is in principle possible, since, if $1 + Y + BY^2 \neq 0$, the equations (1.5) are related to the first-order linear ordinary differential equation

$$\frac{dX}{dY} + P(Y)X = Q(Y),$$

where

$$P(Y) := \frac{-2(1 + BY)}{1 + Y + BY^2} \quad \text{and} \quad Q(Y) := \frac{-2Y}{1 + Y + BY^2}.$$

However, there are simpler procedures for integrating (1.4), and at any rate it is perhaps more useful to obtain phase portraits. In fact, there are five topologically distinct possibilities for (1.4), depending upon the conditions $B < 0$, $B = 0$, $0 < B < \frac{1}{4}$, $B = \frac{1}{4}$ and $B > \frac{1}{4}$. It may be noted that the subcase of (1.4) for which $B = 0$ provides a canonical form for the *quadratic* systems of form (1.1), and that then the origin is always a centre.

In a similar fashion, if to system (1.3) we apply the transformation $x \to X = x^2$, $y \to Y = y^2$ and $t \to T$, where $\frac{dT}{dt} = 2xy$, on each quadrant where $xy \neq 0$, we obtain the linear system

$$\frac{dX}{dT} = -1 + X$$
$$\frac{dY}{dT} = 1 + Y,$$

which is defined for *all* (real) X and Y, and which possesses no critical points for $X \geq 0$ and $Y \geq 0$. Rousseau and Schlomiuk ([22]) show that the same transformation may be applied to all members of their subclass of Type II, the resulting systems being linear.

The system (1.1) is a special case of the more general family of systems of degree $n \geq 1$, which are of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by + xg(x,y) \\ \frac{dy}{dt} &= cx + dy + yg(x,y), \end{aligned} \tag{1.6}$$

where a, b, c and d are (real) constants, and where g is a polynomial in x and y, of degree $n-1$, whose constant term is zero. It is easily shown that, if the origin is a centre of (1.6), then the linearisation matrix $\begin{pmatrix} a & b \\ a & d \end{pmatrix}$ is necessarily nonzero, and has two (complex conjugate) purely imaginary eigenvalues, i.e., without loss of generality, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It follows from this that any centre of system (1.6) must occur at the origin, and is necessarily isochronous, since, in polar coordinates (r, θ), the angle θ satisfies the equation $\frac{d\theta}{dt} = 1$. This constancy of the angular velocity means that the center is "uniformly isochronous" and so it cannot be global, except in the linear case (Conti, [6]). Discussions of isochronous centres have recently been provided by Villarini ([25]) and by Christopher and Devlin ([3]); these works refer to a number of articles on the topic, and include mention of systems such as (1.1).

The plan of the present article is as follows. In Section 2, we first consider systems of degree n, of form (1.6), and obtain a necessary condition on the linearisation matrix, in order for the origin to be a centre. We thenceforth specialize to the case when $n = 3$, thereby leading to a consideration of the system (1.1). The necessary and sufficient conditions (1.2) for the origin to be a centre are then obtained in Theorem 2.2. The proof of the necessity of the conditions (1.2) uses a method which involves the consideration of an analytic integrating factor which is nonzero on some neighbourhood of the origin. This relies on the application of a variant of the criterion of Reeb ([21]), which is also discussed by Mattei and Moussu ([14]) and Moussu ([15]). A statement and an elementary proof of this criterion are provided in Appendix 1 (see Proposition A1.4 and the remarks thereafter). The establishment of the conditions (1.2) leads, in the manner explained, to the systems (1.3) and (1.4), whose phase portraits are given in Section 3, together with a brief discussion of the first integrals and some of their interconnections.

Many of the fundamental computations, which relate to the Liapunov quantities for system (1.1), are more elegantly and economically expressed using tensor notation (for an introduction to this, see Collins, [4]). Since this may be unfamiliar to most readers, those details which are provided here are relegated to Appendix 2. One rather compelling feature is that the tensorial versions of the conditions (1.2), which, by their very nature, are invariant under a nonsingular linear transformation of the dependent variables x and y, are given in a particularly succinct form by equations (A2.9). This is especially useful for determining in Corollary 2.3 the analogous conditions for a centre when the linearisation matrix has not been brought to the standard form. It thereby provides a general practical test for whether or not the origin is a centre in the case of any system of form (1.6) with $n = 3$, while avoiding the possibly inconvenient step of first transforming to suitable new dependent variables.

2. Conditions for a centre. As a preliminary result which is of use in a more detailed investigation of quadratic and cubic systems, we establish the following proposition.

Proposition 2.1. *Consider a polynomial system of degree $n \geq 1$, whose form is given by*

$$\begin{aligned} \frac{dx}{dt} &= ax + by + xg(x,y) \\ \frac{dy}{dt} &= cx + dy + yg(x,y), \end{aligned} \tag{2.1}$$

where a, b, c and d are (*real*) *constants, and g is a polynomial in x and y, of degree $n-1$, whose constant term is zero. If the origin is a centre for* (2.1), *then it is necessary that the linearisation matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ have* (*nonzero*) *complex conjugate eigenvalues with zero real part, i.e., that $a + d = 0$ and $ad - bc > 0$, and hence, without loss of generality, that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.*

Proof. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ possesses two real eigenvalues with either a one- or a two-dimensional eigenspace, then system (2.1) possesses at least one invariant line passing through the origin, which therefore cannot be a centre. The only other possibility is that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ possesses two complex conjugate eigenvalues (with nonzero imaginary parts). If the real part of these eigenvalues is nonzero, then, by standard theory, the origin is a focus, so the only surviving possibility is that the eigenvalues be nonzero and purely imaginary. Then, by means of a linear transformation, we may arrange in the standard way for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be equal to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Remarks. 1. An alternative statement of Proposition 2.1 is that, for systems of form (2.1), a centre at the origin is necessarily nondegenerate, i.e., that, in the terminology of, e.g., Rousseau and Schlomiuk ([22–24]), it is a "weak focus," although some authors reserve this to apply only when, in a suitable neighbourhood of a critical point, the integral curves are topologically equivalent to those of a focus.

2. When the linearisation matrix has been reduced to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the form of the system (2.1) is invariant under a combination of a rotation, a homothety and a

reflection in a coordinate axis (with a concomitant reversal in the direction of t), i.e., it is invariant under the transformation

$$\begin{aligned} x \to x' &= \alpha(x\cos\phi + y\sin\phi),\\ y \to y' &= \alpha\epsilon(-x\sin\phi + y\cos\phi),\\ t \to t' &= \epsilon t\,, \end{aligned} \tag{2.2}$$

where $\epsilon = \pm 1$, $0 \le \phi < 2\pi$ and α is an arbitrary nonzero real number.

3. It follows directly from Proposition 2.1 that, if the origin is a centre, then there are no other (real, finite) critical points of system (2.1); in fact, this is a general feature of uniformly isochronous centres (Conti [6]). It is easily seen that the same conclusion is reached if instead the origin is a nondegenerate focus of (2.1).

4. Villarini ([25]) and Christopher and Devlin ([3]) have shown that any isochronous centre must be nondegenerate. Christopher and Devlin ([3]) and Conti ([6]) point out that if the origin is a centre for any system of form (2.1) in which $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, then the centre is isochronous. Proposition 2.1 provides a rather different result, since it shows that, if there is a centre for system (2.1) at the origin, then it must be nondegenerate, and, following Christopher and Devlin [3] and Conti [6], it must be isochronous.

We now apply Proposition 2.1 to obtain the necessary and sufficient conditions for the origin to be a centre when the system is of form (2.1) with $n = 3$, i.e., for the system

$$\frac{dx}{dt} = -y + x(\alpha x + \beta y + Ax^2 + Bxy + Cy^2) \tag{2.3a}$$

$$\frac{dy}{dt} = x + y(\alpha x + \beta y + Ax^2 + Bxy + Cy^2), \tag{2.3b}$$

where α, β, A, B and C are (real) constants. In the proof of the following theorem, we shall need to recall that, in accordance with the Remarks 2 and 3 following Proposition 2.1, the form of system (2.3) is invariant under the transformation (2.2), and that if the origin is a centre for (2.3), then it is the only (real) critical point of the system.

Theorem 2.2. *The system* (2.3) *possesses a centre at the origin if and only if*

$$A + C = 0 \quad \textit{and} \quad A\alpha^2 + B\alpha\beta + C\beta^2 = 0. \tag{2.4}$$

Proof. (a) **Sufficiency**. We suppose that equations (2.4) hold, and show that then the origin is a centre for system (2.3). If $(\alpha, \beta) = (0, 0)$, we subject the system to a rotation $x' = x\cos\phi + y\sin\phi$, $y' = -x\sin\phi + y\cos\phi$. It is readily calculated that the coefficients of ${x'}^3$ and ${x'}^2 y'$ in the counterpart of (2.3a) are respectively

$$A' = A\cos^2\phi + B\sin\phi\cos\phi + C\sin^2\phi = A\cos 2\phi + \frac{1}{2}B\sin 2\phi$$

and

$$B' = 2(C - A)\sin\phi\cos\phi + B(\cos^2\phi - \sin^2\phi) = -2A\sin 2\phi + B\cos 2\phi.$$

The coefficient C' of $x'y'^2$ in the counterpart of (2.3a) satisfies the condition $A'+C' = 0$. It is therefore clear that, without loss of generality, $A = C = 0$. If $B = 0$, then system (2.3) is linear, and the origin is obviously a centre, so we henceforth assume that $B \neq 0$, in which case the rotational freedom remaining (i.e., preserving the condition that $A = 0$) is subject to the restriction that $\sin 2\phi = 0$ (i.e., $\phi = \dfrac{k\pi}{2}$, where k is an integer), under which circumstances $B' = (-1)^k B$, so it follows that, without loss of generality, $B > 0$. A uniform rescaling of x and y may then be employed to ensure that $B = 1$, and so the system is equivalent to (1.3), which possesses the first integral $\frac{x^2+y^2}{1+y^2} = K$, where K is a nonnegative constant. For values of K satisfying $0 < K < 1$, the integral curves are therefore ellipses centred on the origin, i.e., the origin is a centre. This alternatively follows by noting that system (1.3) is invariant under the reflection $x \to x' = -x$, $y \to y' = y$, $t \to t' = -t$, and using a standard criterion to draw the conclusion.

If $(\alpha, \beta) \neq (0, 0)$, then under a rotation $x' = x\cos\phi + y\sin\phi$, $y' = -x\sin\phi + y\cos\phi$, the coefficients of x'^2 and $x'y'$ in the counterpart of (2.3a) are respectively $\alpha' = \alpha\cos\phi + \beta\sin\phi$, $\beta' = -\alpha\sin\phi + \beta\cos\phi$. Hence, without loss of generality, $\beta = 0$ and $\alpha \neq 0$, and so, by equations (2.4), it follows that $A = C = 0$. The system (2.3) is then invariant under the reflection $x \to x' = -x$, $y \to y' = y$, $t \to t' = -t$, which, by a standard criterion, ensures that the origin is a centre. This concludes the proof of the sufficiency of the conditions (2.4).

(b) **Necessity**. We now suppose that system (2.3) possesses a centre at the origin. For computational purposes, it will be convenient to rewrite the system using the complex dependent variables $z = x + iy$ and $\bar{z} = x - iy$, so that, for example,

$$\frac{dz}{dt} = iz + z(\gamma z + \bar{\gamma}\bar{z} + Dz^2 + Ez\bar{z} + \bar{D}\bar{z}^2), \tag{2.5}$$

where $\gamma = \frac{1}{2}(\alpha - i\beta)$, $D = \frac{1}{4}(A - C - iB)$ and $E = \frac{1}{2}(A + C)$. The conditions (2.4) are then equivalent to the requirements that

$$E = 0 \quad \text{and} \quad D\bar{\gamma}^2 + \bar{D}\gamma^2 = 0$$

and we now proceed to show that these are indeed fulfilled. By Reeb's criterion (see Proposition A1.4 in Appendix 1), if the origin is a centre, then there is an integrating factor, R, which is nonzero and analytic on some neighbourhood of the origin. It will, however, be more convenient to consider instead a function, F, defined by $F = R^\lambda$, where λ is a nonzero real number; clearly F will also be nonzero and analytic in a neighbourhood of the origin. For an analytic system of form $\frac{dx}{dt} = P(x, y)$, $\frac{dy}{dt} = Q(x, y)$, the integrating factor, R, satisfies the condition $\frac{dR}{dt} + R(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}) = 0$

(see equation (A1.4) of Appendix 1). It follows that, for the system consisting of (2.5) and its complex conjugate, the function F satisfies

$$\frac{dF}{dt} + \lambda F\left[3(\gamma z + \bar{\gamma}\bar{z}) + 4(\bar{D}\bar{z}^2 + Ez\bar{z} + Dz^2)\right] = 0. \tag{2.6}$$

Without loss of generality, $F(0,0) = 1$, and so we expand F in the form

$$F = 1 + pz + \bar{p}\bar{z} + qz^2 + rz\bar{z} + \bar{q}\bar{z}^2 + sz^3 + t'z^2\bar{z} + \bar{t}'z\bar{z}^2 + \bar{s}\bar{z}^3 + \cdots, \tag{2.7}$$

where $p, q, r, s, t', \ldots$ are (in general, complex) constants. We then substitute this expansion into (2.6), and employ (2.5) and its conjugate. The result is a power series in z and $\bar{z}$, equated to zero, whose coefficients must therefore all vanish. It may easily be verified that, with some important exceptions, the resulting sequence of equations will provide the determination of any coefficient $(p, q, \ldots)$ of a term of mth degree in the expansion (2.7), as a polynomial in λ, of degree m; the coefficients of the various powers of λ will depend on both the coefficients in (2.7) of terms of degree less than m, and the coefficients $\gamma, \bar{\gamma}, D, \bar{D}$ and E in the system (2.5) of differential equations. The exceptions to this involve the (real) coefficients $(r, \ldots)$ in (2.7) of terms $z\bar{z}, z^2\bar{z}^2, \ldots$. Such coefficients are not determinable; instead, one obtains an algebraic constraint between the coefficients in (2.7) of terms of lower degree. These coefficients of $z\bar{z}, z^2\bar{z}^2$, etc. are not determinable because they are freely specifiable (see Remark 3 following the proof). Any such algebraic constraint between the coefficients in (2.7) is a necessary condition for the origin to be a centre. It is ultimately expressible in terms of coefficients of the system (2.5) alone; it cannot involve λ, which does not appear in (2.5), and which has only been introduced for convenience, nor may it involve the coefficients in (2.7) of terms such as $z\bar{z}, z^2\bar{z}^2$, etc., since these are freely specifiable. As a result, in carrying out computations of necessary conditions, it is justified to ignore such coefficients (i.e., by equating them to zero). This will mean that the polynomials in λ discussed above have zero constant term, and for these computational purposes it is also justified to ignore any term which involves a power of λ in excess of unity (since higher powers are destined to cancel out; the remaining expressions will have a common factor of λ, which will therefore also cancel). In obtaining a sequence of necessary conditions, it is of course also acceptable to incorporate any of those previously encountered. When an equation is valid subject to the above modifications, we shall express it by means of the symbol '$\overset{*}{=}$', rather than '$=$'. Proceeding in the manner indicated, we readily deduce that

$$p = 3i\lambda\gamma \quad \text{and} \quad q \overset{*}{=} -\frac{\lambda}{2}(3\gamma^2 + 4iD). \tag{2.8}$$

In place of an equation for r, the coefficient of $z\bar{z}$ in (2.7), we immediately obtain the constraint

$$E = 0.$$

Proceeding further, we have that

$$s \overset{*}{=} -i\frac{\lambda}{3}\gamma(3\gamma^2 - 7iD) \quad \text{and} \quad t' \overset{*}{=} -i\frac{\lambda}{3}\bar{\gamma}(\gamma^2 - iD). \tag{2.9}$$

In place of an equation for the coefficient of $z^2\bar{z}^2$, we obtain the constraint

$$\frac{1}{3}D\bar{q} + \frac{1}{3}\bar{D}q + \frac{1}{2}(\gamma\bar{t}' + \bar{\gamma}t') = 0,$$

which, using (2.8) and (2.9), is equivalent to the condition

$$D\bar{\gamma}^2 + \bar{D}\gamma^2 = 0,$$

as required, and this completes the proof of the necessity of the conditions (2.4).

Remarks. 1. The conditions for a centre which are provided by Theorem 2.2 are of direct use when the linearisation matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of (2.1) has already been reduced to the standard form $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In practice, it may not be convenient to perform this reduction prior to applying the tests for a centre. The question then arises as to what the associated conditions would be, for the more general system. The answer is provided by the following corollary.

Corollary 2.3. *The system*

$$\begin{aligned} \frac{dx}{dt} &= ax + by + x(\alpha x + \beta y + Ax^2 + Bxy + Cy^2) \\ \frac{dy}{dt} &= cx + dy + y(\alpha x + \beta y + Ax^2 + Bxy + Cy^2)\,, \end{aligned}$$

where $a, b, c, d, \alpha, \beta, A, B$ and C are (real) constants, possesses a centre at the origin if and only if

$$a + d = 0, \quad ad - bc > 0, \quad bA - aB - cC = 0$$

and

$$(b\alpha - a\beta)^2 A - (b\alpha - a\beta)(a\alpha + c\beta)B + (a\alpha + c\beta)^2 C = 0.$$

Proof. The conditions that $a + d = 0$ and $ad - bc > 0$ have already been determined to be necessary for the origin to be a centre. The remaining two conditions are the generalized versions of the conditions (2.4). They may be determined by a laborious, pedestrian computation, but they follow immediately from a tensorial treatment (see Theorem A2.1 of Appendix 2, and the remarks thereafter, which lead to (A2.10)).

2. In the part of the proof concerned with the necessity of the conditions, we showed that when $(\alpha, \beta) \neq (0, 0)$, it is possible to apply a rotation so that $\beta = 0$ and $\alpha \neq 0$, in which case $A = C = 0$. One may further apply a homothety to ensure that $\alpha = 1$, in which case system (2.3) agrees with (1.4).

3. The choice of complex dependent variables in (2.5) is akin to that used by Dulac ([10]) in his study of quadratic vector fields. In Dulac's approach, a power series expansion for a *first integral* is sought (which, in the present treatment, is the limiting case when $\lambda \to 0$; cf. equation (2.6)). This also gives rise to a sequence of equations for determining the associated coefficients, together with a set of constraints, and a set of freely specifiable coefficients. The arbitrariness which we have encountered in the coefficients of $z\bar{z}, z^2\bar{z}^2$, etc. may be traced to the fact that if F is any suitable integrating factor, then so also is $\tilde{F} := F \times [1 + a_1\phi + a_2\phi^2 + \cdots]$, where ϕ is a first integral and the coefficients $a_1, a_2, \ldots$ are arbitrary real numbers (for which the series converges). Following the results of Poincaré ([19]) and Lyapunov ([13]), the first integral, ϕ, may be written in the form $\phi = x^2 + y^2 + \sum_{m=3}^{\infty} \phi_m(x, y)$, where ϕ_m is a homogeneous polynomial in x and y, of degree m, with $m \geq 3$. We may therefore choose a_1 so that the revised value of the coefficient of $z\bar{z}$ in the expansion of $\tilde{F}$ has some desired property (Dulac's choice corresponds to making it unity). Having so fixed this coefficient, any further adjustments to F of the above type must have $a_1 = 0$. Then, one may still adjust a_2 so that the coefficient of $z^2\bar{z}^2$ is fixed in some specified manner (in effect, Dulac makes it zero), which thenceforth restricts a_2 to be zero, and so on.

The employment of an approach which involves an integrating factor ($\lambda \neq 0$), rather than a first integral ($\lambda \to 0$), is of advantage because it reduces the number of steps which are required to generate the necessary conditions for a centre; i.e., it is more efficient than the conventional method. It is also of use in the exploration of situations where the function F is a polynomial, which often occurs when there is a centre; aspects of this are briefly discussed, in the context of the system (1.4), in Appendix 2.

4. The special case of system (2.3) which has quadratic homogeneous nonlinearities (i.e., $A = B = C = 0$) automatically satisfies the conditions (2.4) for there to be a centre at the origin. The special case which has cubic homogeneous nonlinearities (i.e., $\alpha = \beta = 0$) possesses a centre at the origin if and only if $A + C = 0$. Christopher and Devlin ([3]) and Conti ([6]) have obtained necessary and sufficient conditions (in the form of integral expressions) for the origin to be a centre of a system (2.1), where the linearisation matrix is $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the polynomial g is *homogeneous* of arbitrary degree. In the cases when such a system is either quadratic or cubic, it is readily seen that their conditions are equivalent to those discussed here.

3. Exact integration and phase portraits. We have seen in the previous Sections that any (quadratic or cubic) system (1.1) may be reduced to either (1.3) or (1.4), i.e., to either

$$\left\{ \begin{aligned} \frac{dx}{dt} &= -y(1 - x^2) \\ \frac{dy}{dt} &= x(1 + y^2) \end{aligned} \right. \tag{3.1}$$

or

$$\begin{cases} \dfrac{dx}{dt} = -y + x^2 + Bx^2y \\ \dfrac{dy}{dt} = x + xy + Bxy^2. \end{cases} \tag{3.2}$$

The exact integration of system (3.1) is straightforward. By division, we readily obtain the first integral

$$\frac{x^2+y^2}{1+y^2} = K, \tag{3.3}$$

where K is a nonnegative constant; this means that the integral curves are conic sections (on the complement of the origin), being ellipses for $0 < K < 1$, hyperbolas for $K > 1$, and a parallel line pair for $K = 1$. It may be noted that (3.1) is a canonical form for one of the three cases that arise in the classification of those systems which have cubic homogeneous nonlinearities and which possess an isochronous centre at the origin (Pleshkan, [16]). This system was employed by Galeotti and Villarini ([11]), following Conti ([5]), as an example of a case wherein a critical point at infinity need not form the limit set of a trajectory which lies in the finite part of the plane. The first integral (3.3) corresponds to that of the special case where $a - d = b = c = 0$ of Rousseau and Schlomiuk ([22]).

The question of the integration of (3.2) in closed form may be approached in a number of ways. Perhaps the most immediate procedure, which is both more practical and more elegant than that of integrating the related system (1.5), involves the method of Darboux ([7]), which relies on the existence of sufficiently many algebraic invariant curves (or of limiting cases thereof); this procedure will now be discussed.

In order to integrate the system (3.2), we observe that it admits the complex invariant line pair $x \pm iy = 0$, and also, if $B \neq 0$, the parallel line pair $y = \alpha_\pm$, where $\alpha_\pm := \frac{-1\pm(1-4B)^{1/2}}{2B}$ (this last pair being distinct if and only if $B \neq \frac{1}{4}$, and real if and only if $B \leq \frac{1}{4}$). Indeed (cf. (2.5)),

$$\frac{d}{dt}(x+iy) = (x+iy)[i+x+Bxy],$$
$$\frac{d}{dt}(x-iy) = (x-iy)[-i+x+Bxy]$$

and, when $B \neq 0$,

$$\frac{d}{dt}(y-\alpha_\pm) = (y-\alpha_\pm)[Bx(y-\alpha_\mp)],$$

where the terms in square brackets are called "cofactors." If $B(B-\frac{1}{4}) \neq 0$, the linear combination of cofactors

$$B(\alpha_+ - \alpha_-)[i+x+Bxy] + B(\alpha_+ - \alpha_-)[-i+x+Bxy]$$
$$- 2(1+B\alpha_+)[Bx(y-\alpha_-)] + 2(1+B\alpha_-)[Bx(y-\alpha_+)]$$

is identically zero, and so it follows that (on the complement of the lines $y = \alpha_\pm$ in the case when $0 \neq B < \frac{1}{4}$) a first integral is

$$(x^2+y^2)\Big[y + \frac{1-(1-4B)^{\frac{1}{2}}}{2B}\Big]^{-(1-4B)^{-\frac{1}{2}}-1}\Big[y + \frac{1+(1-4B)^{\frac{1}{2}}}{2B}\Big]^{(1-4B)^{-\frac{1}{2}}-1} = K, \tag{3.4}$$

where K is a constant. It is clear that the expression on the left-hand side of (3.4) is always real for $(x, y) \in \mathbb{R}^2$, and it may be noted in passing that it reduces to an especially simple form when $B = 3/16$, which also arises in Appendix 2, as an example in which there is an integrating factor of an especially simple form. It is easily established that, when $B > \frac{1}{4}$, an alternative form of the first integral (3.4) is

$$\frac{x^2+y^2}{By^2+y+1}\exp\Big\{\frac{2}{(4B-1)^{\frac{1}{2}}}\arctan\Big[\frac{(4B-1)^{\frac{1}{2}}}{2By+1}\Big]\Big\} = K', \tag{3.5a}$$

where K' is a nonnegative constant. Similarly, when $0 \neq B < \frac{1}{4}$, alternative forms of (3.4) are

$$\frac{x^2+y^2}{By^2+y+1}\exp\Big\{\frac{2}{(1-4B)^{1/2}}\operatorname{arctanh}\Big[\frac{(1-4B)^{1/2}}{2By+1}\Big]\Big\} = \text{constant} \tag{3.5b}$$

for $|2By+1| > (1-4B)^{1/2}$, and

$$\frac{x^2+y^2}{By^2+y+1}\exp\Big\{\frac{2}{(1-4B)^{1/2}}\operatorname{arctanh}\Big[\frac{2By+1}{(1-4B)^{1/2}}\Big]\Big\} = \text{constant}$$

for $|2By+1| < (1-4B)^{1/2}$.

In the special case of (3.2) with $B = 0$, we have

$$\frac{d}{dt}(x^2+y^2) = 2x(x^2+y^2) \quad \text{and} \quad \frac{d}{dt}(y+1) = x(y+1),$$

and so (by a similar argument, or more simply by division) the first integral on the complement of the line $y = -1$ is

$$\frac{x^2+y^2}{(1+y)^2} = K, \tag{3.6}$$

where K is a nonnegative constant. This means that the integral curves are conic sections (on the complement of the union of the origin and the line $y = -1$), being ellipses if $0 < K < 1$, hyperbolas for $K > 1$, and a parabola if $K = 1$. When $B = 0$, system (3.2) is quadratic, and the first integral (3.6) corresponds to the case where $a = C = A + b = d = 0, b \neq 0$ of Schlomiuk ([24]). It may be noted that, when $B = 0$, (3.2) is a canonical form for one of the four cases that arise in the classification of those quadratic systems which possess an isochronous centre (Loud, [12]; cf. Pleshkan and Sibirskiĭ, [17]).

In the special case of (3.2) with $B = \frac{1}{4}$, we let $w = y + 2$, and obtain

$$\frac{d}{dt}(x^2+y^2) = \frac{1}{2}(x^2+y^2)x(w+2) \quad \text{and} \quad \frac{dw}{dt} = \frac{1}{4}w^2x,$$

whence $\left(\frac{x^2+y^2}{w^2}\right)e^{4/w}$ = constant (if $w \neq 0$), i.e., a first integral on the complement of the line $y = -2$ is

$$\frac{\left(x^2+y^2\right)e^{\frac{4}{2+y}}}{(2+y)^2} = K\,, \tag{3.7}$$

where K is a nonnegative constant.

It should be noticed that the special cases of (3.6) and (3.7), where $B = 0$ and $B = \frac{1}{4}$ respectively, may be recovered from the appropriate limit of the more general expression (3.4). Thus, when $B \to 0$, the second factor in the left-hand side of (3.4) approaches $(y+1)^{-2}$ and, by an application of l'Hôpital's rule, the third factor approaches unity, thus reducing (3.4) to (3.6). If $B \to \frac{1}{4}$, a similar but more complicated limiting procedure when applied to (3.4) leads to (3.7); alternatively, this result may be recovered more directly by applying the limit as $B \to \frac{1}{4}$ to the first integrals when expressed in the forms of (3.5a) and (3.5b). This type of limiting procedure is fairly common in solutions of polynomial systems. It may be used in the case of quadratic vector fields which possess a centre (a compilation of first integrals in this case, apparently without reference to the limiting procedure, has been provided by Schlomiuk, [24], following Dulac, [10]); a similar observation has been made in the case of some cubic vector fields by Rousseau and Schlomiuk ([22]), who note that Christopher ([2]) was also aware of this.

It may be observed that system (3.1) admits six invariant lines, viz., $x \pm iy = 0, x \pm 1 = 0$ and $y \pm i = 0$, from which, using the method of Darboux ([7]), the first integral (3.3) may be alternatively determined, in a variety of possible ways. It is of interest to note that system (3.1) is obtainable as the limiting case when $B \to +\infty$ in system (3.2), as may be seen upon first applying the homothety $x \to x' = \sqrt{B}x, y \to y' = \sqrt{B}y$ to (3.2), and then proceeding to the limit. The first integral (3.3) of system (3.1) is also obtainable by applying the same procedure to the first integral (3.4) of (3.2). A limiting case of system (3.2) when $B \to -\infty$ is similarly obtainable by applying a homothety combined with a rotation, viz., $x \to x' = \sqrt{-B}y$, $y \to y' = -\sqrt{-B}x$, and then proceeding to the limit. This leads once again to system (3.1). When applied to the first integral (3.4), this limiting procedure provides an alternative but equivalent form of the first integral (3.3) of system (3.1). The forms of the first integrals (3.3), (3.4), (3.6) and (3.7) are such that the existence of a centre at the origin is manifest.

The Poincaré compactifications of the phase portraits for systems (3.1) and (3.2) are drawn in Figures 1 and 2 respectively, from which are evident the nonglobal nature of the centre and its uniqueness as a critical point. Figure 1 corresponds to Figure 6 of Rousseau and Schlomiuk ([22]) (cf. also Galeotti and Villarini, [11]). There are five qualitatively distinct possibilities for Figure 2, and these are labeled (a), (b), (c), (d) and (e), according as $B < 0, B = 0$, $0 < B < \frac{1}{4}$, $B = \frac{1}{4}$ and $B > \frac{1}{4}$. The quadratic case (in which $B = 0$) is given by Figure 2(b) and agrees with that of the subcase $b = -1$ in Figure 3 of Schlomiuk ([24]).

We now comment briefly on some features of these phase portraits. There are two (real) invariant lines in the case of Figure 1 and Figures 2(a) and 2(c), there is one (real) invariant line in the case of Figures 2(b) and 2(d), and there are no (real)

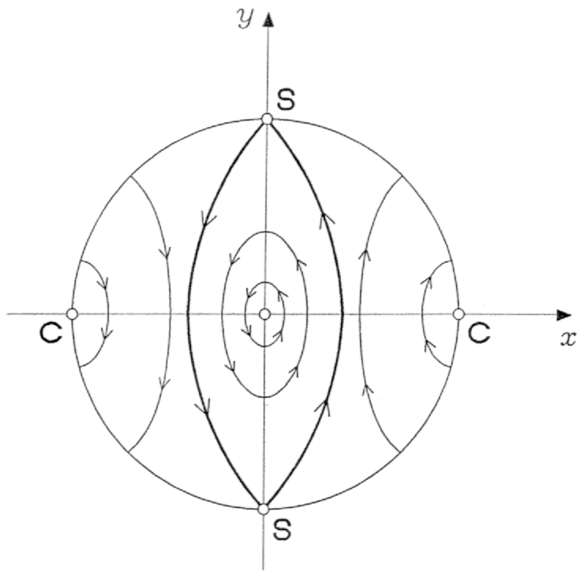

Figure 1. Phase portrait for system (3.1).
Solid curves indicate invariant lines, $S \equiv$ saddle and $C \equiv$ centre.

invariant lines for Figure 2(e); such invariant lines are indicated in the figures by means of heavily drawn curves. Conti ([6]) has shown that for a nonlinear system of degree n which possesses a uniformly isochronous centre, the boundary of the centre region is the union of ν open, unbounded trajectories (and no critical points), where $1 \leq \nu \leq n-1$. For the quadratic system (3.2) with $B=0$, this (single) boundary is given by the parabola $x^2 = 1+2y$, as is evident in Figure 2(b). In Figures 1 and 2(a), the two invariant lines form the boundary of the centre region, so here ν attains its maximal value of 2. In Figures 2(c), 2(d) and 2(e), this boundary consists of exactly one unbounded trajectory, which means that for system (3.2) with $B>0$, ν is strictly less than the maximum possible value for a cubic system. It may be noted that Conti ([6]) has provided, in his Example 4.1, a family of uniformly isochronous systems in which ν attains its upper bound of $n-1$, for arbitrary degree $n>1$; in the cases when $n=2$ and $n=3$, the systems in this example are equivalent to our systems (3.2) with $B=0$ and (3.1) respectively.

In all of the phase portraits, the equator (i.e., the line at infinity) is *not* an invariant curve of the extended vector field; this is related to the fact that, as observed by Poincaré ([18]), for a polynomial system $\frac{dx}{dt}=X$, $\frac{dy}{dt}=Y$ of degree n, the polynomial $xY-yX$ is normally of degree $n+1$, in which case the equator will consist of (the union of) orbits, but if $xY-yX$ is of degree less than $n+1$ (as in the case here), then, except at critical points, the equator is transverse to the flow. In Figure 2(b), when $B=0$ in (3.2), there are *no* critical points at infinity, and the integral curves never meet the equator orthogonally (they meet tangentially on the y-axis; the associated integral curve is the parabola $x^2 = 1+2y$ found by putting $K=1$ in (3.5)). In all other cases, there are critical points at infinity, and these lie on both coordinate

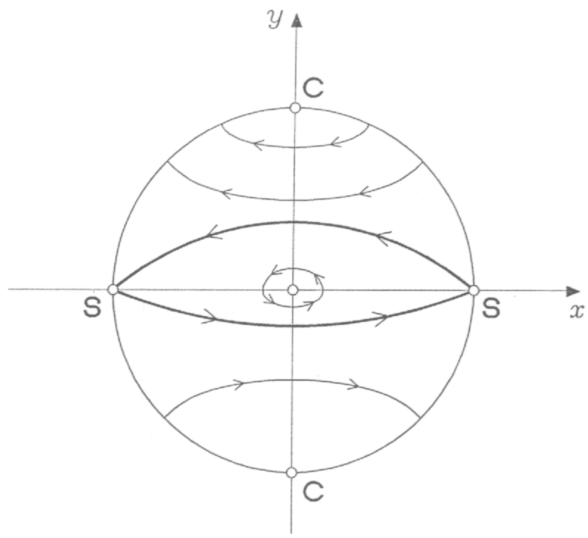

Figure 2-a $B < 0$

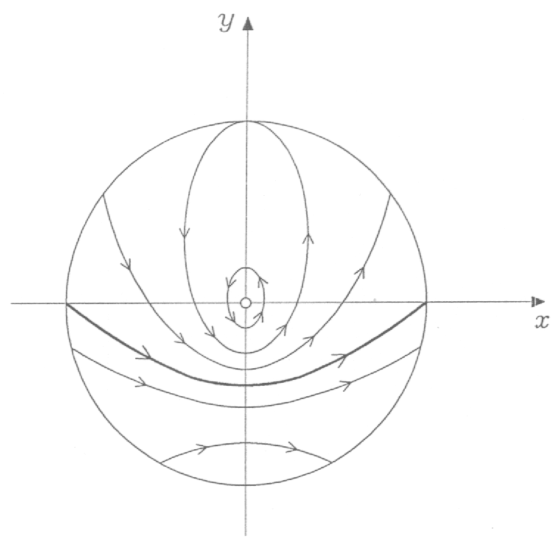

Figure 2-b $B = 0$

axes. The integral curves meet the equator either orthogonally or at a critical point. One of the critical points at infinity is a saddle, and this lies on the y-axis, except in Fig. 2(a) (corresponding to $B < 0$), when it lies on the x-axis. In Figure 2(c) (when $0 < B < \frac{1}{4}$), there is a node on the x-axis at infinity, whereas in Figure 2(d) (corresponding to $B = \frac{1}{4}$), this is replaced by a saddle-node. In Figure 2(e) (where $B > \frac{1}{4}$), the second critical point at infinity, which is on the x-axis, is a focus. In Figures 1 and 2(a), corresponding respectively to system (3.1) and system (3.2) with $B < 0$, the second critical point at infinity is a centre; this lies on the x-axis in Figure 1, and on the y-axis in Figure 2(a). The fact that a centre arises

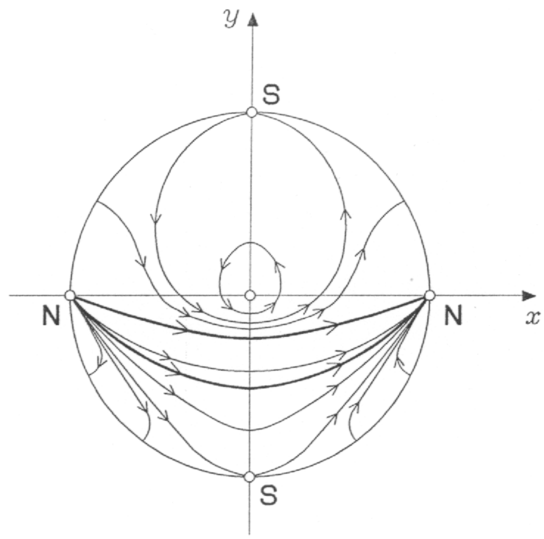

Figure 2-c $0 < B < \frac{1}{4}$

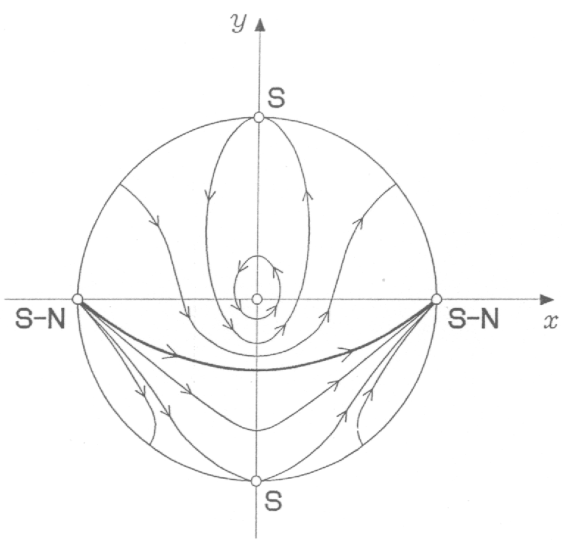

Figure 2-d $B = \frac{1}{4}$

in these cases, as opposed to a focus, may be deduced from a standard reflection criterion (specifically, for system (3.1) with $u = \frac{y}{x}$, $z = \frac{1}{x}$ and $\frac{d\tau}{dt} = \frac{1}{z}$, we obtain a polynomial system whose linearisation matrix, at the critical point $(u, z) = (0, 0)$, possesses nonzero purely imaginary eigenvalues, and the system is invariant under the reflection $u \to -u$, $z \to z$, $\tau \to -\tau$; for system (3.2) with $B < 0$, we put $v = \frac{x}{y}$, $z = \frac{1}{y}$ and $\frac{d\tau}{dt} = \frac{1}{z}$, obtaining a polynomial system whose linearisation matrix, at the critical point $(v, z) = (0, 0)$, has nonzero purely imaginary eigenvalues, and the system is invariant under the reflection $v \to -v$, $z \to z$, $\tau \to -\tau$). The only case

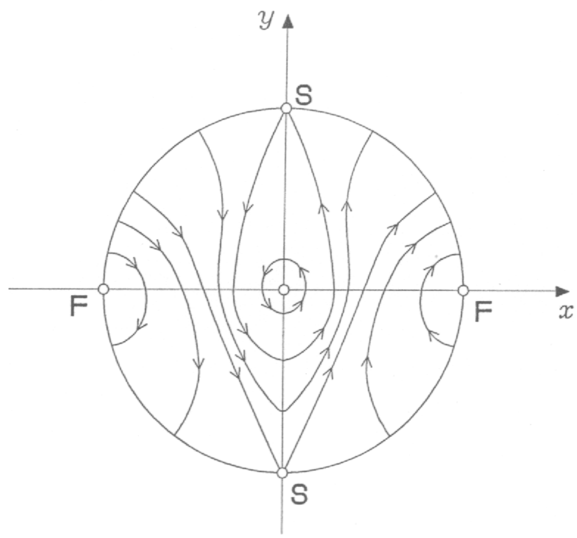

Figure 2-e $B > \frac{1}{4}$

Figure 2. Phase portraits for system (3.2). Solid curves indicate invariant lines, $S \equiv$ saddle, S-$N \equiv$ saddle-node, $N \equiv$ node, and $C \equiv$ centre.

in which a limit cycle at infinity could conceivably occur is that in which $B > \frac{1}{4}$, and this possibility is ruled out upon transforming to suitable variables (e.g., $u = \frac{y}{x}$, $z = \frac{1}{x}$) and then examining the first integral.

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Appendix 1. Throughout, we shall consider real vector fields which are analytic on a neighbourhood of a critical point, which, by a translation, we may take to be at the origin. Let P_k and Q_k be (real) homogeneous polynomials in x and y, of degree k, where k is an integer and $k \geq 2$.

Then the system associated with the vector field is expressible as

$$\begin{cases} \dfrac{dx}{dt} = ax + by + \displaystyle\sum_{k=2}^{\infty} P_k(x,y) \\ \dfrac{dy}{dt} = cx + dy + \displaystyle\sum_{k=2}^{\infty} Q_k(x,y), \end{cases} \tag{A1.1}$$

where a, b, c and d are real numbers. It was shown by Poincaré ([18]) that, in order for the origin to be a centre for system (A1.1), it is necessary that the linearisation matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be either singular or nonzero with purely imaginary eigenvalues. It is possible for the origin to be a centre when the linearisation matrix is singular (e.g., in the case of the Hamiltonian system $\frac{dx}{dt} = -y^3$, $\frac{dy}{dt} = x^3$); we shall henceforth consider only those cases when the critical point in question is *nondegenerate*, i.e., when the linearisation matrix is nonsingular (in which case, the critical point is necessarily isolated).

In the conventional approach of determining necessary and sufficient conditions for a centre, the following theorem is of fundamental importance.

Theorem A1.1 (Poincaré, [19], Lyapunov, [13])**.** *Suppose that, in system* (A1.1), *the matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *has* (*nonzero*) *purely imaginary eigenvalues* (*i.e.,* $a + d = 0$ *and* $ad - bc > 0$)*. Then there is a centre at the origin if and only if the system possesses a nonconstant real analytic first integral in a neighbourhood of the origin.*

In this Appendix, we shall be considering the relation between the existence of a centre and that of an integrating factor, for analytic vector fields. The main result is given by the following theorem, which is analogous to Theorem A1.1.

Theorem A1.2. *Suppose that, in system* (A1.1), *the matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *has* (*nonzero*) *purely imaginary eigenvalues* (*i.e.,* $a + d = 0$ *and* $ad - bc > 0$)*. Then there is a centre at the origin if and only if there is a nonzero analytic integrating factor in a neighbourhood of the origin.*

This theorem will be proved by considering two separate propositions; the first (Proposition A1.3) deals with the sufficient condition for a centre. The second (Proposition A1.4) concerns the necessary condition, and is a special case of the criterion of Reeb ([21]) (see also Mattei and Moussu, [14], and Moussu, [15]). However, it should be noted that the proofs which are presented here employ only elementary methods, unlike previous proofs, and they are therefore somewhat more accessible. In the proofs of these propositions, it will be convenient to introduce a further specialisation, by invoking a linear transformation of x and y and a simultaneous rescaling of the independent variable, t, so that, without loss of generality, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. There is some remaining freedom in the variables, but it is unnecessary to consider this for the present purposes.

The condition that a function, R, which is not identically zero, be an integrating factor for the system

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \tag{A1.2}$$

for suitable functions P and Q, is that it satisfy the divergence requirement

$$\frac{\partial}{\partial x}(RP) + \frac{\partial}{\partial y}(RQ) = 0, \tag{A1.3}$$

which may be alternatively written in the form

$$\frac{dR}{dt} + R\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) = 0. \tag{A1.4}$$

Equation (A1.3) demonstrates explicitly that the vector field (RP, RQ) is Hamiltonian, and so it may well be expected that there is an intimate connection between the existence of an integrating factor and that of a first integral. The results of Poincaré ([19]) and Lyapunov ([13]), given in Theorem A1.1, would then, under appropriate circumstances, exhibit a connection between the conditions for a centre and the existence of an integrating factor.

Indeed, it is known that if the functions P and Q in equation (A1.2) are *polynomials*, then the existence of a first integral, ϕ, requires the existence of an integrating factor, R. A precise formulation of this is provided by Schlomiuk ([23]), but it is not satisfactory for the present purposes, because the resulting integrating factor, R, would not necessarily be analytic in any neighbourhood of the critical point. However, in this formulation, Schlomiuk was not concerned with imposing the additional requirement that the critical point be a centre in the linear approximation (I am indebted to Dr. C.G. Hewitt for calling this point to my attention). It is perhaps also of interest to note in passing that Prelle and Singer ([20]) have shown that, for polynomial vector fields, the existence of an "elementary" first integral (i.e., one which is expressible in terms of exponential, logarithmic and algebraic functions) requires the existence of an algebraic integrating factor. We shall find in Proposition A1.4 that, with the additional requirement that the linearisation matrix have (nonzero) purely imaginary eigenvalues, the analyticity of the integrating factor may be assured, not merely when the functions P and Q are polynomials, but even when they are general analytic functions. The proof of this proposition is straightforward, but it is more convenient first to demonstrate the validity of a result which provides a sufficient condition for a centre.

Proposition A1.3. *Suppose that in system* (A1.1), *the matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *has* (*nonzero*) *purely imaginary eigenvalues* (*i.e.,* $a + d = 0$ *and* $ad - bc > 0$), *and that there is an integrating factor which is nonzero and analytic in a neighbourhood of the origin. Then the origin is a centre.*

Proof. The system (A1.2) may be written in the form of (A1.1), where, as already established, $a = d = 0$ and $c = -b = 1$. Suppose that the integrating factor is R, so that, on a neighbourhood, U, of the origin, equation (A1.3) is satisfied, R is nonzero and analytic, and P and Q are also analytic functions, given by expressions of the form

$$P(x,y) = -y + \sum_{k=2}^{\infty} P_k(x,y) \quad \text{and} \quad Q(x,y) = x + \sum_{k=2}^{\infty} Q_k(x,y),$$

where P_k and Q_k are homogeneous polynomials in x and y, of degree k, with $k \geq 2$. There is therefore a function, ϕ, which satisfies the equations

$$\frac{\partial \phi}{\partial x} = RQ \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -RP,$$

whose compatibility is guaranteed by equation (A1.3). Since P, Q and R are analytic on U, it follows that ϕ is also analytic on U. Now, ϕ is constant along the integral curves of the system (A1.2), because $\frac{d\phi}{dt} = \frac{\partial \phi}{\partial x}P + \frac{\partial \phi}{\partial y}Q = 0$ on U. Furthermore, ϕ is not identically constant on U, since R is nonzero on U, and since the vector field (P, Q) is not identically zero on U. Thus ϕ is a nonconstant, analytic first integral on U, and so it follows from the results of Poincaré ([19]) and Lyapunov ([13]), given in Theorem A1.1, that the origin is a centre.

Remark. There is a statement of Mattei and Moussu ([14]) which is related to Proposition A1.3. When expressed in elementary terms, this is that if there is an analytic vector field possessing a critical point (which we may take to be the origin, O), and if there is an integrating factor which is nonzero and analytic in a neighbourhood of O, then, in a suitable neighbourhood of O, there is a nonconstant analytic first integral of the form

$$\phi = Ax^2 + 2Hxy + By^2 + \sum_{m=3}^{\infty} \phi_m(x,y), \tag{A1.5}$$

where ϕ_m is a homogeneous polynomial in x and y, of degree m, with $m \geq 3$ (cf. equation (A1.7) below). When specialised to the context of Proposition A1.3, the quadratic form $Ax^2+2Hxy+By^2$ is positive definite.

We now consider the existence of an integrating factor as a necessary condition for a centre. This is given by

Proposition A1.4 (Reeb, [21]). *Suppose that an analytic vector field possesses a nondegenerate critical point which is a centre. Then there is an integrating factor which is nonzero and analytic in a neighbourhood of the centre.*

Proof. The proof which is given here proceeds in several stages. It is first shown that the gradient vector field of the first integral, ϕ, is nonzero in a deleted neighbourhood of the centre. It is next demonstrated that, if the vector field is (P, Q), then, in a suitable neighbourhood of the origin, $\frac{\partial \phi}{\partial x}$ is of form RQ, where R is nonzero and analytic. From this, it may be deduced that $\frac{\partial \phi}{\partial y} = -RP$ on this neighbourhood, and hence that R satisfies equation (A1.3); i.e., that R is an integrating factor with the required properties.

As in the proof of Proposition A1.3, we may suppose that the critical point is at the origin, and that the system is given by

$$\begin{aligned} \frac{dx}{dt} &= P(x,y) = -y + \sum_{k=2}^{\infty} P_k(x,y) \\ \frac{dy}{dt} &= Q(x,y) = x + \sum_{k=2}^{\infty} Q_k(x,y), \end{aligned} \tag{A1.6}$$

where P_k and Q_k are homogeneous polynomials in x and y, of degree k, with $k \geq 2$. It is immediately evident from these expressions that, in a sufficiently small neighbourhood, U, of the origin, the functions P and Q vanish simultaneously if and only if $x = y = 0$ (thus demonstrating directly that the origin is an isolated critical point).

The second step is accomplished by recalling, from the results of Poincaré ([19]) and Lyapunov ([13]), as given in Theorem A1.1, that there is, on a suitable neighbourhood, $U_1 \subseteq U$, of the origin, a nonconstant first integral, ϕ, which is analytic on U_1. It is easy to show that ϕ may be expanded in the form

$$\phi = x^2 + y^2 + \sum_{m=3}^{\infty} \phi_m(x,y) \quad (= \text{ constant}), \tag{A1.7}$$

where ϕ_m is a homogeneous polynomial in x and y, of degree m, with $m \geq 3$ (cf., for example, Dulac, [10, Equation (12)]). Thus $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ are analytic on U_1, and their power series expansions about the origin begin with linear terms which are $2x$ and $2y$ respectively. This implies that, in a sufficiently small neighbourhood, $U_2 \subseteq U_1$, of the origin, $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ vanish simultaneously if and only if $x = y = 0$.

Now, since ϕ is a first integral on U_1, it satisfies the relation $\frac{\partial \phi}{\partial x}P + \frac{\partial \phi}{\partial y}Q \equiv 0$ on U_1. Hence, bearing in mind the above, it follows that $\frac{\partial \phi}{\partial x} = 0$ if and only if $Q = 0$ on the deleted neighbourhood $U_2 \setminus \{(0,0)\}$ of the origin. We next introduce the change of variables given by $X = Q(x,y)$ and $Y = y$. This is, in a neighbourhood of the origin, a regular map, which is, moreover, analytic. It follows that there is some neighbourhood, $U_3 \subseteq U_2$, of the origin, on which the inverse map exists and is analytic. We may write this inverse map in the form

$$x = X + r(X,Y), \quad y = Y, \tag{A1.8}$$

where r is an analytic function on U_3, which, when expanded as a power series in X and Y about the origin, consists of terms whose lowest degree is at least two. Using equations (A1.7) and (A1.8), it follows that, on U_3, $\frac{\partial \phi}{\partial x}$ is expressible as a series, in terms of powers of X and Y, whose first term is $2X$. On the deleted neighbourhood $U_3 \setminus \{(0,0)\}$ of the origin, $\frac{\partial \phi}{\partial x} = 0$ if and only if $X = 0$. Consequently, on U_3, $\frac{\partial \phi}{\partial x}$ is of the form XR, where R is analytic on U_3, and $R(0,0) = 2$.

We have therefore shown that there is a neighbourhood, U_3, of the origin, on which $\frac{\partial\phi}{\partial x} = RQ$. Since $U_3 \subseteq U_1$, we have that $\frac{\partial\phi}{\partial x}P + \frac{\partial\phi}{\partial y}Q \equiv 0$ on U_3, and thus $Q(\frac{\partial\phi}{\partial y} + RP) = 0$ on U_3. It follows that $\frac{\partial\phi}{\partial y} + RP = 0$ on the set $U_3 \setminus \{(a,b)|Q(a,b) = 0\}$, which means, by the analyticity of the functions concerned, that $\frac{\partial\phi}{\partial y} + RP \equiv 0$ on a neighbourhood, $U_4 \subseteq U_3$, of the origin. Thus, on U_4, $\frac{\partial\phi}{\partial x} = RQ$ and $\frac{\partial\phi}{\partial y} = -RP$, and so $\frac{\partial}{\partial x}(RP) + \frac{\partial}{\partial y}(RQ) = 0$, i.e., R is an integrating factor for the system (A1.6), which is analytic on the neighbourhood U_4 of the origin, and $R(0,0) = 2$. There is therefore a neighbourhood, $U_5 \subseteq U_4$, of the origin on which R is a nonzero, analytic integrating factor.

Remarks. The criterion of Reeb ([21]) is discussed by Mattei and Moussu ([14]). When expressed in elementary terms, this states that, if there is an analytic vector field with a critical point (which we may take to be the origin, O), and which possesses, in some neighbourhood of O, a nonconstant analytic first integral of form (A1.5), with $AB - H^2 \neq 0$, then, in a suitable neighbourhood of O, there is a nonzero analytic integrating factor. The case when the critical point is a centre is that in which $AB - H^2 > 0$.

In the case of a *quadratic* vector field, if a critical point is a centre, then it is *necessarily* nondegenerate. This was noted by Schlomiuk ([23, 24]), who credited the result to Berlinskiĭ ([1]), Date ([8]) and de Jager ([9]).

Proof of Theorem A1.2. Theorem A1.2 is now an immediate consequence of Propositions A1.3 and A1.4.

Corollary A1.5. *Suppose that a quadratic vector field possesses a centre. Then there is an integrating factor which is nonzero and analytic in a neighbourhood of the centre.*

Proof. The corollary follows immediately from the above remark and the statement of Proposition A1.4.

Appendix 2. We present here a summary of the computations of the necessary conditions for a centre, as discussed in the proof of Theorem 2.2, but using the tensorial treatment (in fact, this is how the results were originally obtained). For the sake of brevity, it will be assumed that the reader is familiar with the content of the discussion in Collins ([4]). System (2.3) is then expressed compactly in the form

$$\frac{dx^i}{dt} = B^i{}_j x^j + x^i\left(\tfrac{2}{3}P_j x^j + \tfrac{3}{4}P_{jk}x^j x^k\right), \qquad (A2.1)$$

where the tensors $\Gamma^i{}_{jk}$ and $\Gamma^i{}_{jk\ell}$, which arise as coefficients of the quadratic and cubic terms respectively, have the property that, when their contravariant indices are lowered with the fundamental antisymmetric tensor f_{ij}, their completely symmetric parts (i.e., Q_{ijk} and $Q_{ijk\ell}$, respectively) are zero. The tensor $B^i{}_j$ in (A2.1) has zero trace and positive determinant, and so the associated tensor B_{ij} is symmetric and has complex conjugate principal directions. Thus B_{ij} is expressible in the form $B_{ij} = b_{(i}\bar{b}_{j)} \neq 0$. In the usual (real) reference frame, with respect to which f_{ij} has nonzero components $f_{12} = -f_{21} = 1$, $B^1{}_2 = -B^2{}_1 = -1$ (cf. (1.1)), and so $B_{ij} = \text{diag}\,(1,1)$, and we may write $b^i = \binom{1}{i}$ and $\bar{b}^i = \binom{1}{-i}$. The vectors $\mathbf{b}$ and $\bar{\mathbf{b}}$ are eigenvectors of $B^i{}_j$, since $B^i{}_j b^j = \frac{1}{2}(\bar{b}_j b^j)b^i$ and $B^i{}_j\bar{b}^j = -\frac{1}{2}(\bar{b}_j b^j)\bar{b}^i$, the associated eigenvalues being purely imaginary, as required. By direct computation, we obtain that $\mathbf{b}$ has associated eigenvalue $-i$, so that the (invariant) quantity $\bar{b}_j b^j = -2i$, a result which may also be verified by direct computation. In a complex frame of reference, in which $b^i = \delta^i_1$ and $\bar{b}^i = \delta^i_2$, the fact that $\bar{b}_j b^j = -2i$ (independently of the reference frame) now means that, in this complex basis, $\bar{b}_1 = -2i$ and $b_2 = 2i$, so the nonzero components of f_{ij} in this frame must be $f_{12} = -f_{21} = -2i$. Furthermore, in this frame, $b_1 = \bar{b}_2 = 0$, and so $B_{11} = B_{22} = 0$, whereas $B_{12} = B_{21} = \frac{1}{2}b_2\bar{b}_1 = 2$. It is readily calculated that $B^1{}_1 = -B^2{}_2 = -i$, $B^1{}_2 = B^2{}_1 = 0$ (which is compatible with (2.5)), and that $B^{11} = B^{22} = 0$ and $B^{12} = B^{21} = \frac{1}{2}$ with respect to this basis. Use of a complex basis provides a considerable advantage, since, for instance, complex conjugation may be employed, which effectively almost halves the number of equations which must be considered. In addition, in a complex basis, the freely specifiable coefficients in the expansion (2.7) (and (A2.2) below) of F are simply those of form $c_{12}, c_{1122}, c_{111222}$, etc., whereas in the real basis, this freedom is more difficult to express.

The expansion of the function F in (2.7) is now given as

$$F = 1 + c_i x^i + c_{ij} x^i x^j + \cdots + c_{i_1 i_2 \cdots i_n} x^{i_1} x^{i_2} \cdots x^{i_n} + \cdots , \tag{A2.2}$$

where, for all $n \geq 2$, the coefficients $c_{i_1 i_2 \cdots i_n}$ are, without loss of generality, symmetric upon interchange of any pair of indices, i.e., $c_{i_1 i_2 \cdots i_n} = c_{(i_1 i_2 \cdots i_n)}$. The tensorial form of equation (2.6) is given by

$$\frac{dF}{dt} + \lambda F (2P_i x^i + 3P_{ij} x^i x^j) = 0. \tag{A2.3}$$

Upon substituting (A2.2) into (A2.3), and employing (A2.1), we find the following sequence of conditions.

$$c_i B^i{}_s = -2\lambda P_s, \tag{A2.4a}$$

$$\frac{2}{3} c_{(s} P_{t)} + 2c_{i(s} B^i{}_{t)} = -\lambda (2P_{(s} c_{t)} + 3P_{st}), \tag{A2.4b}$$

$$\frac{3}{4} c_{(s} P_{tu)} + \frac{4}{3} c_{(st} P_{u)} + 3c_{i(st} B^i{}_{u)} = -\lambda (2P_{(s} c_{tu)} + 3P_{(st} c_{u)}), \tag{A2.4c}$$

etc., the general form of which is

$$\begin{aligned} &\frac{3}{4}(n-2) c_{(s_1 s_2 \cdots s_{n-2}} P_{s_{n-1} s_n)} + \frac{2}{3}(n-1) c_{(s_1 s_2 \cdots s_{n-1}} P_{s_n)} + n c_{i(s_1 s_2 \cdots s_{n-1}} B^i{}_{s_n)} \\ &\quad = -\lambda \left[2P_{(s_i} c_{s_2 s_3 \cdots s_n)} + 3P_{(s_1 s_2} c_{s_3 \cdots s_n)} \right] , \end{aligned} \tag{A2.5}$$

where n is a positive integer, and where we write $c_{i_1 i_2 \cdots i_m} = 1$ when $m = 0$, and $c_{i_1 i_2 \cdots i_m} = 0$ when $m = -1$.

We may now proceed to attempt to compute values of the coefficients $c_{i_1 i_2 \cdots i_n}$, for low values of n. It will be convenient to introduce the vector $V^i = B^{ij} P_j$, whose components, in the complex basis, are $V^1 = \frac{1}{2} P_2$ and $V^2 = \frac{1}{2} P_1$. Thus, by (A2.4a), i.e., (A2.5) with $n = 1$, we have

$$c_i = 2\lambda V_i, \tag{A2.6}$$

which is equivalent to the result that $p = 3i\lambda\gamma$ in Section 2. We now invoke the modifications which were introduced in the proof of Theorem 2.2, i.e., we (justifiably) neglect powers of λ higher than one, and we ignore terms involving $c_{12}, c_{1122}, c_{111222}$, etc. It is then evident from the pattern of equations (A2.4), or of equation (A2.5), that the right-hand sides may be replaced by zero whenever $n \geq 3$, and that when $n = 2$, the term involving $P_{(s} c_{t)}$ may be ignored. Now (A2.4b), i.e., (A2.5) with $n = 2$, shows that

$$c_{11} \stackrel{*}{=} \lambda \big(-\frac{3i}{2} P_{11} - \frac{2}{3} P_1 P_1 \big)$$

whose complex conjugate

$$c_{22} \stackrel{*}{=} \lambda \big(\frac{3i}{2} P_{22} - \frac{2}{3} P_2 P_2 \big)$$

is immediate. These equations are equivalent to (2.8) and its complex conjugate.

Algebraic constraints arise in the sequence (A2.5) for each even value of the integer n, when in the final term on the left-hand side $\frac{n}{2}$ of the indices $s_1 s_2, \ldots, s_n$ are equal to 1, while the other $\frac{n}{2}$ are equal to 2. Thus the first such constraint arises when $n = 2$, when the coefficient of c_{12} is zero, and it then immediately follows, using (A2.6), that

$$P_{12} = 0. \tag{A2.7}$$

Equation (A2.5) with $n = 3$ is used to compute expressions for c_{ijk}, and the next algebraic constraint will arise from the case $n = 4$, when the coefficient of c_{1122} will vanish, and so on. Proceeding in this fashion, we easily find that

$$c_{111} \stackrel{*}{=} \lambda\big(-\frac{7}{6}P_1P_{11} + \frac{8}{27}iP_1P_1P_1\big),$$

so, by complex conjugation,

$$c_{222} \stackrel{*}{=} \lambda\big(-\frac{7}{6}P_2P_{22} - \frac{8}{27}iP_2P_2P_2\big).$$

These equations are equivalent to the equation for the coefficient s in Section 2. Moreover,

$$c_{112} \stackrel{*}{=} \lambda\big(-\frac{1}{6}P_2P_{11} + \frac{8}{27}iP_1P_1P_2\big),$$

and hence

$$c_{122} \stackrel{*}{=} \lambda\big(-\frac{1}{6}P_1P_{22} - \frac{8}{27}iP_1P_2P_2\big),$$

which are equivalent to (2.9) and its complex conjugate. The algebraic constraint resulting from putting $n = 4$ in (A2.5) (with $s_1 = s_2 = 1$, and $s_3 = s_4 = 2$), shows that

$$\frac{3}{2}c_{(11}P_{22)} + 2c_{(112}P_{2)} = 0,$$

from which we discover that

$$P_{11}P_2P_2 + P_{22}P_1P_1 = 0. \qquad (A2.8)$$

Since the two constraints (A2.7) and (A2.8) so far obtained are in fact also sufficient, the subsequent algebraic constraints, obtained by putting $n = 6, 8, 10$, etc. in (A2.5), and choosing an equal number of "1" and "2" indices, will not give rise to further independent conditions.

By comparing (A2.1) with (2.5), we may make the identifications $\gamma = \frac{2}{3}P_2, \bar{\gamma} = \frac{2}{3}P_1$, $D = \frac{3}{4}P_{22}, \bar{D} = \frac{3}{4}P_{11}$ and $E = \frac{3}{2}P_{12}$. The conditions (A2.7) and (A2.8) are therefore equivalent to the constraints $E = 0$ and $D\bar{\gamma}^2 + \bar{D}\gamma^2 = 0$, as obtained using the conventional treatment. Moreover it is clear that, in the complex basis, equations (A2.7) and (A2.8) are equivalent to the conditions

$$P_{ij}B^{ij} = 0 \qquad (A2.9a)$$

and

$$P_{ij}V^iV^j = 0. \qquad (A2.9b)$$

In this latest form, the necessary conditions for a centre appear as tensorial statements concerning the vanishing of two scalars, and so, by the fundamental property of tensors, they are valid in *all* reference frames (which are interconnected by means of linear transformations). We may therefore easily convert the conditions (A2.9) to versions which hold in a real reference frame, so, identifying (A2.1) and (2.3), wherein $B^{ij} = \mathrm{diag}\,(1,1)$, we have $\alpha = \frac{2}{3}P_1$, $\beta = \frac{2}{3}P_2$ (hence $V^1 = \frac{3}{2}\alpha$ and $V^2 = \frac{3}{2}\beta$), $A = \frac{3}{4}P_{11}, B = \frac{3}{2}P_{12}$ and $C = \frac{3}{4}P_{22}$, from which we immediately see that (A2.9a) is equivalent to $A + C = 0$ and (A2.9b) is equivalent to $A\alpha^2 + B\alpha\beta + C\beta^2 = 0$, in agreement with (2.4).

We have therefore established the tensorial equivalent to Theorem 2.2, and a formal statement of this will now be given.

Theorem A2.1. *The system* (A2.1) *possesses a centre at the origin if and only if* $P_{ij}B^{ij} = 0$ *and* $P_{ij}V^iV^j = 0$.

From this, the validity of Corollary 2.3 is readily determined, since there $B^i{}_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and so $B^{ij} = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}$. With $a + d = 0$, and $\alpha = \frac{2}{3}P_1$, $A = \frac{3}{4}P_{11}$, etc. as before, equations (A2.9) become

$$bA - aB - cC = 0 \quad \text{and} \quad (b\alpha - a\beta)^2 A - (b\alpha - a\beta)(a\alpha + c\beta)B + (a\alpha + c\beta)^2 C = 0. \qquad (A2.10)$$

It is instructive to compare the ease with which the more general form of the conditions (A2.10) is obtained to the lengthy and tedious calculation using elementary algebra.

It is apparent that the possibility of simple expansions (A2.2) for F, for specific values of λ, may also readily be explored. For instance, one may wish to know circumstances under which there is an integrating factor $R = F^{1/\lambda}$, where F is a polynomial. It is then no longer justified to neglect powers of λ higher than one, nor to ignore c_{12}, c_{1122}, etc., in the sequence of equations (A2.5). For example, it is easily shown that if we insist that $F = 1 + c_i x^i$, where $c_i \neq 0$, then (A2.5) gives rise to the following sequence of (necessary and sufficient) conditions: $c_i = 2\lambda V_i$, $4(\lambda + \frac{1}{3})P_{(i}V_{j)} = -3P_{ij}$ and $(\lambda + \frac{1}{4})P_{ij} = 0$. From these, one may easily convert to a real basis to obtain the possibilities (a) $\lambda = -\frac{1}{3}$, $B = 0$ in (3.2), when the integrating factor is $(1 + y)^{-3}$, and (b) $\lambda = -\frac{1}{4}$, $B = \frac{3}{16}$ in (3.2), when the integrating factor is $(1 + \frac{3}{4}y)^{-4}$. As discussed in Section 2, both possibilities give rise to first integrals of a particularly simple form.

In summary, there are numerous advantages to the tensor approach. These include an economy of notation, an improved "book-keeping" notation, a compactness of the expressions (for instance, compare (A2.9) with (A2.10)), and a highlighting of patterns which develop. The exploration of the possibility of particularly simple integrating factors (and hence first integrals) is also facilitated. It is hoped that this approach may be used effectively in other computations of conditions for a centre. It should also be noted that the calculation of transformed coefficients, as occurs, for instance, in the proof of the sufficiency of the conditions in Theorem 2.2, is trivial when the tensor approach is employed. Similarly, as exemplified by the development of equations (A2.10), the transition to expressions in a general reference frame are made with ease, thereby providing an avoidance of the possibly inconvenient step of first transforming the system from one whose linearisation matrix in $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a + d = 0$ and $ad - bc > 0$, to one with linearisation matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This aspect would be equally valid in any analogous treatment of systems which are more complicated than those discussed herein.