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SMALL SOLUTIONS OF CERTAIN BOUNDARY VALUE PROBLEMS*

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Abstract. Existence and nonexistence results of small solutions are derived for certain boundary value problems.

1. Introduction. In [2], [4], the existence of small solutions of the following boundary value problem is investigated:

$$\mathcal{L}y := Ly + y^3 = f, \quad M_1(y) = M_2(y) = 0\,, \tag{1.1}$$

where L, the linear part of $\mathcal{L}$, is of the form $Ly = y'' + p(x)\cdot y' + q(x)\cdot y$, p, q are integrable on $[a,b]$, f is small and

$$M_1(y) = \alpha_1 y(a) + \alpha_2 y(b) + \alpha_3 y'(a) + \alpha_4 y'(b)\,,$$
$$M_2(y) = \beta_1 y(a) + \beta_2 y(b) + \beta_3 y'(a) + \beta_4 y'(b)\,,$$

α_i and β_i are real numbers.

The operator $\mathcal{L}$ is defined on the domain

$$BC \equiv \Big\{ y \in C^1\big([a,b],\mathbb{R}\big) : y' \text{ is absolutely continuous on } [a,b] \text{ such that } M_1(y) = M_2(y) = 0,\ y'' \in L^1\big([a,b],\mathbb{R}\big) \Big\}\,.$$

We assume that L has a one-dimensional kernel $\ker L$ spanned by φ. We know by the proof of Lemma 1.1 in [4] that there is an $\omega \in L^\infty\big([a,b],\mathbb{R}\big)$ such that $h \in \operatorname{im} L$ (the range of L) if and only if $\int_a^b h(t)\omega(t)\,dt = 0$. We assume $\omega \neq 0$, so codim $\operatorname{im} L = 1$. As a consequence of [3, Lemma 3.2] it follows that if $\varphi^3 \notin \operatorname{im} L$, then 0 is an isolated solution of $\mathcal{L}y = 0$ and thus $\mathcal{L}y = f$ has a small solution for f small. The following theorems are proved in [2, 4].

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Theorem A. *Suppose* $Lw = \varphi^3$ *with* $w \perp \varphi$, *(i.e.,* $\int_a^b w(t)\varphi(t)\,dt = 0$*), but* $Ly = w \cdot \varphi^2$ *has no solutions in* BC*. Then* $\mathcal{L}y = f$ *has at least one small solution for each* $f \in L^1\big([a,b],\mathbb{R}\big)$ *small.*

Theorem B. *Suppose* $Lw = \varphi^3$ *with* $w \perp \varphi$, *(i.e.,* $\int_a^b w(t)\varphi(t)\,dt = 0$*) and* $w\varphi^2 = Lv$ *with* $v \perp \varphi$*, but* $Ly = w^2\varphi + 3v\varphi^2$ *has no solutions in* BC*. Then* $\mathcal{L}y = f$ *has at least one small solution for each* $f \in L^1\big([a,b],\mathbb{R}\big)$ *small.*

The proofs of these theorems in [2] are based on the Lyapunov-Schmidt procedure. Since $\dim\ker L = \operatorname{codim}\operatorname{im} L = 1$, the existence of small solutions of (1.1) is reduced to a one-dimensional bifurcation equation

$$F(c,z) = d\,,$$

where F is a C^∞-smooth mapping, $c, d \in \mathbb{R}$ are small and z is small from a certain Banach space. It has been shown in [2] that

1. If $\varphi^3 \notin \operatorname{im} L$ then $F(c,0) = a_1c^3 + O(c^4)$ as $c \to 0$ with $a_1 \neq 0$;
2. If $Lw = \varphi^3$ with $w \perp \varphi$ but $w \cdot \varphi^2 \notin \operatorname{im} L$ then $F(c,0) = a_2c^5 + O(c^6)$ as $c \to 0$ with $a_2 \neq 0$;
3. If $Lw = \varphi^3$ with $w \perp \varphi$ and $w\varphi^2 = Lv$ with $v \perp \varphi$ but $w^2\varphi + 3v\varphi^2 \notin \operatorname{im} L$ then $F(c,0) = a_3c^7 + O(c^8)$ as $c \to 0$ with $a_3 \neq 0$.

Now it is clear that the implications 1, 2, 3 ensure the solvability of $F(c,z) = d$ in c small for any d, z small. So the validity of Theorems A, B follows immediately from the implications 2, 3.

Hence the bifurcation mapping $F(c,0)$ has the form

$$F(c,\,0) = a \cdot c^i + O(c^{i+1})\,, \quad a \neq 0\,,$$

where either $i = 3$ or $i = 5$ or $i = 7$. It has been pointed out in [2] that this property does not hold by chance, but it follows from the following fact: The map $\mathcal{L}$ is equivariant by the group $\mathbb{Z}_2$, since $\mathcal{L}(-y) = -\mathcal{L}y$ and we can easily derive that $F(c,\,0)$ has this property as well, thus $F(-c,\,0) = -F(c,\,0)$ for c small. Hence it generally holds

$$F(c,\,0) = a \cdot c^{2i+1} + O(c^{2i+3}) \qquad a \neq 0$$

when F is not flat, i.e. $\frac{\partial^i}{\partial c^i}F(0,0) \neq 0$ for some i, and in this case the equation $\mathcal{L}y = f$ has at least one small solution for each f small. Summarizing we see that (1.1) is "generically" always solvable for f small. We show in this paper that the following Fredholm-like alternative result immediately follows from the considerations of [2].

Theorem 1. *Assume* $\dim\ker L = \operatorname{codim}\operatorname{im} L = 1$ *for* (1.1)*. Then either* $y = 0$ *is an isolated solution of* $\mathcal{L}y = 0$ *and then* $\mathcal{L}y = f$ *has a small solution for any small* $f \in L^1\big([a,b],\mathbb{R}\big)$*, or the set* $\big\{y \in BC : \mathcal{L}y = 0\big\}$ *near* $y = 0$ *is an analytic,* *nonconstant curve crossing* $y = 0$ *and then for any* $h \notin \operatorname{im} L$, *i.e.*, $h \in L^1\big([a,b],\mathbb{R}\big)$ *and* $\int_a^b h(t)\omega(t)\,dt \neq 0$, *there are positive constants* K_1, K_2 *such that* $\mathcal{L}y = c\cdot h$, $c \in \mathbb{R}$, $0 < |c| < K_1$ *implies* $\sup |y| > K_2$.

In order to apply the above Theorem 1, one has to verify the fact that $y = 0$ is an isolated solution of $\mathcal{L}y = 0$, and this is usually a difficult task.

The purpose of this paper is also to study "nongeneric" cases of (1.1), i.e., when the bifurcation mapping $F(c, 0)$ is flat at $c = 0$ (that is, $\frac{\partial^i}{\partial c^i}F(0,0) = 0$ for all i), as well as some generalizations in that direction. We are motivated by the example [4]: consider the problem

$$\begin{aligned} &y'' + y^3 = f, \quad -1 \le x \le 1\\ &y(1) = y(-1), \quad y'(1) + y'(-1) = 0\,. \end{aligned} \tag{1.2}$$

In this example, $Ly = y''$ and $\varphi \equiv 1$, since $L\varphi = 0$ and $\varphi = 1$ satisfies the boundary conditions. Furthermore, $\varphi^3 = 1$ is in $\operatorname{im} L$, since $Lw = 1$ with $w = \frac{x^2}{2} - \frac{1}{6} \in BC$ and $w \perp \varphi$ as well as $w\varphi^2 = \frac{x^2}{2} - \frac{1}{6} \in \operatorname{im} L$, since $Lv = \frac{x^2}{2} - \frac{1}{6}$ with $v = \frac{x^4}{24} - \frac{x^2}{12} + \frac{7}{360} \in BC$ and $v \perp \varphi$. Moreover, $w^2\varphi + 3v\varphi^2 = \frac{3x^4}{8} - \frac{5x^2}{12} + \frac{31}{360} \in \operatorname{im} L$, since $Lu = \frac{3x^4}{8} - \frac{5x^2}{12} + \frac{31}{360}$ with $u = \frac{x^6}{80} - \frac{5x^4}{144} + \frac{31x^2}{720} \in BC$. So Theorems A and B do not apply. The following result is proved in [4].

Proposition A. *There are functions* $f \in L^1\big([-1,1],\mathbb{R}\big)$ *with arbitrarily small norms such that any solutions* y *of* (1.2) *must have* $\sup |y| \ge 1/\sqrt{6}$.

Proposition A implies that the corresponding bifurcation mapping $F(c, 0)$ is flat at $c = 0$. Theorem 1 gives much more information about (1.2) than Proposition A. The proof of this proposition in [4] is ad hoc. We show that Proposition A follows from a symmetry of (1.2).

The plan of this paper is as follows. In Section 2, we study the existence of small solutions for abstract operator equations. Theorem 1 is a consequence of these results. We also globalize these results in the sense that we drop the assumption of the smallness of desired solutions. By considering equivariantness of these operator equations according to linear representations of compact Lie groups, we generalize Proposition A; i.e., we show that Proposition A follows from a certain general symmetric principle. In Section 3, we apply abstract results of Section 2 for certain boundary value problems. Section 4 is devoted to concluding remarks.

2. Abstract results. Let X, Y be Banach spaces with norms $|\cdot|_X$, $|\cdot|_Y$, respectively. Consider a C^1-smooth mapping $T\colon X \to Y$.

Theorem 2.1. *Assume* $T(0) = 0$, $DT(0)$ *is a Fredholm operator with index* 0 *and* T *is odd, i.e.*, $T(-x) = -T(x)\,\forall\, x \in X$. *If* 0 *is an isolated solution of* $T(x) = 0$ *then* $T(x) = f$ *has a small solution for any* $f \in Y$ *sufficiently small.*

Proof. We apply the standard Lyapunov-Schmidt procedure by taking continuous projections $P : X \to X$, $Q : Y \to Y$ such that $\operatorname{im} P = \ker L$, and $\operatorname{im} Q = \operatorname{im} L$ with $L = DT(0)$. We modify $T(x) = f$ in the form

$$\begin{aligned} &QT(x_1 + x_2) = z, \quad z \in \operatorname{im} L \\ &(\mathbb{I} - Q)T(x_1 + x_2) = f_2, \quad f_2 \in \operatorname{im}(\mathbb{I} - Q) \\ &x_1 \in \ker L, \quad x_2 \in \operatorname{im}(\mathbb{I} - P)\,. \end{aligned} \tag{2.1}$$

We solve $x_2 = x_2(x_1, z)$, by the implicit function theorem, from the first equation of (2.1) for x_1, z small. Putting this solution into the second one, we arrive at the bifurcation equation

$$F(x_1, z) \equiv (\mathbb{I} - Q)T\big(x_1 + x_2(x_1, z)\big) = f_2\,.$$

Since $-QT(x) = QT(-x)\,\forall\, x \in X$, we have $-x_2(x_1, 0) = x_2(-x_1, 0)$. So

$$\begin{aligned} -F(x_1, 0) &= (\mathbb{I} - Q)\Big(- T\big(x_1 + x_2(x_1, 0)\big)\Big) \\ &= (\mathbb{I} - Q)T\big(- x_1 + x_2(-x_1, 0)\big) = F(-x_1, 0)\,. \end{aligned}$$

Hence $F(x_1, 0)$ is an odd mapping. Furthermore, $x_1 = 0$ is an isolated solution of $F(x_1, 0) = 0$ and $\dim\ker L = \dim\operatorname{im}(\mathbb{I} - Q)$. Hence the Borsuk-Ulam theorem implies $\deg\big(F(\cdot, 0), B_\delta, 0\big) \neq 0$ for a small ball B_δ around $0 \in \ker L$. Here deg is the Brouwer degree. This implies that $\deg\big(F(\cdot, z) - f_2, B_\delta, 0\big) \neq 0$ for any $z \in \operatorname{im} L$, $f_2 \in \operatorname{im}(\mathbb{I} - Q)$ small. The solvability of $F(x_1, z) = f_2$ in x_1 small is proved for any z, f_2 small. The proof is finished.

Now we prove a Fredholm-like alternative result for the equation $T(x) = f$ near $x = 0$ and $f \in Y$ small.

Theorem 2.2. *Assume in addition to the hypotheses of Theorem* 2.1 *that T is analytic, i.e., $T \in C^\omega$, and $\dim\ker DT(0) = 1$. Then either* 0 *is an isolated solution of $T(x) = 0$ and then $T(x) = f$ has a small solution for any $f \in Y$ small, or the set $\big\{x \in X : T(x) = 0\big\}$ near $x = 0$ is an analytic, nonconstant curve crossing $x = 0$ and then for any $h \notin \operatorname{im} DT(0)$, there are positive constants K_1, K_2 such that $T(x) = c \cdot h$, $c \in \mathbb{R}$, $0 < |c| < K_1$ implies $|x|_X > K_2$.*

Proof. The first alternative has been already proved in Theorem 2.1. It remains to prove the second one. We take Q in the proof of Theorem 2.1 such that $\operatorname{im}(\mathbb{I} - Q) = \operatorname{span} h$. So $f_2 = c \cdot h$, $c \in \mathbb{R}$ and $z = 0$ in (2.1), and the equation $T(x) = c \cdot h$, $c \in \mathbb{R}$ for c, x small is equivalent to $F(x_1, 0) = c \cdot h$ with $c \in \mathbb{R}$, $x_1 \in \ker DT(0)$ small. We know that $F(x_1, 0)$ is analytic in x_1 for all $x_1 \in \ker DT(0)$ small. Hence if $x_1 = 0$ is not an isolated solution of $F(x_1, 0) = 0$, then $F(x_1, 0) \equiv 0$ for all $x_1 \in \ker DT(0)$ small. This immediately implies the existence of an analytic one-parametric family of solutions of $T(x) = 0$ crossing $x = 0$. This result also gives that $0 = F(x_1, 0) = c \cdot h$ for $c \neq 0$ small has no solutions for x_1 small. The last part of the assertion of this theorem follows from this fact. The proof is finished.

Remark 2.3. 1. We note that the constants K_1, K_2 from Theorem 2.2 depend on $h \notin \operatorname{im} DT(0)$.

2. Assume $T(x) = Lx + S(x)$ instead of $T \in C^1$, where $L\colon X \to Y$ is a continuous, linear, Fredholm operator with index 0 and $S\colon X \to Y$ is a Lipschitz-continuous mapping near $x = 0$ such that $|S(x)|_Y = O\big(|x|_X^2\big)$ as $|x|_X \to 0$. So we assume that the local Lipschitz constant of S tends to zero as $|x|_X \to 0$. Then Theorem 2.1 still holds. Indeed, we can solve (2.1) similarly as above, but now we use the Banach fixed-point theorem instead of the implicit function theorem (see the arguments over Theorem 2.7, below, for more details).

Consider $\dim \ker L = 1$. So $\ker L = \operatorname{span} \varphi$. Then

$$G(c, 0) \equiv F(c\varphi, 0) = (\mathbb{I} - Q)S\big(c\varphi + x_2(c\varphi, 0)\big)$$

for $c \in \mathbb{R}$ small. We note that $x_2(c\varphi, 0) = O(c^2)$ as $c \to 0$. By assuming the existence of positive constants α, β such that

$$\liminf_{c \to 0} |(\mathbb{I} - Q)S\big(c(\varphi + \gamma(c))\big)|_Y / |c|^\alpha > \beta \tag{C1}$$

for any continuous curve $\gamma\colon (-\beta_1, \beta_1) \to X$ satisfying $|\gamma(c)|_Y/|c| \to 0$ as $c \to 0$, where $\beta_1 > 0$ is a positive number depending on the curve γ, we see that

$$\liminf_{c \to 0} |G(c, 0)|_Y / |c|^\alpha > \beta.$$

So $x_1 = 0$ is an isolated solution of $F(x_1, 0) = 0$. This criterion is useful when S presents only a nondifferentiable Nemitskii operator. For instance, we can take for fixed $\eta = \pm 1$ the nonlinearity

$$y \to \begin{cases} \eta y^3 & \text{for} \quad y \geq 0 \\ \eta y^7 & \text{for} \quad y \leq 0 \end{cases}$$

instead of $y \to y^3\ \forall y \in \mathbb{R}$ in (1.1). Similar problems are studied in [5].

3. We see from the second part of the proof of Theorem 2.2 that if the oddness of T is dropped in this theorem and 0 is not an isolated solution of $T(x) = 0$, then the set $\{x \in X : T(x) = 0\}$ near $x = 0$ is an analytic, nonconstant curve crossing $x = 0$ and for any $h \notin \operatorname{im} DT(0)$, there are positive constants K_1, K_2 such that $T(x) = c \cdot h$, $c \in \mathbb{R}$, $0 < |c| < K_1$ implies $|x|_X > K_2$.

4. If Y is compactly imbedded into X then coincidence degree arguments of [3] can be applied to replace the Lipschitz continuity of the mapping S in paragraph 2 of this remark by the ordinary continuity. To illustrate this application (see also [3,

Lemma 3.2]), we assume $\dim\ker L = 1$ (i.e., $\ker L = \operatorname{span}\varphi$), $|S(x)|_Y = O\big(|x|_X^2\big)$ as $x \to 0$ and $S = S_1 + S_2$, when there are constants $\alpha > 0$, $\beta > 0$ such that

$$\begin{aligned}&\liminf_{n\to\infty} |(\mathbb{I}-Q)S_1\big(c_n\varphi + c_n^2 z_n\big)|_Y/|c_n|^\alpha > \beta \\ &|S_2(x)|_Y = o\big(|x|_X^\alpha\big) \quad \text{as} \quad |x|_X \to 0\end{aligned} \tag{C2}$$

for any $c_n \in \mathbb{R}\setminus\{0\}$, $z_n \in X$ satisfying $c_n \to 0$, $|z_n|_X \le 1$ as $n \to \infty$.

It is clear that condition (C2) implies that the point 0 is an isolated solution for the equation $(\mathbb{I}-Q)S_1|_{\ker L} = 0$ when this map is considered as a map from a small neighborhood of $0 \in \mathbb{R} \simeq \ker L$ into $\mathbb{R} \simeq \operatorname{im}(\mathbb{I}-Q)$.

We claim that then $|\deg\big(L+S, B_\delta, 0\big)| = |\operatorname{ind}\big((\mathbb{I}-Q)S_1|_{\ker L}, 0\big)|$ for a small ball B_δ around $0 \in X$, where deg is the coincidence degree and ind is the index of an isolated zero point. To prove this claim, we show that there is a constant $\delta > 0$ such that

$$\begin{aligned}&Lx_2 + tQS(x_1+x_2) + (\mathbb{I}-Q)\Big(S_1(x_1+tx_2) + tS_2(x_1+x_2)\Big) \ne 0 \\ &\forall\, t \in [0,1], \quad \forall\, x \in X, \quad |x|_X = \delta, \quad x = x_1 + x_2, \quad x_1 \in \ker L, \quad x_2 \in \operatorname{im}(\mathbb{I}-P)\,.\end{aligned}$$

Indeed, if it is not true then there would be two sequences $X \ni x_n \to 0$, $x_n \ne 0$, and $[0,1] \ni t_n \to t_0$ such that

$$Lx_{n2} + t_nQS(x_{n1}+x_{n2}) + (\mathbb{I}-Q)\Big(S_1(x_{n1}+t_nx_{n2}) + t_nS_2(x_{n1}+x_{n2})\Big) = 0\,.$$

Hence

$$\begin{aligned}&Lx_{n2} + t_nQS(x_{n1}+x_{n2}) = 0 \\ &(\mathbb{I}-Q)\Big(S_1(x_{n1}+t_nx_{n2}) + t_nS_2(x_{n1}+x_{n2})\Big) = 0 \\ &x_{n1} = c_n\varphi, \quad c_n \in \mathbb{R}\,.\end{aligned}$$

So

$$|x_{n2}|_X \le c|S(c_n\varphi + x_{n2})|_Y \le c|c_n\varphi + x_{n2}|_X^2 \le 2c|c_n|^2|\varphi|_X^2 + 2c|x_{n2}|_X^2$$

for a constant $c > 0$. This implies $|x_{n2}|_X = O\big(c_n^2\big)$ and hence $c_n \ne 0$ for any n large, because of $x_n \ne 0$. Furthermore, for n sufficiently large, we have

$$\begin{aligned}0 &= \big|(\mathbb{I}-Q)\big(S_1(c_n\varphi + t_nx_{n2}) + t_nS_2(c_n\varphi + x_{n2})\big)\big|_Y/|c_n|^\alpha \\ &\ge \big|(\mathbb{I}-Q)S_1(c_n\varphi + t_nx_{n2})\big|_Y/|c_n|^\alpha - t_n\big|(\mathbb{I}-Q)S_2(c_n\varphi + x_{n2})\big|_Y/|c_n|^\alpha \\ &\ge \beta/2 + o(1)\,.\end{aligned}$$

This contradiction proves the above claim, since then

$$\begin{aligned}|\deg\big(L+S,B_\delta,0\big)| &= |\deg\big(L+(\mathbb{I}-Q)S_1P,B_\delta,0\big)|\\ &= |\deg\big((\mathbb{I}-Q)S_1,B_\delta\cap\ker L,0\big)|\\ &= |\mathrm{ind}\big((\mathbb{I}-Q)S_1/_{\ker L},0\big)|\,.\end{aligned}$$

We note that if in addition S_1 is odd then clearly $\mathrm{ind}\big((\mathbb{I}-Q)S_1|_{\ker L},0\big)\neq 0$. Finally, if $\deg\big(L+S,B_\delta,0\big)\neq 0$ then $L+S=f$ has a solution in B_δ for any $f\in Y$ sufficiently small.

5. Owing to point 4 of this remark we see that the first alternative of the statement of Theorem 2.2 is still valid, when T is continuous near $x=0$ possessing a decomposition $T(x)=L+L_mx^m+o\big(|x|_X^m\big)$ for x small and an odd natural number $m>1$, where L_m is a continuous, symmetric, multilinear map of order m, L has the above properties of paragraph 2 of this remark and $\ker L=\mathrm{span}\,\varphi$ such that $L_m\varphi^m\notin\mathrm{im}\,L$. Indeed, we take $S_1(x)=L_mx^m$ and easily see that then $\mathrm{ind}\big((\mathbb{I}-Q)S_1|_{\ker L},0\big)\neq 0$ (see also [3, Lemma 3.2]). The result 1 from the Introduction concerning (1.1) is related to these arguments.

Now we assume that T is equivariant. So let $\mathcal{L}(X)$, $\mathcal{L}(Y)$ be the space of all linear continuous mappings from X, respectively Y, into itself. Let $\mathcal{G}$ be a compact Lie group. We assume that there are linear representations of $\mathcal{G}$ in X, Y, respectively, denoted by

$$\big\{R_g^X\big\}_{g\in\mathcal{G}}\subset\mathcal{L}(X),\quad \big\{R_g^Y\big\}_{g\in\mathcal{G}}\subset\mathcal{L}(Y)\,.$$

We recall that T is said to be $\mathcal{G}$-equivariant according to the above linear representations of $\mathcal{G}$ if the following holds:

$$R_g^Y\circ T=T\circ R_g^X\quad\forall\,g\in\mathcal{G}\,.$$

It is clear that if T is $\mathcal{G}$-equivariant and $T(0)=0$, then both $\ker DT(0)$ and $\mathrm{im}\,DT(0)$ are $\mathcal{G}$-equivariant, i.e.,

$$R_g^Y\big(\mathrm{im}\,DT(0)\big)=\mathrm{im}\,DT(0),\quad R_g^X\big(\ker DT(0)\big)=\ker DT(0)\quad\forall\,g\in\mathcal{G}\,.$$

We know by [6] that there are $\mathcal{G}$-equivariant continuous projections $P\colon X\to\ker DT(0)$ and $Q\colon Y\to\mathrm{im}\,DT(0)$ according to the above linear representations of $\mathcal{G}$.

Theorem 2.4. *Assume $T(0)=0$, $DT(0)$ is a Fredholm operator with index 0 and T is $\mathcal{G}$-equivariant according to the above linear representations of $\mathcal{G}$. Assume, further, there is a $g\in\mathcal{G}$ such that*

$$\begin{aligned}&\big\{v\in ker\,DT(0):R_g^Xv=v\big\}\neq\{0\}\\ &\big\{w\in im\,(\mathbb{I}-Q):R_g^Yw=w\big\}=\{0\}\,,\end{aligned}$$

and set $m = dim\{v \in ker DT(0) : R_g^X v = v\}$. *Then there exist neighborhoods* U_1, U_2 *of* $0 \in X$, $0 \in Y$, *respectively, satisfying*

a) The equation $T(x) = f$ *has a* m*-parametric, continuous,* R_g^X*-invariant family of solutions in* U_1 *for any* $f \in U_2$ *such that* $R_g^Y Qf = Qf$ *and* $(\mathbb{I} - Q)f = 0$. *This family depends continuously on* f*;*

b) The equation $T(x) = f$ *has no solutions in* $U_1 \cap \{x \in X : R_g^X Px = Px\}$ *for any* $f \in U_2$ *such that* $R_g^Y Qf = Qf$ *and* $(\mathbb{I} - Q)f \neq 0$.

Proof. We follow some ideas of [6, page 96] together with the proof of Theorem 2.1. As in the proof of this theorem, we reduce the study of the equation $Tx = f$ by means of the Lyapunov-Schmidt method, to system (2.1), and let $x_2(\cdot, \cdot)$ be the solution to the first of (2.1). As Q is chosen to satisfy

$$R_g^Y QT(x) = QT\big(R_g^X x\big) \quad \forall\, x \in X\,,$$

we easily get

$$R_g^X x_2(x_1, z) = x_2\big(R_g^X x_1, R_g^Y z\big)$$

for all $x_1 \in \ker X$, $z \in \operatorname{im} DT(0)$ small. So we have

$$\begin{aligned} R_g^Y F(x_1, z) &= (\mathbb{I} - Q)\big(R_g^Y T\big(x_1 + x_2(x_1, z)\big)\big) \\ &= (\mathbb{I} - Q)T\big(R_g^X x_1 + x_2(R_g^X x_1, R_g^Y z)\big) = F\big(R_g^X x_1, R_g^Y z\big)\,. \end{aligned}$$

Hence, if $R_g^X x_1 = x_1$ and $R_g^Y z = z$ we have

$$F(x_1, z) = F\big(R_g^X x_1, R_g^X z\big) = R_g^Y F(x_1, z)\,.$$

But $F(x_1, z) \in \operatorname{im}(\mathbb{I} - Q)$, and hence the assumptions of this theorem imply that $F(x_1, z) = 0$. So we obtain

$$R_g^X x_1 = x_1 \quad \text{and} \quad R_g^X z = z \quad \text{imply} \quad F(x_1, z) = 0.$$

Now the assumptions $R_g^Y Qf = Qf$ and $(\mathbb{I} - Q)f = 0$ give $R_g^Y z = z$ and $f_2 = 0$ in (2.1) while $R_g^Y Qf = Qf$ and $(\mathbb{I} - Q)f \neq 0$ give $R_g^X z = z$ and $f_2 \neq 0$ in (2.1). The above implication proves both the statements *a*) and *b*). The proof is finished.

Corollary 2.5. *Assume* $T(0) = 0$, $DT(0)$ *is a Fredholm operator with index* 0 *and* T *is* $\mathcal{G}$*-equivariant according to the above linear representations of* $\mathcal{G}$*. Let* $m = dim\, ker DT(0)$ *and consider*

$$W = \big\{f \in Y : R_g^Y Qf = Qf\big\}\,.$$

If there is a $g \in \mathcal{G}$ *such that*

$$\begin{aligned} &R_g^X v = v \quad \forall\, v \in ker DT(0) \\ &R_g^Y w \neq w \quad \forall\, w \in im(\mathbb{I} - Q),\, w \neq 0\,, \end{aligned}$$

then there are neighborhoods U_1, U_2 *of* $0 \in X$, *respectively* $0 \in Y$, *satisfying*

a) The equation $T(x) = f$ *has exactly an* m*-parametric, continuous,* R_g^X*-invariant family of solutions in* U_1 *for any* $f \in U_2 \cap W$ *such that* $(\mathbb{I} - Q)f = 0$. *This family depends continuously on such* f;

b) The equation $T(x) = f$ *has no solutions in* U_1 *for any* $f \in U_2 \cap W$ *such that* $(\mathbb{I} - Q)f \neq 0$.

Remark 2.6. 1. We have under the assumptions of Corollary 2.5 that $T(x) = f$ has a small solution for a small $f \in W$ if and only if $(\mathbb{I} - Q)f = 0$ and then such solutions present a continuous, nonconstant, m-parametric family. So the equation $T(x) = f$ behaves near $x = 0$ for any small $f \in W$ like its linearization $DT(0)x = f$, $f \in Y$. Furthermore, it is clear according to the $\mathcal{G}$-equivariantness of Q that $\{f \in Y : R_g^Y f = f\} \subset W$.

2. There are only two unitary, linear mappings on $\mathbb{R}$: $r \mapsto r$ and $r \mapsto -r$. Hence, for the case $m = 1$ in Corollary 2.5, we generally have

$$\text{either} \quad R_g^X v = v \quad \forall\, v \in \ker DT(0) \quad \text{or} \quad R_g^X v = -v \quad \forall\, v \in \ker DT(0)\,,$$

similarly for $\operatorname{im}(\mathbb{I} - Q)$. So the assumptions of Corollary 2.5 are not very restrictive for the case $m = 1$.

3. The assertions of Corollary 2.5 for $m = 1$ remind one of the second alternative of Theorem 2.2. But these assertions follow from the equivariantness of T in Corollary 2.5, while the analyticity of T is crucial in Theorem 2.2.

Now let us consider the equation $0 = T(x) = Lx + S(x)$, where $L\colon X \to Y$ is a continuous linear, Fredholm operator with index 0 and $S\colon X \to Y$ is a globally Lipschitz-continuous mapping with a sufficiently small Lipschitz constant L_S. Then by applying the Banach fixed-point theorem, we can globalize some of the above results in the following way. We have to solve the first equation of (2.1) in $x_2 \in \operatorname{im}(\mathbb{I} - P)$ for any $z \in \operatorname{im} L$ and $x_1 \in \ker L$ to obtain the global bifurcation equation. We solve it by the Banach fixed-point theorem, since now this equation has the form

$$Lx_2 + QS(x_1 + x_2) = z$$

or

$$x_2 + L^{-1}QS(x_1 + x_2) = L^{-1}z$$

where $L^{-1}\colon \operatorname{im} L \to \operatorname{im}(\mathbb{I} - P)$ is the inverse of $L\colon \operatorname{im}(\mathbb{I} - P) \to \operatorname{im} L$. As for any x_{21}, $x_{22} \in \operatorname{im}(\mathbb{I} - P)$ the following holds:

$$\left|L^{-1}QS(x_1 + x_{21}) - L^{-1}QS(x_1 + x_{22})\right|_X \leq ||L^{-1}|| \cdot ||Q|| \cdot L_S \cdot |x_{21} - x_{22}|_X\,;$$

we can take $L_S < 1/\big(||L^{-1}|| \cdot ||Q||\big)$. So we can apply the Banach fixed-point theorem provided the projection Q satisfies $||Q|| < 1/\big(||L^{-1}||L_S\big)$. Since $||Q|| \geq 1$, we note that the inequality $L_S < 1/||L^{-1}||$ is necessary. We obtain in this way a global bifurcation equation $F(x_1, z) = f_2$ as in the proof of Theorem 2.1.

Assuming in addition that S is analytic and $\dim\ker L = 1$, we follow the proof of Theorem 2.2 by taking Q such that $Qh = 0$, so $\operatorname{im}(\mathbb{I} - Q) = \operatorname{span} h$ for $h \notin \operatorname{im} L$. We claim that

$$||Q|| = \left(\inf\left\{|rh - z|_Y : r \in \mathbb{R},\, z \in \operatorname{im} L,\, |z|_Y = 1\right\}\right)^{-1}$$

(of course, the above equation on the norm of Q is independent of S). To prove this claim, we note that for any $\varepsilon > 0$ small, say $0 < \varepsilon < ||Q||$, there are $r_0 \in \mathbb{R}$, $z_0 \in \operatorname{im} L$ such that $r_0 h - z_0 \neq 0$ and

$$|r_0 h - z_0|_Y \big(||Q|| - \varepsilon\big) \leq |z_0|_Y;$$

hence $z_0 \neq 0$, and we have

$$\Big|\frac{r_0}{|z_0|_Y} h - \frac{z_0}{|z_0|_Y}\Big|_Y \leq \big(||Q|| - \varepsilon\big)^{-1}.$$

On the other hand, as

$$|rh - z|_Y ||Q|| \geq |z|_Y$$

for any $r \in \mathbb{R}$, $z \in \operatorname{im} L$, we obtain

$$|rh - z|_Y \geq 1/||Q||$$

for any $r \in \mathbb{R}$, $z \in \operatorname{im} L$, $|z|_Y = 1$. The claim is proved. Hence we see that if $h \in \mathcal{W}$, where the set $\mathcal{W}$ (see Appendix) is defined by

$$\mathcal{W} = \Big\{h \in Y \setminus \operatorname{im} L : \exists\, \delta > 0,\, |rh - z|_Y > ||L^{-1}|| L_S + \delta \quad \forall\, r \in \mathbb{R}, \quad \forall\, z \in \operatorname{im} L, \quad |z|_Y = 1\Big\},$$

then (2.1) is reduced to the study of the global bifurcation equation $F(x_1, z) = ch$, $c \in \mathbb{R}$. Moreover, if $T(x) = 0$ has a nonisolated solution then $F(x_1, 0) = 0\, \forall x_1 \in \ker L$ by the proof of Theorem 2.2. This result immediately implies that then the set $\big\{x \in X : T(x) = 0\big\}$ is an analytic curve $\gamma\colon \mathbb{R} \to X$ such that $|\gamma(r)|_X \to \infty$ as $r \to \pm\infty$. Furthermore, we also see that then $F(x_1, 0) = ch$, $c \neq 0$ has no solutions, i.e., we obtain $\operatorname{im} T \subset Y \setminus \mathcal{W}$.

Summarizing, we obtain

Theorem 2.7. *Let $T(x) = Lx + S(x)$, where $L\colon X \to Y$ is a continuous linear, Fredholm operator with index 0 and $S\colon X \to Y$ is a globally Lipschitz-continuous mapping with a Lipschitz constant L_S. Assume that $L_S ||L^{-1}|| < 1$, and, moreover, that S is analytic and dim ker$L = 1$. If $T(x) = 0$ has a nonisolated solution then* *the set* $\{x \in X : T(x) = 0\}$ *is an analytic curve* $\gamma\colon \mathbb{R} \to X$ *such that* $|\gamma(r)|_X \to \infty$ *as* $r \to \pm\infty$. *Moreover, then*

$$\mathcal{W} \equiv \Big\{h \in Y \setminus im\, L : \exists\, \delta > 0,\ |rh - z|_Y > ||L^{-1}||L_S + \delta \\ \forall\, r \in \mathbb{R}, \quad \forall\, z \in im\, L, \quad |z|_Y = 1\Big\} \subset Y \setminus im\, T\,.$$

We note

$$\mathcal{W} = \Big\{h \in Y \setminus im\, L : \textit{the projection} \quad Q\colon Y \to Y,\ im\, Q = im\, L,\ ker\, Q = span\, h \\ \textit{satisfies} \quad ||Q|| < (||L^{-1}||L_S)^{-1}\Big\}\,.$$

Theorem 2.8. *In Theorem* 2.4 *and Corollary* 2.5, *if* $L_S \cdot ||L^{-1}|| \cdot ||Q|| < 1$, *where* $Q\colon Y \to im\, L$ *is a* $\mathcal{G}$*-equivariant projection, then we can take* $U_1 = X$ *and* $U_2 = Y$. *Similarly, the assumptions of smallness in Remark* 2.6.1 *can be dropped provided* $L_S \cdot ||L^{-1}|| \cdot ||Q|| < 1$ *and* Q *is the same projection.*

Finally, Theorem 2.7 can be restated in the following way.

Corollary 2.9. *If, in addition to the assumptions of Theorem* 2.7, *there is a continuous functional* $\phi\colon Y \to \mathbb{R}$ *such that* $im\, L = \{y \in Y : \phi(y) = 0\}$ *and* L_S *is smaller than in Theorem* 2.7, *namely* $||L^{-1}||L_S < 1/2$, *then there is a constant* $c = 2||\phi|| \cdot ||L^{-1}||$ *such that*

$$im\, T \subset \big\{h \in Y : |\phi(h)| \le c L_S |h|_Y\big\}\,.$$

Proof. For any $h \notin \operatorname{im} L$, we have

$$Qy = y - \phi(y)/\phi(h) \cdot h\,.$$

So

$$||Q|| \le 1 + ||\phi|| \cdot |h|_Y/|\phi(h)|\,.$$

We know, from Theorem 2.7, that

$$\operatorname{im} T \subset \big\{h \in Y \setminus \operatorname{im} L : ||Q|| \ge (||L^{-1}||L_S)^{-1}\big\} \cup \operatorname{im} L\,.$$

Now, from $||Q|| \le 1 + ||\phi|| \cdot |h|_Y/|\phi(h)|$, we see that if $||Q|| \ge (||L^{-1}||L_S)^{-1}$ then certainly

$$||\phi|| \cdot |h|_Y/|\phi(h)| \ge (||L^{-1}||L_S)^{-1} - 1\,.$$

So we get for $||L^{-1}||L_S < 1/2$

$$|\phi(h)| \le \frac{||\phi|| \cdot ||L^{-1}|| \cdot L_S \cdot |h|_Y}{1 - ||L^{-1}||L_S} \le 2||\phi|| \cdot ||L^{-1}|| \cdot L_S \cdot |h|_Y\,.$$

The proof is finished.

3. Applications to BVP's. First of all, Theorem 1 from the Introduction is a consequence of Theorem 2.2. Furthermore, according to Remark 2.3, item 2, and [5], the first alternative of Theorem 1 $\big($*i.e.,* $y = 0$ *is an isolated solution of* $\mathcal{L}y = 0$ *and* $\mathcal{L}y = f$ *has a small solution for any small* $f \in L^1\big([a,b],\mathbb{R}\big)\big)$ is valid when L is self-adjoint and the nonlinearity $y \to y^3$ in (1.1) is replaced by $y \to q(y)$ such that $q \in C(\mathbb{R},\mathbb{R})$, $\lim_{y\to 0_\pm} q(y)/y^{m_\pm} \neq 0$ for odd natural numbers $m_\pm > 1$ and these limits have the same sign, i.e., both are either nonnegative or nonpositive.

On the other hand, we claim that this alternative also holds provided φ changes its sign on $[a,b]$ and there are odd natural numbers $m_\pm > 1$ such that

1.
$$\int_{\{\varphi\geq 0\}} \omega(x)\varphi(x)^{\min\{m_-,m_+\}}\,dx \neq 0, \quad \int_{\{\varphi\leq 0\}} \omega(x)\varphi(x)^{\min\{m_-,m_+\}}\,dx \neq 0\,;$$

2. $y \to y^3$ in (1.1) is replaced by $q \in C(\mathbb{R},\mathbb{R})$ satisfying
$$q(y) = \tilde{q}(y) + o\big(|y|^{\min\{m_-,m_+\}}\big) \quad \text{as} \quad y \to 0\,,$$

where $\tilde{q}(y) = c_1 y^{m_+}$ for $y \geq 0$ and $\tilde{q}(y) = c_2 y^{m_-}$ for $y \leq 0$ when c_1, c_2 are nonzero constants with the same sign.

To prove this claim, we use Remark 2.3, item 4, by taking $S_1(y)(\cdot) = \tilde{q}\big(y(\cdot)\big)$ and we show that such S_1 satisfies (C2) with $\alpha = \min\{m_-,m_+\}$. We note that

$$(\mathbb{I} - Q)y = \int_a^b \omega(x)y(x)\,dx\Big/\int_a^b \omega^2(x)\,dx \cdot \omega\,.$$

For simplicity we assume $c_1 = c_2 = 1$. Since $\big(\tilde{q}(y)\big)' = O\big(|y|^{\min\{m_-,m_+\}-1}\big)$ for $y \neq 0$ small, we have

$$\Big|\int_a^b \omega(x)\Big(\tilde{q}\big(c_n\varphi(x) + c_n^2 z_n(x)\big) - \tilde{q}\big(c_n\varphi(x)\big)\Big)\,dx\Big| = O\big(|c_n|^{\min\{m_-,m_+\}+1}\big)$$

for $c_n \in \mathbb{R}\setminus\{0\}$, $c_n \to 0$ as $n \to \infty$ and $z_n \in C\big([a,b],\mathbb{R}\big)$ such that $\sup_{[a,b]}|z_n| \leq 1$. Furthermore, it is clear that

$$\begin{aligned}&\liminf_{n\to\infty}\Big|\int_a^b \omega(x)\tilde{q}\big(c_n\varphi(x)\big)\,dx\Big|\Big/|c_n|^{\min\{m_-,m_+\}}\\ &\geq \min\Big\{\Big|\int_{\{\varphi\geq 0\}} \omega(x)\varphi(x)^{\min\{m_-,m_+\}}\,dx\Big|, \quad \Big|\int_{\{\varphi\leq 0\}} \omega(x)\varphi(x)^{\min\{m_-,m_+\}}\,dx\Big|\Big\}\,.\end{aligned}$$

So the above S_1 satisfies (C2) and the claim is proved. If in addition L is self-adjoint then $\omega = \varphi$ and the above inequalities of condition 1. are automatically valid. We note that the above condition 2. is equivalent to the existence of an odd natural number $m > 1$ for which the limits $\lim_{y\to 0_\pm} q(y)/y^m$ exist and have the same sign and at least one of them is nonzero.

The second alternative of Theorem 1 (*i.e., the set* $\{y \in BC : \mathcal{L}y = 0\}$ *near* $y = 0$ *is an analytic, nonconstant curve crossing* $y = 0$ *and for any* $h \notin \operatorname{im} L$, *i.e.,* $h \in L^1([a,b],\mathbb{R})$ *and* $\int_a^b h(t)\omega(t)\,dt \neq 0$, *there are positive constants* K_1, K_2 *such that* $\mathcal{L}y = c\cdot h$, $c \in \mathbb{R}$, $0 < |c| < K_1$ *implies* $\sup|y| > K_2$) may be valid according to Remark 2.3, item 3, for any $q\colon \mathbb{R}\to\mathbb{R}$, instead of $y \to y^3$, which is analytic satisfying $q(y) = O(y^2)$ as $y \to 0$.

Now we deal with generalizations of the example (1.2) of [4] mentioned in the Introduction.

Theorem 3.1. *Consider*

$$\begin{aligned} &y'' + g(y,x) = f, \quad -1 \le x \le 1\\ &y(1) = y(-1), \quad y'(1) + y'(-1) = 0\,, \end{aligned} \tag{3.1}$$

where $g \in C^1\big(\mathbb{R}\times[-1,1],\mathbb{R}\big)$ *satisfies*

$$\begin{aligned} &g(y,-x) = g(y,x) \quad \forall\,(y,x) \in \mathbb{R}\times[-1,1]\\ &g(0,\cdot) = 0, \quad \frac{\partial}{\partial y} g(0,\cdot) = 0\,. \end{aligned}$$

Then

a) For any $f \in C([-1,1],\mathbb{R})$ *sufficiently small such that* $f(-x) = f(x)\,\forall\, x \in [-1,1]$, *the equation* (3.1) *has exactly a nonconstant, one-parametric family of symmetric (i.e.,* $y(x) = y(-x)$*) small solutions;*

b) For any $f \in C([-1,1],\mathbb{R})$ *sufficiently small such that* $f(x) = cx + h(x)$, $0 \neq c \in \mathbb{R}$, $h(-x) = h(x)\,\forall\, x \in [-1,1]$, *the equation* (3.1) *has no small solutions.*

Proof. We apply Corollary 2.5 by taking

$$\begin{aligned} &X = \big\{y \in C^2([-1,1],\mathbb{R}) : y(1) = y(-1),\ y'(1) + y'(-1) = 0\big\},\\ &Y = C([-1,1],\mathbb{R}), \quad T(y) = y'' + g(y,\cdot)\\ &\mathcal{G} = \mathbb{Z}_2 = \{0,1\}, \quad (R_1^X y)(x) = y(-x) \quad (R_1^Y y)(x) = y(-x) \quad \forall\, x \in [-1,1]\,. \end{aligned}$$

We already know that $\ker DT(0) = \operatorname{span}\varphi$, $\varphi \equiv 1$ and by [4] we have

$$\operatorname{im} DT(0) = \Big\{y \in C\big([-1,1],\mathbb{R}\big) : \int_{-1}^1 y(x)x\,dx = 0\Big\}\,.$$

So

$$\begin{aligned} &\big(Qy\big)(x) = y(x) - \frac{3}{2}x\int_{-1}^1 y(s)s\,ds \quad \forall\, x \in [-1,1]\\ &\operatorname{im}(\mathbb{I} - Q) = \operatorname{span}\omega,\ \omega(x) = x\,. \end{aligned}$$

Furthermore, we note that $y(-x) = y(x) \, \forall \, x \in [-1,1]$ implies $Qy = y$. Hence we have (see also Remark 2.6, item 1)

$$\big\{y \in C\big([-1,1],\mathbb{R}\big) : y(-x) = y(x) \, \forall \, x \in [-1,1]\big\} \subset W \cap \operatorname{im} Q\,.$$

Finally, $R_1^X\varphi = \varphi$, $R_1^Y\omega = -\omega$. The conclusion follows from Corollary 2.5.

Theorem 3.2. *Assume in addition to Theorem* 3.1 *that g is globally analytic in y uniformly for x; i.e.,*

$$g(y,x) = \sum_{i=0}^{\infty} g_i(x)y^i$$

with $g_i \in C\big([-1,1],\mathbb{R}\big)$ and $\lim_{i\to\infty} \sqrt[i]{|g_i(x)|} = 0$ uniformly for $x \in [-1,1]$. Then for any $h \in C\big([-1,1],\mathbb{R}\big)$ such that $\int_{-1}^{1} h(x)x\,dx \neq 0$, there are positive constants K_1, K_2 depending on h such that any solution of (3.1) *with $f = ch$, $c \in \mathbb{R}$, $0 < |c| < K_1$ satisfies $|y|_{C^2} > K_2$.*

Proof. We apply Theorem 2.2 in the framework of the proof of Theorem 3.1.

Remark 3.3. The difference between Theorems 3.1 and 3.2 is the following: the smallness of f and corresponding solutions y in Theorem 3.1 is uniform, while the constants K_1, K_2 in Theorem 3.2 depend on h.

Since $y'' = 0$, $y(-1) + y(1) = 0$, $y'(-1) = y'(1)$ is the adjoint problem to $y'' = 0$, $y(-1) = y(1)$, $y'(-1) + y'(1) = 0$, the following results hold.

Theorem 3.4. *Consider*

$$\begin{aligned} &y'' + g(y,x) = f, \quad -1 \le x \le 1 \\ &y'(1) = y'(-1), \quad y(1) + y(-1) = 0\,, \end{aligned} \tag{3.2}$$

where $g \in C^1\big(\mathbb{R}\times[-1,1],\mathbb{R}\big)$ satisfies

$$\begin{aligned} &g(-y,-x) = -g(y,x) \quad \forall\,(y,x) \in \mathbb{R}\times[-1,1] \\ &g(0,\cdot) = 0 \quad \frac{\partial}{\partial y} g(0,\cdot) = 0\,. \end{aligned}$$

Then

a) For any $f \in C\big([-1,1],\mathbb{R}\big)$ sufficiently small such that $f(-x) = -f(x) \, \forall \, x \in [-1,1]$, the equation (3.2) *has exactly a one-parametric, symmetric (i.e., $y(-x) = -y(x)$) family of nonconstant small solutions;*

b) For any $f \in C\big([-1,1],\mathbb{R}\big)$ sufficiently small such that $f(x) = c + h(x)$, $0 \neq c \in \mathbb{R}$, $h(-x) = -h(x) \, \forall \, x \in [-1,1]$, the equation (3.2) *has no small solutions.*

Proof. We apply Corollary 2.5 by taking

$$\begin{aligned} &X = \big\{y \in C^2\big([-1,1],\mathbb{R}\big) : y'(1) = y'(-1),\, y(1) + y(-1) = 0\big\}, \\ &Y = C\big([-1,1],\mathbb{R}\big), \quad T(y) = y'' + g(y,\cdot) \\ &\mathcal{G} = \mathbb{Z}_2 = \{0,1\}, \quad \big(R_1^X y\big)(x) = -y(-x) \;\; \big(R_1^Y y\big)(x) = -y(-x), \quad \forall\, x \in [-1,1]\,. \end{aligned}$$

We have that $\ker DT(0) = \operatorname{span}\varphi$, $\varphi(x) = x$ and

$$\operatorname{im} DT(0) = \left\{ y \in C\big([-1,1],\mathbb{R}\big) : \int_{-1}^{1} y(x)\,dx = 0 \right\}.$$

So

$$\big(Qy\big)(x) = y(x) - \frac{1}{2}\int_{-1}^{1} y(s)\,ds \quad \forall\, x \in [-1,1]$$
$$\operatorname{im}(\mathbb{I} - Q) = \operatorname{span}\omega,\ \omega \equiv 1\,.$$

Furthermore, we note that $y(-x) = -y(x)\,\forall\, x \in [-1,1]$ implies $Qy = y$. Hence we have (see also Remark 2.6, item 1)

$$\left\{ y \in C\big([-1,1],\mathbb{R}\big) : y(-x) = -y(x)\,\forall\, x \in [-1,1] \right\} \subset W \cap \operatorname{im} Q\,.$$

Finally, $R_1^X\varphi = \varphi$, $R_1^Y\omega = -\omega$. The proof is finished by Corollary 2.5.

Theorem 3.5. *Assume in addition to Theorem* 3.4 *that g is globally analytic in y uniformly for x. Then for any $h \in C\big(\mathbb{R}\times[-1,1],\mathbb{R}\big)$ such that $\int_{-1}^{1} h(x)\,dx \neq 0$, there are positive constants K_1, K_2 depending on h such that any solution of* (3.2) *with $f = ch$, $c \in \mathbb{R}$, $0 < |c| < K_1$ satisfies $|y|_{C^2} > K_2$.*

Proof. We apply Theorem 2.2 in the framework of the proof of Theorem 3.4.

Remark 3.6. Remark 3.3 can be modified for (3.2).

Theorem 3.7. *Assume that $g \in C\big(\mathbb{R}\times[-1,1],\mathbb{R}\big)$ is globally Lipschitz continuous in y uniformly for $x \in [-1,1]$ in* (3.1) *and* (3.2) *with a sufficiently small Lipschitz constant L_g, as well as that g satisfies*

$$g(y,-x) = g(y,x) \quad \forall\,(y,x) \in \mathbb{R}\times[-1,1]$$

for (3.1) *and*

$$g(-y,-x) = -g(y,x) \quad \forall\,(y,x) \in \mathbb{R}\times[-1,1]$$

for (3.2). *Then the assumption of smallness of f and y can be dropped in Theorems* 3.1 *and* 3.4*; i.e., f can be arbitrarily large and y is a corresponding, not necessarily small, solution. Moreover, when g is in addition globally analytic in y uniformly for x, then there is a constant $\Omega > 0$ such that* (3.1), *respectively* (3.2), *has no solutions for any $f \in C\big([-1,1],\mathbb{R}\big)$ satisfying $|\int_{-1}^{1} f(x)x\,dx| > \Omega L_g |f|_C$, respectively $|\int_{-1}^{1} f(x)\,dx| > \Omega L_g |f|_C$.*

Proof. We apply Theorem 2.8 and Corollary 2.9 in the framework of the above theorems, since $\phi(y) = \int_{-1}^{1} f(x)x\,dx$ for the case (3.1), and $\phi(y) = \int_{-1}^{1} f(x)\,dx$ for the case (3.2).

The following result is motivated by an example of [1], where it was shown that

$$y'' + g(t+y) = c = \text{constant}$$

has no 2π-periodic solutions if $c \neq 0$ and if $c = 0$ has a continuum of such solutions provided $g \in C^1\big(\mathbb{R},\mathbb{R}\big)$ is 2π-periodic satisfying additional assumptions.

Corollary 3.8. *Consider*

$$y'' + g(y) = c = constant\,, \tag{3.3}$$

where $g\colon \mathbb{R} \to \mathbb{R}$ *is odd and globally Lipschitz continuous. Then there is a* $t_0 > 0$ *such that for any fixed* $t \in (0, t_0)$, *the equation* (3.3) *considered on the interval* $[-t, t]$ *with the boundary value conditions*

$$\begin{aligned}&y'(t) = y'(-t), \quad y(t) + y(-t) = 0\\ &\big(\textit{respectively} \quad y'(t) + y'(-t) = 0, \quad y(t) = y(-t)\big)\end{aligned}$$

has no solutions for any $c \neq 0$. *While for* $c = 0$ (*respectively arbitrary* c) *this boundary problem has exactly a one-parametric family of solutions which really presents a continuum of odd* (*respectively even*) *solutions of* (3.3) *defined on the whole of* $\mathbb{R}$.

Proof. By taking the change $y(xt) = z(x)$, $x \in [-1, 1]$, consider the problems

$$z'' + t^2 g(z) = t^2 c, \quad z'(1) = z'(-1), \quad z(1) + z(-1) = 0 \tag{3.4}$$

and

$$z'' + t^2 g(z) = t^2 c, \quad z(1) = z(-1), \quad z'(1) + z'(-1) = 0\,. \tag{3.5}$$

By applying the first part of Theorem 3.7 to both (3.4) and (3.5) for t small, we see that (3.4) has a solution only if $c = 0$ as well as that (3.4) with $c = 0$, respectively (3.5) with arbitrary c, possesses a continuum of odd, respectively even, solutions. Since g is odd and globally Lipschitz continuous, these solutions determine the desired two continuums of solutions of (3.3). The proof is finished.

Remark 3.9. The upper bound of the smallness of both the global Lipschitz constant L_g in Theorem 3.7 and number t_0 in Corollary 3.8 can be derived from Theorem 2.8 and Corollary 2.9.

Instead of the above second-order problems, we can consider $2n$-order ones of the forms

$$y^{(2n)} + g(y, x) = 0, \quad y^{(i)}(-1) + (-1)^{i+1} y^{(i)}(1) = 0, \quad i = 0, 1, \dots, 2n-1\,, \tag{3.7}$$

$$y^{(2n)} + g(y, x) = 0, \quad y^{(i)}(-1) + (-1)^{i} y^{(i)}(1) = 0, \quad i = 0, 1, \dots, 2n-1\,, \tag{3.8}$$

where $y^{(i)} = \frac{d^i}{dt^i} y$ and $g \in C^1\big(\mathbb{R} \times [-1, 1], \mathbb{R}\big)$ has the properties of Theorem 3.1, respectively Theorem 3.4, for (3.7), respectively (3.8). It is easy to see that the problem adjoint to

$$y^{(2n)} = 0, \quad y^{(i)}(-1) + (-1)^{i+1} y^{(i)}(1) = 0, \quad i = 0, 1, \dots, 2n-1 \tag{3.9}$$

has the form

$$y^{(2n)} = 0, \quad y^{(i)}(-1) + (-1)^i y^{(i)}(1) = 0, \quad i = 0, 1, \dots, 2n-1\,. \tag{3.10}$$

Furthermore, the solutions of (3.9) is the set

$$\ker_1 \equiv \Big\{ \sum_{i=0}^{n-1} a_{2i} x^{2i} : a_{2i} \in \mathbb{R} \Big\}$$

and for (3.10) is the set

$$\ker_2 \equiv \Big\{ \sum_{i=0}^{n-1} a_{2i+1} x^{2i+1} : a_{2i+1} \in \mathbb{R} \Big\}\,.$$

We see that $\ker_1$, $\ker_2$ contain only even, odd functions, respectively. So the proof of Theorem 3.1, respectively Theorem 3.4, can be repeated for equations (3.7), (3.8) respectively, to obtain the following results.

Theorem 3.10. *If $g \in C^1\big(\mathbb{R} \times [-1,1], \mathbb{R}\big)$ satisfies the assumptions of Theorem* 3.1 *then*

a) For any $f \in C\big([-1,1], \mathbb{R}\big)$ sufficiently small such that $f(-x) = f(x)\, \forall\, x \in [-1,1]$, the equation (3.7) *has exactly a nonconstant, n-parametric family of small solutions;*

b) For any $f \in C\big([-1,1], \mathbb{R}\big)$ sufficiently small such that

$$f(x) = \sum_{i=0}^{n-1} a_{2i+1} x^{2i+1} + h(x), \quad \sum_{i=0}^{n-1} a_{2i+1}^2 \neq 0,\ h(-x) = h(x)\ \forall\, x \in [-1,1],$$

the equation (3.7) *has no small solutions.*

Theorem 3.11. *If $g \in C^1\big(\mathbb{R} \times [-1,1], \mathbb{R}\big)$ satisfies the assumptions of Theorem* 3.4 *then*

a) For any $f \in C\big([-1,1], \mathbb{R}\big)$ sufficiently small such that $f(-x) = -f(x)\, \forall\, x \in [-1,1]$, the equation (3.8) *has exactly a nonconstant, n-parametric family of small solutions;*

b) For any $f \in C\big([-1,1], \mathbb{R}\big)$ sufficiently small such that

$$f(x) = \sum_{i=0}^{n-1} a_{2i} x^{2i} + h(x), \quad \sum_{i=0}^{n-1} a_{2i}^2 \neq 0,\ h(-x) = -h(x)\ \forall\, x \in [-1,1],$$

the equation (3.8) *has no small solutions.*

Theorem 3.7 can be extended for (3.7) and (3.8).

Theorem 3.12. *Assume that $g \in C\big(\mathbb{R}\times[-1,1],\mathbb{R}\big)$ is globally Lipschitz continuous in y uniformly for $x \in [-1,1]$ in* (3.7) *and* (3.8) *with a sufficiently small Lipschitz constant as well as that g satisfies*

$$g(y,-x) = g(y,x) \quad \forall\,(y,x) \in \mathbb{R} \times [-1,1]$$

for (3.7) *and*

$$g(-y,-x) = -g(y,x) \quad \forall\,(y,x) \in \mathbb{R} \times [-1,1]$$

for (3.8)*. Then the assumption of smallness of f and y can be dropped in Theorems* 3.10 *and* 3.11*; i.e., f can be arbitrarily large and y is a corresponding, not necessarily small, solution.*

4. Concluding remarks.

Remark 4.1. Theorems 3.1, 3.4, 3.10, 3.11 and 3.12 as well as the first part of Theorem 3.7 concerning Theorems 3.1 and 3.4 can be extended to higher-dimensional cases, i.e. when y is vector valued in (3.1), (3.2), (3.7) and (3.8). More concretely, if y is m-dimensional, i.e., $y \in \mathbb{R}^m$, then there are exactly m-parametric families of solutions in Theorems 3.1 and 3.4 instead of one-parametric ones, while in Theorems 3.10 and 3.11, there are exactly nm-parametric families of solutions instead of n-parametric ones.

Remark 4.2. Multipoint value problems can be similarly studied. For instance, let us consider the problem

$$\begin{aligned} &y'' + g(y,x) = f, \quad -1 \le x \le 1 \\ &y(-1) = y(1), \quad y'(0) = 0\,, \end{aligned} \tag{4.1}$$

where $g \in C^1\big(\mathbb{R} \times [-1,1],\mathbb{R}\big)$ has the properties of Theorem 3.1. It is easy to see that the proof of Theorem 3.1 can be repeated, where now we put

$$X = \Big\{ y \in C^2\big([-1,1],\mathbb{R}\big) : y(1) = y(-1),\ y'(0) = 0 \Big\}\,,$$

to obtain the validity of the statement of this theorem for (4.1). It is interesting to note that in this case

$$\operatorname{im} DT(0) = \Big\{ y \in C\big([-1,1],\mathbb{R}\big) : \int_{-1}^{1} y(x)\tau(x)\,dx = 0 \Big\}\,,$$

where

$$\tau(x) = \begin{cases} -x+1 & \text{for} \quad x \in (0,1] \\ -x-1 & \text{for} \quad x \in [-1,0)\,. \end{cases}$$

So τ is odd possessing a discontinuity at $x = 0$.

We also note that the "adjoint" problem to (4.1) has the form

$$\begin{aligned} & y'' + g(y,x) = f, \quad -1 \le x \le 1 \\ & y'(-1) = y'(1), \quad y(0) = 0\,, \end{aligned} \tag{4.2}$$

where $g \in C^1\big(\mathbb{R} \times [-1,1], \mathbb{R}\big)$ has the properties of Theorem 3.4. Again, the proof of Theorem 3.4 can be repeated, where now we put

$$X = \Big\{ y \in C^2\big([-1,1], \mathbb{R}\big) : y(0) = 0,\, y'(1) = y'(-1) \Big\}\,,$$

to obtain the validity of the statement of this theorem for (4.2).

Similarly, the equations (3.7), (3.8) can be replaced by

$$y^{(2n)} + g(y,x) = 0, \quad y^{(2i)}(-1) = y^{(2i)}(1), \quad y^{2i+1}(0) = 0, \quad i = 0,1,\dots,n-1\,, \tag{4.3}$$

$$y^{(2n)} + g(y,x) = 0, \quad y^{(2i+1)}(-1) = y^{(2i+1)}(1), \quad y^{2i}(0) = 0, \quad i = 0,1,\dots,n-1\,, \tag{4.4}$$

to obtain the validity of the statements of Theorems 3.10 and 3.11 for (4.3) and (4.4), respectively.

Remark 4.3. Finally, we note that the ideas of the proofs of the problems studied in Section 3 of this paper can be extended for more general ordinary differential equations with certain functional conditions invariant either under the transformation $y(x) \to y(-x)$ or $y(x) \to -y(-x)$.

Appendix. We are not able to show that the set $\mathcal{W}$ of Theorem 2.7 is nonempty for arbitrary L_S satisfying $0 < L_S < 1/||L^{-1}||$. On the other hand, we have

Theorem A1. *If $L_S > 0$ is such that $L_S||L^{-1}|| < 1/2$, then $\mathcal{W}$ is nonempty.*

Proof. We know by Theorem 2.7 that

$$\mathcal{W} = \Big\{ h \in Y \setminus \operatorname{im} L : \text{the projection} \quad Q\colon Y \to Y,\, \operatorname{im} Q = \operatorname{im} L,\, \ker Q = \operatorname{span} h \\ \text{satisfies} \quad ||Q|| < \big(||L^{-1}||L_S\big)^{-1} \Big\}\,.$$

Since $\operatorname{im} L$ is a closed subspace of Y with codimension 1, there is a continuous functional $\phi\colon Y \to \mathbb{R}$ such that $\operatorname{im} L = \{y \in Y : \phi(y) = 0\}$. Then by the proof of Corollary 2.9, we obtain

$$Qy = y - \phi(y)/\phi(h) \cdot h, \quad ||Q|| \le 1 + ||\phi|| \cdot |h|_Y / |\phi(h)|\,.$$

For any small $\epsilon > 0$ there is a $h_\epsilon \notin \operatorname{im} L$ satisfying

$$0 < \big(||\phi|| - \epsilon\big)|h_\epsilon|_Y \le |\phi(h_\epsilon)| \le ||\phi|| \cdot |h_\epsilon|_Y\,.$$

Hence

$$||Q|| \le 1 + \frac{||\phi||}{||\phi|| - \epsilon} = 2 + \frac{\epsilon}{||\phi|| - \epsilon}\,,$$

where Q corresponds to h_ϵ. Since $||L^{-1}||L_S < 1/2$, there is a small $\epsilon_0 > 0$ such that

$$2 < 2 + \frac{\epsilon_0}{||\phi|| - \epsilon_0} < \frac{1}{||L^{-1}||L_S}$$

and then $h_{\epsilon_0} \in \mathcal{W}$. The proof is finished.

Theorem A2. *If Y is a Hilbert space then $\mathcal{W}$ is nonempty for any $0 < L_S < 1/||L^{-1}||$.*

Proof. Let $h_0 \ne 0$ be orthogonal to $\operatorname{im} L$. Then the orthogonal projection $Q\colon Y \to \operatorname{im} L$ satisfies $||Q|| = 1$, so for any $0 < L_S < 1/||L^{-1}||$, we see that $rh_0 \in \mathcal{W}$, $\forall\, r \in \mathbb{R} \setminus \{0\}$. The proof is finished.

Finally, as in the proof of Theorem 2.7, we have

$$||\mathbb{I} - Q|| = \Big(\inf\Big\{\Big|\frac{h}{|h|_Y} - z\Big|_Y : z \in \operatorname{im} L\Big\}\Big)^{-1}$$

for any $h \notin \operatorname{im} L$ and Q is the usual projection. Consequently,

$$\operatorname{dist}\,(h/|h|_Y, \operatorname{im} L) = ||\mathbb{I} - Q||^{-1}\,.$$

Since $(\mathbb{I} - Q)y = \phi(y)/\phi(h)\cdot h$, where ϕ is as in the proof of Theorem A1, we obtain, arguing as in the proof of the same theorem, that for any $1 > \epsilon > 0$, there is a $h_\epsilon \notin \operatorname{im} L$ such that $||\mathbb{I} - Q|| \le 1 + \epsilon$, hence

$$\operatorname{dist}\,(h_\epsilon/|h_\epsilon|_Y, \operatorname{im} L) \ge \frac{1}{1+\epsilon}\,.$$

The last inequality is known as a lemma of Riesz (see [7]). On the other hand, since $||Q|| \le ||\mathbb{I} - Q|| + 1$, Riesz's lemma implies Theorem A1.

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