Differential and Integral Equations Volume 10, Number 2, March 1997, pp. 201 – 217.

H_∞-CALCULUS FOR ELLIPTIC OPERATORS WITH NONSMOOTH COEFFICIENTS*

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(Submitted by: Herbert Amann)

Abstract. We show that general systems of elliptic differential operators have a bounded H_∞-functional calculus in L_p spaces, provided the coefficients satisfy only minimal regularity assumptions.

1. Introduction. In this paper, we consider general systems of elliptic operators on $\mathbb{R}^n$ and on closed manifolds. We prove the existence of a bounded holomorphic functional calculus in L_p spaces under minimal regularity assumptions on the coefficients. In fact, we shall show that the same conditions guaranteeing the fundamental resolvent estimates are also sufficient for proving the existence of a bounded functional calculus. This provides a considerable improvement on recent results contained in [2], [14]. Results under weak regularity assumptions have important applications in the field of nonlinear partial differential equations; see the introduction in [2], [14].

Our approach makes use of results and techniques from harmonic analysis, e.g. Caldéron-Zygmund theory, and the $T(1)$-theorem for singular integral operators. We rely on the well-known paper of David and Journé ([6]); see in particular Sections V and VI.

The existence of a bounded functional calculus—and the weaker property of bounded imaginary powers—for elliptic operators has been investigated by many authors, using different methods.

In [15], see also [16], complex powers of elliptic operators on compact closed manifolds have been studied. In [17], general systems of boundary value problems are considered and the author proves the boundedness of imaginary powers. This result was later extended by [8] to give a bounded holomorphic functional calculus. Both authors work in the C^∞-category and their methods rely on the theory of

Received for publication December 1995.
*The first author was supported by a Macquarie University Research Fellowship. The second author was supported by the Swiss National Foundation.
AMS Subject Classifications: 47F05, 47A60, 35J45, 42B20.

pseudo-differential operators and on careful estimates on the expansion of the resolvent.

In [2], general systems of elliptic operators on $\mathbb{R}^n$ and on compact closed manifolds are studied. A bounded holomorphic calculus is obtained, provided that the coefficients are Hölder continuous. The authors use pseudo-differential operators, the technique of symbol-smoothing, and localization and perturbation arguments. However, in the context of partial differential equations, the case of Hölder-continuous coefficients is still considered as the "smooth" case.

If we stay with scalar second-order elliptic operators in divergence form, a bounded holomorphic functional calculus can be proved under weak regularity assumptions on the coefficients; see [10]. The results apply to elliptic operators on $\mathbb{R}^n$ and also to boundary value problems on bounded domains with Dirichlet or Neumann boundary conditions. In this case, a functional calculus of Hörmander type can be obtained as well; see [10] and [9].

It has been shown in [14] that scalar second-order elliptic operators in non-divergence form have bounded imaginary powers. The authors assume that the top-order coefficients are Hölder continuous and they impose conditions at infinity. Elliptic operators on $\mathbb{R}^n$ and also boundary value problems with Dirichlet conditions on bounded and unbounded domains are considered. Their approach relies on a general perturbation theorem, on commutator estimates, and on localization arguments.

In this paper we can show that the restrictive commutator condition contained in the perturbation result [14] Proposition 3.1 is not needed, when applied to differential operators on $\mathbb{R}^n$. The arguments in [14] combined with our new perturbation result show that the Hölder condition imposed in [14] can be removed.

It has been proved in [6], by using the $T(1)$-theorem and the method of multilinear expansion, that small perturbations of $-\Delta$ on $\mathbb{R}^n$ have a bounded holomorphic calculus. We will use the same methods to prove that small perturbations of general systems of differential operators with constant coefficients on $\mathbb{R}^n$ have a bounded holomorphic calculus.

In Section 2 we review the functional calculus introduced by McIntosh ([13]). Section 3 deals with elliptic systems with constant coefficients. We provide some modifications of results contained in [2] to cover the case when 0 belongs to the spectrum. It should be observed that we use a very general ellipticity condition for systems. These results are used in Section 4, where we add small perturbations. Theorem 4.2 is our main result and its proof is carried out in Section 5. In Section 6 we prove that general systems of elliptic operators on $\mathbb{R}^n$ have a bounded holomorphic calculus, provided that the top-order coefficients are bounded and uniformly continuous. For getting this result, we use the device of localization and perturbation introduced in [2]. Finally, Section 7 deals with elliptic systems acting on sections of vector bundles over compact closed manifolds.

Acknowledgments. We would like to express our gratitude to Alan McIntosh for valuable suggestions. The second author thanks Michael Christ, Chun Li, and

Joachim Escher for helpful discussions.

2. Operators of type ω and their functional calculus. In this section we review some facts of the functional calculus as introduced by McIntosh ([13]); see also [5], [1]. Let $0 \le \omega < \pi$ be given. Then

$$S_\omega := \{z \in \mathbb{C} : |\arg z| \le \omega\} \cup \{0\}$$

denotes the closed sector of angle ω and S_ω^0 denotes its interior, while $\dot{S}_\omega := S_\omega \setminus \{0\}$. An operator A on some Banach space E is said to be of type ω if A is closed and densely defined, $\sigma(A) \subset S_\omega$, and for each $\theta \in (\omega, \pi]$ there exists a constant C_θ such that

$$|\lambda| \, \|(\lambda I - A)^{-1}\|_{L(E)} \le C_\theta, \quad \lambda \in -\dot{S}_{\pi-\theta}.$$

If $\mu \in (0, \pi]$, then

$$H_\infty(S_\mu^0) := \{f : S_\mu^0 \to \mathbb{C}\,;\, f \text{ is holomorphic and } \|f\|_{H_\infty} < \infty\}, \tag{2.1}$$

where $\|f\|_{H_\infty} := \sup\{|f(z)|\,;\, z \in S_\mu^0\}$. In addition, we define

$$\Psi(S_\mu^0) := \{g \in H_\infty(S_\mu^0)\,;\, \exists\, s > 0\,, \exists\, c \ge 0 : |g(z)| \le c\,\frac{|z|^s}{1 + |z|^{2s}}\}. \tag{2.2}$$

Let $\omega < \theta < \mu$ and let Γ be the oriented contour given by

$$\gamma(t) := \begin{cases} -te^{-i\theta} & \text{for} \quad t < 0, \\ \ \ te^{i\theta} & \text{for} \quad t \ge 0. \end{cases}$$

If A is of type ω and $g \in \Psi(S_\mu^0)$, we define $g(A) \in L(E)$ by

$$g(A) := -\frac{1}{2\pi i} \int_\Gamma (\lambda I - A)^{-1} g(\lambda)\, d\lambda\,. \tag{2.3}$$

If, in addition, A is one-one and has dense range and if $f \in H_\infty(S_\mu^0)$, then

$$f(A) := [h(A)]^{-1}(fh)(A), \tag{2.4}$$

where $h(z) := z(1+z)^{-2}$. It can be shown that $f(A)$ is a well-defined linear operator in E and that this definition is consistent with the previous one for $f \in \Psi(S_\mu^0)$. The definition of $f(A)$ can even be extended to encompass unbounded holomorphic functions; see [13] for details.

Given $N \geq 1$ and $\mu \in (0,\pi]$, we say that an operator A has a *bounded* H_∞-*calculus* if A is of type ω for some $\omega \in [0,\mu)$, if A is one-one and has dense range, and if $f(A) \in L(E)$ with

$$\|f(A)\|_{L(E)} \leq N\|f\|_{H_\infty}, \quad f \in H_\infty(S_\mu^0), \tag{2.5}$$

for some constant $N \geq 1$. We then denote the class of operators A of type ω which satisfy (2.5) by

$$\mathcal{H}_\infty(E;N,\mu).$$

It is evident that (2.5) imposes a serious restriction upon operators A of type ω. Let us add the following very useful observation, which is based on the Convergence Lemma of McIntosh ([13]). It shows that in order to prove (2.5) we can restrict our attention to functions $g \in \Psi(S_\mu^0)$. This has the advantage that we can deal with absolutely convergent Dunford-Taylor integrals; see the definition (2.3).

Lemma 2.1. *Let A be of type ω and assume in addition that A is one-one with dense range. Then there exists $\kappa \geq 1$ such that the following statement holds*: *if*

$$\|g(A)\|_{L(E)} \leq N\,\|g\|_{H_\infty}, \quad g \in \Psi(S_\mu^0) \tag{2.6}$$

for some $N \geq 1$, then $A \in \mathcal{H}_\infty(E;\kappa N,\mu)$.

Proof. We refer to [2] Lemma 2.1; see also [13], [1]. □

We add the following useful perturbation result.

Lemma 2.2. *Let A be of type ω and assume that A is one-one and has dense range. Suppose that $A \in \mathcal{H}_\infty(E;N,\mu)$. Then $sI + A \in \mathcal{H}_\infty(E,N,\mu)$ for all $s \geq 0$.*

Proof. Let $f \in H_\infty(S_\mu)$ be given and set $\tau_s f(z) := f(s+z)$. Then $\tau_s f \in H_\infty(S_\mu)$ for $s \geq 0$ and it can be shown that $\tau_s f(A) = f(sI + A)$. This implies that $sI + A \in \mathcal{H}_\infty(E;N,\mu)$ for all $s \geq 0$.

3. Elliptic operators with constant coefficients. This section is devoted to the study of elliptic systems on $\mathbb{R}^n$ with constant coefficients. We introduce the notion of a homogeneous (M,ω)-elliptic system and we prove that its L_p-realization is of type ω and has a bounded functional calculus for $1 < p < \infty$. It will be important for later purposes to show that certain quantities do not depend on the individual differential operators, but hold uniformly for the class of (M,ω)-elliptic operators. Throughout the remainder of this paper, $H := (H,|\cdot|)$ denotes a finite-dimensional complex Banach space. Let

$$\mathcal{A} := \mathcal{A}(D) := \sum_{|\alpha|=\ell} a_\alpha D^\alpha \tag{3.1}$$

be a homogeneous differential operator of order $\ell \in 2\dot{\mathbb{N}}$ with coefficients $a_\alpha \in L(H)$. We use the notation $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ for each multi-index $\alpha \in \mathbb{N}^n$, and $D_j = -i\partial_j$ for $1 \le j \le n$. Let

$$a(\xi) := \sum_{|\alpha|=\ell} a_\alpha \xi^\alpha, \qquad \xi \in \mathbb{R}^n \tag{3.2}$$

be the principal symbol of $\mathcal{A}$ and note that $a(\xi) \in L(H)$ for each $\xi \in \mathbb{R}^n$. Let $\sigma(a(\xi))$ denote the spectrum, that is, the eigenvalues, of $a(\xi)$. We will now impose an ellipticity condition on $\mathcal{A}$ which is sufficient (and also necessary) to guarantee that its $L_p(\mathbb{R}^n)$-realization is an operator of type ω.

Let $M \ge 1$ and $\omega \in [0, \pi)$ be given. Following [2] Section 7, $\mathcal{A}$ is called (M, ω)-*elliptic* if

$$\sum_{|\alpha|=\ell} |a_\alpha| \le M, \quad \sigma(a(\xi)) \subset \dot{S}_\omega, \quad |a(\xi)^{-1}| \le M, \quad |\xi| = 1. \tag{3.3}$$

Here, $|a(\xi)^{-1}|$ denotes the operator norm of $a(\xi)^{-1} \in L(H)$. We will often write L_p or $L_p(\mathbb{R}^n)$ for the Lebesgue space $L_p(\mathbb{R}^n, H)$ of vector-valued functions on $\mathbb{R}^n$. Analogously, W_p^ℓ and $W_p^\ell(\mathbb{R}^n)$ will stand for the Sobolev space $W_p^\ell(\mathbb{R}^n, H)$ of vector-valued functions on $\mathbb{R}^n$. Let A denote the L_p-realization of the differential operator $\mathcal{A}$, defined on $W_p^\ell(\mathbb{R}^n)$.

For the rest of this section, $p \in (1, \infty)$ is fixed. We are ready to prove the following proposition.

Proposition 3.1. *Let $M \ge 1$ and $\theta \in (\omega, \pi)$ be fixed. Then $-\dot{S}_{\pi-\theta} \subset \rho(A)$ and there is a positive constant c such that*

$$|\lambda|\, \|(\lambda I - A)^{-1}\|_{L(L_p)} \le c, \quad \lambda \in -\dot{S}_{\pi-\theta}, \tag{3.4}$$

for all homogeneous (M, ω)-elliptic differential operators $\mathcal{A}$ with constant coefficients. A is of type ω and A is one-one with dense range.

Proof. Let $\lambda \in -\dot{S}_{\pi-\theta}$ and $\mu(\xi) \in \sigma(a(\xi))$ be given. It is then easy to verify that

$$|\lambda - \mu(\xi)| \ge |\lambda| \sin(\theta - \omega) \quad \text{and} \quad |\lambda - \mu(\xi)| \ge |\mu(\xi)| \sin(\theta - \omega).$$

It follows from (3.3) and the fact that a is positive homogeneous of degree ℓ that

$$\sigma(a(\xi)) \subset S_\omega \cap [|z| \ge |\xi|^\ell / M], \quad \xi \in \mathbb{R}^n.$$

This implies the existence of a constant $r = r(M)$ such that

$$\sigma(\lambda - a(\xi)) \subset [|z| \ge r \sin(\theta - \omega)], \quad (\lambda, \xi) \in -\dot{S}_{\pi-\theta} \times \mathbb{R}^n, \quad |\lambda| + |\xi|^\ell = 1.$$

We deduce from [2] Lemma 4.1 that there exists a constant $c = c(M,\theta)$ such that

$$|(\lambda - a(\xi))^{-1}| \le c, \quad (\lambda,\xi) \in -\dot{S}_{\pi-\theta} \times \mathbb{R}^n, \quad |\lambda| + |\xi|^\ell = 1. \tag{3.5}$$

Next, observe that

$$\big(\lambda - a(\xi)\big)^{-1} = (|\lambda| + |\xi|^\ell)^{-1}\big((|\lambda| + |\xi|^\ell)^{-1}\lambda - a((|\lambda| + |\xi|^\ell)^{-1/\ell}\xi)\big)^{-1},$$

where $(\lambda,\xi) \in -\dot{S}_{\pi-\theta} \times \mathbb{R}^n$. It follows from (3.5) that

$$|(\lambda - a(\xi))^{-1}| \le c\,(|\lambda| + |\xi|^\ell)^{-1}, \quad (\lambda,\xi) \in -\dot{S}_{\pi-\theta} \times \mathbb{R}^n. \tag{3.6}$$

Let $\lambda \in -\dot{S}_{\pi-\theta}$ be fixed. Then $[\xi \mapsto (\lambda - a(\xi))^{-1}] \in C^\infty(\mathbb{R}^n, GL(H))$ and the derivatives $\partial^\beta(\lambda - a)^{-1}$ are given by finite linear combinations of terms of the form

$$(\lambda - a)^{-1}(\partial^{\beta_1} a)(\lambda - a)^{-1} \cdots (\lambda - a)^{-1}(\partial^{\beta_r} a)(\lambda - a)^{-1}, \tag{3.7}$$

where $\beta_1 + \cdots + \beta_r = \beta$ and $\beta_i \in \mathbb{N}^n$. (3.6) and (3.7) yield

$$|\xi|^{|\beta|}|\partial^\beta(\lambda - a(\xi))^{-1}| \le c_\beta\,(|\lambda| + |\xi|^\ell)^{-1}, \quad (\lambda,\xi) \in -\dot{S}_{\pi-\theta} \times \dot{\mathbb{R}}^n, \tag{3.8}$$

where $\beta \in \mathbb{N}^n$ is arbitrary and $c_\beta = c_\beta(M,\theta)$. Let $(\lambda - a)^{-1}(D)$ denote the Fourier multiplier operator with symbol $(\lambda - a)^{-1}$. If follows from (3.8) and Mikhlin's multiplier theorem, see [18], p. 96, that $(\lambda - a)^{-1}(D) \in L(L_p)$ and

$$\|(\lambda - a)^{-1}(D)\|_{L(L_p)} \le c\,|\lambda|^{-1}, \quad \lambda \in -\dot{S}_{\pi-\theta},$$

with $c = c(M,\theta,p)$. Let $\lambda \in -\dot{S}_{\pi-\theta}$ be fixed. We infer from (3.8) and Leibniz's rule that $|\xi|^{|\beta|}|\partial^\beta\big((1 + |\xi|^2)^{\ell/2}(\lambda - a(\xi))^{-1}\big)| \le c_\beta(\lambda)$ for $\xi \in \mathbb{R}^n$ and $\beta \in \mathbb{N}^n$. Hence $(\lambda - a)^{-1}(D) \in L(L_p, W_p^\ell)$ by Mikhlin's multiplier theorem and a well-known characterization of Sobolev spaces. We conclude that

$$(\lambda I - A)^{-1} = (\lambda - a)^{-1}(D) \tag{3.9}$$

and (3.4) is now proved. It remains to show that A is one-one and has dense range. Let us assume that $Au = 0$ for some $u \in W_p^\ell$. It follows that $a\hat{u} = 0$ in $\mathcal{S}'$, the space of tempered distributions, where $\hat{u}$ denotes the Fourier transform of u. Hence, $\hat{u}$ has support contained in $\{0\}$ and we conclude that $u = 0$.

Let $a^*(\xi) := [a(\xi)]^* \in L(H)$ be the transpose of $a(\xi)$. Observe that a^* satisfies the same conditions in (3.3) as a does. A duality argument then shows that A has dense range and the proof is now completed. □

We show that homogeneous (M,θ)-elliptic differential operators with constant coefficients have a bounded H_∞-functional calculus. This result will be used later on for an induction argument.

Proposition 3.2. *Let $M \geq 1$ and $\mu \in (\omega, \pi)$ be fixed. Then there exists a constant $N \geq 1$ such that $A \in \mathcal{H}_\infty(L_p; N, \mu)$ for all homogeneous (M, ω)-elliptic differential operators $\mathcal{A}$ with constant coefficients.*

Proof. Let $\theta \in (\omega, \pi)$ be fixed and let Γ be the contour introduced in Section 2. Given $g \in \Psi(S^0_\mu)$ we set

$$g(a)(\xi) := -\frac{1}{2\pi i} \int_\Gamma g(\lambda)(\lambda - a(\xi))^{-1}\, d\lambda, \quad \xi \in \mathbb{R}^n.$$

It follows from (2.2) and (3.6) that $g(a)(\xi) \in L(H)$ is well defined. Similar arguments as in the proof of [2] Lemma 5.4 show that

$$|\xi|^{|\beta|} |\partial^\beta g(a)(\xi)| \leq c_\beta \, \|g\|_{H_\infty}, \quad \xi \in \dot{\mathbb{R}}^n, \quad \beta \in \mathbb{N}^n. \tag{3.10}$$

Moreover, it is easy to verify that $g(A)$ is a Fourier multiplier operator with symbol $g(a)$. Hence (3.10) and Mikhlin's multiplier theorem yield $\|g(A)\|_{L(L_p)} \leq c\|g\|_{H_\infty}$. Lemma 2.1 then shows that $A \in \mathcal{H}_\infty(L_p; N, \mu)$ for some $N = N(M, \mu, p)$. □

4. A perturbation result. This section is the core of our paper. We will show that small perturbations of homogeneous (M, ω)-elliptic operators with constant coefficients have a bounded H_∞-functional calculus.

We consider homogeneous differential operators with variable coefficients of the form

$$\mathcal{L} = \mathcal{A} + \mathcal{B} := \sum_{|\alpha| = \ell} a_\alpha D^\alpha + \sum_{|\alpha| = \ell} b_\alpha D^\alpha, \tag{4.1}$$

with $\ell \in 2\dot{\mathbb{N}}$, where the coefficients a_α are constant, whereas the coefficients b_α are assumed to be measurable and (essentially) bounded, that is,

$$b_\alpha \in L_\infty(\mathbb{R}^n, L(H)), \qquad \alpha \in \mathbb{N}^n, \quad |\alpha| = \ell.$$

Let

$$\|b\|_\infty := \sum_{|\alpha| = \ell} \|b_\alpha\|_\infty, \tag{4.2}$$

where $\|b_\alpha\|_\infty$ is the essential supremum of $b_\alpha \in L_\infty(\mathbb{R}^n, L(H))$. As in the last section, A, B and L denote the L_p-realizations of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{L}$, respectively, defined on the Sobolev space W^ℓ_p. Let $1 < p < \infty$ be fixed.

Lemma 4.1. *Let $M \geq 1$ and $\theta \in (\omega, \pi]$ be given. Then there are positive constants ε_0 and c such that $-\dot{S}_{\pi-\theta} \subset \rho(L)$ and*

$$|\lambda| \, \|(\lambda I - L)^{-1}\|_{L(L_p)} \leq c, \quad \lambda \in -\dot{S}_{\pi-\theta}, \tag{4.3}$$

for all homogeneous (M, ω)-elliptic differential operators $\mathcal{A}$ with constant coefficients and all differential operators $\mathcal{B}$ with $\|b\|_\infty \leq \varepsilon_0$. Moreover, L is one-one and has dense range.

Proof. Let $\alpha \in \mathbb{N}^n$ with $|\alpha| = \ell$ be given. It follows from (3.8) and Leibniz's rule that

$$|\xi|^{|\beta|}|\partial^\beta\big(\xi^\alpha(\lambda - a(\xi))^{-1}\big)| \le c_\beta, \quad (\lambda,\xi) \in (-S_{\pi-\theta} \times \mathbb{R}^n)^{\cdot}, \tag{4.4}$$

where $\beta \in \mathbb{N}^n$ and $c_\beta = c_\beta(M,\theta)$. Hence there is a constant $\varepsilon_0 = \varepsilon_0(M,\theta,p)$ such that

$$\|B(\lambda I - A)^{-1}\|_{L(L_p)} \le 1/2, \quad \lambda \in -\dot{S}_{\pi-\theta}, \quad \|b\|_\infty \le \varepsilon_0, \tag{4.5}$$

owing to Mikhlin's multiplier theorem. This implies (4.3) by the usual Neumann series argument. Next, observe that

$$D^\alpha u = (k_\alpha a)(D)u = k_\alpha(D)\mathcal{A}(D)u = k_\alpha(D)Au, \quad u \in W_p^\ell,$$

where $k_\alpha(D)$ is a Fourier multiplier with symbol $k_\alpha(\xi) := \xi^\alpha a(\xi)^{-1}$ for $\xi \in \dot{\mathbb{R}}^n$. (4.4) and Mikhlin's multiplier theorem yield $k_\alpha(D) \in L(L_p)$ and $\|k_\alpha(D)\|_{L(L_p)} \le c$. It follows that $\|Bu\| \le 1/2\|Au\|$ for all $u \in W_p^\ell$, provided $\|b\|_\infty \le \varepsilon_0$, with ε_0 small enough. Let us assume that $(A+B)u = 0$ for some $u \in W_p^\ell$. Then

$$\|Au\| = \|(A+B)u - Bu\| = \|Bu\| \le 1/2\|Au\|$$

and we conclude that $Au = 0$. Proposition 3.1 yields $u = 0$. Hence, $L = A + B$ is one-one. Let $K := \sum_{|\alpha|=\ell} b_\alpha k_\alpha(D)$. It follows that $K \in L(L_p)$ with $\|K\|_{L(L_p)} \le 1/2$, and that $BA^{-1} \subset K$, where BA^{-1} is defined on $R(A)$. Next, note that $(I+K)$ is an isomorphism on L_p which is a bijection from $R(A)$ onto $R(A+B)$. Since $R(A)$ is dense in L_p, which has been established in Proposition 3.1, we see that $R(A+B)$ is dense in L_p, too. Thus, L has dense range. □

We are now ready for the main theorem of this section.

Theorem 4.2. *Let $M \ge 1$ and $\mu \in (\omega,\pi]$ be given. Then there are constants $\varepsilon_0 > 0$ and $N \ge 1$ such that*

$$A + B \in \mathcal{H}_\infty(L_p; N, \mu)$$

for all homogeneous (M,ω)-elliptic differential operators $\mathcal{A}$ with constant coefficients and all differential operators $\mathcal{B}$ with $\|b\|_\infty \le \varepsilon_0$.

Proof. We fix $\theta \in (\omega,\mu)$. Lemma 4.1 shows that $\dot{\Gamma} \subset \rho(A+B)$, and that the spectrum of $A+B$ lies to the right of $\dot{\Gamma}$. Here, Γ denotes the contour introduced in Section 2. If $g \in \Psi(S_\mu^0)$, then $g(A+B) \in L(L_p)$ is well defined, thanks to (2.3) and (4.3). Moreover, the proof of Lemma 4.1 shows that

$$g(A+B) = -\frac{1}{2\pi i}\sum_{m=0}^\infty \int_\Gamma g(\lambda)(\lambda I - A)^{-1}[B(\lambda I - A)^{-1}]^m \, d\lambda\,.$$

In order to prove that $A+B$ has a bounded H_∞-functional calculus, it suffices to establish (2.6), since we have proved that $A+B$ is one-one and has dense range. Let $g \in \Psi(S_\mu^0)$ be fixed and let

$$T_m := \int_\Gamma g(\lambda)(\lambda I - A)^{-1}[B(\lambda I - A)^{-1}]^m \, d\lambda\,, \quad m \in \mathbb{N}.$$

The main estimate in the proof of Theorem 4.2 is the following.

Proposition 4.3. *Suppose that $1 < p < \infty$. Then there exists a constant C_p such that*

$$\|T_m\|_{L(L_p)} \le C_p^{m+1} \|b\|_\infty^m \|g\|_{H_\infty}, \quad m \in \mathbb{N}. \tag{4.6}$$

The estimate holds uniformly for all homogeneous (M, ω)-elliptic differential operators $\mathcal{A}$ and all differential operators $\mathcal{B}$.

We will postpone the proof of Proposition 4.3 and finish the proof of Theorem 4.2. It follows immediately from the proposition that $\|g(A+B)\|_{L(L_p)} \le C_p \|g\|_{H_\infty}$ for all $g \in \Psi(S_\mu^0)$, whenever $\|b\|_\infty \le 1/(2C_p) \wedge \varepsilon_0$. Thanks to Lemma 2.1, the proof of Theorem 4.2 is now completed. □

5. Proof of Proposition 4.3. In this section we study the operators T_m by means of Caldéron-Zygmund techniques. We refer to [6], [19] and [4] for the theoretical background on singular integral operators and the $T(1)$-theorem. We will rely on [6], where a functional calculus for small perturbations of $-\Delta$ is studied.

Let $\tilde{t} := sgn(t)\, t^{-\ell}$ for $t \in \dot{\mathbb{R}}$ and set

$$S_t := \tilde{t}\,(\gamma(\tilde{t}) - A)^{-1}, \qquad R_t := D^\alpha(\gamma(\tilde{t}) - A)^{-1}, \quad t \in \dot{\mathbb{R}}, \tag{5.1}$$

where $\alpha \in \mathbb{N}^n$ with $|\alpha| = \ell$ is fixed. In the sequel, σ_a denotes dilation of a function (or a distribution) by $a \in \dot{\mathbb{R}}$. We note the following properties of S_t and R_t.

Lemma 5.1. *Suppose $0 < t < \infty$.*

a) *There exists a matrix-valued function $\varphi \in C^\infty(\dot{\mathbb{R}}^n) \cap L_1(\mathbb{R}^n)$ such that $S_t = \varphi_t *$, where $\varphi_t := (1/t)^n \sigma_{1/t}\,\varphi$ and where $*$ denotes convolution. φ satisfies*

$$|\varphi(x)| \le C|x|^{1-n}, \quad |\nabla\varphi(x)| \le C|x|^{-n}, \quad |x| \le 1, \quad x \ne 0, \tag{5.2}$$

and

$$|\varphi(x)| \le C_k|x|^{-k}, \quad |\nabla\varphi(x)| \le C_k|x|^{-k}, \quad |x| \ge 1, \quad k \in \mathbb{N}. \tag{5.3}$$

The same assertions hold for S_{-t}, $(S_t)^*$ and $(S_{-t})^*$.

b) *R_t is bounded on $L_p(\mathbb{R}^n, H)$ for all $1 < p < \infty$, with a norm independent of t. R_t admits a decomposition $R_t = \rho_t + \pi_t$, where the distribution kernel of ρ_t is supported in $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n\,;\, |x-y| \le t\}$ and π_t is the convolution by a matrix-valued function $\omega_t := (1/t)^n \sigma_{1/t}\,\omega$, where $\omega \in C^\infty(\mathbb{R}^n) \cap L_1(\mathbb{R}^n)$ satisfies*

$$|\omega(x)| + |\nabla\omega(x)| \le c\,(1 + |x|)^{-(n+1)}, \quad x \in \mathbb{R}^n. \tag{5.4}$$

The same assertions remain true for R_{-t}, $(R_t)^*$ and $(R_{-t})^*$.

Proof. a) Let $t>0$. (3.9) and the homogeneity of a show that S_t is a Fourier multiplier operator with symbol $\sigma_t(e^{i\theta}-a)^{-1}$. Thus, $S_t=(\mathcal{F}^{-1}\sigma_t(e^{i\theta}-a)^{-1})*$ and we obtain that

$$S_t=\big((1/t)^n\sigma_{1/t}\,\mathcal{F}^{-1}(e^{i\theta}-a)^{-1}\big)*\ ,$$

see [12] p. 167, or [3], p. 132. It follows from (3.6)–(3.7) that

$$|\partial^\beta(e^{i\theta}-a(\xi))^{-1}|\le c_\beta\,\langle\xi\rangle^{-\ell-|\beta|},\quad \xi\in\mathbb{R}^n,\quad \beta\in\mathbb{N}^n, \tag{5.5}$$

where $\langle\xi\rangle:=(1+|\xi|^2)^{1/2}$ for $\xi\in\mathbb{R}^n$. Indeed, observe that

$$|\partial^\gamma a(\xi)|\le c\,|\xi|^{\ell-|\gamma|}\le c\,\langle\xi\rangle^{\ell-|\gamma|},\quad \xi\in\mathbb{R}^n,\quad \gamma\in\mathbb{N}^n,$$

since $\partial^\gamma a$ vanishes for $|\gamma|>\ell$. (5.5) is then a consequence of (3.6) and (3.7). We conclude from (5.5), or from the weaker estimate (3.8), that $\varphi:=\mathcal{F}^{-1}(e^{i\theta}-a)^{-1}\in C^\infty(\dot{\mathbb{R}}^n,L(H))$; see [19], p. 245. Next, note that (5.5) yields

$$\partial^\beta(e^{i\theta}-a)^{-1}\in L_1(\mathbb{R}^n,L(H))$$

whenever $\ell+|\beta|\ge n+1$. Since $\ell\ge 2$, this is always satisfied if $|\beta|\ge n-1$. Therefore,

$$|x^\beta\varphi(x)|=|(\mathcal{F}^{-1}\partial^\beta(e^{i\theta}-a)^{-1})(x)|\le c_\beta,\quad |\beta|\ge n-1, \tag{5.6}$$

owing to Riemann-Lebesgue's lemma. Leibniz's rule and an analogous argument as above show that $|\partial^\beta(\xi^\gamma(e^{i\theta}-a(\xi))^{-1})|\le c_{\beta,\gamma}\,\langle\xi\rangle^{-\ell-|\beta|+|\gamma|}$ for all $\xi\in\mathbb{R}^n$ and all $\beta,\gamma\in\mathbb{N}^n$. We conclude that

$$|x^\beta\partial^\gamma\varphi(x)|=|(\mathcal{F}^{-1}\partial^\beta(\xi^\gamma(e^{i\theta}-a)^{-1}))(x)|\le c_\beta,\quad |\beta|\ge n,\quad |\gamma|=1. \tag{5.7}$$

The assertions in (5.2) and (5.3) are now consequences of (5.6)–(5.7). It is easy to see that S_{-t} is a Fourier multiplier operator with symbol $-\sigma_t(e^{-i\theta}-a)^{-1}$, whereas $(S_t)^*$ has symbol $\sigma_t(e^{i\theta}-a^*)^{-1}$ and $(S_{-t})^*$ has symbol $-\sigma_t(e^{-i\theta}-a^*)^{-1}$. The remaining statements can now be proved in the same way as above.

b) If $t>0$, R_t is a Fourier multiplier operator with symbol $\xi^\alpha(t^{-\ell}e^{i\theta}-a)^{-1}$. It follows from (4.4) and Mikhlin's multiplier theorem that R_t is bounded on L_p, with a norm independent of t. Note that

$$\xi^\alpha(t^{-\ell}e^{i\theta}-a(\xi))^{-1}=\sigma_t(\xi^\alpha(e^{i\theta}-a(\xi))^{-1}),\quad \xi\in\mathbb{R}^n.$$

We conclude that $R_t=\big((1/t)^n\sigma_{1/t}\,\mathcal{F}^{-1}\xi^\alpha(e^{i\theta}-a)^{-1}\big)*$. Let $\zeta\in\mathcal{D}(\mathbb{R}^n,[0,1])$ be a smooth cut-off function with $\zeta\equiv 1$ on $\frac{1}{2}\bar{\mathbb{B}}(0,1)$, and with support contained in $\mathbb{B}(0,1)$. Then $R_t=\rho_t+\pi_t$, where

$$\rho_t:=\big(\frac{1}{t^n}\,\sigma_{1/t}\,\zeta\mathcal{F}^{-1}\xi^\alpha(e^{i\theta}-a)^{-1}\big)*,\quad \pi_t:=\big(\frac{1}{t^n}\,\sigma_{1/t}\,(1-\zeta)\mathcal{F}^{-1}\xi^\alpha(e^{i\theta}-a)^{-1}\big)*\ .$$

Note that

$$\big\langle (1/t)^n \sigma_{1/t}\, \zeta \mathcal{F}^{-1}(e^{i\theta}-a)^{-1}, \phi \big\rangle = \big\langle \mathcal{F}^{-1}(e^{i\theta}-a)^{-1}, \zeta \sigma_t\, \phi \big\rangle = 0,$$

whenever $\phi \in \mathcal{D}(\mathbb{R}^n)$ has its support contained in $(\bar{\mathbb{B}}(0,t))^c$. It follows that the distribution kernel of ρ_t is supported in $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n\,;\, |x-y| \le t\}$.

We will now consider the operator π_t. Similar arguments as above show that

$$|(\partial^\gamma \mathcal{F}^{-1} \xi^\alpha (e^{i\theta}-a)^{-1})(x)| \le c\,(1+|x|)^{-(n+1)}, \quad |x| \ge 1/2, \quad |\gamma| \le 1.$$

It follows that $\omega := (1-\zeta)\mathcal{F}^{-1}\xi^\alpha(e^{i\theta}-a)^{-1}$ satisfies all the properties of Lemma 5.1b). Similar arguments prove the remaining statements. And so, the proof of Lemma 5.1 is now completed. □

To simplify the notation, we will assume that the differential operator $\mathcal{B}$ has the form $\mathcal{B} = b\, D^\alpha$, with $b \in L_\infty(\mathbb{R}^n, L(H))$, where $\alpha \in \mathbb{N}^n$ is fixed as above. The general case consists of a finite sum of expressions of this type.

For the remainder of this section, B will now denote the multiplication operator induced by the function b. Let

$$\mu(t) := -\ell\, g(\gamma(\tilde{t}))\gamma'(\tilde{t}), \quad t \in \dot{\mathbb{R}}.$$

With these notations,

$$T_m = \int_{-\infty}^{\infty} S_t (BR_t)^m \mu(t)\, \frac{dt}{t}\,, \quad m \in \mathbb{N}.$$

We will use the decomposition

$$R_t = (I - Q_t - W_t)R\,, \tag{5.8}$$

where Q_t and W_t are Fourier multiplier operators with symbols

$$\gamma(\tilde{t})(\gamma(\tilde{t})-a(\xi))^{-1} - (\gamma(\tilde{t})(\gamma(\tilde{t})+|\xi|^\ell)^{-1})^N I \quad \text{and} \quad (\gamma(\tilde{t})(\gamma(\tilde{t})+|\xi|^\ell)^{-1})^N I,$$

respectively, where I is the identity map on H and where $N \in \dot{\mathbb{N}}$ is fixed. Moreover, R has symbol $-\xi^\alpha a(\xi)^{-1}$. Note that (4.4) and Mikhlin's multiplier theorem imply that R is a bounded operator on $L_p(\mathbb{R}^n, H)$. Let $0 < t < \infty$. The proof of Lemma 5.1a) shows that there exists a function ψ satisfying (5.2)–(5.3), such that Q_t is given as a convolution operator by the function $\psi_t := (1/t)^n \sigma_{1/t}\, \psi$. It is important to note that $\int \psi(x)\, dx = 0$. Indeed, this follows from $\hat{\psi}(\xi) = e^{i\theta}(e^{i\theta}-a(\xi))^{-1} - (e^{i\theta}(e^{i\theta}+|\xi|^\ell)^{-1})^N I$ and from $\int \psi(x)\, dx = \hat{\psi}(0) = 0$. It is easy to see that $W_t = \omega_t\, *$, where ω satisfies the condition (5.4), provided N is chosen large enough. Analogous statements hold for $t \in (-\infty, 0)$. Finally, note that

$$sgn(t) e^{sgn(t)\, i\theta} S_t = Q_t + W_t, \quad t \in \dot{\mathbb{R}}. \tag{5.9}$$

We will now prove Proposition 4.3 by induction on m. The proof follows the lines of [6], pp. 387–389. It can be verified that Proposition 2 in [6] remains valid in our situation, since its proof only uses estimates of the type derived in Lemma 5.1.

We will first show that

$$\|T_m\|_{L(L_2)} \le C^{m+1}\|b\|_\infty^m\,\|g\|_{H_\infty}. \tag{5.10}$$

Observe that the case $m=0$ has been proved in Proposition 3.2. Let $m \ge 1$ and suppose that (5.10) has been established for all operators of order $m-1$. Then,

$$T_m = T_{m-1}BR - T_m^1 R - T_m^2 R \tag{5.11}$$

with

$$T_m^1 := \int_{-\infty}^{\infty} S_t(BR_t)^{m-1}BQ_t\mu(t)\,\frac{dt}{t}, \quad T_m^2 := \int_{-\infty}^{\infty} S_t(BR_t)^{m-1}BW_t\mu(t)\,\frac{dt}{t},$$

owing to (5.8). It remains to show that T_m^1 and T_m^2 are L_2-bounded, thanks to the induction hypothesis and the fact that B and R are bounded on L_2.

Let us first consider the operator T_m^2. Here we follow [6], where the $T(1)$-theorem and [6], Proposition 2 are used to derive

$$\|T_m^2\|_{L(L_2)} \le C^{m+1}\|b\|_\infty^m\,\|g\|_{H_\infty}. \tag{5.12}$$

We shall now turn our attention to the operator T_m^1. We will no longer need the analyticity of the function g and we can consider separately the cases $t>0$ and $t<0$ in the integral. Let us concentrate on the case $t>0$. It follows from (5.9) that

$$e^{i\theta}\int_0^\infty S_t(BR_t)^{m-1}BQ_t\mu(t)\,\frac{dt}{t} = T_m^3 + T_m^4\,,$$

with

$$T_m^3 := \int_0^\infty Q_t(BR_t)^{m-1}BQ_t\mu(t)\,\frac{dt}{t}\,, \quad T_m^4 := \int_0^\infty W_t(BR_t)^{m-1}BQ_t\mu(t)\,\frac{dt}{t}\,.$$

In order to prove that T_m^3 is bounded on L_2, we will use the cancellation contained in $\int \psi(x)\,dx = 0$. A standard argument then shows that

$$\|T_m^3\|_{L(L_2)} \le C^{m+1}\|b\|_\infty^m\,\|g\|_{H_\infty};$$

see [6], Lemma 7. Next, observe that the operator $(T_m^4)^*$ satisfies [6], formula (41). It follows with an induction argument, see the proof of [6], Theorem 4, that $(T_m^4)^*$

is L_2-bounded and $\|(T_m^4)^*\|_{L(L_2)} \le C^{m+1}\|b\|_\infty^m \|g\|_{H_\infty}$. Since T_m^4 satisfies the same estimate we have proved that

$$\|T_m^1\|_{L(L_2)} \le C^{m+1}\|b\|_\infty^m \|g\|_{H_\infty}. \tag{5.13}$$

(It should be mentioned that the proof of the L_2-boundedness of T_m^1 given in [6] seems to contain a gap. Indeed, observe that the authors assume in Theorem 4 and thereafter that S_t has cancellation, which is not satisfied for the operator S_t under consideration.)

We have now proved (5.10). Since the kernel of T_m satisfies [6], formula (34), we obtain that

$$\|T_m\|_{L(L_\infty,\,\mathrm{BMO})} \le C^{m+1}\|b\|_\infty^m \|g\|_{H_\infty}; \tag{5.14}$$

see [19], p. 178, for instance. We will now use that

$$[\,\mathrm{BMO}(\mathbb{R}^n), L_2(\mathbb{R}^n)]_{2/p} \doteq L_p, \quad 2 < p < \infty; \tag{5.15}$$

see [11], or [4], [19]. Here, $[\cdot,\cdot]_\theta$ denotes the complex interpolation method and $\doteq$ means equivalent norms. We conclude from (5.12) and (5.14)–(5.15) that

$$\|T_m\|_{L(L_p)} \le C_p\, C^m \|b\|_\infty^m \|g\|_{H_\infty}, \quad 2 \le p < \infty. \tag{5.16}$$

It remains to prove (4.6) for $p \in (1,2)$. Observe that [6], Proposition 2 applies to $(T_m^1)^*$ and $(T_m^2)^*$, thanks to Lemma 5.1. Since the L_2 estimates (5.12)–(5.13) remain valid for the duals of T_m^j, $j = 1,2$, we infer that

$$\|(T_m^j)^*\|_{L(L_\infty,\,\mathrm{BMO})} \le C^{m+1}\|b\|_\infty^m \|g\|_{H_\infty}, \quad j = 1,2.$$

(5.15) and the dual versions of (5.12)–(5.13) yield

$$\|T_m^j\|_{L(L_p)} \le C_p\, C^m \|b\|_\infty^m \|g\|_{H_\infty}, \quad 1 < p < 2, \quad j = 1,2. \tag{5.17}$$

We will now proceed by induction on m. Let $p \in (1,2)$ be fixed. The L_p-boundedness of T_0 follows from Lemma 3.2 (or from (5.10) and [6], Proposition 2, applied to $(T_0)^*$). Let $m \ge 1$ and suppose that (4.6) holds for all operators of order $m-1$. Then, (5.11), (5.17) and the induction hypothesis yield (4.6). This completes the proof of Proposition 4.3.

Remark 5.2. We have proved a stronger result than stated in Proposition 4.3. An inspection of the proof shows that

$$\|T_m\|_{L(L_p)} \le C_p\, C^m \,\|b\|_\infty^m \|g\|_{H_\infty}, \quad 2 \le p < \infty.$$

Note, however, that the argument after (5.17) gives a weaker estimate in the case $1 < p < 2$.

6. Elliptic operators on $\mathbb{R}^n$. In this section we consider general elliptic systems on $\mathbb{R}^n$. We show the existence of a bounded H_∞-calculus, provided that the top-order coefficients are bounded and uniformly continuous. In order to prove our result, we will use the device of localization and approximation introduced in [2], Section 3. The underlying strategy is to localize, to use perturbation results for the "localized" operators, and then to patch together. For this, we introduce a suitable partition of unity of the space $\mathbb{R}^n$.

As before, $H = (H, |\cdot|)$ denotes a finite-dimensional complex Banach space. We consider differential operators

$$\mathcal{A} := \sum_{|\alpha|\le\ell} a_\alpha D^\alpha \tag{6.1}$$

of order $\ell \in 2\dot{\mathbb{N}}$ with variable coefficients $a_\alpha : \mathbb{R}^n \to L(H)$. Let

$$\mathcal{A}_\pi(x,\xi) := \sum_{|\alpha|=\ell} a_\alpha(x)\xi^\alpha, \quad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \tag{6.2}$$

be the principal symbol associated with $\mathcal{A}$. Let $M \ge 1$ and $\omega \in [0,\pi)$ be given. Then $\mathcal{A}$ is *uniformly (M,ω)-elliptic* if

$$\sum_{|\alpha|=\ell} \|a_\alpha\|_\infty \le M, \quad \sigma(\mathcal{A}_\pi(x,\xi)) \subset \dot{S}_\omega, \quad |(\mathcal{A}_\pi(x,\xi))^{-1}| \le M, \quad x \in \mathbb{R}^n,\ |\xi| = 1.$$

Let us assume that the coefficients of (6.1) satisfy

$$a_\alpha \in \begin{cases} \mathrm{BUC}\,(\mathbb{R}^n, L(H)) & \text{if} \quad |\alpha| = \ell, \\ L_\infty(\mathbb{R}^n, L(H)) & \text{if} \quad |\alpha| \le \ell - 1, \end{cases} \tag{6.3}$$

and

$$\sum_{|\alpha|\le\ell} \|a_\alpha\|_\infty \le M. \tag{6.4}$$

We can now prove the following result.

Theorem 6.1. *Suppose that $1 < p < \infty$. Let $M \ge 1$ and $0 \le \omega < \mu < \pi$ be given. Then there exist constants $s > 0$ and $N \ge 1$ such that*

$$sI + A \in \mathcal{H}_\infty(L_p; N, \mu) \tag{6.5}$$

for all uniformly (M,ω)-elliptic operators $\mathcal{A}$ whose coefficients satisfy (6.3)–(6.4).

Proof. In the following, $\mathcal{Q}$ denotes the cube $(-1,1)^n$ of $\mathbb{R}^n$. Let $\rho \in (0,1]$ be fixed and let $\{U_j\,;\, j \in \mathbb{N}\}$ be an enumeration of the open covering

$$\{(\rho/2)\,(z/2 + \mathcal{Q})\,;\, z \in \mathbb{Z}^n\}$$

of $\mathbb{R}^n$. Note that the covering $\{U_j\,;\,j\in\mathbb{N}\}$ depends on ρ. To keep the notation simple, we do not indicate this dependence. For each $j\in\mathbb{N}$, let x_j be the center of the cube U_j. Observe that the covering $\{U_j\,;\,j\in\mathbb{N}\}$ has finite multiplicity, meaning that no point of $\mathbb{R}^n$ is contained in more than m cubes of the covering $\{U_j\,;\,j\in\mathbb{N}\}$ for an appropriate fixed $m\in\mathbb{N}$. Observe that

$$\varphi_j(x) := (2/\varepsilon)(x-x_j), \quad x\in\mathbb{R}^n,$$

is a smooth diffeomorphism from U_j onto $\mathcal{Q}$. Next let $\pi\in\mathcal{D}(Q,[0,1])$ be given such that $\pi\equiv 1$ on $(1/2)\mathcal{Q}$ and set

$$\pi_j := (\pi\circ\varphi_j)\Bigl(\sum\nolimits_i(\pi\circ\varphi_i)\Bigr)^{-1/2}.$$

Then $\pi_j\in\mathcal{D}(U_j,[0,1])$ and $\{\pi_j^2\,;\,j\in\mathbb{N}\}$ is a smooth partition of unity subordinated to the covering $\{U_j\,;\,j\in\mathbb{N}\}$ of $\mathbb{R}^n$. In particular,

$$\sum_{j\in\mathbb{N}}\pi_j^2(x)=1, \quad x\in\mathbb{R}^n. \tag{6.6}$$

Moreover, let $\chi_j\in\mathcal{D}(U_j,[0,1])$ be such that $\chi_j\equiv 1$ on the support of π_j. (The functions χ_j can be constructed in the same way as π_j.)

a) Assume that $a_\alpha=0$ for all $|\alpha|\le\ell-1$. Define

$$A_j := \mathcal{A}_j+\mathcal{B}_j := \sum_{|\alpha|=\ell}a_\alpha(x_j)D^\alpha+\sum_{|\alpha|=\ell}\chi_j(a_\alpha-a_\alpha(x_j))D^\alpha, \quad j\in\mathbb{N}. \tag{6.7}$$

Note that $\mathcal{A}_j$ is, for each $j\in\mathbb{N}$, a uniformly (M,ω)-elliptic homogeneous differential operator with constant coefficients. Given $\varepsilon_0>0$ there exists $\rho_0\in(0,1]$ such that

$$\sum_{|\alpha|=\ell}\|\chi_j(a_\alpha-a_\alpha(x_j))\|_\infty\le\varepsilon_0, \quad j\in\mathbb{N}, \tag{6.8}$$

thanks to (6.4), and the fact that each cube U_j has diameter less than $\sqrt{n}\,\rho$. Let ε_0 be small enough and fix ρ_0 such that (6.8) is satisfied. If follows from (6.7)–(6.8), and from Lemma 2.2 and Theorem 4.2, that there exists a constant $N\ge 1$ such that

$$sI+A_j\in\mathcal{H}_\infty(L_p;N,\mu), \quad s\ge 0, \quad j\in\mathbb{N}. \tag{6.9}$$

Next, observe that

$$A(\pi_j u)=\sum_{|\alpha|=\ell}a_\alpha(x_j)D^\alpha(\pi_j u)+\sum_{|\alpha|=\ell}\chi_j(a_\alpha-a_\alpha(x_j))D^\alpha(\pi_j u)=A_j(\pi_j u) \tag{6.10}$$

for all $j \in \mathbb{N}$ and all $u \in W_p^\ell$, owing to the fact that $\chi_j \equiv 1$ on the support of π_j. It follows from (6.6) and (6.10) that

$$(zI + A)u = \sum_j \Big(\pi_j (zI + A_j)\pi_j u + [A, \pi_j]\pi_j u \Big), \quad z \in \mathbb{C}, \quad u \in W_p^\ell, \tag{6.11}$$

where $[A, \pi_j]v := A(\pi_j v) - \pi_j(Av)$ for $v \in W_p^\ell$. (6.11) can be rewritten as

$$zI + A = r(zI + \mathbb{A} + r^c C)r^c,$$

where the operators r, r^c and $\mathbb{A}, C$ are introduced in [2], Sections 3, 9. We are now in a similar situation as in the proof of [2], Theorem 9.4 and we skip the details. Let us mention, however, that the inverse of $\lambda I + sI + A$ exists for all $\lambda \in -S_{\pi-\theta}$, where $\theta > \omega$ is fixed, and

$$(\lambda I + sI + A)^{-1} = r(\lambda I + sI + \mathbb{A} + r^c C)^{-1} r^c, \quad \lambda \in -S_{\pi-\theta},$$

provided $s > 0$ is large enough. It follows from (6.9) and [2], Proposition 3.2 that there are constants $s > 0$ and $N \geq 1$ such that $sI + A \in \mathcal{H}_\infty(L_p; N, \mu)$.

b) It remains to remove the assumption made in a). The general case is now a consequence of the perturbation result [2], Theorem 2.6. $\square$

Corollary 6.2. *Suppose that $1 < p < \infty$. Let $M \geq 1$ and $0 \leq \omega < \mu < \pi$ be given. Then there exists constants $s > 0$ and $N \geq 1$ such that*

$$\|(sI + A)^{it}\|_{L(L_p)} \leq N e^{\mu |t|}, \quad t \in \mathbb{R}, \tag{6.12}$$

for all uniformly (M, ω)-elliptic operators $\mathcal{A}$ whose coefficients satisfy (6.3)–(6.4).

7. Elliptic operators on closed manifolds. We show that elliptic operators with continuous coefficients, acting on sections of vector bundles over compact closed manifolds, have a bounded H_∞-calculus. We use the same terminology as introduced in [2], Section 10.

Let X be a compact closed n-dimensional C^ℓ-manifold, and let $G := (G, \pi, X)$ be a complex C^ℓ-vector bundle over X of rank N with fiber H.

Let

$$\mathcal{A} : W_p^\ell(X, G) \to L_p(X, G)$$

be a linear differential operator of order $\ell \in 2\dot{\mathbb{N}}$ with continuous coefficients and let

$$\mathcal{A}_\pi : T(X)^* \to \mathrm{End}(G)$$

be its principal symbol; see [7], Section 23.15.6, for instance. Given $\omega \in [0, \pi)$, the operator $\mathcal{A}$ is *ω-elliptic* if

$$\sigma(\mathcal{A}_\pi(\xi_x^*)) \subset \dot{S}_\omega, \quad \xi_x^* \in [T_x(X)^*]^{\cdot}, \quad x \in X.$$

We can now state the following theorem.

Theorem 7.1. *Suppose that* $1 < p < \infty$. *Let* $0 \le \omega < \mu < \pi$ *be given and assume that* $\mathcal{A}$ *is* ω*-elliptic. Then there exist* $s > 0$ *and* $N \ge 1$ *such that*

$$sI + A \in \mathcal{H}_\infty(L_p(X,G); N, \mu).$$

Proof. This follows from Theorem 6.1 and from analogous arguments as in [2], Section 10. □

Corollary 7.2. *Assume that* $\mathcal{A}$ *satisfies the assumptions of Theorem* 7.1. *Then there exist* $s > 0$ *and* $N \ge 1$ *such that* $\|(sI + A)^{it}\|_{L(L_p)} \le N e^{\mu|t|}, \quad t \in \mathbb{R}$.

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