Differential and Integral Equations Volume 10, Number 2, March 1997, pp. 265 – 272.

NECESSARY AND SUFFICIENT CONDITIONS FOR INTERMITTENT STABILIZATION OF LINEAR OSCILLATORS BY LARGE DAMPING

László Hatvani† and Tibor Krisztin‡
Bolyai Institute, Szeged University, Aradi vértanúk tere 1, H-6720 Szeged, Hungary

(Submitted by: James Serrin)

Abstract. The oscillator

$$x'' + h(t)x' + x = 0$$

is considered, where the damping $h : \mathbb{R}_+ \to \mathbb{R}_+$ is piecewise continuous and large in the sense

$$\liminf_{t\to\infty} \int_t^{t+\delta} h > 0 \quad \text{for every} \quad \delta > 0.$$

The problem of intermittent damping, initiated by P. Pucci and J. Serrin, is investigated. Let a sequence $\{I_n = [\alpha_n, \beta_n]\}$ of disjoint intervals be given such that $\alpha_n \to \infty$ as $n \to \infty$. A necessary and sufficient condition is given for $\{I_n\}$ and h on $I := \bigcup_{n=1}^{\infty} I_n$ guaranteeing $x(t) \to 0$, $x'(t) \to 0$ as $t \to \infty$ for every solution x, any way h may be defined out of I.

1. Introduction. Consider the damped linear oscillator of one degree of freedom described by the second-order differential equation

$$x'' + h(t)x' + x = 0, \qquad (t \in \mathbb{R}_+ := [0, \infty)), \tag{1.1}$$

where the damping coefficient $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a piecewise-continuous function. The equilibrium $x = x' = 0$ is stabilized by the friction if it is *asymptotically stable*, which means that for every solution x of (1.1)

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0 \tag{1.2}$$

holds.

It is well-known ([2–8]) that the equilibrium can be asymptotically stable only if the damping coefficient h is "not too small" and "not too large" in mean. The stabilization is *intermittent* if h is controlled from above or from below (or from both sides) only on a sequence of disjoint intervals I_n of $\mathbb{R}_+$ ("control set"). From

---

Received for publication in revised form December 1995.
†Research was supported by the Hungarian Foundation for Scientific Research, Grant No. T/016367.
‡Research was supported by the Hungarian Foundation for Scientific Research, Grant No. T/01367 and T/7298.
AMS Subject Classifications: 34D20 .

a mechanical point of view, the intermittent stabilization occurs if the oscillator is positively damped in the time intervals I_n, but has its damping either switched off or unrestricted at other times. The terminology of "intermittent damping" and "control set" was introduced for more general systems by P. Pucci and J. Serrin ([7]).

Definition. Let $\{I_n = [\alpha_n, \beta_n]\}_{n=1}^{\infty}$ be a sequence of disjoint intervals in $\mathbb{R}_+$ such that $\alpha_n \to \infty$ as $n \to \infty$. We say that *the equilibrium $x = x' = 0$ is stabilized on the control set $I = \bigcup_{n=1}^{\infty} I_n$* if the zero solution $x = x' = 0$ of (1.1) is asymptotically stable any way h may be defined out of I.

The problem is to find conditions on the control set I and the damping coefficient h guaranteeing the equilibrium to be stabilized on I. The first result in this problem was

Theorem A (R.A. Smith, [8]). *The equilibrium $x = x' = 0$ is stabilized on the control set $I = \bigcup_{n=1}^{\infty} I_n$ if*

$$\sum_{n=1}^{\infty} \underline{h}_n |I_n| \big[\min\{|I_n|;\ \frac{1}{1+\overline{h}_n}\}\big]^2 = \infty, \tag{1.3}$$

where $|I_n|$ is the length of I_n, and

$$\underline{h}_n := \inf_{t\in I_n} h(t); \qquad \overline{h}_n := \sup_{t\in I_n} h(t). \tag{1.4}$$

Later on it turned out ([2–7]) that two cases have to be distinguished: the case of "small damping" when the damping coefficient h is bounded above, and the case of "large damping" which means that h is *integrally positive*; i.e.,

$$\liminf_{t\to\infty} \int_t^{t+\delta} h(s)\,ds > 0 \quad \text{for every} \quad \delta > 0. \tag{1.5}$$

These two cases require essentially different techniques. In the case of small damping, to guarantee the stabilization the friction has to preclude the possibility of oscillation not tending to the equilibrium ([2, 4, 6–8]). In the case of large damping the phenomenon of overdamping has to be precluded. The *overdamping* means that $x'(t) \to 0$, $x(t) \to c \neq 0$, as $t \to \infty$, for some solutions ([1–9]). P. Pucci and J. Serrin ([6–7]) extended Smith's result to very general nonlinear quasi-variational systems of many degrees of freedom for both cases of small and large damping. The consequence of their main result for equation (1.1) in the case of large damping is

Theorem B (P. Pucci and J. Serrin, [7, Theorem 2]). *Suppose that h is integrally positive; i.e.,* (1.5) *is satisfied. The equilibrium $x = x' = 0$ is stabilized on the control set $I = \bigcup_{n=1}^{\infty} I_n$ if there is a constant c such that*

$$\sum_{n=1}^{\infty} |I_n|^2 \min\{1, \frac{c}{\int_{I_n} h}\} = \infty. \tag{1.6}$$

(It is worth noticing that here, and in the sequel, the stabilization on I means that h may be arbitrary out of I but it must be integrally positive on the whole of $\mathbb{R}_+$; i.e., (1.5) is satisfied for all $t \geq 0$.) This theorem is suitable to deduce some earlier results ([1–2, 4–5, 8–9]) as special cases. It is clear that in the case of fast-varying damping coefficients h the effect of the damping over I_n can be characterized by the integral of h over I_n more precisely than by the use of infimum $\underline{h}_n$ and supremum $\overline{h}_n$. For this reason, Theorem B is essentially more effective than Theorem A.

Our purpose in this paper is to show that a precise analysis of the integral of the damping coefficient h can yield a necessary and sufficient condition for the intermittent stabilization.

Theorem 1.1. *Suppose that h is integrally positive; i.e., (1.5) holds. Let the control set $I = \bigcup_{n=1}^{\infty} I_n$ and a constant $c > 0$ be given, and introduce the notation*

$$t_{n,k} := \inf\{t \geq \alpha_n : \int_{\alpha_n}^{t} h = kc\}, \qquad s_{n,k} := \min\{t_{n,k}; \beta_n\} \tag{1.7}$$

for $n = 1, 2, \dots$; $k = 0, 1, 2, \dots$. The equilibrium $x = x' = 0$ is stabilized on I if and only if

$$\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}(s_{n,k} - s_{n,k-1})^2\Big) = \infty. \tag{1.8}$$

It is easy to see that Smith's condition is not necessary; e.g., the damping

$$h(t) = \begin{cases} 1 & \text{for } n \leq t \leq n + 1/\sqrt{n} \\ n^2 & \text{for } n + 1/\sqrt{n} < t < n + 1 \quad (n = 1, 2, \dots) \end{cases}$$

stabilizes the equilibrium on $\bigcup_{n=1}^{\infty}[n, n + 1/\sqrt{n}]$ by Theorem B, but (1.3) is not satisfied with any $\{I_n\}$.

We will show by an example that Theorem B does not give a necessary condition either in its original form. However, using Theorem 1.1 we will also show that (1.6) *is* necessary in the sense that if the equilibrium is stabilized on a control set $I = \bigcup_{n=1}^{\infty} I_n$, then there exists a control set $I' = \bigcup_{n=1}^{\infty} I'_n$ such that $I' \subset I$ and (1.6) is satisfied with $\{I'_n\}$ instead of $\{I_n\}$.

2. The Proof of Theorem 1.1.

We will deduce Theorem 1.1 from

Theorem C (L. Hatvani, T. Krisztin, and V. Totik, [3])**.** *Suppose that h in (1.1) is integrally positive; i.e., (1.5) holds, and $c > 0$ is a fixed number. Then the equilibrium $x = x' = 0$ of (1.1) is asymptotically stable if and only if*

$$\sum_{k=1}^{\infty}(t_k - t_{k-1})^2 = \infty, \tag{2.1}$$

where

$$t_k := \inf\{t \in \mathbb{R}_+ : \int_0^t h = kc\} \quad (k = 0, 1, \dots). \tag{2.2}$$

(See Theorem 2.2 and condition (2.4) in [3].)

Proof of Theorem 1.1. (Sufficiency) By Theorem C it is enough to prove that (1.8) implies (2.1).

Introduce the notation

$$P := \{n : \text{ there exists } t_k \text{ such that } t_k \in [\alpha_n, \alpha_{n+1})\}.$$

If $n \notin P$, then $\int_{\alpha_n}^{\beta_n} h < c$ and $s_{n,0} = \alpha_n, s_{n,1} = s_{n,2} = \cdots = \beta_n$.

We distinguish two cases according as $\sum_{n\notin P} |I_n|^2$ is finite or not.

Case 1: $\sum_{n\notin P} |I_n|^2 = \infty$. In this case

$$\sum_{n\notin P} \Big(\sum_{k=1}^{\infty} (s_{n,k} - s_{n,k-1})^2 \Big) = \sum_{n\notin P} |I_n|^2.$$

For any $n \notin P$ there is a k such that $I_n \subset [t_{k-1}, t_k]$. Then $(t_k - t_{k-1})^2 \geq \sum |I_n|^2$, where the summation goes for those indices $n \in P$ for which $I_n \subset [t_{k-1}, t_k]$ holds. Consequently, $\sum_{k=1}^{\infty} (t_k - t_{k-1})^2 = \infty$.

Case 2: $\sum_{n\notin P} |I_n|^2 < \infty$. Then $\sum_{n\in P} \big(\sum_{k=1}^{\infty} (s_{n,k} - s_{n,k-1})^2 \big) = \infty$. If $n \in P$, then we may define

$$k(n) := \min\{k : t_k \in [\alpha_n, \alpha_{n+1})\},$$

$$m(n) := \begin{cases} 0 & \text{if } \ t_{k(n)} > \beta_n \\ \max\{j : t_{k(n)+j} \leq \beta_n\} & \text{if } \ t_{k(n)} \leq \beta_n. \end{cases}$$

For $n \in P$, the members of the sequences $\{t_k\}$ and $\{s_{n,k}\}$ separate one another on I_n in the following sense:

$$\begin{aligned} t_{k(n)-1} < s_{n,0} \leq t_{k(n)} < s_{n,1} \leq t_{k(n)+1} < \cdots < s_{n,j-1} \leq t_{k(n)+j-1} < s_{n,j} \leq \cdots \\ < s_{n,m(n)} \leq t_{k(n)+m(n)} \leq s_{n,m(n)+1} = s_{n,m(n)+2} = \cdots = \beta_n < t_{k(n)+m(n)+1}. \end{aligned}$$

So we have

$$\begin{aligned} &\sum_{j=1}^{\infty} (s_{n,j} - s_{n,j-1})^2 = \sum_{j=1}^{m(n)+1} (s_{n,j} - s_{n,j-1})^2 \\ &\leq \sum_{j=1}^{m(n)+1} (t_{k(n)+j} - t_{k(n)+j-1} + t_{k(n)+j-1} - t_{k(n)+j-2})^2 \leq 4 \sum_{j=k(n)}^{k(n)+m(n)+1} (t_j - t_{j-1})^2. \end{aligned}$$

Hence, by using that any $[t_{j-1}, t_j]$ may intersect at most two $[\alpha_n, \alpha_{n-1})$ with $n \in P$, we obtain

$$\sum_{n=1}^{\infty}(t_n - t_{n-1})^2 \geq \frac{1}{2}\sum_{n\in P}\Big(\sum_{j=k(n)}^{k(n)+m(n)+1}(t_j - t_{j-1})^2\Big)$$
$$\geq \frac{1}{8}\sum_{n\in P}\Big(\sum_{j=1}^{\infty}(s_{n,j} - s_{n,j-1})^2\Big) = \infty,$$

and the proof of the sufficiency is completed.

(Necessity) Use the method of contradiction. Suppose, that the equilibrium is stabilized on I (which means that it is asymptotically stable any way h may be defined on $J := \mathbb{R}_+ \setminus I$) and suppose that (1.8) is not true. We will show that h can be redefined on J so that (2.1) is not true either, which is a contradiction because, by Theorem C, (2.1) is a necessary condition for the asymptotic stability.

Let us change the damping from h to $\widetilde{h}$ so that

a) $h \equiv \widetilde{h}$ on $I = \bigcup_{n=1}^{\infty}[\alpha_n, \beta_n]$;

b) $\widetilde{h}(t) \geq cn^2(\alpha_{n+1} - \beta_n + 1)$ for all $n = 1, 2, \ldots$ and $t \in (\beta_n, \alpha_{n+1})$;

c) $(1/c)\int_0^{\alpha_n}\widetilde{h}$ is an integer for every n;

d) $\widetilde{h}$ is integrally positive.

We can guarantee properties b) and c) by adding an appropriate constant to h on the interval (β_n, α_{n+1}) for every n.

Let $\{\widetilde{t}_k\}$ be the sequence corresponding to $\widetilde{h}$ defined analogously to (2.2). Let $q(n)$ denote the smallest one of the indices i for which $\widetilde{t}_i \in [\beta_n, \alpha_{n+1}]$, and let $r(n) = (1/c)\int_0^{\alpha_n}\widetilde{h}$. It is easy to see that $r(n) < q(n) \leq r(n+1)$ for all n. We will show that the sum $\sum_{n=1}^{\infty} S_n$ is convergent, where

$$S_n = \sum_{i=r(n)+1}^{r(n+1)}(\widetilde{t}_i - \widetilde{t}_{i-1})^2 = \sum_{i=r(n)+1}^{q(n)}\cdots + \sum_{i=q(n)+1}^{r(n+1)}\cdots = S_n^{(1)} + S_n^{(2)}.$$

Since $\beta_n \in (\widetilde{t}_{q(n)-1}, \widetilde{t}_{q(n)}]$, we have

$$(\widetilde{t}_{q(n)} - \widetilde{t}_{q(n)-1})^2 \leq 2(\beta_n - \widetilde{t}_{q(n)-1})^2 + 2(\widetilde{t}_{q(n)} - \beta_n)^2.$$

Property b) implies $\widetilde{t}_{q(n)} - \beta_n \leq 1/n$. Taking into account also the identity $\widetilde{t}_{n,k} - \widetilde{t}_{n,k-1} \equiv t_{n,k} - t_{n,k-1}$ we obtain

$$S_n^{(1)} \leq 3\sum_{k=1}^{\infty}(s_{n,k} - s_{n,k-1})^2 + \frac{2}{n^2}.$$

Now we estimate $S_n^{(2)}$ in the case $q(n) < r(n+1)$. By property b) we have

$$0 < \widetilde{t}_i - \widetilde{t}_{i-1} \le \frac{1}{n^2(\alpha_{n+1} - \beta_n)} \quad \text{for} \quad i = q(n)+1, \dots, r(n+1).$$

On the other hand, the definition of $q(n)$ shows that

$$\sum_{i=q(n)+1}^{r(n+1)} (\widetilde{t}_i - \widetilde{t}_{i-1}) \le \alpha_{n+1} - \beta_n.$$

From the last two inequalities we get $S_n^{(2)} \le 1/n^2$. Consequently, the infinite series

$$\sum_{n=1}^{\infty} S_n = \sum_{n=1}^{\infty} S_n^{(1)} + \sum_{n=1}^{\infty} S_n^{(2)} \le 3 \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{\infty} (s_{n,k} - s_{n,k-1})^2 \Big) + 3 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because (1.8) does not hold by our indirect assumption. This means that the necessary condition of the asymptotic stability is not satisfied for the damping $\widetilde{h}$.

This contradiction completes the proof of Theorem 1.1.

3. Remarks. Let us compare Theorem B of P. Pucci and J. Serrin and our Theorem 1.1, i.e., conditions (1.6) and (1.8).

The advantage of (1.6) is that it needs only the lengths of the intervals of the control set and the integrals of the damping coefficient over these intervals. In other words, condition (1.6) controls only the total damping effects $\int_{I_n} h$. However, the necessary and sufficient condition (1.8) shows that not only the total damping effects $\int_{I_n} h$ are important but their distributions on I_n's, as well. We prove that Theorem B is a consequence of Theorem 1.1, but the converse is not true.

Proposition 3.1. *Condition* (1.6) *implies condition* (1.8).

Proof. For a fixed n we distinguish two cases:

$$\text{a)} \quad \int_{I_n} h < c; \qquad \text{b)} \quad \int_{I_n} h \ge c.$$

In case a) we have $s_{n,0} = \alpha_n$, $s_{n,1} = s_{n,2} = \cdots = \beta_n$. Therefore,

$$\sum_{k=1}^{\infty} (s_{n,k} - s_{n,k-1})^2 = |I_n|^2.$$

In case b) let $j(n) \ge 1$ denote the smallest integer for which $s_{n,j(n)} = \beta_n$. Applying the Cauchy inequality we obtain

$$\sum_{k=1}^{\infty} (s_{n,k} - s_{n,k-1})^2 = \sum_{k=1}^{j(n)} (s_{n,k} - s_{n,k-1})^2 \ge \frac{1}{j(n)} \Big(\sum_{k=1}^{j(n)} (s_{n,k} - s_{n,k-1}) \Big)^2 =$$

$$= \frac{1}{j(n)} (\beta_n - \alpha_n)^2 \ge |I_n|^2 \frac{1}{1 + \int_{I_n} h/c} \ge \frac{1}{2} |I_n|^2 \frac{c}{\int_{I_n} h}.$$

The proposition is proved.

Proposition 3.2. *Condition* (1.8) *does not imply condition* (1.6); *i.e., condition* (1.6) *in Theorem B is not necessary for the stabilization on I.*

Proof. Define

$$h(t) := \begin{cases} 2\sqrt{n^3} & \text{for } n \le t < n + 1/2\sqrt{n}, \\ 2\sqrt{n} & \text{for } n + 1/2\sqrt{n} \le t < n + 1/\sqrt{n}, \quad (n = 1, 2, \dots) \\ 1 & \text{otherwise.} \end{cases}$$

Taking $c = 1$ we obviously have

$$s_{n,n} = n + \frac{1}{2\sqrt{n}}, \quad s_{n,n+1} = n + \frac{1}{\sqrt{n}}$$

for all $n \ge 1$, so (1.8) is satisfied and the equilibrium is stabilized on $\bigcup_{n=1}^{\infty}[n, n + 1/\sqrt{n}]$.

On the other hand, $|I_n| = 1/\sqrt{n}$ and $\int_{I_n} h = n + 1$, so, for any c, (1.8) is not satisfied. This example proves the proposition.

Analyzing the above example one can observe that the same damping stabilizes the equilibrium on $I' = \bigcup_{n=1}^{\infty}[n + 1/2\sqrt{n}, n + 1/\sqrt{n}]$. A similar result is true in the general case:

Proposition 3.3. *Suppose that the equilibrium is stabilized on I. Then there is a control set $I' = \bigcup_{m=1}^{\infty} I'_m \subset I$ such that condition (1.6) with $\{I'_m\}$ is satisfied.*

Proof. By Theorem 1.1, condition (1.8) holds for $\{I_n = [\alpha_n, \beta_n]\}$. Consequently, condition (1.6) also holds with

$$\{I'_m\} = \left\{ \{[s_{n,k-1}, s_{n,k}]\}_{k=0}^{j(n)} \right\}_{n=1}^{\infty},$$

where $j(n)$ denotes the smallest integer for which $s_{n,j(n)} = \beta_n$. The intervals in the sequence $\{I'_m\}$ are not necessarily disjoint. However, it is easy to see that they can be shrunk to get a new sequence of disjoint intervals (1.6) still holds for. The proposition is proved.

REFERENCES

[1] Zvi Artstein and E. F. Infante, *On the asymptotic stability of oscillators with unbounded damping,* Quart. Appl. Mech., 34 (1976), 195–199.

[2] R. J. Ballieu and K. Peiffer, *Attractivity of the origin for the equation $\ddot{x} + f(t, x, \dot{x})|\dot{x}|^{\alpha}\dot{x} + g(x) = 0$,* J. Math. Anal. Appl., 65 (1978), 321–332.

[3] L. Hatvani, T. Krisztin and V. Totik, *A necessary and sufficient condition for the asymptotic stability of the damped oscillator,* J. Differential Equations, 119 (1995), 209–223.

[4] L. Hatvani and V. Totik, *Asymptotic stability of the equilibrium of the damped oscillator,* Differential Integral Equations, 6 (1993), 835–848.

[5] S. Murakami, *Asymptotic behaviour of solutions of ordinary differential equations,* Tôhoku Math. J., 34 (1982), 559–574.

[6] P. Pucci and J. Serrin, *Precise damping conditions for global asymptotic stability for nonlinear second order systems,* Acta Math., 170 (1993), 257–307.

[7] P. Pucci and J. Serrin, *Asymptotic stability for intermittently controlled nonlinear oscillators,* SIAM J. Math. Anal., 25 (1994), 815–835.

[8] R. A. Smith, *Asymptotic stability of* $x'' + a(t)x' + x = 0$, Quart. J. Math. Oxford Ser. (2), 12 (1961), 123–126.

[9] L.H. Thurston and J.S.W. Wong, *On global asymptotic stability of certain second order differential equations with integrable forcing terms,* SIAM J. Appl. Math., 24 (1973), 50–61.