Differential and Integral Equations Volume 10, Number 2, March 1997, pp. 273 – 287.

ASYMPTOTIC BEHAVIOUR FOR SYMMETRIC HYPERBOLIC DISSIPATIVE SYSTEMS*

BARBARA LAZZARI
Dipartimento di Matematica, Università degli Studi di Bologna
Piazza di Porta San Donato 5, 40127 Bologna, Italy

ELENA VUK
Dipartimento di Automazione per l'Elettronica, Università degli Studi di Brescia, Italy

(Submitted by: Luigi Salvadori)

Abstract. In this paper we study the existence, uniqueness and asymptotic behaviour of the solution for a class of first-order symmetric hyperbolic systems introduced by Friedrichs ([6]), in a bounded domain with an energy-absorbing boundary. It is carried out using the Fourier transform method. An energy decay rate result is obtained.

1. Introduction. This paper deals with the study of the following evolutive problem inside a bounded domain $\Omega \subset \mathbb{R}^3$:

$$\mathcal{E}\frac{\partial}{\partial t}\mathcal{U} = \frac{\partial}{\partial x_i}(A^i\mathcal{U}) + B\mathcal{U} + f; \qquad i = 1,2,3 \tag{1.1}$$

with $\mathcal{E}$ and A^i $m \times m$ symmetric matrices with real coefficients, $\mathcal{E}$ positive definite, satisfying the condition

$$\sup\Big\{ -\frac{A^i(x)\nu_i\xi\cdot\xi}{\mathcal{E}(x)\xi\cdot\xi} \quad : \quad x\in\Omega, \quad \xi\in\mathbb{R}^m, \quad \sum_{i=1}^{3}\nu_i = 1\Big\} = c < \infty. \tag{1.2}$$

Interesting examples of symmetric hyperbolic systems (1.1) satisfying the condition (1.2) are given by classical equations of mathematical physics (e.g. linear elasticity, electromagnetism, etc.). The quantities $e = \frac{1}{2}\mathcal{E}\mathcal{U}\cdot\mathcal{U}$ and $\Sigma^i = -\frac{1}{2}A^i\mathcal{U}\cdot\mathcal{U}$ can be interpreted respectively as "energy density" and as the components of Poynting's vector on the boundary $\partial\Omega$, describing the flow of power.

The particular feature of the system yields

$$\frac{d}{dt}\int_\Omega e(x,t)\,dx = \int_{\partial\Omega}\Sigma^i(x,t)n_i(x)\,ds + \int_\Omega f(x,t)\cdot\mathcal{U}(x,t)\,dx,$$

Received for publication November 1995.
*This work has been performed under the auspices of G.N.F.M.-C.N.R. and partially supported by Italian M.U.R.S.T. through the 40% Project.
AMS Subject Classifications: 35B40, 35Q20, 73H10, 78A25.

where $n = (n_1, n_2, n_3)$ is the outward normal to $\partial\Omega$.

The aim of this paper is to study the asymptotic behaviour of the solutions of the initial-boundary value problem related to the differential system (1.1) when Ω has a dissipative boundary, i.e., in regions with a boundary $\partial\Omega$ made up of partly a nontrapping surface and partly an energy-absorbing surface.

For domains subject to local dissipative boundary conditions, which do not permit the energy to increase inside Ω in absence of source terms, Wilcox ([18–20]) obtained an a priori "domain of dependence" inequality for solutions of (1.1), voicing the fact that energy propagates inside Ω with finite speed c defined by (1.2). This inequality yields existence, uniqueness and continuous dependence theorems for system (1.1) in bounded time intervals.

Classical conservative boundary conditions (for instance Dirichlet or Neumann conditions) fall within the class of local dissipative boundary conditions defined by Wilcox, thus the possible dissipative properties of $\partial\Omega$ are not necessary in order to obtain existence, uniqueness and continuous dependence in bounded time intervals. On the other hand, it is easy to see that the attractivity of the null solution fails if the model does not exhibit an effective dissipation property. The damping mechanism can be achieved so that a portion of the boundary $\partial\Omega$ is required to be an *energy-absorbing surface*.

We shall prove that inside domains with energy-absorbing boundary, the energy decay allows us to obtain information on the asymptotic behaviour of solutions which yield an asymptotic stability result and the exponential energy decay in absence of source terms.

The rest of this paper is organized as follows. Section 2 defines the problem and recalls the main result obtained by Wilcox ([20]) in domains subject to local dissipative boundary conditions. The study of the asymptotic behaviour of the solution in domains with boundary exhibiting an effective dissipation property is scrutinized in Section 3 where, by means of the Fourier time-transform method, information on solutions of the evolutive problem in the space-time domain $\Omega \times \mathbb{R}^+$ are derived through the study of the properties of time-harmonic solutions with a fixed frequency of the same problem. Results obtained in this section serve as a basis to obtain in Section 4 the exponential energy decay inside Ω, in absence of source terms, via the method of semigroups of contraction operators ([3]). Finally, in Section 5, results of Sections 3 and 4 are applied to two classical examples:

a) The evolution of the electromagnetic field inside a rigid, inhomogeneous, anisotropic linear dielectric with a boundary which is a good (not perfect) conductor obeying the linear impedance condition of Schelkunoff-Graffi ([7], [16])

$$E^t = \lambda H^t \times n \tag{1.3}$$

between the tangential components of electric and magnetic fields E and H, where λ is the impedance coefficient, and results dealing with existence and uniqueness for time harmonic solutions have been obtained by [5].

b) The evolution of a linear inhomogeneous, anisotropic elastic body with viscous boundary obeying the linear condition

$$Sn + \alpha v = 0 \tag{1.4}$$

between stress tensor S and velocity v, and results dealing with exponential decay rate of the energy have been obtained by [2] and [11], if the boundary is subject to certain additional geometric restrictions.

2. Statement and the main result. We consider the first-order linear symmetric hyperbolic system with variable coefficients

$$\begin{aligned} &\mathcal{E}(x)\frac{\partial}{\partial t}\mathcal{U}(x,t) = \frac{\partial}{\partial x_i}[A^i(x)\mathcal{U}(x,t)] + B(x)\mathcal{U}(x,t) + f(x,t), \\ &\mathcal{U}(x,0) = \mathcal{U}_0(x) \end{aligned} \tag{2.1}$$

in a bounded open connected set $\Omega \subset \mathbb{R}^3$ whose boundary $\partial\Omega$ is a smooth surface. The unknown function $\mathcal{U}$ is a vector of $\mathbb{R}^m$ as the source term f and $\mathcal{U}_0$, distribution of $\mathcal{U}$ at time $t = 0$, while $\mathcal{E}$, A^i and B are $m \times m$ matrices whose entries are smooth functions of x in Ω, with $\mathcal{E}$ symmetric and positive definite, A^i symmetric and $(B + B^T) = -\frac{\partial A^i}{\partial x_i}$.

In order to give a precise formulation of the problem in an appropriately generalized form we observe that the operator

$$A\mathcal{U} = \frac{\partial}{\partial x_i}(A^i\mathcal{U}) - \frac{1}{2}\frac{\partial A^i}{\partial x_i}\mathcal{U}$$

and its formal adjoint, $-A$, map $C_0^1(\Omega)$ into $L^2(\Omega)$ and give the following:

Definition 2.1. Let $\mathcal{U} \in L^2(\Omega)$; then $A\mathcal{U}$ is said to exist in $L^2(\Omega)$ if and only if a function $\mathcal{V} \in L^2(\Omega)$ exists such that:

$$\int_\Omega A(x)\phi(x)\cdot\mathcal{U}(x)\,dx + \int_\Omega \phi(x)\cdot\mathcal{V}(x)\,dx = 0 \qquad \forall\phi \in C_0^1(\Omega); \tag{2.2}$$

in this case we write $A\mathcal{U} = \mathcal{V}$.

The linear subspace of $L^2(\Omega)$ defined by

$$L^2(A,\Omega) = \{\mathcal{U} \in L^2(\Omega) : A\mathcal{U} \in L^2(\Omega)\},$$

is a Hilbert space with the following inner product:

$$(\mathcal{U},\mathcal{V})_A = \int_\Omega [A(x)\mathcal{U}(x)\cdot A(x)\mathcal{V}(x) + \mathcal{U}(x)\cdot\mathcal{V}(x)]\,dx.$$

Definition 2.2. A boundary condition Γ for A is a closed linear subspace of $L^2(A,\Omega)$ so that $C_0^1(\Omega) \subset \Gamma$. Γ is said to be a local boundary condition for A provided that $\phi\mathcal{U} \in \Gamma$ whenever $\phi \in C_0^1(\mathbb{R}^3)$ and $\mathcal{U} \in \Gamma$. Γ is said to be a *dissipative boundary condition* for A if

$$\int_\Omega A(x)\mathcal{U}(x)\cdot\mathcal{U}(x)\,dx \le 0, \qquad \forall\mathcal{U} \in \Gamma. \tag{2.3}$$

The space Γ is a separable Hilbert space relative to the inner product of $L^2(A,\Omega)$. If I denotes the temporal interval $(0,T)$, the space $L^2(I,\Gamma)\cap H^1(I,L^2(\Omega))$ defines a class of functions $\mathcal{U}$ for which $\mathcal{E}\frac{\partial}{\partial t}\mathcal{U}$, $\frac{\partial}{\partial x_i}(A^i\mathcal{U})$ and $B\mathcal{U}$ exist in $\Omega\times I$ and satisfy the boundary condition Γ. The domain of dependence inequality states (see [20])

Theorem 2.1. *Let Γ be a local dissipative boundary condition for A; then the speed c defined by* (1.2) *is finite and every $\mathcal{U} \in L^2(I,\Gamma)\cap H^1(I,L^2(\Omega))$ satisfies*

$$\begin{aligned}\int_{\Omega\cap\mathcal{B}(x_0,r)} \mathcal{E}(x)\mathcal{U}(x,T)\cdot\mathcal{U}(x,T)\,dx &\le \int_{\Omega\cap\mathcal{B}(x_0,r+cT)} \mathcal{E}(x)\mathcal{U}(x,0)\cdot\mathcal{U}(x,0)\,dx \\ &+ 2\int_0^T\int_{\Omega\cap\mathcal{B}(x_0,r+c(T-t))} f(x,t)\cdot\mathcal{U}(x,t)\,dx\,dt\end{aligned} \tag{2.4}$$

with $\mathcal{B}(x_0,r) = \{x\colon |x-x_0|\le r\}$ and $f = \mathcal{E}\frac{\partial}{\partial t}\mathcal{U} - \frac{\partial}{\partial x_i}(A^i\mathcal{U}) - B\mathcal{U}$.

Choosing r large enough (2.4) becomes

$$\begin{aligned}&\int_\Omega \mathcal{E}(x)\mathcal{U}(x,T)\cdot\mathcal{U}(x,T)\,dx \\ &\le \int_\Omega \mathcal{E}(x)\mathcal{U}(x,0)\cdot\mathcal{U}(x,0)\,dx + 2\int_0^T\int_\Omega f(x,t)\cdot\mathcal{U}(x,t)\,dx\,dt\end{aligned}$$

and by means of a Gronwall-type inequality ([1]) and of classical methods on the uniform convergence of Cauchy sequences it is easy to obtain the following energy inequality:

$$\operatorname*{ess\,sup}_{t\in[0,T)}\|\mathcal{U}(t)\|^2 \le k\big(\|\mathcal{U}(0)\|^2 + \int_0^T \|f(t)\|^2 dt\big) \tag{2.5}$$

where the constant k depends on $\mathcal{E}$ and T.

The energy inequality allows one to prove existence, uniqueness and continuous dependence theorems for initial-value problem (2.1) in cylindrical space time $\Omega\times(0,T)$, with $T < +\infty$, subject to a local boundary dissipative condition Γ. Moreover, when the constant k does not depend on T, inequality (2.5) assures that the solution of the problem (2.1) is bounded in $(0,\infty)$ whenever the source f belongs to $L^2(\mathbb{R}^+, L^2(\Omega))$.

Now we confine our attention to a class of symmetric hyperbolic systems which satisfy some additional conditions. Firstly, we specify the smoothness of the domain Ω and of the real-valued coefficients of (2.1):

$\boldsymbol{\mathcal{P}}_1$ - Ω satisfies the cone property ([17]) and the boundary $\partial\Omega$ of Ω is a piecewise C^1 surface, the entries of $\mathcal{E}$ and A^i belong to $C^1(\bar{\Omega})$, while the entries of B belong to $C(\bar{\Omega})$.

Secondly, we observe that any symmetric matrix A^i has an unique decomposition $A^i = \Delta^i + (\Delta^i)^T$, where Δ^i is the lower triangular matrix of components

$$\Delta^i_{\alpha\beta} = \begin{cases} A^i_{\alpha\beta} & \text{if } \alpha > \beta \\ \frac{1}{2} & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha < \beta \end{cases}$$

and we require:

$\boldsymbol{\mathcal{P}}_2$ - The operator A is a *natural operator* ([18]); i.e., there exists a partition $P = \{1, \ldots, p\}$, $Q = \{p+1, \ldots, m\}$ of the dependent variables so that $\Delta^i_{\alpha\beta} n_i = 0$ for $\alpha \in P$, $1 \leq \beta \leq m$ and for $\beta \in Q$, $1 \leq \alpha \leq m$, identically in n_i.

For sufficiently regular functions, namely for $\mathcal{U}$, $\mathcal{V} \in L^2(A, \Omega)$, the divergence theorem yields

$$\begin{aligned} \int_\Omega A(x)\mathcal{U}(x) \cdot \mathcal{V}(x)\, dx = \tfrac{1}{2} \int_{\partial\Omega} A^i(x) n_i(x) \mathcal{U}(x) \cdot \mathcal{V}(x) d\,\sigma \\ + \tfrac{1}{2} \int_\Omega [A^i(x) \frac{\partial}{\partial x_i} \mathcal{U}(x) \cdot \mathcal{V}(x) - A^i(x)\mathcal{U}(x). \frac{\partial}{\partial x_i} \mathcal{V}(x)]\, dx, \end{aligned}$$

so that the boundary condition (2.3) can be rewritten as

$$\int_{\partial\Omega} A^i(x) n_i(x) \mathcal{U}(x) \cdot \mathcal{U}(x)\, d\sigma \leq 0. \tag{2.6}$$

It is easy to prove that the *natural boundary conditions*

$$\Delta^i \mathcal{U} n_i = 0 \quad \text{or} \quad \mathcal{U}(\Delta^i)^T n_i = 0 \quad \text{on } \partial\Omega \tag{2.7}$$

satisfy inequality (2.6). They represent the classical Dirichlet or Neumann boundary conditions which are conservative. Because in this paper we consider strongly dissipative boundaries, we must give a definition which leaves out conditions of this kind. Since A satisfies $\boldsymbol{\mathcal{P}}_2$, we write $\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2$, where $\mathcal{U}_1 = \{u_1, \ldots, u_p, 0, \ldots, 0\}$, $\mathcal{U}_2 = \{0, \ldots, 0, u_{p+1}, \ldots, u_m\}$ and we consider the following family of boundary conditions:

$$(A^i n_i)\mathcal{U}_1 = (\Delta^i n_i)^T \Lambda (\Delta^i n_i) \mathcal{U}_2 \quad \text{on } \partial\Omega \tag{2.8}$$

where Λ is a symmetric matrix. For conditions of type (2.8) we have

$$\int_\Omega A(x)\mathcal{U}(x)\cdot\mathcal{U}(x)\,dx = \int_{\partial\Omega} A^i(x)n_i(x)\mathcal{U}_1(x)\cdot\mathcal{U}_2(x)\,d\sigma \tag{2.9}$$
$$= \int_{\partial\Omega} \Lambda(x)[\Delta^i(x)n_i(x)]\mathcal{U}_2(x)\cdot[\Delta^i(x)n_i(x)]\mathcal{U}_2(x)\,d\sigma$$

and we require:

$\boldsymbol{\mathcal{P}}_3$ - the symmetric matrix Λ has entries which belong to $C^1(\partial\bar{\Omega})$; moreover, the boundary condition (2.8) defined an energy-absorbing boundary, i.e., a positive constant γ exists such that

$$-\int_{\partial\Omega} \Lambda(x)[\Delta^i(x)n_i(x)]\mathcal{U}_2(x)\cdot[\Delta^i(x)n_i(x)]\mathcal{U}_2(x)\,d\sigma \geq \gamma\|A^i n_i\mathcal{U}_2\|^2_{L^2(\partial\Omega)}. \tag{2.10}$$

The formulation of the initial-boundary value problem (2.1)–(2.8) in the space-time domain $\Omega\times\mathbb{R}^+$ makes use of several functional spaces ([10]) which are defined below:

$$\mathcal{L}(\Omega) = \text{closure in } L^2 \text{of the set } \mathcal{P} = \{\mathcal{U}\colon \mathcal{U} = \frac{\partial}{\partial x_i}(A^i\phi) + B\phi, \quad \phi\in C^\infty(\bar{\Omega})\};$$
$$\mathcal{L}_{\mathcal{E}}(\Omega) = \{\mathcal{U}\in L^2(\Omega)\colon \mathcal{E}\mathcal{U}\in\mathcal{L}(\Omega)\};$$
$$\mathcal{R}(\Omega) = \{\mathcal{U}\in\mathcal{L}_{\mathcal{E}}(\Omega)\cap L^2(A,\Omega), (A^i n_i)\mathcal{U}_1 = (\Delta^i n_i)^T\Lambda(\Delta^i n_i)\mathcal{U}_2 \quad \text{on } \partial\Omega\};$$
$$\mathcal{H}(\mathbb{R}^+,\Omega) = H^1(\mathbb{R}^+,\mathcal{L}_{\mathcal{E}}(\Omega))\cap L^2(\mathbb{R}^+,\mathcal{R}(\Omega)).$$

The space $\mathcal{H}(\mathbb{R}^+,\Omega)$ is a Hilbert space with the following inner product:

$$[\mathcal{U},\mathcal{V}]_A = \int_{\mathbb{R}^+}\int_\Omega \Big[\mathcal{E}(x)\frac{\partial}{\partial t}\mathcal{U}(x,t)\cdot\frac{\partial}{\partial t}\mathcal{V}(x,t) \tag{2.11}$$
$$+ A(x)\mathcal{U}(x,t)\cdot\mathcal{V}(x,t) + \mathcal{U}(x,t)\cdot\mathcal{V}(x,t)\Big]\,dx\,dt.$$

Now it is possible to give the definition of the solution.

Definition 2.3. The function $\mathcal{U}(x,t)\in L^2(\mathbb{R}^+,\mathcal{L}_{\mathcal{E}}(\Omega))$ defines a *weak solution* of the problem (2.1)–(2.8), with supply $f(x,t)\in L^2(\mathbb{R}^+,\mathcal{L}(\Omega))$ and initial data $u_0(x)\in\mathcal{L}_{\mathcal{E}}(\Omega)$ if

$$\int_{\mathbb{R}^+}\int_\Omega \mathcal{U}(x,t)\cdot\Big\{\mathcal{E}(x)\frac{\partial}{\partial t}\phi(x,t) - \frac{\partial}{\partial x_i}[A^i(x)\phi(x,t)] - B(x)\phi(x,t)\Big\}\,dx\,dt$$
$$+\int_\Omega \mathcal{E}(x)\mathcal{U}_0(x)\cdot\phi_0(x)\,dx + \int_{\mathbb{R}^+}\int_\Omega f(x,t)\cdot\phi(x,t)\,dx\,dt = 0 \tag{2.12}$$

for every $\phi\in\mathcal{H}(\mathbb{R}^+,\Omega)$.

The first result of this paper which will be proven in the next section is Theorem 2.2. It allows us to extend the results of Wilcox to the temporal interval $(0,\infty)$ and to prove the *attractivity* of the null solution.

Theorem 2.2. *Under the constitutive hypotheses* $\boldsymbol{\mathcal{P}}_1$–$\boldsymbol{\mathcal{P}}_3$ *the problem* (2.1)–(2.8) *with supply* $f \in L^2(\mathbb{R}^+, \mathcal{L}(\Omega))$ *and initial data* $\mathcal{U}_0 \in \mathcal{L}_{\mathcal{E}}(\Omega)$ *admits one and only one weak solution* (*with finite energy*) $\mathcal{U} \in L^2(\mathbb{R}^+, \mathcal{L}_{\mathcal{E}}(\Omega))$ *according to the Definition* 2.3.

Theorem 2.2 along with the inequality (2.5) allows us to improve an asymptotic stability property for this class of linear hyperbolic dissipative systems in the sense that the $\mathcal{L}_{\mathcal{E}}(\Omega)$ norm of the solution of the problem (2.1)–(2.8) tends in mean to zero for $t \to +\infty$.

3. Transform problem. In order to prove Theorem 2.2 we apply the Fourier time-transform to the problem (2.1) and we study the elliptic system obtained. Let $g : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function; then

$$\hat{g}(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}^+} e^{-i\omega t} g(t)\, dt \quad \omega \in \mathbb{R}$$

denotes the Fourier time-transform, in the distributional sense of the causal function[1] g, and if $g \in L^2(\mathbb{R}^+)$, then $\hat{g} \in L^2(\mathbb{R})$.

Let $L^2_{\mathcal{F}}(\mathbb{R})$ be the subset of $L^2(\mathbb{R})$ of the functions which are Fourier transforms of causal functions. $L^2_{\mathcal{F}}(\mathbb{R})$ is a closed subspace of $L^2(\mathbb{R})$ with the usual norm, and the properties of Fourier transform for causal functions yield ([13])

Remark 3.1. If $\phi \in \mathcal{H}(\mathbb{R}^+, \Omega)$ then

$$\text{i)} \quad \hat{\phi} \in L^2_{\mathcal{F}}(\mathbb{R}, \mathcal{R}(\Omega));$$

$$\text{ii)} \quad \phi_0(x) = \phi(x, 0) = \frac{1}{\pi} \int_{\mathbb{R}} \hat{\phi}(x, \omega)\, d\omega;$$

$$\text{iii)} \quad (i\omega\hat{\phi}(x, \omega) - \phi_0(x)) \in L^2_{\mathcal{F}}(\mathbb{R}, \mathcal{L}_{\mathcal{E}}(\Omega)).$$

Let $\mathcal{H}_{\mathcal{F}}(\mathbb{R}, \Omega)$ be the set of functions $\hat{\phi}$, Fourier transforms of functions ϕ belong respectively to $\mathcal{H}(\mathbb{R}^+, \Omega)$. The Plancherel theorem for Fourier transforms applied to (2.11) defines in a natural manner an inner product on $\mathcal{H}_{\mathcal{F}}(\mathbb{R}, \Omega)$:

$$[\hat{\mathcal{U}}, \hat{\mathcal{V}}]_{A_{\mathcal{F}}} = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\Omega} \{\mathcal{E}(x)[i\omega\hat{\mathcal{U}}(x, w) - \mathcal{U}_0(x)] \cdot \overline{[i\omega\hat{\mathcal{V}}(x, w) - \mathcal{V}_0(x)]}$$
$$+ A(x)\hat{\mathcal{U}}(x, w) \cdot \overline{\hat{\mathcal{V}}}(x, w) + \hat{\mathcal{U}}(x, w) \cdot \overline{\hat{\mathcal{V}}}(x, w)\}\, dx\, d\omega,$$

where the superpose bar denotes the function complex conjugate and so a natural isomorphism is defined between $\mathcal{H}_{\mathcal{F}}(\mathbb{R}, \Omega) \to \mathcal{H}(\mathbb{R}^+, \Omega)$ ([17]).

[1]A function will be called causal if it equals 0 for negative t. A function defined on $\mathbb{R}^+$ can be identified with a function on $\mathbb{R}$, which vanishes identically on $(-\infty, 0)$.

The Plancherel theorem makes it possible to rewrite (2.12) in terms of the Fourier transform:

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\Omega}\hat{\mathcal{U}}(x,w)\cdot\{\mathcal{E}(x)\overline{[i\omega\hat{\phi}(x,\omega)-\phi_0(x)]} \\ -\frac{\partial}{\partial x_i}[A^i(x)\overline{\hat{\phi}}(x,\omega)]-B(x)\overline{\hat{\phi}}(x,\omega)\}\,dx\,d\omega \\ +\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\Omega}\hat{f}(x,\omega)\cdot\overline{\hat{\phi}}(x,\omega)\,dx\,d\omega+\int_{\Omega}\mathcal{E}(x)\mathcal{U}_0(x)\cdot\phi_0(x)\,dx=0, \end{aligned} \tag{3.1}$$

and the isomorphism between $\mathcal{H}_{\mathcal{F}}(\mathbb{R},\Omega)$ and $\mathcal{H}(\mathbb{R}^+,\Omega)$ gives

Lemma 3.1. *A function $\hat{\mathcal{U}}\in L^2_{\mathcal{F}}(\mathbb{R},\mathcal{L}_{\mathcal{E}}(\Omega))$ is the Fourier transform of a weak solution of the initial boundary value problem* (2.1)–(2.8) *in the sense of Definition* 2.3, *if and only if the equality* (3.1) *holds $\forall\hat{\phi}\in\mathcal{H}_{\mathcal{F}}(\mathbb{R},\Omega)$.*

Let $a\colon L^2_{\mathcal{F}}(\mathbb{R},\mathcal{L}_{\mathcal{E}}(\Omega))\times\mathcal{H}_{\mathcal{F}}(\mathbb{R},\Omega)\to\mathbb{C}$ be the integral form

$$\begin{aligned} a(\hat{\mathcal{U}},\hat{\phi})=-\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\Omega}\hat{\mathcal{U}}(x,\omega)\cdot\{\mathcal{E}(x)\overline{[i\omega\hat{\phi}(x,\omega)-\phi_0(x)]} \\ -\frac{\partial}{\partial x_i}[A^i(x)\overline{\hat{\phi}}(x,\omega)]-B(x)\overline{\hat{\phi}}(x,\omega)\}\,dx\,d\omega. \end{aligned} \tag{3.2}$$

In terms of a the existence and uniqueness theorem becomes

Theorem 3.1. *For any $\hat{f}\in L^2_{\mathcal{F}}(\mathbb{R},\mathcal{L}(\Omega))$, $\mathcal{U}_0\in\mathcal{L}_{\mathcal{E}}(\Omega)$ there exists one and only one function $\hat{\mathcal{U}}\in L^2_{\mathcal{F}}(\mathbb{R},\mathcal{L}_{\mathcal{E}}(\Omega))$ satisfying*

$$a(\hat{\mathcal{U}},\hat{\phi})=-\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\Omega}\hat{f}(x,\omega)\cdot\overline{\hat{\phi}}(x,\omega)\,dx\,d\omega+\int_{\Omega}\mathcal{E}(x)\mathcal{U}_0(x)\cdot\phi_0(x)dx \tag{3.3}$$

for all $\hat{\phi}\in\mathcal{H}_{\mathcal{F}}(\mathbb{R},\Omega)$.

Proof. (Uniqueness) Due to the linearity of the problem, our attention can be restricted to the uniqueness of the solution for the homogeneous problem, namely, to prove that if $\hat{\mathcal{U}}\in L^2_{\mathcal{F}}(\mathbb{R},\mathcal{L}_{\mathcal{E}}(\Omega))$ and

$$a(\hat{\mathcal{U}},\hat{\phi})=0\quad\forall\hat{\phi}\in\mathcal{H}_{\mathcal{F}}(\mathbb{R},\Omega), \tag{3.4}$$

necessarily $\hat{\mathcal{U}}=0$.

If the text function $\hat{\phi}$ is specialised to have the form $\hat{\phi}(x,\omega)=\varphi(x)\hat{\psi}(\omega)$ where $\hat{\psi}$ is a scalar function and $\hat{\psi}$, $i\omega\hat{\psi}\in L^2_{\mathcal{F}}(\mathbb{R})$, then (3.4) becomes

$$\int_{\mathbb{R}}\overline{i\omega\hat{\psi}(\omega)}\int_{\Omega}\hat{\mathcal{U}}(x,\omega)\cdot\big\{\mathcal{E}(x)\varphi(x)+\frac{1}{i\omega}\big[\frac{\partial}{\partial x_i}[A^i(x)\varphi(x)]+B(x)\varphi(x)\big]\big\}\,dx\,d\omega=0, \tag{3.5}$$

and because equality (3.5) holds for every $i\omega\hat{\psi}(\omega) \in L^2_{\mathcal{F}}(\mathbb{R})$, then, almost everywhere in $\mathbb{R}$,

$$\int_\Omega \hat{\mathcal{U}}(x,\omega)\cdot\big\{\mathcal{E}(x)\varphi(x) + \frac{1}{i\omega}[\frac{\partial}{\partial x_i}[A^i(x)\varphi(x)] + B(x)\varphi(x)]\big\}\,dx = 0, \quad \forall\varphi \in \mathcal{R}(\Omega). \tag{3.6}$$

It is easy to see that relation (3.6) is satisfied if, for any fixed real ω, the problem[2]

$$\begin{aligned} i\omega\mathcal{E}(x)\hat{\mathcal{U}}(x,\omega) &= \frac{\partial}{\partial x_i}[A^i(x)\hat{\mathcal{U}}(x,\omega)] + B(x)\hat{\mathcal{U}}(x,\omega) + \hat{f}(x,\omega) \\ (A^i n_i)\hat{\mathcal{U}}_1(x,\omega) &= (\Delta^i n_i)^T\Lambda(\Delta^i n_i)\hat{\mathcal{U}}_2(x,\omega) \qquad \text{on} \quad \partial\Omega \end{aligned} \tag{3.7}$$

with source $\hat{f}(\cdot,\omega) \in \mathcal{L}(\Omega)$, has an unique *weak solution*, i.e., *a function* $\hat{\mathcal{U}}(\cdot,\omega) \in \mathcal{L}_{\mathcal{E}}(\Omega))$ *such that*

$$\int_\Omega \hat{\mathcal{U}}(x,\omega)\cdot\big\{i\omega\mathcal{E}(x)\varphi(x) + \frac{\partial}{\partial x_i}[A^i(x)\varphi(x)] + B(x)\varphi(x)\big\}\,dx = \int_\Omega \hat{f}(x,\omega)\cdot\varphi(x)\,dx \tag{3.8}$$

for any $\varphi \in \mathcal{R}(\Omega)$.

The first step in obtaining the uniqueness theorem for weak solutions of (3.7) is to prove that if $\hat{\mathcal{U}}(\cdot,\omega) \in \mathcal{L}_{\mathcal{E}}(\Omega)$ is a weak solution of (3.7), then $\hat{\mathcal{U}}(\cdot,\omega) \in \mathcal{R}(\Omega)$.[3] Let $\hat{\mathcal{U}}(\cdot,\omega)$ be a weak solution of (3.7); then (3.8) gives

$$\begin{aligned} &\int_\Omega \frac{\partial}{\partial x_i}[A^i(x)\varphi(x)]\cdot\hat{\mathcal{U}}(x,\omega)\,dx \\ &= \int_\Omega \big\{\hat{f}(x,\omega) - [i\omega\mathcal{E}(x) - B(x)]\hat{\mathcal{U}}(x,\omega)\big\}\cdot\varphi(x)\,dx, \quad \forall\varphi \in C^1_0(\Omega). \end{aligned} \tag{3.9}$$

Since $\big\{\hat{f}(\cdot,\omega) - [i\omega\mathcal{E}(\cdot) - B(\cdot)]\hat{\mathcal{U}}(\cdot,\omega)\big\} \in \mathcal{L}(\Omega)$, then $\hat{\mathcal{U}}(\cdot,\omega) \in \mathcal{L}_{\mathcal{E}}(\Omega) \cap L^2(A,\Omega)$ and equation (3.7_1) holds almost everywhere in Ω. Moreover, because $\hat{\mathcal{U}}$ satisfies (3.8), we have

$$\int_{\partial\Omega} A^i(x)n_i(x)\hat{\mathcal{U}}(x,\omega)\cdot\varphi(x)\,d\sigma = 0 \qquad \forall\varphi \in \mathcal{R}(\Omega),$$

or equivalently

$$\int_{\partial\Omega}[(A^i n_i)\hat{\mathcal{U}}_1(x,\omega) - (\Delta^i n_i)^T\Lambda(\Delta^i n_i)\hat{\mathcal{U}}_2(x,\omega)]\cdot\varphi_1(x)\,d\sigma = 0 \qquad \forall\varphi_1 \in L^2(\partial\Omega) \tag{3.10}$$

and equation (3.7_2) holds almost everywhere on $\partial\Omega$.

[2]For $\omega \neq 0$, system (3.7) is the problem which arises when we are interested in finding time-harmonic solutions on whole real time axis of (2.1_1)–(2.8) with time-harmonic sources, while for $\omega = 0$, when we are interested in finding static solutions on whole real-time axis.

[3]This result was found by ([19], Lemma 3) for time-harmonic solutions of Maxwell equations.

Let $\hat{\mathcal{U}}(\cdot,\omega)$ be a solution of (3.7); if we multiply equation (3.7$_1$) by $\overline{\hat{\mathcal{U}}}(\cdot,\omega)$ and integrate on Ω, we obtain

$$-2\int_{\partial\Omega}\Lambda(x)[(\Delta^i n_i(x))]\hat{\mathcal{U}}_2(x,\omega)\cdot[(\Delta^i n_i(x))]\overline{\hat{\mathcal{U}}}_2(x,\omega)\,d\sigma$$
$$=\mathcal{R}e\int_{\Omega}\hat{f}(x,\omega)\cdot\overline{\hat{\mathcal{U}}}(x,\omega)\,dx. \tag{3.11}$$

For $\hat{f}(\cdot,\omega)=0$, because of (2.8) and (2.10), equality (3.11) yields

$$A^i(x)n_i(x)\hat{\mathcal{U}}_1(x,\omega)=0,\quad A^i(x)n_i(x)\hat{\mathcal{U}}_2(x,\omega)=0\quad\text{on}\quad\partial\Omega. \tag{3.12}$$

If $\omega\neq 0$, under the constitutive assumption $\boldsymbol{\mathcal{P}}_1$, which assures the smoothness of the entries of $\mathcal{E}$, A^i and B, the problem

$$\begin{aligned}
&i\omega\mathcal{E}(x)\hat{\mathcal{U}}(x,\omega)=\frac{\partial}{\partial x_i}[A^i(x)\hat{\mathcal{U}}(x,\omega)]+B(x)\hat{\mathcal{U}}(x,\omega)\\
&A^i(x)n_i(x)\hat{\mathcal{U}}_1(x,\omega)=0\\
&A^i(x)n_i(x)\hat{\mathcal{U}}_2(x,\omega)=0
\end{aligned}\tag{3.13}$$

admits only the solution $\hat{\mathcal{U}}(\cdot,\omega)=0$ (see for example [8], [12]).

If $\omega=0$, because $\hat{\mathcal{U}}(\cdot,0)\in\mathcal{L}(\Omega)$ we can pose $\hat{\mathcal{U}}(\cdot,0)=\frac{\partial}{\partial x_i}(A^i\mathcal{V})+B\mathcal{V}$, and substituting in (3.13) obtain

$$\begin{aligned}
&\frac{\partial}{\partial x_j}A^j(x)[\frac{\partial}{\partial x_i}(A^i(x)\mathcal{V}(x))+B(x)\mathcal{V}(x)]\\
&\qquad+B(x)[\frac{\partial}{\partial x_i}(A^i(x)\mathcal{V}(x))+B(x)\mathcal{V}(x)]=0\\
&A^j(x)n_j(x)[\frac{\partial}{\partial x_i}(A^i(x)\mathcal{V}(x))+B(x)\mathcal{V}(x)]_1=0\\
&A^j(x)n_j(x)[\frac{\partial}{\partial x_i}(A^i(x)\mathcal{V}(x))+B(x)\mathcal{V}(x)]_2=0
\end{aligned}\tag{3.14}$$

multiplication of equation (3.14$_1$) by $\overline{\mathcal{V}}(\cdot,\omega)$ and integration on Ω yield

$$\int_{\Omega}\big|\frac{\partial}{\partial x_i}[A^i(x)\mathcal{V}(x)]+B(x)\mathcal{V}(x)\big|^2\,dx=\int_{\Omega}\big|\hat{\mathcal{U}}(x,0)\big|^2\,dx=0;$$

(Existence) Recalling the definition of a, if

$$\hat{\mathcal{U}}\in\mathcal{H}_{\mathcal{F}}(\mathbb{R},\Omega),\qquad\text{and}\qquad\mathcal{U}_0(x)=\frac{1}{\pi}\int_{\mathbb{R}}\hat{\mathcal{U}}(x,\omega)\,d\omega,$$

a straightfoward calculation yields

$$a(\hat{\mathcal{U}},\hat{\phi})-\int_\Omega \mathcal{E}(x)\mathcal{U}_0(x)\cdot\phi_0(x)\,dx=\frac{1}{2\pi}\int_{\mathbb{R}}\int_\Omega\{\mathcal{E}(x)[i\omega\hat{\mathcal{U}}(x,\omega)-\mathcal{U}_0(x)] \\ -\frac{\partial}{\partial x_i}[A^i(x)\hat{\mathcal{U}}(x,\omega)]-B(x)\hat{\mathcal{U}}(x,\omega)\}\cdot\overline{\hat{\phi}}(x,\omega)\,dx\,d\omega, \tag{3.15}$$

so (3.3) becomes

$$\int_{\mathbb{R}}\int_\Omega \hat{f}(x,\omega)\cdot\overline{\hat{\phi}}(x,\omega)\,dx\,d\omega=\int_{\mathbb{R}}\int_\Omega\{\mathcal{E}(x)[i\omega\hat{\mathcal{U}}(x,\omega)-\mathcal{U}_0(x)] \\ -\frac{\partial}{\partial x_i}[A^i(x)\hat{\mathcal{U}}(x,\omega)]-B(x)\hat{\mathcal{U}}(x,\omega)\}\cdot\overline{\hat{\phi}}(x,\omega)\,dx\,d\omega. \tag{3.16}$$

Now, to prove the existence is the same as asserting that the set

$$\mathcal{N}=\{(\hat{f},\mathcal{U}_0)\in L^2_{\mathcal{F}}(\mathbb{R},\mathcal{L}(\Omega))\times\mathcal{L}_{\mathcal{E}}(\Omega):\quad \hat{\mathcal{U}}\in\mathcal{H}_{\mathcal{F}}(\mathbb{R},\Omega), \\ \mathcal{U}_0(x)=\frac{1}{\pi}\int_{\mathbb{R}}\hat{\mathcal{U}}(x,\omega)\,d\omega,\quad \text{and (3.16) holds}\quad \forall\hat{\phi}\in L^2_{\mathcal{F}}(\mathbb{R},\mathcal{L}_{\mathcal{E}}(\Omega))\}$$

is dense in $L^2_{\mathcal{F}}(\mathbb{R},\mathcal{L}(\Omega))\times\mathcal{L}_{\mathcal{E}}(\Omega)$.

Let $(\hat{g},\mathcal{V})\in L^2_{\mathcal{F}}(\mathbb{R},\mathcal{L}(\Omega))\times\mathcal{L}_{\mathcal{E}}(\Omega)\setminus \mathrm{Cl}\{\mathcal{N}\}$. If $(\hat{g},\mathcal{V})\neq 0$, then the Hann-Banach theorem ensures the existence of $(\hat{\phi}^*,\mathcal{W})\in L^2_{\mathcal{F}}(\mathbb{R},\mathcal{L}(\Omega))\times\mathcal{L}_{\mathcal{E}}(\Omega)$:

$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_\Omega \hat{g}(x,\omega)\cdot\overline{\hat{\phi}^*}(x,\omega)\,dx\,d\omega+\int_\Omega\mathcal{E}(x)\mathcal{V}(x)\cdot\mathcal{W}(x)\,dx\neq 0 \tag{3.17}$$

while

$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_\Omega \hat{f}(x,\omega)\cdot\overline{\hat{\phi}^*}(x,\omega)\,dx\,d\omega+\int_\Omega\mathcal{E}(x)\mathcal{U}_0(x)\cdot\mathcal{W}(x)\,dx=0\quad\forall(\hat{f},\mathcal{U}_0)\in\mathcal{N}. \tag{3.18}$$

Because $(\hat{f},\mathcal{U}_0)\in\mathcal{N}$, then (3.18) asserts that

$$\int_{\mathbb{R}}\int_\Omega\{\mathcal{E}(x)[i\omega\hat{\mathcal{U}}(x,\omega)-\mathcal{U}_0(x)]+\frac{\partial}{\partial x_i}[A^i(x)\hat{\mathcal{U}}(x,\omega)]+B(x)\hat{\mathcal{U}}(x,\omega)\} \\ \times\overline{\hat{\phi}^*}(x,\omega)\,dx\,d\omega=0\qquad\forall\hat{\mathcal{U}}\in\mathcal{H}_{\mathcal{F}}(\mathbb{R},\Omega), \tag{3.19}$$

but (3.19) represents the weak formulation for the formal adjoint of (3.3). Repeating the same technique used in the first part of this theorem we obtain $\hat{\phi}^*=0$, and this contradicts (3.17).

4. Exponential decay. In the previous section we have shown that the dissipation property of the boundary allows us to obtain an existence and uniqueness theorem of the solution for the evolutive problem (2.1)–(2.8) in the space-time domain $\Omega \times \mathbb{R}^+$. As a consequence of this theorem we have information about the asymptotic behaviour of solutions. In this section we study the asymptotic behaviour of the energy and prove its exponential decay inside Ω, in absence of source terms.

Let $\mathcal{U}$ be a solution (2.1)–(2.8); then the function

$$\eta(t) = \frac{1}{2}\int_\Omega \mathcal{E}(x)\mathcal{U}(x,t)\cdot\mathcal{U}(x,t)\,dx \tag{4.1}$$

is the energy of the solution $\mathcal{U}$ at time t. If $\mathcal{U}$ is a weak solution in the sense of Definition 2.3, then $\mathcal{U} \in L^2(\mathbb{R}^+, \mathcal{L}_\mathcal{E}(\Omega)$, and η results an integrable function on $\mathbb{R}^+$. This property makes it possible to obtain the exponential decay rate of the energy by means of the application of the theory of semigroups of contraction operators. For this purpose the mixed initial boundary value problem (2.1)–(2.8) must be reformulated as an abstract Cauchy problem:

$$\frac{d}{dt}\mathcal{U}(t) = L\mathcal{U}(t) + \mathcal{E}^{-1}f(t), \quad \mathcal{U}(0) = \mathcal{U}_0, \tag{4.2}$$

where $L = \mathcal{E}^{-1}\big(A^i\frac{\partial}{\partial x_i} - B^T\big)$ is an operator with domain $\mathcal{R}(\Omega) \subset \mathcal{L}_\mathcal{E}(\Omega)$.

The inner product

$$(\mathcal{U},\mathcal{V})_\eta = \int_\Omega \mathcal{E}(x)\mathcal{U}(x)\cdot\mathcal{V}(x)\,dx, \quad \mathcal{U},\mathcal{V} \in \mathcal{L}_\mathcal{E}(\Omega) \tag{4.3}$$

is equivalent to the usual inner product of L^2, but the associated norm is the energy norm. The new formulation of the problem allows us to prove the following result ([14]):

Lemma 4.1. *Under the hypotheses $\boldsymbol{\mathcal{P}}_1$–$\boldsymbol{\mathcal{P}}_3$ the operator $L : \mathcal{R}(\Omega) \to \mathcal{L}_\mathcal{E}(\Omega)$ is a maximal dissipative operator on $\mathcal{L}_\mathcal{E}(\Omega)$; i.e.,*

a) $(L\mathcal{U},\mathcal{U})_\eta \le 0 \quad \forall \mathcal{U} \in \mathcal{R}(\Omega)$,

b) *the range of $L - \mathbb{I}$ is $\mathcal{L}_\mathcal{E}(\Omega)$, where $\mathbb{I}$ is the identity operator.*

Proof. a) If $\mathcal{U} \in \mathcal{R}(\Omega)$, then $L\mathcal{U} \in \mathcal{L}_\mathcal{E}(\Omega)$; moreover, (4.3), (2.9) and (2.10) yield

$$(L\mathcal{U},\mathcal{U})_\eta = \int_\Omega A(x)\mathcal{U}(x)\cdot\mathcal{U}(x)\,dx = \frac{1}{2}\int_{\partial\Omega} A^i(x)n_i(x)\mathcal{U}(x)\cdot\mathcal{U}(x)\,ds \le 0. \tag{4.4}$$

Property *b)* is equivalent to the existence theorem for the problem

$$L\mathcal{U} - \mathcal{U} = g, \quad g \in \mathcal{L}_\mathcal{E}(\Omega),$$

in the space $\mathcal{R}(\Omega)$. The positive definiteness of $\mathcal{E}$ allows us to reformulate the existence theorem for the equivalent problem

$$(A^i\frac{\partial}{\partial x_i} - B^T)\mathcal{U} - \mathcal{E}\mathcal{U} = \mathcal{E}g$$

and the result is obtained with techniques similar to Theorem 3.1.

Theorem 4.1. *Under the constitutive hypotheses* $\boldsymbol{\mathcal{P}}_1$–$\boldsymbol{\mathcal{P}}_3$ *the energy inside* Ω, *related to the solution of the problem* (2.1)–(2.8) *with supply* $f = 0$ *and initial data* $\mathcal{U}_0 \in \mathcal{L}_{\mathcal{E}}(\Omega)$, *satisfies*

$$\begin{aligned}2\eta(t) &= \int_\Omega \mathcal{E}(x)\mathcal{U}(x,t)\cdot\mathcal{U}(x,t)\,dx \\ &\le Ne^{-kt}\int_\Omega \mathcal{E}(x)\mathcal{U}_0(x)\cdot\mathcal{U}_0(x)\,dx = 2Ne^{-kt}\eta(0)\end{aligned} \tag{4.5}$$

with N *and* k *positive constants.*

Proof. Properties a) and b) of Lemma 4.1 imply, by virtue of the Lumer-Phillips Theorem (see [15]), that L generates a semigroup of linear contractions $\mathcal{S}(t)$ on $\mathcal{L}_{\mathcal{E}}(\Omega)$ which is strongly continuous with respect to the energy norm.

In the absence of source terms, the solution of (4.2) can be written as $\mathcal{U}(t) = \mathcal{S}(t)\mathcal{U}_0$, the related energy $\eta(t) = (\mathcal{S}(t)\mathcal{U}_0, \mathcal{S}(t)\mathcal{U}_0)_\eta$ and, in this formalism, Theorem 2.2 affirms that,

$$\int_{\mathbb{R}^+} (\mathcal{S}(t)\mathcal{U}_0, \mathcal{S}(t)\mathcal{U}_0)_\eta dt < \infty. \tag{4.6}$$

On the other hand, for a classical result on semigroup theory in Hilbert spaces ([4]), (4.6) assures that there exist N and k positive constants, so that

$$(\mathcal{S}(t)\mathcal{U}_0, \mathcal{S}(t)\mathcal{U}_0)_\eta \le Ne^{-kt}(\mathcal{U}_0,\mathcal{U}_0)_\eta \quad \forall \mathcal{U}_0 \in \mathcal{L}_{\mathcal{E}}. \tag{4.7}$$

5. Applications. In this section our approach is used to discuss the asymptotic behaviour of solutions of two classical physical models.

Electromagnetism. As a first application consider the Maxwell equations for a linear dielectric material inside a bounded domain Ω in absence of charges:

$$\begin{aligned}&\varepsilon(x)\frac{\partial}{\partial t}E(x,t) = \nabla\times H(x,t) + J(x,t) &(5.1)\\ &\mu(x)\frac{\partial}{\partial t}H(x,t) = -\nabla\times E(x,t)\\ &\nabla\cdot(\varepsilon(x)E(x,t)) = 0\\ &\nabla\cdot(\mu(x)H(x,t)) = 0\end{aligned}$$

where the dielectric tensor ε and the magnetic permeability tensor μ belong to $C^1(\bar{\Omega})$ and are real symmetric and positive definite, while J represents the electric impressed current. The linear impedance relation

$$E^t(x,t) = \lambda(x)H^t(x,t)\times n(x) \quad \text{on} \quad \partial\Omega \tag{1.3}$$

can be used to characterize a boundary $\partial\Omega$ which is a good (nonperfect) conductor and the damping mechanism is induced by the fact that the electromagnetic field is not entirely reflected, but is partially refracted by the surface $\partial\Omega$ ([9]). The impedance coefficient $\lambda \in C^1(\partial\overline{\Omega})$, with $\lambda(x) \geq \lambda_0 > 0$, depends on the electrical conductivity of the boundary and tends to zero when this conductivity tends to infinity; i.e., when $\partial\Omega$ is a perfect conductor. With the notation

$$\mathcal{U} = \begin{bmatrix} E \\ H \end{bmatrix}; \quad \mathcal{E} = \begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix}; \quad A^i \frac{\partial}{\partial x_i} = \begin{bmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{bmatrix}; \quad f = \begin{bmatrix} J \\ 0 \end{bmatrix},$$

(5.1) assumes the form (2.1) with $B = 0$, when $\mathcal{U} \in L^2(\mathbb{R}^+, \mathcal{J}_\varepsilon(\Omega) \times \mathcal{J}_\mu(\Omega))$, where

$$\mathcal{J}_\varepsilon(\Omega) = \{E \in L^2(\Omega); \int_\Omega \varepsilon(x)E(x)\cdot\nabla\phi(x)\,dx = 0, \quad \forall\phi \in C^\infty(\overline{\Omega})\}$$

$$\mathcal{J}_\mu(\Omega) = \{H \in L^2(\Omega), \int_\Omega \mu(x)H(x)\cdot\nabla\phi(x)\,dx = 0, \quad \forall\phi \in C^\infty(\overline{\Omega})\}.$$

Moreover, for sufficiently regular electromagnetic fields the boundary condition (2.9) becomes

$$\int_\Omega A(x)\mathcal{U}(x)\cdot\mathcal{U}(x)\,dx = -\int_{\partial\Omega} \lambda(x)\left|H^t(x)\right|^2 d\sigma.$$

Elasticity. As a second application consider the evolutive problem in linear elasticity inside a bounded domain Ω:

$$\rho(x)\frac{\partial^2}{\partial t^2}u(x,t) = \nabla\cdot S(x,t) + b(x,t) \tag{5.2}$$

where u is the displacement vector, b the body force, while the body density $\rho \in C^1(\overline{\Omega})$, with $\rho(x) \geq \rho_0 > 0$, the stress tensor S is defined by the constitutive equation $S(x,t) = G(x)\nabla u(x,t)$ and the fourth-order elasticity tensor G is symmetric and positive definite. The boundary condition

$$G(x)\nabla u(x,t)n(x) + \alpha(x)v(x,t) = 0 \quad \text{on} \quad \partial\Omega \tag{1.4}$$

represents the viscous characteristic feature of the surface of the elastic body and the coefficient $\alpha \in C^1(\partial\overline{\Omega})$, with $\alpha(x) \geq \alpha_0 > 0$. Since equation (5.2) is equivalent to the first-order system

$$\begin{aligned} &\rho(x)\frac{\partial}{\partial t}v(x,t) = \nabla\cdot[G(x)\nabla u(x,t)] + b(x,t) \\ &G(x)\frac{\partial}{\partial t}\nabla u(x,t) = G(x)\nabla v(x,t) \end{aligned} \tag{5.3}$$

with the notation

$$\mathcal{U} = \begin{bmatrix} v \\ \nabla u \end{bmatrix}; \quad \mathcal{E} = \begin{bmatrix} \rho\mathbb{I} & 0 \\ 0 & G \end{bmatrix}; \quad A^i \frac{\partial}{\partial x_i} = \begin{bmatrix} 0 & G\nabla \\ G\nabla & 0 \end{bmatrix};$$

$$B = \begin{bmatrix} 0 & 0 \\ -\nabla \cdot G & 0 \end{bmatrix}; \quad f = \begin{bmatrix} b \\ 0 \end{bmatrix},$$

(5.3) assumes the form (2.1). Moreover the boundary condition (2.9) becomes

$$\int_\Omega A(x)\mathcal{U}(x) \cdot \mathcal{U}(x)\, dx = -\int_{\partial\Omega} \alpha(x)\big|\nabla u(x)\big|^2\, d\sigma.$$

REFERENCES

[1] C. Baiocchi, *Soluzioni ordinarie e generalizzate del problema di Cauchy per equazioni differenziali astratte non lineari del secondo ordine in spazi di Hilbert*, Ricerche Mat. **16** (1967), 27–95.

[2] L. Chen, *Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain*, Math. Pures et Appl. **58** (1979), 249–273.

[3] C.M. Dafermos, *Contraction semigroups and trends to equilibrium in continuum mechanics*, Proc. IUTAM/IMU, Lectures Notes in Mathematics 503, 1975.

[4] R. Dakto, *Extending a theorem of A.M. Liapunov to Hilbert space*, J. Math. Anal. and Appl. **32** (1970), 610–616.

[5] M. Fabrizio and B. Lazzari, *Esistenza e unicità per i campi stazionari con condizioni al contorno dissipative*, Annali Mat. Pura e Applicata CXXXV (1983), 73–83.

[6] K.O. Friedrichs, *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math. **7** (1954), 345–393.

[7] D. Graffi, *Sulle condizioni al contorno approssimate nell'elettromagnetismo*, Acc. Scienza Ist. Bologna, Rendiconti, Serie XI **5** (1958), 581–591.

[8] D. Guidetti, *On elliptic boundary value problems with dynamic boundary conditions of parabolic type* (to appear).

[9] D.S. Jones, *The Theory of Electromagnetism*, Pergamon Press, 1964.

[10] O.A. Ladyzenskaya, *The mathematical theory of viscous incompressible flow*, Mathematics and its Applications, Gordon and Breach, New York-London, 1964.

[11] I. Lagnese, *Decay of solution of wave equation in a bounded region with boundary dissipation*, J. Diff. Equations **50** (1983), 163–182.

[12] C. Muller, *Foundations of the Mathematical Theory of Electromagnetic Waves*, Springer-Verlag, 1969.

[13] A. Papoulis, *The Fourier Integral and its Applications*, Mc Graw-Hill Book Company, Inc., New York, 1962.

[14] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Lecture Notes in Mathematics 10, University of Maryland, 1974.

[15] R.S. Phillips, *Dissipative operators and hyperbolic systems of partial differential equations*, Trans. Amer. Math. Soc. **90** (1958), 193–254.

[16] S.I. Schelkunoff, *Electromagnetic Waves*, D. Van Nostrand Company Inc., New York, 1958.

[17] F. Treves, *Basic Linear Partial Differential Equations*, Academic Press, New York, 1975.

[18] C.H. Wilcox, *The Domain of Dependence Inequality and Initial Boundary Value Problems for Symmetric Hyperbolic Systems*, MRC Technical Summary Report n.333, 1962.

[19] C.H. Wilcox, *The steady-state diffraction of electromagnetic radiation by an obstacle in an inhomogeneous anisotropic conducting medium*, Arch. Rational Mech. Anal. **14** (1963), 326–336.

[20] C.H. Wilcox, *The domain of dependence inequality for symmetric hyperbolic systems, Bull. Amer. Math. Soc.* **70** (1964), 149–154.