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PERIODIC MOTIONS OF A LATTICE OF PARTICLES WITH SINGULAR FORCES

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Abstract. We prove the existence of periodic motions of prescribed energy E or period T for an autonomous Hamiltonian system consisting of a lattice of N particles with one fixed endpoint. Each interparticle force is defined over a bounded interval of the real line, the potentials of such forces having a barrier at the endpoints of the interval; a physical model for this system is suggested. The proofs are based on variational methods and by suitable sequences of approximating problems that involve strong-force terms: this technique also allows one to obtain the existence of periodic bounce trajectories in a rectangular billiard.

1. Introduction. In this paper we consider an autonomous Hamiltonian system consisting of a lattice of N ($N \geq 2$) particles; the state of the lattice at time t is represented by a vector $q(t) = \{q_1(t), \dots, q_N(t)\}$, where $q_i(t)$ represents the state of the i-th particle; we study lattices with one fixed endpoint, i.e., $q_0 \equiv 0$. In our system each particle of the lattice interacts only with the nearest neighbors; denote by V_{i+1} the potential of the interaction between the i-th and the $(i+1)$-th particle, then if such potentials are smooth the equations governing the motion of the lattice are

$$\begin{aligned} \ddot{q}_i(t) &= V_i'[q_{i-1}(t) - q_i(t)] - V_{i+1}'[q_i(t) - q_{i+1}(t)] \qquad i = 1, \dots, N-1 \\ \ddot{q}_N(t) &= V_N'[q_{N-1}(t) - q_N(t)]\ . \end{aligned} \tag{1}$$

The starting point of studies of motions in lattices is the celebrated numerical experiment of Fermi-Pasta-Ulam ([17]); their model represents a chain of disks which are allowed to rotate around their axes, the variables $q_i(t)$ being the values of the angle of rotation; the nearest neighbor disks are connected with a spring which raises a force, which is linear plus a perturbing term. They observed that if the perturbation is small enough, then the system does not converge to equipartition of energy. The idea that a strongly nonlinear dynamical system could lack ergodicity was supported by the existence of periodic motions for lattices of particles with exponential interactions proved by Toda ([25]); in that case the system is completely integrable and admits explicit periodic and soliton solutions. Later, this problem has been studied by variational methods for a class of large perturbations of the Toda interactions and

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for given period T in [21], where the case of periodic lattices is also studied. Finally, the existence of periodic motions of finite energy have been proved in the case of an infinite lattice ([6, 7, 8]). In this case a lack of compactness of the system appears (the Palais-Smale condition does not hold) and different variational techniques are used. In the above-cited references only the prescribed period problem is considered. To improve the knowledge about this model we thought that it could be interesting to study the case where the interparticle potentials have repulsive singularities at the endpoints of their (bounded) intervals of definition: imagine these disks are free to rotate until a given position from which they are bounced back in an elastic way; in this case a suitable definition of generalized solution of (1) is needed. As the intervals of definition of the potentials are bounded any two disks cannot move too far away from each other. We assume $N \geq 2$ because if $N = 1$ the problem becomes that of the one-dimensional bounce ([13, 14, 15]).

In the last few years periodic solutions of systems with singular potentials have been studied by means of the calculus of variations: the results concern both the existence of periodic solutions of prescribed period T or of prescribed energy E. It looks difficult to give complete references of the results existing in literature; however, let us make an attempt to indicate the ones which are more closely related to our problem; see [3] for a survey of results and wider references.

The fixed period problem is somewhat older and our interest is mainly about the papers of Benci-Giannoni ([9]) and Terracini ([24]), indeed the problem of constrained rotating disks is similar to that of the billiard and of the pinball. In [9] it is proved that if T is small enough then there exists a periodic bounce trajectory in the billiard (i.e., a closed bounded subset of $\mathbf{R}^N$ with boundary of class C^2) for which it is possible to estimate the number of bounces. The proof is performed on an approximation scheme which uses a penalization method involving a strong force term (see [18]). A bounce trajectory is obtained as a limit of regular trajectories; to estimate the number of bounces a Morse index argument is used. In [24] a fixed-period problem with repulsive singularities is also studied. The case of exterior disconnected domains is considered and the model suggested is that of the pinball. An existence and a multiplicity result of periodic solutions having small period and high energy are obtained by similar techniques as in [9] with some further difficulties. The penalization term must also modify the behaviour of the potential at infinity. Further references can be found in the above-cited papers.

To our knowledge for the fixed energy case only Hamiltonian systems having attractive singularities have been studied; see, e.g., [2, 5, 10, 23] and references therein; all these papers have the common feature of finding periodic solutions in the presence of weak forces by means of the penalization technique involving strong forces outlined above. These solutions correspond to critical points of the Maupertuis-like functional defined in [26]. The same procedure has been used in [1] for a different choice of the action functional while in [4] the approximation of the system is obtained by means of smooth potentials instead of the ones having a strong force.

We will prove that (1) admits a (possibly) generalized periodic solution of prescribed energy E or period T; for both problems we consider the case of a potential barrier, with potentials bounded in their interval of definition: when the boundary of such an interval is reached the disks collide against the barrier and bounce back in an elastic way, conserving energy. This result is obtained by means of a sequence of approximated problems having a strong force term and by taking the limit. Our problem is somewhat similar to that considered in [9] and the same techniques apply; however, there are two basic differences in our approach. The first one is that it also describes bounces when three (or more) disks collide against a barrier at the same time: in the model suggested in [9] these multiple collisions correspond to bounces in the (right) corners of a billiard, i.e., when its boundary is no longer of class C^2. The second difference is that we also consider the prescribed energy case which can be studied with slight modifications and for which the case of a repulsive potential barrier seems not to have been studied before. The techniques involved in the proofs of our results also apply to obtain the existence of periodic bounce trajectories in a rectangular billiard; note that a general result about periodic bounce trajectories in a billiard having any kind of angles seems not to be possible; see Section 5 where further models are suggested. The assumptions on the potentials V_i and the exact statements of our results are given in next section.

2. Statements of the results. For all $i = 1, \dots, N$ let $a_i, b_i \in \mathbf{R}$ with $a_i < b_i$ and assume that the potentials V_i are defined on (a_i, b_i); in the sequel, we denote by $\mathcal{U}(a_i)$ (respectively $\mathcal{U}(b_i)$) a generic right (respectively left) neighborhood of a_i (respectively b_i). Define the sets

$$\Lambda_T := \{q \in H_T : q_i(t) - q_{i+1}(t) \in (a_i, b_i) \;\; \forall i \; \forall t \in [0,T]\} \;,$$

where H_T is the Hilbert space of absolutely continuous T-periodic functions $q : [0,T] \to \mathbf{R}^N$ with square-integrable derivative endowed with the scalar product

$$\forall q, p \in H_T \qquad (q,p) = \int_0^T \dot{q}(t) \cdot \dot{p}(t)\; dt + \Big(\int_0^T q(t)\; dt\Big)\Big(\int_0^T p(t)\; dt\Big).$$

By letting $\Phi(q) := \sum_{i=1}^N V_i(q_{i-1} - q_i)$ we can also write (1) in vectorial form

$$\ddot{q} = -\Phi'(q) \;. \tag{2}$$

Set $d_i(t) := q_{i-1}(t) - q_i(t)$ and

$$\Omega = (a_1, b_1) \times \cdots \times (a_N, b_N) \subset \mathbf{R}^N \;; \tag{3}$$

if we define the potential $\mathbf{V} : \Omega \to \mathbf{R}$ by

$$\forall d = (d_1, \dots, d_N) \in \Omega \qquad \mathbf{V}(d) = \Phi(q) = \sum_{i=1}^N V_i(d_i) \;,$$

our problem becomes similar to that studied in [9] except for the facts that $\partial\Omega$ is not of class C^2 (it has right corners) and that we will also consider the prescribed energy case.

For all $i = 1, \dots, N$ we assume the potential V_i to satisfy

$$V_i \in C^2[a_i, b_i] \; ; \tag{T}$$

the assigned energy problem becomes more interesting if we require a slightly weaker condition. We assume that at the endpoints a_i and b_i the potentials V_i have a potential barrier or blow up at infinity:

$$V_i \in C^2(a_i, b_i) \text{ either } \lim_{x \to a_i^+} V_i(x) = +\infty \text{ or } V_i \in C^2[a_i, b_i] \quad \text{(and similarly in } b_i). \tag{E}$$

Note that if $\mathbf{V}$ is bounded and continuous in Ω we can extend its definition to $\bar{\Omega}$.

Let us explain what we mean by generalized solution.

Definition 1. A generalized T-periodic solution of energy E of equation (1) is a function $q \in \bar{\Lambda}_T$ such that

(i) there exist N finite Borel measures $\mu_i(d_i)$ on $[0, T]$ $(i = 1, \dots, N)$ such that $B_i := \text{supp}[\mu_i(d_i)] \subset \{t \in [0, T]; d_i(t) \in \{a_i, b_i\}\}$

(ii) there exists $i \in \{1, \dots, N\}$ such that $B_i \neq \emptyset$

(iii) if $\mu(q) := \sum_i \mu_i(d_i)$ and $B(q) := \bigcup_i B_i$ then $|B(q)| = 0$, $q \in C^2[0, T] \setminus B(q)$ and the equation $\ddot{q} = -\Phi'(q) - \mu(q)$ is satisfied in distributional sense; i.e.,

$$\int_0^T \dot{q}\dot{p}\, dt - \int_0^T \Phi'(q) p\, dt = \int_{B(q)} p\, d\mu(q) \qquad \forall p \in C_T^\infty,$$

where C_T^∞ is the space of smooth T-periodic functions with values in $\mathbf{R}^N$

(iv) if $t \in B_i$ then $\dot{d}_i$ has a right and left limit in t and

$$\lim_{s \to t^-} \dot{d}_i(s) = -\lim_{s \to t^+} \dot{d}_i(s)$$

(v) $\frac{1}{2} \sum_{i=1}^N \dot{q}_i^2(t) + \sum_{i=1}^N V_i[d_i(t)] = E$ for all $t \in [0, T] \setminus B(q)$.

Condition (ii) in the previous definition requires the existence of at least one bounce instant $t \in [0, T]$ and of the corresponding bounce point $q(t)$; if no bounces occur then $\mu_i(d_i) \equiv 0$ for all i and a generalized solution of (1) becomes a classical (C^2) solution.

For the prescribed energy case we will prove (see Section 3):

Theorem 1. *Assume* (E)*; then for almost all* $E > 0$ *there exists* $T > 0$ *such that system* (1) *admits a* (*possibly*) *generalized* T*-periodic solution of energy* E*. Moreover*

- *if* $V_i \in C^2[a_i, b_i]$ *for all* i*, there exists* $E_0 = E_0(V_i) \geq 0$ *such that for almost all* $E > E_0$ *the generalized solution has at least one bounce; in particular, if* $V_i \equiv 0$ *for all* i*, then* $E_0 = 0$
- *if* $\lim_{x \to a_i, b_i} V_i(x) = +\infty$ *for all* i*, then the solution is a classical solution.*

Define the set

$$\mathcal{E} := \mathbf{R}^+ \setminus \{E > 0 : \exists d \in \Omega,\ \mathbf{V}(d) = E,\ \mathbf{V}'(d) = 0\}\ ; \tag{4}$$

by Sard's Lemma almost every $E > 0$ belongs to $\mathcal{E}$. The statement of Theorem 1 holds for all $E \in \mathcal{E}$.

For the prescribed period case we will prove (see Section 4):

Theorem 2. *Assume* (*T*)*; then for all* $T > 0$ *there exists* $E > 0$ *such that system* (1) *admits a generalized* T*-periodic solution of energy* E*.*

Let us now turn to the billiard ball problem in case of billiards having right corners. Let $-\infty < a_i < b_i < +\infty$ for all $i = 1, \dots, N$ and let $\Omega \subset \mathbf{R}^N$ be defined as in (3); moreover, let $S^1_T = [0,T]/\{0,T\}$. Consider a loop $\gamma : S^1_T \to \bar{\Omega}$ and assume that $\exists t \in [0,T]$ such that $\gamma(t) \in \partial\Omega$; then there exist a certain number of sides C_j of $\partial\Omega$ (i.e., a certain number of hyperplanes of equation $x = a_i$ or $x = b_i$) to which $\gamma(t)$ belongs. Let ν_j denote the unit outer normal to C_j, then we define the *index of* $\gamma(t)$ to be the nonnegative integer $\alpha[\gamma(t)]$ corresponding to the number of sides C_j for which $\lim_{s\to t} \gamma'(s) \cdot \nu_j > 0$ and we call *positive sides* such C_j. Following Benci-Giannoni ([9]) we consider *periodic bounce trajectories* with respect to the potential $\mathbf{V}$:

Definition 2. A loop $\gamma : S^1_T \to \bar{\Omega}$ is called a T-periodic bounce trajectory of energy E with respect to the potential $\mathbf{V}$ if the following statements hold:

(i) $\gamma \in C^2(S^1_T)$ except for at most a finite number $k > 0$ of instants $t_1, \dots, t_k \in [0,T]$ for which $\gamma(t_j) \in \partial\Omega$
(ii) $\gamma''(t) = -\mathbf{V}'[\gamma(t)]$ for all $t \notin \{t_1, \dots, t_k\}$
(iii) for every $t \in \{t_1, \dots, t_k\}$ there exist the limits $\lim_{s\to t^\pm} \gamma'(s) =: \gamma'_\pm(t)$
(iv) for all $t \in \{t_1, \dots, t_k\}$ such that $\alpha[\gamma(t)] = m \geq 1$ denote by C_j ($j = 1, \dots, m$) the positive sides of Ω to which $\gamma(t)$ belongs and by ν_j the unit outer normal of C_j; then

$$\gamma'_+(t) - \sum_{j=1}^m \left(\gamma'_+(t) \cdot \nu_j[\gamma(t)]\right) \nu_j[\gamma(t)] = \gamma'_-(t) - \sum_{j=1}^m \left(\gamma'_-(t) \cdot \nu_j[\gamma(t)]\right) \nu_j[\gamma(t)]\ ,$$
$$\gamma'_+(t) \cdot \nu_j[\gamma(t)] = -\gamma'_-(t) \cdot \nu_j[\gamma(t)] \qquad \forall j = 1, \dots, m$$

(v) $\frac{1}{2}\dot{\gamma}^2(t) + \mathbf{V}[\gamma(t)] = E$ for all $t \in [0,T] \setminus \{t_1, \dots, t_k\}$.

By using completely similar techniques as for Theorems 1 and 2 we will prove (see Section 5):

Theorem 3. *Let* $\Omega \subset \mathbf{R}^N$ *be as in* (3) *and let* $\mathbf{V} \in C^2(\bar{\Omega}, \mathbf{R})$*. Then*

(i) *for all* $E \in \mathcal{E}$ *there exists* $T > 0$ *and a* T*-periodic* (*possibly*) *bounce trajectory of energy* E *with respect to the potential* $\mathbf{V}$*; moreover, there exists* $E_0 \geq 0$ *such that if* $E > E_0$ *such trajectory has at least one bounce*
(ii) *for all* $T > 0$ *there exists* $E > 0$ *and a* T*-periodic bounce trajectory of energy* E *with respect to the potential* $\mathbf{V}$*.*

Remarks. Theorems 1 and 2 provide a generalized solution q of (1) which may be a classical solution if the corresponding set $B(q)$ is empty. Theorem 3 says nothing about the existence of a bounce in a corner; it only explains how the ball bounces back if it does hit the corner. In the literature ([4, 9, 23, 24]) an upper bound of the number of bounces is obtained by means of the Morse index of the critical points of the approximating functionals; the same techniques can be applied to our problem yielding at most $N+1$ bounces. For this reason we will not wonder about the measure of the set $B(q)$. An interesting open problem is to establish if a bounce in a corner contributes to the Morse index by means of its index α.

3. Generalized solutions of prescribed energy. In this section we prove Theorem 1; without loss of generality we assume that $a_i = -1$, $b_i = 1$ and that $V_i \geq 0$ for all $i = 1, \dots, N$. We first prove Theorem 1 in the presence of weak forces; we require that

$$\lim_{x \to -1^+} V_i(x) = +\infty \quad \textit{and} \quad \lim_{x \to 1^-} V_i(x) = +\infty \quad \forall i = 1, \dots, N\ . \tag{E'}$$

If the potentials V_i satisfy both (E) and (E'), periodic solutions of (1) of prescribed energy $E \in \mathcal{E}$ ($\mathcal{E}$ as in (4)) correspond to critical points at positive level of the functional $J : \Lambda_1 \to \mathbf{R}$ defined by (see [26])

$$J(q) = \frac{1}{2} \int_0^1 |\dot{q}|^2 \, dt \big[E - \int_0^1 \Phi(q) \, dt\big].$$

One can easily prove that the functional J is of class $C^2(\Lambda_1, \mathbf{R})$ and its derivative in $q \in \Lambda_1$ is the linear operator $J'(q)$ defined by

$$J'(q)[p] = \int_0^1 \dot{q}\dot{p} \, dt \big[E - \int_0^1 \Phi(q) \, dt\big] - \frac{1}{2} \int_0^1 |\dot{q}|^2 \, dt \int_0^1 \Phi'(q) p \, dt \quad \forall p \in H_1;$$

a critical point q of J at positive level is a periodic solution of period 1 of the equation

$$\ddot{q} = -T^2 \Phi'(q)\ , \tag{5}$$

where

$$T^2 = \frac{\frac{1}{2} \int_0^1 |\dot{q}|^2}{E - \int_0^1 \Phi(q)}\ , \tag{6}$$

hence it is a T-periodic solution of (2) of energy E, after a time scaling. The proposition below is by now well-known; see [4, 11]. We outline the proof for completeness and because a very simple linking argument can be used.

Proposition 1. *Assume* (E) *and* (E'); *then for all* $E \in \mathcal{E}$ ($\mathcal{E}$ *as in* (4)) *there exists* $T > 0$ *such that system* (1) *admits a solution* $q \in \Lambda_T$ *satisfying*

$$\frac{1}{2} \sum_{i=1}^N \dot{q}_i^2(t) + \sum_{i=1}^N V_i[d_i(t)] = E \qquad \forall t \in [0, T]\ .$$

Proof. The Palais-Smale condition for J may not hold: for this reason we add a strong force term; that is, we modify the potentials V_i in the neighborhoods $\mathcal{U}(-1)$ and $\mathcal{U}(1)$ where $V_i(x) \geq 2E$ obtaining new potentials W_i having a quadratic growth when $d_i \to -1$ or $d_i \to 1$ and we define the global potential $\Psi(q) := \sum_i W_i(d_i)$; using the strong force term one then proves that if $\{q^n\} \subset \Lambda_1$ is a sequence such that $q^n \rightharpoonup q \in \partial\Lambda_1$, then $\int_0^1 \Psi(q^n) \to +\infty$. We consider the functional

$$\tilde{J}(q) = \frac{1}{2}\int_0^1 |\dot{q}|^2\,dt\big[E - \int_0^1 \Psi(q)\,dt\big]$$

which has the same regularity as J. By modifying the potential Ψ outside its sublevel E and by the same reasoning as in [11] (see also [4]) one proves that for all $c > 0$, the functional $\tilde{J}$ satisfies the Palais-Smale condition at level c; the limit of a Palais-Smale sequence cannot be on $\partial\Lambda_1$ by the above strong force arguments. To prove the existence of a nonzero critical level b we use a linking technique (see [20]). Consider the orthogonal decomposition of the space H_1 into the N-dimensional subspace of constants H_c and the infinite-dimensional subspace of functions with zero average H_0; by the definition of the functional, $\tilde{J}|_{H_c} \equiv 0$. Let $w = (\sin(2\pi t), 0, \dots, 0) \in H_0$, and let $\{a_i^k\}$ and $\{b_i^k\}$ ($i = 1, \dots, N$ and $k \in \mathbf{N}$) be $2N$ strictly monotone sequences of real numbers such that $a_i^k < b_i^k$, $a_i^k \downarrow -1$ and $b_i^k \uparrow 1$ for all i. Define $Q_k \subset H_c \oplus \mathbf{R}^+ w$ by

$$Q_k := \{q \in H_c \oplus \mathbf{R}^+ w : \min_t[d_i(t)] \geq a_i^k,\ \max_t[d_i(t)] \leq b_i^k\ \forall i\}\ .$$

Q_k is a compact, convex subset of Λ_1 for all integers k; by (E'), if k is large enough, we have $\tilde{J}(q) \leq 0$ for all $q \in \partial Q_k$: from now on we assume that such choice of k has been made. Let $S_\rho := \{q \in H_0 : \|q\| = \rho\}$, where $\rho > 0$ is chosen so small that $S_\rho \subset \Lambda_1$ (such a choice is always possible because of the imbedding $H^1 \subset L^\infty$); as $\tilde{J}(q) \geq c\|q\|^2$ when q is in a suitable neighborhood of 0 in H_0, if ρ is small enough, then $\inf_{q \in H_0 \cap S_\rho} \tilde{J}(q) = \beta > 0$. Moreover, ∂Q_k and S_ρ link: all the assumptions of the linking theorem are satisfied, therefore $\tilde{J}$ admits a critical level defined by

$$b := \inf_{\gamma \in \Gamma} \max_{q \in Q_k} \tilde{J}(\gamma(q)) \geq \beta\ ,$$

where $\Gamma = \{\gamma \in C(Q_k, \Lambda_1) : \gamma|_{\partial Q_k} \equiv Id\}$.

To complete the proof note that the solutions obtained with the modified potentials W_i are solutions of the original problem as well, as the system is autonomous and the region of the phase space where $\Phi > E$ is not reached: the critical point of $\tilde{J}$ previously obtained is also a critical point of J. $\square$

To prove Theorem 1 we have to drop assumption (E'); from now on we only assume (E). We introduce a sequence of approximating problems, each satisfying the hypotheses of Proposition 1. To this purpose we add a suitable term to the potentials. For each i we have four possible cases: if both $\lim_{x\to -1^+} V_i(x) < +\infty$ and $\lim_{x\to 1^-} V_i(x) < +\infty$ let

$$U_i(x) = \begin{cases} \frac{1}{(x+1)^2} & \text{for } -1 < x < -1/2 \\ \frac{1}{(x-1)^2} & \text{for } 1/2 < x < 1 \\ C^2 \text{ -continuation} & \text{for } -1/2 \le x \le 1/2 \ , \end{cases}$$

if $\lim\limits_{x\to -1^+} V_i(x) = +\infty$ and $\lim\limits_{x\to 1^-} V_i(x) = +\infty$ let $U_i(x) \equiv 0$, if $\lim\limits_{x\to -1^+} V_i(x) = +\infty$ and $\lim\limits_{x\to 1^-} V_i(x) < +\infty$ let

$$U_i(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{(x-1)^2} & \text{for } 1/2 < x < 1 \\ C^2 \text{ -continuation} & \text{for } 0 \le x \le 1/2 \ , \end{cases}$$

the remaining case being defined similarly; we also require the C^2-continuation to satisfy

$$U_i(x) \ge 0 \quad \text{and} \quad U_i'(x)x \ge 0 \quad \forall x \in (-1,1) \ . \tag{7}$$

Let $\varepsilon > 0$, consider the potential

$$\Phi_\varepsilon(q) := \sum_{i=1}^N [V_i(d_i) + \varepsilon U_i(d_i)]$$

and define the functional $J_\varepsilon : \Lambda_1 \to \mathbf{R}$ by

$$J_\varepsilon(q) = \frac{1}{2}\int_0^1 |\dot{q}|^2\,dt\big[E - \int_0^1 \Phi_\varepsilon(q)\,dt\big];$$

for all $\varepsilon > 0$ the potential Φ_ε satisfies the assumptions of Proposition 1, therefore for almost all $E > 0$ the functional J_ε admits a nontrivial critical point q^ε. Let

$$T_\varepsilon^2 = \frac{\frac{1}{2}\int_0^1 |\dot{q}^\varepsilon|^2}{E - \int_0^1 \Phi_\varepsilon(q^\varepsilon)} \qquad \text{and} \qquad F_\varepsilon = \frac{1}{2}|\dot{q}^\varepsilon(t)|^2 + T_\varepsilon^2 \Phi_\varepsilon[q^\varepsilon(t)].$$

Note that F_ε is independent of t because it represents the energy of the solution q^ε before the time scaling. To take the limit as $\varepsilon \to 0$ we need estimates of $\|q^\varepsilon\|$, T_ε and F_ε:

Lemma 1. *There exist $\bar{\varepsilon}, c_1, c_2 > 0$ such that $c_1 \le \|q^\varepsilon\| \le c_2$, $c_1 \le T_\varepsilon \le c_2$ and $c_1 \le F_\varepsilon \le c_2$ for all $\varepsilon \in (0, \bar{\varepsilon})$.*

Proof. Let $q \in \Lambda_1$ and $\varepsilon_1 > \varepsilon_2 > 0$; then $J_{\varepsilon_1}(q) \le J_{\varepsilon_2}(q)$, therefore by the variational characterization of the critical level we can choose $\bar{\varepsilon} > 0$ such that a positive lower bound for the sequence $\{J_\varepsilon(q^\varepsilon)\}$ exists, whenever $\varepsilon < \bar{\varepsilon}$. Maintaining the same notations as in the proof of Proposition 1, set

$$Q := \{q \in H_c \oplus \mathbf{R}^+ w : \min_t [d_i(t)] > -1,\ \max_t [d_i(t)] < 1\ \forall i\}\ ;$$

note that $Q_j \subset Q_k \subset Q$ for all $j < k$ and that the set $Q^+ = \{q \in Q,\ J_0(q) \geq 0\}$ is compact; let k_ε be the smallest integer such that $J_\varepsilon \leq 0$ on $\partial Q_{k_\varepsilon}$, then

$$\inf_{\gamma \in \Gamma} \max_{q \in Q_{k_\varepsilon}} J_\varepsilon(\gamma(q)) \leq \max_{q \in Q^+} J_\varepsilon(q) \leq \max_{q \in Q^+} J_0(q)\ ,$$

therefore the sequence $\{J_\varepsilon(q^\varepsilon)\}$ is upper bounded. An upper bound of $\|q^\varepsilon\|$ can be obtained as in Lemma 4.2 in [4]; if $\|q^\varepsilon\| \to 0$, then $J_\varepsilon(q^\varepsilon) \to 0$, therefore $\|q^\varepsilon\|$ is lower bounded as well. The other estimates follow directly from the estimates on $J_\varepsilon(q^\varepsilon)$ and $\|q^\varepsilon\|$. $\square$

By Lemma 1, up to a subsequence, we have $q^\varepsilon \rightharpoonup q \in \bar{\Lambda}_1$ and $T_\varepsilon \to T > 0$; let $d_i^\varepsilon(t) = q_{i-1}^\varepsilon(t) - q_i^\varepsilon(t)$ and note that $d_i^\varepsilon \to d_i$ uniformly. If $q \notin \partial\Lambda_1$, by standard techniques we infer the strong convergence of q^ε to q (since $\varepsilon U_i'(d_i^\varepsilon) \to 0$ uniformly) and the existence of a periodic solution at energy E follows. Otherwise we prove the existence of a generalized solution: by the same device as in [9] we can prove

Proposition 2. *For all $\varepsilon > 0$ let q^ε be the critical point of J_ε given by Proposition 1 and let q denote the weak limit of the sequence $\{q^\varepsilon\}$, then*

(i) *$q^\varepsilon \to q$ up to a subsequence*

(ii) *there exist N finite Borel measures $\mu_i(d_i)$ on $[0,1]$ $(i = 1, \dots, N)$ such that $\varepsilon U_i'(d_i^\varepsilon) \to \mu_i(d_i)$ in the weak* measure topology, up to a subsequence*

(iii) *$B_i := \operatorname{supp}[\mu_i(d_i)] \subset \{t \in [0,1];\ d_i(t) = \pm 1\}$ for all $i = 1, \dots, N$*

(iv) *if $\mu(q) := \sum_i \mu_i(d_i)$ and $B(q) := \bigcup_i B_i$ then the equation $T^{-2}\ddot{q} = -\Phi'(q) - \mu(q)$ is satisfied in distributional sense, i.e.,*

$$\int_0^1 \dot{q}\dot{p}\ dt - \int_0^1 \Phi'(q)p\ dt = \int_{B(q)} p\ d\mu(q) \qquad \forall p \in C_1^\infty$$

(v) *after the time scaling $\bar{q}(t) = q(t/T)$ we have*

$$\frac{1}{2}\sum_{i=1}^N \dot{\bar{q}}_i^{\,2}(t) + \sum_{i=1}^N V_i[\bar{d}_i(t)] = E \qquad \text{for all} \quad t \in [0,T] \setminus B(q)\ .$$

Proof. (i) By Lemma 1 we have $|\dot{q}_i^\varepsilon(t)|^2 \leq c$ and $V_i[d_i^\varepsilon(t)] \leq c$ for all t, hence $d_i^\varepsilon(t)$ is far from the asymptotes of V_i (if $V_i \notin C^2[-1,1]$), and $V_i'(d_i^\varepsilon)$ is also uniformly bounded. Next note that

$$J_\varepsilon'(q^\varepsilon)[q^\varepsilon] = \int_0^1 |\dot{q}^\varepsilon|^2\ dt - \int_0^1 \sum_i V_i'(d_i^\varepsilon)d_i^\varepsilon\ dt - \varepsilon\int_0^1 \sum_i U_i'(d_i^\varepsilon)d_i^\varepsilon\ dt = 0\ ;$$

therefore, $\varepsilon\int_0^1\sum_i U_i'(d_i^\varepsilon)d_i^\varepsilon$ is uniformly bounded as well: by (7) we infer that

$$\exists K>0 \qquad \varepsilon\|U_i'(d_i^\varepsilon)\|_1\le K\ . \tag{8}$$

As q^ε is a critical point of J_ε, then by well-known results $q^\varepsilon\in C^2([0,1],\mathbf{R}^N)$ and for all $t\in[0,1]$ we have

$$\begin{cases} T_\varepsilon^{-2}\ddot q_i^\varepsilon(t)=V_i'[d_i^\varepsilon(t)]+\varepsilon U_i'[d_i^\varepsilon(t)]-V_{i+1}'[d_{i+1}^\varepsilon(t)]-\varepsilon U_{i+1}'[d_{i+1}^\varepsilon(t)] \\ \qquad\qquad i=1,\dots,N-1 \\ T_\varepsilon^{-2}\ddot q_N^\varepsilon(t)=V_N'[d_N^\varepsilon(t)]+\varepsilon U_N'[d_N^\varepsilon(t)] \end{cases};$$

hence, by (8), $\{\ddot q_i^\varepsilon\}$ is bounded in L^1 and $q^\varepsilon\to q$ in H_1, up to another subsequence, by the compact imbedding $W^{2,1}\subset H^1$.

(ii) By (8) the sequence $\{\varepsilon U_i'(d_i^\varepsilon)\}$ (as a sequence of measures) converges weakly* to a measure $\mu_i(d_i)$, up to a subsequence.

(iii) The measure $\mu_i(d_i)$ has support in $\{t\in[0,1];\ d_i(t)=\pm1\}$; indeed, $\forall\bar t\notin B_i$ there exists a neighborhood $\mathcal{V}(\bar t)$ such that $\varepsilon U_i'(d_i^\varepsilon)\to0$ uniformly in $\mathcal{V}(\bar t)$.

(iv) follows from (i) and (iii) by taking the limit for $\varepsilon\to0$ in the equation $J'(q^\varepsilon)[p]=0$.

(v) For all i there exists a function w_i such that for almost every t we have $\varepsilon U_i[d_i^\varepsilon(t)]\to w_i(t)$, because $\{\varepsilon U_i[d_i^\varepsilon(t)]\}$ is bounded in L^∞. We claim that $w_i(t)=0$ for almost every t; indeed, let d_i be the uniform limit of $\{d_i^\varepsilon\}$: if $d_i(t)\ne\pm1$, then $\{U_i[d_i^\varepsilon(t)]\}$ is bounded and $\varepsilon U_i[d_i^\varepsilon(t)]\to0$, therefore $w_i(t)>0$ implies either $d_i(t)=-1$ or $d_i(t)=1$. Let $E_i\subset[0,1]$ be the set where $w_i(t)>0$: by the definition of U_i, the uniform convergence of $\{d_i^\varepsilon\}$ to d_i and the previous observation we infer that for ε small and $t\in E_i$ either

$$\varepsilon U_i'[d_i^\varepsilon(t)]=-\frac{2\varepsilon U_i[d_i^\varepsilon(t)]}{d_i^\varepsilon(t)+1}\quad\text{or}\quad \varepsilon U_i'[d_i^\varepsilon(t)]=\frac{2\varepsilon U_i[d_i^\varepsilon(t)]}{1-d_i^\varepsilon(t)}$$

and therefore $|\varepsilon U_i'[d_i^\varepsilon(t)]|\to+\infty$ for all $t\in E_i$. If the Lebesgue measure of E_i is positive, we contradict (8) by Fatou's Lemma.

By Lemma 1, $F_\varepsilon\to F>0$; furthermore, $V_i[d_i^\varepsilon(t)]+\varepsilon U_i[d_i^\varepsilon(t)]\to V_i[d_i(t)]$ and $\dot q^\varepsilon(t)\to\dot q(t)$ for all i and almost every t, therefore q satisfies

$$T^{-2}\sum_i\dot q_i^2(t)+\sum_i V_i[d_i(t)]=F\quad\text{for a.e.}\quad t\in[0,1]$$

and (v) follows by redefining $|\dot q|$ on an negligible set. $\square$

Note that $d_i\in H^1[0,1]$ and therefore the kinetic energy is almost everywhere equal to a continuous function. To represent q we can modify it on a set of Lebesgue measure 0 to obtain $q\in C[0,1]$ and similarly, we can modify $\dot q^2$ to obtain $\dot q^2\in C[0,1]$. This fact shows that if two particles bounce there is conservation of the energy; i.e., the bounce is elastic.

We have $q \in C^2[0,T] \setminus B(q)$ because $\text{supp}\mu(q) = B(q)$. If $t \in B_i \setminus \bigcup_{j\neq i} B_j$, then the "reflection law" ((iv) in Definition 1) follows from the reflection law of Benci-Giannoni (see (2.10) in [9]); if $t \in B_i \cap B_j$ with $j \neq i$, it follows from Proposition 2 (iv).

To complete the proof of Theorem 1 it remains to prove that if the potentials V_i are bounded and E is large enough the solution found above has at least one bounce. Assume the converse, then (5) holds pointwise on $[0,1]$. Multiply it by q, integrate over $[0,1]$ and take into account (6) to obtain

$$2\big[E - \int_0^1 \Phi(q)\big] = \int_0^1 \Phi'(q)q \ ;$$

therefore, $2E = \sum_i \int_0^1 [V_i'(d_i)d_i + 2V_i(d_i)]$ and we get a contradiction if $E > E_0$ where

$$E_0 = \sum_{i=1}^N \big(\frac{1}{2}\max_{[-1,1]} |V_i'| + \max_{[-1,1]} V_i\big) \ .$$

On the other hand, if V_i is unbounded in $\mathcal{U}(-1)$ and in $\mathcal{U}(1)$ for all i, the generalized solution of (1) previously obtained is a classical solution as $V_i[d_i(t)] \leq E$ for all i and $t \in [0,1]$.

4. The prescribed period problem. In this section we prove Theorem 2; without restrictions we assume that $a_i = -1$, $b_i = 1$ and $V_i \geq 0$ for all $i = 1, \dots, N$. We first prove the existence of a classical T-periodic solution of (1) with a different assumption on the potentials: for all $i = 1, \dots, N$ assume that

$$\begin{cases} V_i \in C^2(-1,1) \quad \exists \alpha_i, \beta_i > 0 \quad \text{such that} \\ x \leq -\frac{1}{2} \Rightarrow V_i(x) = \frac{\alpha_i}{(x+1)^2} \quad \text{and} \quad x \geq \frac{1}{2} \Rightarrow V_i(x) = \frac{\beta_i}{(x-1)^2} \ . \end{cases} \tag{T$'$}$$

Consider the functional $I : \Lambda_T \to \mathbf{R}$ defined by

$$I(q) = \frac{1}{2}\int_0^T |\dot{q}|^2 \ dt - \int_0^T \Phi(q) \ dt \ ;$$

by standard arguments one infers that the functional I is of class $C^2(\Lambda_T, \mathbf{R})$ and its first derivative in $q \in \Lambda_T$ is the linear operator $I'(q)$ defined by

$$I'(q)[p] = \int_0^T \dot{q}\dot{p} \ dt - \int_0^T \Phi'(q)p \ dt \qquad \forall p \in H_T \ .$$

Moreover, the functional I satisfies the Palais-Smale condition:

Lemma 2. *Assume* (T′)*; then for all* $T>0$ *the corresponding functional* I *satisfies the Palais-Smale condition in* Λ_T*.*

Proof. Let $\{q^n\}\subset\Lambda_T$ be a Palais-Smale sequence for I; we claim that $\{q^n\}$ is bounded. By contradiction, if $\|q^n\|\to\infty$ then, as $I(q^n)$ is bounded, we have

$$\int_0^T \Phi(q^n)\asymp \|q^n\|^2\ ,$$

where the symbol $\asymp$ means that the two quantities have the same rate of growth at infinity for $n\to\infty$; moreover, as $I'(q^n)[q^n]=\varepsilon_n\|q^n\|$ (with $\varepsilon_n\to 0$ as $n\to\infty$) we also have

$$\int_0^T \Phi'(q^n)q^n\asymp \|q^n\|^2\ ,$$

and finally,

$$\int_0^T \Phi(q^n)\asymp \int_0^T \Phi'(q^n)q^n\ . \tag{9}$$

As $\int_0^T\Phi(q^n)\to\infty$, we know that for n large enough there exist $i_n\in\{1,\dots,N\}$ and $t_n\in[0,T]$ such that $K_n:=\min[d^n_{i_n}(t_n)+1,1-d^n_{i_n}(t_n)]<\frac{1}{2}$ and $K_n\to 0$ as $n\to\infty$. We can make the reasoning that follows for each index i_n satisfying the above property. Let

$$E_n^+:=\{t\in[0,T]:d^n_{i_n}(t)>\frac{1}{2}\}\quad E_n^-:=\{t\in[0,T]:d^n_{i_n}(t)<-\frac{1}{2}\}$$
$$E_n^o:=\{t\in[0,T]:d^n_{i_n}(t)\in[-\frac{1}{2},\frac{1}{2}]\}\ ;$$

by the previous remark, $|E_n^+\cup E_n^-|>0$. Note that $(C_k>0)$

$$\int_0^T V_{i_n}(d^n_{i_n})=\int_{E_n^+}+\int_{E_n^-}+\int_{E_n^o}\le C_1\Big[\int_{E_n^+}\frac{1}{(1-d^n_{i_n})^2}+\int_{E_n^-}\frac{1}{(1+d^n_{i_n})^2}\Big]+C_2$$

and similarly

$$\int_0^T V'_{i_n}(d^n_{i_n})d^n_{i_n}\ge C_1\Big[\int_{E_n^+}\frac{1}{(1-d^n_{i_n})^3}+\int_{E_n^-}\frac{1}{(1+d^n_{i_n})^3}\Big]-C_3\ .$$

Hence, by Hölder's inequality, we obtain $(C_k>0)$

$$\int_0^T V_{i_n}(d^n_{i_n})\le C_4\Big\|\frac{1}{d^n_{i_n}-1}\Big\|^2_{L^3(E_n^+)}+C_4\Big\|\frac{1}{d^n_{i_n}+1}\Big\|^2_{L^3(E_n^-)}+C_2\ ;$$

moreover,

$$\int_0^T V'_{i_n}(d^n_{i_n})d^n_{i_n}\ge C_1\Big\|\frac{1}{d^n_{i_n}-1}\Big\|^3_{L^3(E_n^+)}+C_1\Big\|\frac{1}{d^n_{i_n}+1}\Big\|^3_{L^3(E_n^-)}-C_3$$

which contradicts (9) and proves that $\|q^n\|$ is bounded. Therefore, $\exists q \in H$ such that, up to a subsequence, $q^n \rightharpoonup q$; by standard arguments one can deduce that the convergence $q^n \to q$ is in the norm topology. Moreover, $q \in \Lambda_T$; indeed, recall that a sequence $\{q^n\}$ such that $q^n \rightharpoonup q \in \partial\Lambda_T$ has the property that $\int_0^T \Phi(q^n) \to +\infty$.

Remark. The previous result can be easily extended to potentials of the kind

$$V_i(x) = \frac{\alpha_i}{(x+1)^2} + \frac{\beta_i}{(x-1)^2} + \omega_i(x) \ , \tag{10}$$

where $\omega_i \in C^2[-1,1]$ is such that $V_i(x) \geq 0$ in $[-1,1]$.

The proof of Theorem 2 is also obtained as in the prescribed energy case: only a minor change in the definition of the strong force term is needed. Define

$$U_i(x) = \begin{cases} \frac{1}{(x+1)^2} & \text{for } -1 < x \leq -3/4 \\ \frac{1}{(x-1)^2} & \text{for } 3/4 \leq x < 1 \\ 0 & \text{for } -1/2 \leq x \leq 1/2 \\ C^2\text{-continuation} & \text{elsewhere ,} \end{cases}$$

with the C^2-continuation satisfying (7). From now on we assume that (T) holds. Consider the potential

$$\Phi_\varepsilon(q) := \sum_{i=1}^{N} [V_i(d_i) + \varepsilon U_i(d_i)]$$

and the functional $I_\varepsilon : \Lambda_T \to \mathbf{R}$ defined by

$$I_\varepsilon(q) = \frac{1}{2}\int_0^T |\dot{q}|^2 \ dt - \int_0^T \Phi_\varepsilon(q) \ dt \ ;$$

for all $\varepsilon > 0$ the potential Φ_ε satisfies the assumptions of Lemma 2 (in fact it is of the kind of (10)). By using similar arguments as in [9] we can prove:

Lemma 3. *There exists $\bar{T} > 0$ such that if $T < \bar{T}$ then for all $\varepsilon > 0$ the functional I_ε admits a critical point q^ε. Moreover, there exists $\bar{\varepsilon} > 0$ such that if $\varepsilon < \bar{\varepsilon}$ then the following estimates hold* ($c_i > 0$ *are independent of* ε) :

$$c_1 \leq I_\varepsilon(q^\varepsilon) \leq c_2 \qquad c_3 \leq \|q^\varepsilon\| \leq c_4 \qquad E(q^\varepsilon) \leq c_5 \ .$$

Proof. We proceed as in the proof of Proposition 1, the only differences being the facts that $I_\varepsilon|_{H_c} \leq 0$ (instead of $I_\varepsilon|_{H_c} \equiv 0$) and that in order to satisfy the hypotheses of the linking theorem with the functional I_ε we need the period T to be small enough; see Proposition 3.4 in [9]. Let

$$M := \max_{[-1,1]} \Big[\sum_{i=1}^{N} V_i(x)\Big] \ , \qquad M' := \sum_{i=1}^{N} \max_{[-1,1]} |V_i'(x)| \ ;$$

choose
$$\bar{T} = \frac{1}{4\sqrt{2(M+M')}} \tag{11}$$
and assume that $T < \bar{T}$. Note that
$$\|q\|_\infty \leq \frac{1}{4} \Rightarrow \|d_i\|_\infty \leq \frac{1}{2} \Rightarrow \sum_{i=1}^{N} U_i[d_i(t)] \equiv 0\ ;$$
hence, by Wirtinger's inequality, for all $q \in H_0$ such that $\|q\| \leq \frac{1}{4\sqrt{T}}$ we have $\Phi_\varepsilon[q(t)] = \Phi[q(t)]$ pointwise and for all $\varepsilon > 0$. Therefore, for all $q \in H_0$ such that $\|q\| = \frac{1}{4\sqrt{T}}$ we have
$$I_\varepsilon(q) = \frac{1}{2}\int_0^T |\dot{q}|^2 - \int_0^T \Phi(q) \geq \frac{1}{32 \cdot T} - T \cdot M = \alpha > 0.$$
If we take the same set Q_k as in the proof of Proposition 1 and we make use of the same variational characterization we obtain $q^\varepsilon \in \Lambda_T$ such that $I'_\varepsilon(q^\varepsilon) = 0$; moreover, $I_\varepsilon(q^\varepsilon) > \alpha > 0$ for all $\varepsilon > 0$ and by reasoning as in the proof of Lemma 1 we obtain $c_2 < \infty$ such that $I_\varepsilon(q^\varepsilon) \leq c_2$. As in the proof of Lemma 2, if $\|q^\varepsilon\| \to \infty$ we contradict the fact that $\int_0^T \Phi_\varepsilon(q^\varepsilon) \asymp \int_0^T \Phi'_\varepsilon(q^\varepsilon)q^\varepsilon$; therefore the existence of c_4 is proved. The existence of c_3 follows directly from $I_\varepsilon(q^\varepsilon) > \alpha$. Finally, the above estimates yield $c_5 > 0$ such that $E(q^\varepsilon) \leq c_5$ by integrating over $[0,T]$ the identity $E(q^\varepsilon) = \frac{1}{2}|\dot{q}^\varepsilon(t)|^2 + \Phi_\varepsilon[q^\varepsilon(t)]$. □

If $T < \bar{T}$, we know that $\exists q$ such that, up to a subsequence, $q^\varepsilon \rightharpoonup q$ in H_T and $q^\varepsilon \to q$ uniformly when $\varepsilon \to 0$, by the compact imbedding $H^1 \subset L^\infty$. We claim that q is a nontrivial generalized T-periodic solution of (1).

Proposition 3. *Assume that $T < \bar{T}$ and for all $\varepsilon > 0$ let q^ε be the critical point of I_ε given by Lemma* 3*; then*

(i) *$q^\varepsilon \to q$ up to a subsequence;*

(ii) *there exist N finite Borel measures $\mu_i(d_i)$ on $[0,T]$ $(i = 1, \dots, N)$ such that $\varepsilon U'_i(d_i^\varepsilon) \to \mu_i(d_i)$ in the weak* measure topology, up to a subsequence;*

(iii) *$B_i := \operatorname{supp}[\mu_i(d_i)] \subset \{t \in [0,T];\ d_i(t) = \pm 1\}$ for all $i = 1, \dots, N$;*

(iv) *if $\mu(q) := \sum_i \mu_i(d_i)$ and $B(q) := \bigcup_i B_i$ then the equation $\ddot{q} = -\Phi'(q) - \mu(q)$ is satisfied in distributional sense; i.e.,*
$$\int_0^T \dot{q}\dot{p}\ dt - \int_0^T \Phi'(q)p\ dt = \int_{B(q)} p\ d\mu(q) \qquad \forall p \in C_T^\infty$$

(v) *there exists $E > 0$ such that*
$$\frac{1}{2}\sum_{i=1}^{N} \dot{q}_i^2(t) + \sum_{i=1}^{N} V_i[d_i(t)] = E \qquad \text{for all} \quad t \in [0,T] \setminus B(q)$$
and therefore the kinetic energy of the system is a continuous function with respect to t.

The proof of the above result can be obtained as the proof of Proposition 2, with minor changes.

Next, remark that in the proof of Lemma 3 we saw that $I_\varepsilon(q^\varepsilon) > \alpha$ which implies $\frac{1}{2}\int_0^T |\dot{q}^\varepsilon|^2 > \alpha$; note also that if $T < \bar{T}$ ($\bar{T}$ as in (11)) then $\alpha > T\cdot M' \geq \int_0^T \Phi'(q^\varepsilon)q^\varepsilon$; therefore, as $q^\varepsilon \to q$,

$$\frac{1}{2}\int_0^T |\dot{q}|^2 \geq \int_0^T \Phi'(q)q \ . \tag{12}$$

Assume that q is a classical solution of (1). Then we can multiply (2) by q and integrate by parts to contradict (12); we conclude that if $T < \bar{T}$ then the solution q has at least one bounce.

To obtain an upper bound of the number of bounces we should restrict to "small" periods T (i.e., $T < \bar{T}$) and proceed as in [9]; however, as remarked in Section 2 we do not wonder here about this upper bound (which is in fact $N+1$). Therefore, a T-periodic generalized solution of (1) exists for all $T > 0$ because a T-periodic solution is also a kT-periodic solution for all $k \in \mathbf{N}$; if $T \in [k\bar{T}, (k+1)\bar{T})$, such a solution has at least k and at most $k(N+1)$ bounces. Note that if $V_i \equiv 0$ for all i, then $\bar{T} = +\infty$.

5. Some billiards with corners. We start this section with the

Proof of Theorem 3. This proof can be obtained by arguments similar to those already used in Sections 3 and 4. Consider the equation

$$\gamma''(t) = -\mathbf{V}'[\gamma(t)] \ ; \tag{13}$$

denote by V_i' (respectively γ_i) the components of $\mathbf{V}'$ (respectively γ) in the canonical basis of $\mathbf{R}^N$, then we can project equation (13) onto the 1-dimensional subspaces of $\mathbf{R}^N$ obtaining a system similar to (1):

$$\gamma_i''(t) = -V_i'[\gamma(t)] \qquad i = 1, \dots, N \ . \tag{14}$$

To each equation of (14) add a strong force term $\varepsilon U_i'[\gamma_i(t)]$ and prove that for all $\varepsilon > 0$ this system admits a classical solution γ^ε. Next, let $\varepsilon \to 0$; reasoning as for Proposition 2 we can prove that $\varepsilon U_i'(\gamma_i)$ converges weakly* to a measure $\mu_i(\gamma_i)$ whose support is contained in the set $\{t \in [0,T];\ \gamma_i(t) \in \{a_i, b_i\}\}$. The limit equation is then satisfied in the distributional sense; by adding the measures and by the results of [9] we see that the "reflection law" ((iv) in Definition 2) is fulfilled: the measure that describes the bounce in a corner is the sum of the measures which describe the bounces on all the positive sides of Ω to which such corner belongs. $\square$

Let us make an attempt to explain why a general treatment of the billiard ball problem seems not to be possible if the boundary of the billiard is only Lipschitz continuous. In case of Lipschitz-continuous boundary $\partial\Omega$ one can prove that the bounce is elastic (see [19]) but, generally, one cannot describe the reflection law. For simplicity assume that $\mathbf{V} \equiv 0$. To understand the reflection law of a bounce in a corner we approximate the incidence trajectory by means of parallel ones and we figure out how these approximating bounces behave. With this approximating procedure, we find some kind of indeterminacy. A discontinuity with respect to the bounce point appears in the reflection law; see also [22]. However, in 2D domains, if the measure θ of the angle is a submultiple of the angle $\frac{\pi}{2}$ ($\theta = \frac{\pi}{2n}$, $n \in \mathbf{N}$, $n \geq 1$) such discontinuity is eliminable; the reflected trajectory of a trajectory hitting the corner has $2n$ reflecting components (only $2n - 1$ if the trajectory is tangent to one of the sides). If $\theta = \frac{\pi}{2n}$, to eliminate the discontinuity the bounce in a corner must be described by an inversion of the velocity; that is, if τ is the bounce instant and $\gamma'(t)$ is the velocity at time t then $\gamma'(\tau^+) = -\gamma'(\tau^-)$. As we will see, for other kinds of angles the discontinuity is not eliminable. For more general problems about billiards see also [12, 16] and references therein.

On the other hand, if one wishes to describe the bounce in a corner by a procedure involving approximating domains, we believe that the limit of this procedure depends on the approximating sequence. More precisely, approximate a corner (i.e., a function of the kind $f(x) = \alpha|x|$ for some $\alpha > 0$) by a sequence $\{f_n\}$ of C^2-functions such that $f_n \to f$ uniformly; for every $n \in \mathbf{N}$ it is possible to describe the reflection law by means of the results of [9]. The limit reflection law depends on the functions f_n considered.

We deal with a 2D domain having a corner and we study only three particular measures of the angle θ as they are quite illustrative. Obviously different kinds of problems appear for other measures of the angle θ and in higher dimensions.

• In case of an inward angle, i.e., a nonconvex domain as in Figure 1, the Cauchy problem seems to be ill-posed (the existence of a solution is guaranteed for smooth domains Ω by Remark 2.14 in [9]); it looks difficult to obtain a generalized solution by an approximating procedure. Indeed, hit the angle with trajectories a and b and in both cases reach point C. It is impossible to determine the bounce trajectory starting from C and hitting the inward corner; here the discontinuity is of the second kind.

• The case of an obtuse angle has a different problem; hit the corner of Figure 2 with a trajectory having an angle of width $\frac{\pi}{3}$ with side a (and b). If we approximate such trajectory by trajectories hitting before side a (Figure $2'$) we obtain no reflection component from side b while if we approximate it with trajectories hitting before side b (Figure $2''$) we obtain the opposite result. In this case we have a discontinuity of first kind.

• In the case of Figure 3, the same arguments as above lead to the following discontinuity: if we approximate the trajectory which hits the vertex along the bisecting line by hitting first side a we obtain an exit trajectory parallel to a while the converse holds if we first hit side b.

To conclude this paper, we introduce two models that involve different kinds of corners and for which it seems difficult to describe the reflection law.

• If instead of fixing only one endpoint of the lattice we fix both, i.e. $q_0 \equiv q_{N+1} \equiv 0$, a different model is obtained; indeed, this assumption gives the system a certain rigidity, described by

$$\sum_{i=0}^{N}[q_i(t) - q_{i+1}(t)] = 0 \ .$$

Let $d_i(t) := q_{i-1}(t) - q_i(t)$ and $\mathbf{V}(d) = \sum_{i=1}^{N+1} V_i(d_i)$; then the potential $\mathbf{V}$ is defined on the set

$$\Omega := \Big\{ d = (d_1, \dots , d_{N+1}) \in (a_1, b_1) \times \cdots \times (a_{N+1}, b_{N+1});\ \sum_{i=1}^{N+1} d_i = 0 \Big\} \subset \mathbf{R}^{N+1} \ .$$

Consider the simple case $N = 2$ with $a_i = -1$, $b_i = 1$ $(i = 1, 2, 3)$. Here $\Omega \subset \mathbf{R}^3$ becomes the plane regular hexagon whose vertices are the points $(\pm1, \mp1, 0)$, $(\pm1, 0, \mp1)$ and $(0, \pm1, \mp1)$. Hence, we deal with angles whose amplitude is $\frac{2\pi}{3}$ for which we have already remarked above it seems hard to describe the reflection law. On the other hand, the natural constraint in the $\{q_1, q_2\}$-plane can be represented by an irregular hexagon; see Figure 4: the two right angles (for which we can describe the reflection law) correspond to the double bounce of q_1 and q_2 against their barrier with respect to the two different endpoints while the other four angles (for which it is not possible to describe such law) correspond to the double bounce of q_1 and q_2 against the barrier of the same endpoint.

• Another interesting model is that of a set of N marbles which can roll on a closed guide-rail and cannot swap their positions: each marble can bounce against the two adjacent neighbors in an elastic way. We identify such marbles with a material point and we assume that the endpoints A and B correspond to the coordinates 0 and 1 on the real line; see Figure 5; denote by x_i the coordinate of the i-th marble; then the natural constraint of this system is $0 \le x_1 \le \cdots \le x_N \le 1$. Consider the particular case $N = 2$, then such constraint can be represented in the $\{x_1, x_2\}$-plane by means of a right-angled isosceles triangle; see Figure 6. Also in this case the right angle (the good one) represents the double bounce of x_1 against A and of x_2 against B. The two other corners are "bad" (acute) angles and correspond to the double bounce of x_1 and x_2 against the same endpoint. In this case the acute angle is an even submultiple of π and one can still describe the bouncing law; the case of $N \ge 3$ marbles seems much more complicated.

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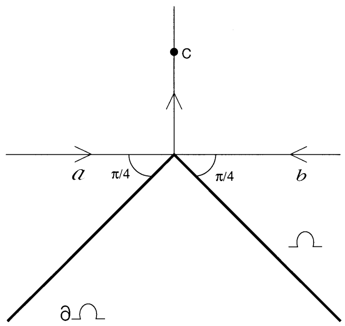


Figure 1. $(\theta = \frac{3\pi}{2})$

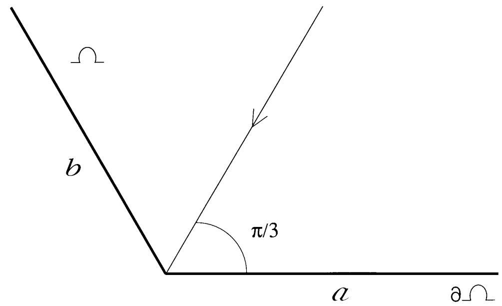
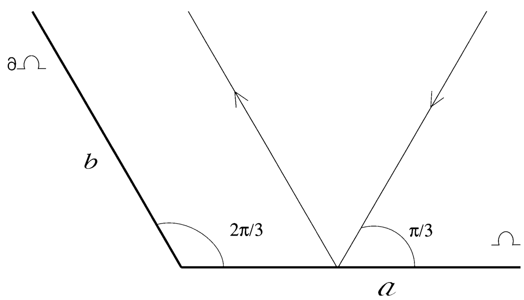


Figure 2. $(\theta = \frac{2\pi}{3})$



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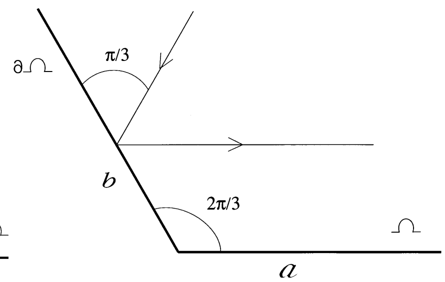

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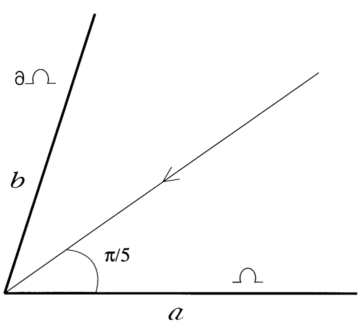
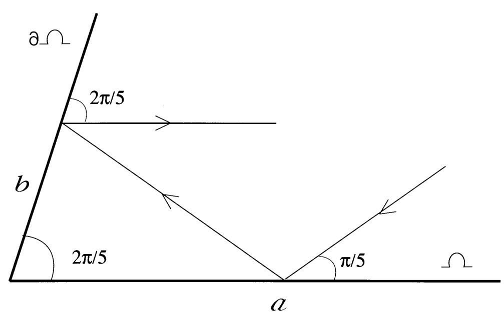


Figure 3. $(\theta = \frac{2\pi}{3})$



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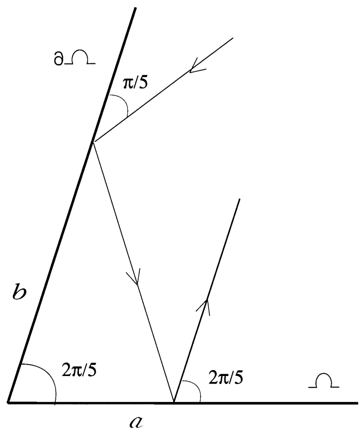

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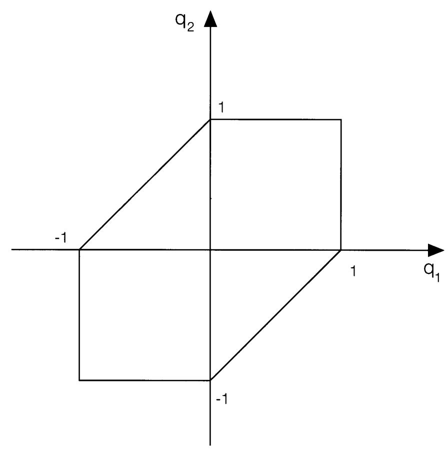

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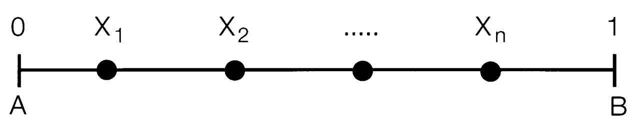

Figure 5.

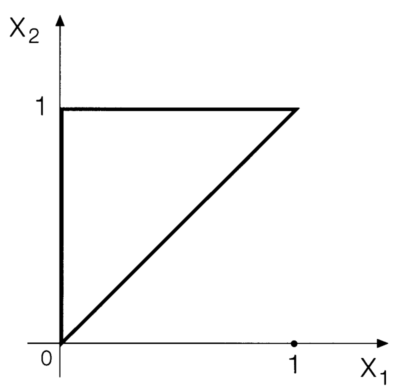

Figure 6.