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ON THE GENERALIZED KORTEWEG-DE VRIES-TYPE EQUATIONS

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Abstract. We show well-posedness results for the generalized Korteweg-de Vries equation with nonlinear term $F(u)\partial_x u$. We assume $F(u)$ is a C^4 function and $F(0) = 0$. Using a version of the chain rule for fractional derivatives and some estimates on the evolution group, we prove existence, uniqueness and regularity properties of the solution of the equation when the space of the initial data is $H^s(\mathbb{R})$, $s > 1/2$. The theorem we prove is sharp. We obtain all the above results also for a mixed KdV and Schrödinger type equation proposed as a model for the propagation of a signal in an optic fiber.

1. Introduction. In this paper we study the initial value problem, IVP, for the generalized Korteweg-de Vries equation:

$$\partial_t u + \partial_x^3 u + F(u)\partial_x u = 0, \quad u(x,0) = u_0(x) \;\; x,t \in \mathbb{R}, \tag{1}$$

and for an equation of mixed KdV and Schrödinger type:

$$\begin{aligned} &i\partial_t u + \alpha\partial_x^2 u + i\beta\partial_x^3 u + \gamma(u)|u|^2 u + i\delta(u)|u|^2 u_x + i\epsilon(u)u^2\overline{u}_x = 0, \\ &u(0,x) = u_0(x), \;\; x,t \in \mathbb{R}, \end{aligned} \tag{2}$$

with α, $\beta \in \mathbb{R}$, $\beta \neq 0$.

We are interested in local well-posedness results for data in classical Sobolev spaces $H^s(\mathbb{R})$, with small index s. Here the notion of well-posedness includes existence, uniqueness and continuous dependence of the solution upon the initial data. Before stating the main theorems we give a short review of the history of the two problems.

The equation in (1) with $F(u) = u$ was first derived by Korteweg and de Vries in [17] as a model for long waves propagating in a channel. Well-posedness results for (1) can be found in the work of several people, like Bona and Scott ([2]), Bona and Smith ([3]), T. Kato ([11]) and Saut and Temam ([20]). The tool they use to attack problem (1), in this special case, is the classical energy method. Another classical

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approach is by the scattering method ([7], [8], [10] and [19]). Kenig, Ponce and Vega considered the problem from a different point of view. They used estimates on oscillatory integrals ([12], [14]), and later a fixed-point argument on the modified Sobolev space $X_{s,b}$ (see (5) in Section 2), first introduced by Bourgain in [4]. Their most recent result is local well-posedness in $H^s(\mathbb{R})$, $s > -3/4$ ([13], [15]).

The modified KdV with $F(u) = u^2$ has been actively studied especially because many hyperbolic models can be reduced to this equation. Also in this case the energy method and the scattering method constitute the classical approach to prove well-posedness results. Kenig, Ponce and Vega considered the problem in the more general situation in which $F(u) = u^k$ ([14]). Their result can be summarized in the table below, where in the left-hand side we list the order of the nonlinearity and in the right-hand side the indices of the Sobolev spaces in which they prove local well-posedness:

$$\begin{aligned} &\text{for } k = 2, \quad s \geq 1/4 \\ &\text{for } k = 3, \quad s \geq 1/12 \\ &\text{for } k \geq 4, \quad s \geq (k-4)/2k. \end{aligned}$$

For $k \geq 4$ the result is sharp. In fact, Kenig, Ponce and Vega, in collaboration with Bernir and Svandstedt, proved that the IVP (1) with $F(u) = u^k$ is ill-posed for $s < s_k$ ([1]).

A more general nonlinearity has been considered by Christ and Weinstein in [6]. They assume that $F(u)$ is smooth and with a polynomial behavior near the origin; precisely, $F(u) = \mathcal{O}(|u|^\alpha)$ where $\alpha > 3$. They prove global well-posedness for initial data $u_0 \in H^1(\mathbb{R})$ and $\|u_0\|_{H^1}$ small ([6]). The periodic version of the IVP (1) has been extensively studied by Bourgain. His result is global well-posedness for data $u_0 \in H^s(\mathbb{T})$, $s > 3/2$ and $\|u_0\|_{H^s}$ small ([4], [5]).

In this paper we assume that $F(u)$ is a smooth function and we prove that the IVP (1) is well-posed in any interval of time $[-T, T]$ for data u_0 in H^s, $s > 1/2$, with small norm (see Theorem 3.2). This is a sharp result in view of the remark we made above for $F(u) = u^k$, $k \geq 4$.

We now consider the IVP (2). The equation in (2), where $\alpha, \beta, \gamma, \delta, \epsilon$ are real numbers and $\beta \neq 0$, was first proposed by A. Hasegawa and Y. Kodama in [9] as a model for the propagation of a signal in a optic fiber (see also [16]). We first notice that when $\alpha = \gamma = \epsilon = 0$ (2) reduces to (1). The original equation of Hasegawa and Kodama was studied by C. Laurey who proved local existence and uniqueness results for the IVP (2) in $H^s(\mathbb{R})$, $s > 3/4$ ([18]). In this paper we show that this result can be improved: the IVP (2), with γ, δ, ϵ constant, is locally well-posed for $s \geq 1/4$ (see Theorem 3.3). Moreover we show existence and uniqueness results for (2) with γ, δ, ϵ smooth function of u (see Theorem 3.4).

We would like to thank C. Kenig for many useful and interesting conversations that we had during the preparation of this paper.

2. Some notation. In order to treat functions defined in a space-time domain we introduce mixed norm spaces. If $1 \le p, q < \infty$, then $f \in L^p_x L^q_T$ if

$$\|f\|_{L^p_x L^q_T} = \Big(\int_{-\infty}^{\infty} \Big(\int_{-T}^{T} |f(x,t)|^q \, dt\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}} < \infty. \tag{3}$$

Similarly, we say that $g \in L^q_T L^p_x$ if

$$\|g\|_{L^q_T L^p_x} = \Big(\int_{-T}^{T} \Big(\int_{-\infty}^{\infty} |g(x,t)|^p \, dx\Big)^{\frac{q}{p}} dt\Big)^{\frac{1}{q}} < \infty. \tag{4}$$

If $p = \infty$ or $q = \infty$ we have the obvious definition involving the essential supremum.

We also use the modified Sobolev space $X_{s,b}$, first introduced by Bourgain in [4] in order to prove well-posedness for KdV-type problems and then used by Kenig, Ponce and Vega to prove well-posedness for the classical KdV problem in Sobolev spaces with negative indices.

Definition 2.1. For $s, b \in \mathbb{R}$, $s \ge 0$ we define the space $X_{s,b}$ to be the completion of the Schwartz space with respect to the norm

$$\|f\|_{X_{s,b}} = \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1 + |\tau - \xi^3|)^{2b} (1 + |\xi|)^{2s} |\hat{f}(\xi,\tau)|^2 \, d\xi \, d\tau\Big)^{\frac{1}{2}}. \tag{5}$$

If $\phi \in C_0^\infty(\mathbb{R}), \phi = 1$ on $[-1,1]$ and $supp\, \phi \subseteq [-2,2]$, then for any $T > 0$ we denote

$$\mu^T_{0,s}(u) = \|\phi(T^{-1}t) u(t,x)\|_{X_{s,b}}. \tag{6}$$

Below we often use the symbol D^s_x to indicate the differential operator $(-\Delta)^{s/2}$ and $\mathcal{F}$ to indicate the Fourier transform operator.

3. Statement of the results.

Theorem 3.1. *If $F(u) = u^2 g(u)$ and $g \in C^2_b(\mathbb{R})$,[†] then for any $u_0 \in H^s(\mathbb{R})$, $s \ge 1/4$, there exists $T = T(\|u_0\|_{H^{1/4}}) > 0$ (with $T(\rho) \to \infty$ as $\rho \to 0$) and a unique strong solution $u(t)$ of the IVP (1) satisfying*

$$u \in C([-T,T] : H^s(\mathbb{R})), \tag{7}$$

$$\|u\|_{L^\infty_T L^2_x} + \|D^s_x u\|_{L^\infty_T L^2_x} < \infty, \tag{8}$$

$$\|\frac{\partial u}{\partial x}\|_{L^\infty_x L^2_T} + \|D^s_x \frac{\partial u}{\partial x}\|_{L^\infty_x L^2_T} < \infty, \tag{9}$$

$$\|u\|_{L^5_x L^{10}_T} + \|D^s_x u\|_{L^5_x L^{10}_T} < \infty, \tag{10}$$

$$\|u\|_{L^4_x L^\infty_T} < \infty, \tag{11}$$

$$\|D_x u\|_{L^{20}_x L^{\frac{5}{2}}_T} < \infty. \tag{12}$$

[†] $C^2_b(\mathbb{R})$ denotes the space of functions bounded together with their first and second derivatives.

Moreover for any $T' \in [-T, T]$ *there exists a neighborhood* $\mathcal{U}$ *of* $u_0 \in H^s(\mathbb{R})$ *such that the map* $\tilde{u}_0 \to \tilde{u}(t)$ *from* $\mathcal{U}$ *into the class defined by* (7)–(12)*, with* T' *instead of* T*, is Lipschitz.*

The next theorem treats a more general nonlinearity $F(u)$.

Theorem 3.2. (i) *If* $F(u) \in C^4(\mathbb{R})$ *and* $F(0) = 0$*, then for any interval of time* $[-T, T]$ *there exists* $\epsilon > 0$ *such that for any* $u_0 \in H^s(\mathbb{R})$*,* $s > 1/2$ *and* $\|u_0\|_{H^s} \leq \epsilon$*, there exists a unique solution of the IVP* (1) *satisfying* (7)–(12) *and*

$$\|\phi(T^{-1}t)u\|_{X_{s,b}} < \infty \tag{13}$$

for some $b \in (\frac{1}{2}, \frac{3}{4})$*. Moreover, for any* $T' \in [-T, T]$ *there exists a neighborhood* $\mathcal{U}$ *of* $u_0 \in H^s(\mathbb{R})$ *such that the map* $\tilde{u}_0 \to \tilde{u}(t)$ *from* $\mathcal{U}$ *into the class defined by* (7)–(12) *with* T' *instead of* T *is Lipschitz.*

(ii) *If* $F(u)$ *is as above and has a polynomial growth, then for any* $u_0 \in H^s(\mathbb{R})$*,* $s \geq s_0 > 1/2$*, there exists* $T = T(\|u_0\|_{H^{s_0}}) > 0$ *(with* $T(\rho) \to \infty$ *as* $\rho \to 0$*) and a unique strong solution* $u(t)$ *of the IVP* (1) *satisfying the same properties listed in part* (i).

We now consider the IVP (2). The first theorem treats the case when γ, δ, ϵ are constants.

Theorem 3.3. *Assume* γ, δ, ϵ *are complex numbers and let* $s \geq 1/4$*. Then for any* $u_0 \in H^s(\mathbb{R})$ *there exist an interval of time* $[-T, T]$*, such that* $T = T(\|u_0\|_{H^{1/4}}) > 0$*, with* $T(\rho) \to \infty$ *as* $\rho \to 0$*, and a unique solution* $u(t)$ *of the IVP* (2)*, satisfying* (7)–(12)*. Moreover, for any* $T' \in (0, T)$ *there exists a neighborhood* $\mathcal{U}$ *of* u_0 *in* $H^s(\mathbb{R})$ *such that the map* $\tilde{u_0} \to \tilde{u}(t)$ *from* $\mathcal{U}$ *into the class defined by* (7)–(12)*, with* T' *instead of* T*, is Lipschitz.*

The next theorem treats the more general situation when $\gamma = \gamma(u)$, $\delta = \delta(u)$ and $\epsilon = \epsilon(u)$ are smooth functions of the variable u itself.

Theorem 3.4. (i) *If* $\gamma, \delta, \epsilon \in C_b^2(\mathbb{R})$*, then the IVP* (2) *is locally well-posed for initial data in* $H^s, s \geq 1/4$*, and the same conclusion stated in Theorem* 3.1 *holds.*

(ii) *If* $\gamma, \delta, \epsilon \in C^4(\mathbb{R})$*, then for any interval of time* $[0, T]$*, the IVP* (2) *is well-posed for initial data in* H^s*,* $s > 1/2$*, provided* $\|u_0\|_{H^s}$ *is small and the same conclusion stated in part* (i) *of Theorem* 3.2 *holds.*

(iii) *If* γ, δ, ϵ *are as in* (ii) *and have a polynomial growth, then the IVP* (2) *is locally well-posed for initial data in* H^s*,* $s > 1/2$ *and the same conclusion stated in part* (ii) *of Theorem* 3.2 *holds.*

Remark 1. Theorem 3.2 and 3.4 are sharp.

By a recent result due to Birnir, Kenig, Ponce, Svanstedt and Vega, see [1], Theorem 3.2 is sharp, in the sense that well-posedness in general fails for initial data in H^s for $s < 1/2$. In fact, as we mentioned in the introduction, these authors proved that the IVP (1), with $F(u) = u^k$, is in general ill-posed in H^s for $s < s_k = 1/2-2/k$ ([1]). In Theorem 3.2 we allow $F(u)$ to be any polynomial of any order k, hence $\lim_{k\to\infty} s_k = 1/2$ becomes the critical index for well-posedness of the IVP (1). On the other had because the IVP (2) is a generalization of (1), as we mentioned in the introduction, also Theorem 3.4 is sharp.

4. Preparation for the proof. We first present a set of linear estimates for IVP (1) and (2). For (1) we introduce the unitary group $\{W(t)\}_{-\infty}^{\infty}$ describing the solution of the linear IVP associated with (1)

$$\partial_t v + \partial_x^3 v = 0, \quad v(x,0) = u_0(x) \;\; x,t \in \mathbb{R}; \tag{14}$$

that is, $v(x,t) = W(t)u_0(x) = S_t * u_0(x)$ and $S_t(\cdot)$ is defined by the oscillatory integral

$$S_t(x) = c\int_{-\infty}^{\infty} \exp\{ix\xi\}\exp\{it\xi^3\}\, d\xi.$$

For the IVP (2) instead, we denote with $\{U(t)\}_{-\infty}^{\infty}$ the unitary group which defines the solution of

$$iu_t + \alpha u_{xx} + i\beta u_{xxx} = 0, \quad u(x,0) = u_0(x), \;\; t,x \in \mathbb{R}, \tag{15}$$

and hence $u(x,t) = U(t)u_0(x) = \tilde{S}_t * u_0(x)$, where

$$\tilde{S}_t(x) = c\int_{-\infty}^{\infty} \exp\{ix\xi\}\exp\{it\phi(\xi)\}\, d\xi$$

and $\phi(\xi) = \alpha\xi^2 + \beta\xi^3$.

Below we introduce a compact notation for a class of norms of mixed type:

$$\mu_{1,s}^T(w) = \|w\|_{L_T^\infty L_x^2} + \|D_x^s w\|_{L_T^\infty L_x^2} \tag{16}$$
$$\mu_{2,s}^T(w) = \|\partial_x w\|_{L_x^\infty L_T^2} + \|D_x^s \partial_x w\|_{L_x^\infty L_T^2} \tag{17}$$
$$\mu_3^T(w) = \|D_x w\|_{L_x^{20} L_T^{\frac{5}{2}}} \tag{18}$$
$$\mu_{4,s}^T(w) = \|w\|_{L_x^5 L_T^{10}} + \|D_x^s w\|_{L_x^5 L_T^{10}} \tag{19}$$
$$\mu_5^T(w) = \|w\|_{L_x^4 L_T^\infty} \tag{20}$$

We have the following results:

Theorem 4.1. (i) *If* $u_0 \in H^s(\mathbb{R})$, $s \geq 1/4$ *and* $b > 1/2$, *then*

$$\mu_{0,s}^T(W(t)u_0) \leq CT^{\frac{1-2b}{2}}||u_0||_{H^s} \tag{21}$$

(*see* (6) *for the definition of* $\mu_{0,s}^T(\cdot)$);

$$\mu_{1,s}^T(W(t)u_0) = C\|u_0\|_{H^s} \tag{22}$$

(*Conservation law for the* H^s*-norm*);

$$\mu_{2,s}^T(W(t)u_0) \leq C\|u_0\|_{H^s} \tag{23}$$

(*Smoothing effect*);

$$\mu_3^T(W(t)u_0) \leq C\|u_0\|_{H^s}, \tag{24}$$

$$\mu_{4,s}^T(W(t)u_0) \leq C\|u_0\|_{H^s}. \tag{25}$$

(ii) *If* $g \in L_x^1 L_T^2$, *then*

$$\sup_{t\in[-T,T]} \Big\| \frac{\partial}{\partial x} \int_0^t W(t-t')g(\cdot,t')\,dt' \Big\|_2 \leq C\|g\|_{L_x^1 L_T^2}. \tag{26}$$

(iii) *If* $u_0 \in H^{1/4}(\mathbb{R})$, *then*

$$\mu_5^T(W(t)u_0) \leq C\|D^{1/4}u_0\|_{L^2} \tag{27}$$

(*Maximal function inequality*).

All the estimates (22)–(27) *above hold if we replace the group* $\{W(t)\}_{-\infty}^{\infty}$ *with* $\{U(t)\}_{-\infty}^{\infty}$.

Proof. The proof of the theorem concerning the group $\{W(t)\}_{-\infty}^{\infty}$ is due to Kenig, Ponce and Vega and can be found in [14] and [13]. To prove the estimates for $U(t)$ we generalize the arguments presented in [14] and [13]. The estimate (22) is a group property. In order to prove (23) we take a function $\theta \in C_0^\infty$ such that $supp\,\theta \subseteq [-4|\alpha|/3|\beta|, 4|\alpha|/3|\beta|]$ and we observe that

$$\forall \xi \in (supp\,\theta)^c \Longrightarrow |\phi'(\xi)| \geq (3/2)|\beta|\xi^2.$$

We write

$$\frac{\partial}{\partial x}U(t)f(x) = i\int_{-\infty}^{\infty} \exp\{ix\xi + it\phi(\xi)\}\xi(1-\theta)(\xi)\hat{f}(\xi)\,d\xi$$
$$+ i\int_{-\infty}^{\infty} \exp\{ix\xi + t\phi(\xi)\}\xi\theta(\xi)\hat{f}(\xi)\,d\xi = I_1 + I_2.$$

By 4.3 of Theorem 4.1 in [12]

$$\|I_1\|_{L^\infty_x L^2_T} \le C\Big(\int_{B^c(0,\frac{4|\alpha|}{3|\beta|})} |\xi|^2 \frac{|\hat{f}(\xi)|^2}{|\phi'(\xi)|}\, d\xi\Big)^{\frac{1}{2}} \le C\Big(\int_{\mathbb{R}} |\hat{f}(\xi)|^2\, d\xi\Big)^{\frac{1}{2}} = C\|f\|_{L^2}.$$

For the second term I_2 we prove the dual inequality

$$\Big\|\int_{-T}^{T} \frac{\partial}{\partial x} U_\theta(-t')g(\cdot,t')\, dt'\Big\|_{L^2_x} \le c\|g\|_{L^1_x L^2_t}, \tag{28}$$

where

$$U_\theta(t)g(x) = \int_{-\infty}^{\infty} \exp\{ix + it\phi(\xi)\}\theta(\xi)\hat{g}(\xi)\, d\xi.$$

Observe that

$$\frac{\partial}{\partial x} U_\theta(-t')g(\cdot,t') = \mathcal{F}^x\big(\exp\{-it'\phi(\xi)\}i\xi\theta(\xi)\hat{g}^x(\xi,t')\big) = K_{-t'} * g(\cdot,t')$$

where

$$K_{-t'}(x) = \int_{-\infty}^{\infty} \exp\{-it'\phi(\xi) - ix\xi\}i\xi\theta(\xi)\, d\xi.$$

By Proposition 2.6 in [12]

$$|K_{-t'}(x)| \le K^T(x) = \begin{cases} c & \text{if } |x| \le c_1(T+1) \\ (c/|x|)^{\frac{1}{2}} & \text{if } c_1(T+1) \le |x| \le c_2(T+1) \\ 1/(1+|x|^2) & \text{if } |x| \ge c_2(T+1), \end{cases}$$

and by Young's inequality

$$\begin{aligned} \Big\|\int_{-T}^{T} K_{-t'} * g(\cdot,t')\Big\|_{L^2_x} &\le \Big\|K^T * \int_{-T}^{T} g(\cdot,t')\Big\|_{L^2_x} \le c\|K^T\|_{L^1_x}\Big\|\int_{-T}^{T} g(\cdot,t')\, dt'\Big\|_{L^2_x} \\ &\le C\|g\|_{L^1_x L^2_T}. \end{aligned}$$

The estimate (26) follows from (23) using duality arguments (see [14], page 353). In order to prove (27) we introduce again a cut-off function $\theta \in C_c^\infty(\mathbb{R})$ such that $supp\,\theta \subseteq I = [-\alpha/(3|\beta|) - |\alpha|/(3|\beta|), -\alpha/(3|\beta|) + |\alpha|/(3|\beta|)]$, and we notice that

$$\forall \xi \in I^c \Longrightarrow \Big|\frac{\phi'(\xi)}{\phi''(\xi)}\Big| \le C|\xi|.$$

Using the notation we introduced in order to prove (23) and Theorem 2.5 in [12], we obtain

$$\|U_{(1-\theta)}(t)u_0\|_{L^4_x L^\infty_T} \leq \Big(\int_{I^c} |\hat{u}_0(\xi)|^2 \Big|\frac{\phi'(\xi)}{\phi''(\xi)}\Big|^{\frac{1}{2}} d\xi\Big)^{\frac{1}{2}}$$
$$\leq c\Big(\int_{-\infty}^{\infty} |\hat{u}_0(\xi)|^2 |\xi|^{\frac{1}{2}} d\xi\Big)^{\frac{1}{2}} \leq c\|u_0\|_{H^{\frac{1}{4}}}.$$

Next we consider $U_\theta(t)u_0$. Again by duality it is enough to prove that

$$\Big\|\int_{-T}^{T} U_\theta(-t')g(\cdot,t')\,dt'\Big\|_{L^2_x} \leq c \parallel g \parallel_{L^{\frac{4}{3}}_x L^1_T}.$$

Observe that

$$U_\theta(-t')g(\cdot,t') = \mathcal{F}^x\big(\exp\{-it'\phi(\xi)\}\theta(\xi)\hat{g}^x(\xi,t')\big) = H_{-t'} * g(\cdot,t'),$$

where

$$H_{-t'}(x) = \int_{-\infty}^{\infty} \exp\{-it'\phi(\xi) - ix\xi\}\theta(\xi)\,d\xi.$$

Again by Proposition 2.6 in [12], $|H_{-t'}(x)| \leq K^T(x)$. Then by Young's inequality

$$\Big\|\int_{-T}^{T} H_{-t'} * g(\cdot,t')\,dt'\Big\|_{L^2_x} \leq \Big\|K^T * \int_{-T}^{T} g(\cdot,t')\,dt'\Big\|_{L^2_x}$$
$$\leq \|K^T\|_{L^{\frac{4}{3}}_x} \Big\|\int_{-T}^{T} g(\cdot,t')\,dt'\Big\|_{L^{\frac{4}{3}}_x} \leq C\|g\|_{L^{\frac{4}{3}}_x L^1_T}.$$

The estimate (25) comes from (23) and (27) using the same argument as in Corollary 3.8 in [14], while (24) comes again from (23) and (27) using complex interpolation (see Proposition 3.17 in [14]). □

The next lemma establishes a connection between the space $X_{s,b}$ and classical Sobolev spaces.

Lemma 4.2. *Let $f \in C_0^\infty([-T,T]\times\mathbb{R})$ and $1/2 < b < 1$, then*

$$\|f\|_{X_{s,b-1}} \leq CT^{(1-b)}[\|f\|_{L^2_x L^2_T} + \|D^s_x f\|_{L^2_x L^2_T}]. \tag{29}$$

Proof. Using the definition of the $X_{s,b-1}$-norm, we write

$$\|f\|_{X_{s,b-1}} = \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{f}(\xi,\tau)|^2 \frac{(1+|\xi|)^{2s}}{(1+|\xi^3-\tau|)^{2(1-b)}}\,d\xi\,d\tau\Big)^{\frac{1}{2}}$$
$$= \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{f}(\xi,\xi^3-\lambda)|^2 \frac{(1+|\xi|)^{2s}}{(1+|\lambda|)^{2(1-b)}}\,d\xi\,d\lambda\Big)^{\frac{1}{2}}$$
$$= \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big[|\mathcal{F}(W(t)J^s f)(\xi,\lambda)|(1+|\lambda|)^{-(1-b)}\big]^2\,d\xi\,d\lambda\Big)^{\frac{1}{2}}$$

where

$$(J^s h)^\wedge(\xi) = (1+|\xi|)^s \hat{h}(\xi) = \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(I_\alpha W(t) J^s f)(x,t)|^2\,dx\,dt\Big)^{\frac{1}{2}}$$

and $I_\alpha(f) = (-\Delta)^{-\frac{\alpha}{2}}(f)$ is the Riesz Potential of index $\alpha = (1-b)$. By Theorem 1, page 119 of [21],

$$\|I_\alpha(f)\|_q \le C\|f\|_p, \tag{30}$$

where $1/q = 1/p - \alpha$. If we use (30) with $1/p = 1/2 + (1-b) = 3/2 - b$, we can continue our chain of inequalities with

$$\begin{aligned}
&\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(I_\alpha W(t) J^s f)(x,t)|^2\,dx\,dt\Big)^{\frac{1}{2}}\\
&\le \Big(\int_{-\infty}^{\infty}\Big(\int_{-T}^{T} |W(t)J^s f(x,t)|^p\,dt\Big)^{\frac{2}{p}} dx\Big)^{\frac{1}{2}}\\
&\text{and by Hölder's inequality } \le T^{(1-b)}\Big(\int_{-\infty}^{\infty}\Big(\int_{-T}^{T} |W(t)J^s f(x,t)|^2\,dt\Big)dx\Big)^{\frac{1}{2}}\\
&= T^{(1-b)}\Big(\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} |W(t)J^s f(x,t)|^2\,dx\Big)dt\Big)^{\frac{1}{2}}
\end{aligned}$$

and by the invariance properties of the group $W(t)$ (22)

$$= T^{(1-b)}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |J^s f(x,t)|^2\,dx\,dt\Big)^{\frac{1}{2}}.$$

If μ_j^T for $j = 1,\dots,5$ are the norms we introduced in (16)–(20) and μ_0^T is as in (6) with $b > 1/2$, then we have

Lemma 4.3. *Let $f(x,t) \in C_0^\infty$, then*

$$\mu_j^T(f(x,t)) \le C\mu_0^T(f(x,t)) \quad j = 1,\dots,5. \tag{31}$$

Proof. First of all we observe that without loss of generality we can assume that

$$f(x,t) = \psi(T^{-1}t) f(x,t), \tag{32}$$

where ψ is a smooth characteristic function of the interval $[-1,1]$. Then we write

$$\begin{aligned}
f(x,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp i(x\xi + t\tau)\hat{f}(\xi,\tau)\,d\xi\ d\tau\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp i(x\xi + t\tau)\frac{g(\xi,\tau)}{(1+|\xi|)^s(1+|\xi^3-\tau|)^b}\,d\xi\ d\tau
\end{aligned}$$

where

$$\begin{aligned} g(\xi,\tau) &= \hat{f}(\xi,\tau)(1+|\xi|)^s(1+|\xi^3-\tau|)^b \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp i(x\xi+t\xi^3)\exp(-i\lambda t)\frac{g(\xi,-\lambda+\xi^3)}{(1+|\xi|)^s(1+|\lambda|)^b}\,d\xi\;d\lambda \\ &= \int_{-\infty}^{\infty}\exp(-i\lambda t)\big[\int_{-\infty}^{\infty}\exp i(x\xi+t\xi^3)\frac{g(\xi,-\lambda+\xi^3)}{(1+|\xi|)^s(1+|\lambda|)^b}\,d\xi\big]\;d\lambda. \end{aligned}$$

If we define

$$\widehat{v_\lambda}(\xi) = \frac{g(\xi,-\lambda+\xi^3)}{(1+|\xi|)^s(1+|\lambda|)^b},$$

then by Minkowski's inequality, the definition of $W(t)$ and Theorem 4.1, it follows

$$\begin{aligned} \mu_j^T(f(x,t) &\le \int_{-\infty}^{\infty}\mu_j^T(W(t)v_\lambda)\,d\lambda \\ &\le C\int_{-\infty}^{\infty}\|v_\lambda\|_{H^s}\,d\lambda = C\int_{-\infty}^{\infty}\big(\int_{-\infty}^{\infty}\frac{|g(\xi,\xi^3-\lambda)|^2}{(1+|\lambda|)^{2b}}\,d\xi\big)^{\frac{1}{2}}\,d\lambda \\ &\le C\big(\int_{-\infty}^{\infty}(1+|\lambda|)^{-2b}\big)^{\frac{1}{2}}\big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(\xi,\xi^3-\lambda)|^2\,d\xi\,d\lambda\big)^{\frac{1}{2}} \\ &= C\|f\|_{X_{s,b}} = C\mu_0^T(f). \end{aligned}$$

The next lemma relates the spaces $X_{s,b-1}$ and $X_{s,b}$ through the group $W(t)$.

Lemma 4.4. *Let $w(x,t)\in X_{s,b}$ and $b>1/2$, then*

$$\mu_0^T\big(\int_0^t W(t-t')w(t')\,dt'\big) \le CT^{\frac{1-2b}{2}}||w||_{X_{s,b-1}}. \tag{33}$$

This lemma is proved as Lemma 3.3 in [13].

The next lemma, due to Kenig, Ponce and Vega, states the key inequality they used to prove well-posedness for the classical KdV problem in Sobolev spaces with negative indices.

Lemma 4.5. *Assume $s>-3/4$. There exists $\theta_0>0$, small, such that if $b>1/2$ and $\theta_0+1-2b>0$, then for any $w(x,t)\in X_{s,b}$, with $t\in[-T,T]$*

$$||\partial_x w^2(x,t)||_{X_{s,b-1}} \le CT^{\theta_0}\mu_0^T(w)^2. \tag{34}$$

For the proof see Lemma 3.4 in [15].

Now we need to recall a version of a chain rule for fractional derivatives due to Kenig, Ponce and Vega.

Lemma 4.6. *Let $1 < r < \infty$, $0 < s < 1$, $\alpha > 1$ and $h \in L^{sr}_{loc}(\mathbb{R})$. Then*

$$||D^s F(u)h||_r \leq C||F'(u)||_\infty ||D^s(u) M(h^{\alpha r})^{\frac{1}{\alpha r}}||_r, \tag{35}$$

where M is the maximal functional of Littlewood-Paley.

The proof can be found in [14], where the lemma is stated as Theorem A.7.

We end this section with the following Sobolev-type inequality:

Lemma 4.7. *Let $w \in C_0^\infty$, then*

$$\|w\|_{L^5_T L^\infty_x} \leq CT^\alpha [\|w\|_{L^5_x L^{10}_T} + \|D_x^{\frac{1}{4}} w\|_{L^5_x L^{10}_T}] \tag{36}$$

for some $\alpha > 0$.

Proof. By definition

$$\|w\|_{L^5_T L^\infty_x} = \Big(\int_{-T}^{T} \|w\|^5_{L^\infty_x}\, dt\Big)^{1/5}$$

and by the Sobolev's theorem for fractional derivatives (see [21]),

$$\|w\|_{L^5_T L^\infty_x} \leq C\Big(\int_{-T}^{T} [\|w\|^5_{L^5_x} + \|D_x^{1/4} w\|^5_{L^5_x}]\, dt\Big)^{1/5} = C[\|w\|_{L^5_x L^5_T} + \|D_x^{1/4} w\|_{L^5_x L^5_T}]$$

and by Hölder's inequality

$$\leq C\big[T^\beta \int_{-\infty}^{\infty} \Big(\int_{-T}^{T} |D_x^{1/4} w|^{10}\, dt\Big)^{1/2} dx\big]^{1/5} + C\big[T^\beta \int_{-\infty}^{\infty} \Big(\int_{-T}^{T} |w|^{10}\, dt\Big)^{1/2} dx\big]^{1/5}$$
$$= CT^\alpha [\|w\|_{L^5_x L^{10}_T} + \|D^{1/4} w\|_{L^5_x L^{10}_T}].$$

5. Proof of the results.

Proof of Theorem 3.1. The argument we present is a generalization of the ideas used by Kenig, Ponce and Vega in order to prove Theorem 2.3 in [14]. We restrict ourselves to the most interesting case, $s = 1/4$. The proof of the general case $s \geq 1/4$ follows by interpolation. From (16)–(20), where we drop the index s, we define $\Omega^T(w) = \max_{j=1,\dots,5} \mu_j^T(w)$, and $Y_T = \{w \in C([-T,T] : H^{\frac{1}{4}}(\mathbb{R}))/\Omega^T(w) < \infty\}$. For any fixed $u_0 \in H^{1/4}(\mathbb{R})$, we denote by $\Phi(v) = \Phi_{u_0}(v)$ the solution of the linear inhomogeneous IVP

$$\partial_t u + \partial_x^3 u + v^2 g(v) \partial_x v = 0, \quad u(x,0) = u_0(x) \;\; x,t \in \mathbb{R}, \tag{37}$$

where $v \in Y^a_T$,

$$Y^a_T = \{w \in C([-T,T] : H^{1/4}(\mathbb{R}))/\Omega^T(w) \leq a\}. \tag{38}$$

We show that there exist $a > 0$ and $T > 0$ (depending only on $\|u_0\|_{H^{1/4}}$) such that if $v \in Y_T^a$, then $u = \Phi(v) \in Y_T^a$ and $\Phi : Y_T^a \to Y_T^a$ is a contraction. By the Duhamel principle the IVP (37) is equivalent to the integral equation

$$u(t) = W(t)u_0 - \int_0^t W(t-t')\big(v^2 g(v)\frac{\partial v}{\partial x}\big)(t')\,dt', \tag{39}$$

hence the solution u of the IVP (1) is a fixed point for the operator Φ defined above.

Assume now that $v \in Y_T$. By Lemma 4.1

$$\Omega^T(\Phi(w)) \le C\|u_0\|_{H^s} + \Omega^T\big(\int_0^\infty W(t-t')v^2 g(v)\frac{\partial v}{\partial x}\,dt'\big). \tag{40}$$

The heart of the proof is in establishing the following inequality:

$$\Omega^T\big(\int_0^\infty W(t-t')v^2 g(v)\frac{\partial v}{\partial x}\,dt'\big) \le CT^\gamma(\|g\|_\infty + \|g'\|_\infty)(\Omega^T(v))^k, \tag{41}$$

for some $\gamma > 0$, $k \in \mathbb{N}$. We set

$$w(t) = \int_0^t W(t-t')\big(v^2 g(v)\frac{\partial v}{\partial x}\big)(t')\,dt'.$$

Again by Lemma 4.1 we have

$$\mu_j^T(w) \le \int_{-T}^T \big(\big\|D^{\frac{1}{4}}\big(v^2 g(v)\frac{\partial v}{\partial x}\big)\big\|_2 + \big\|v^2 g(v)\frac{\partial v}{\partial x}\big\|_2\big)\,dt', \tag{42}$$

for $j = 1, \dots, 5$. We reduced our problem to finding appropriate estimates for

$$\int_{-T}^T \big\|D^{\frac{1}{4}}\big(v^2 g(v)\frac{\partial v}{\partial x}\big)\big\|_2\,dt', \tag{43}$$

$$\int_{-T}^T \big\|v^2 g(v)\frac{\partial v}{\partial x}\big\|_2\,dt'. \tag{44}$$

For (44) we use

$$\begin{aligned}
\big\|v^2 g(v)\frac{\partial v}{\partial x}\big\|_{L^2_T L^2_x} &\le \big(\int_{-\infty}^\infty \int_{-T}^T \big|v^2 g(v)\frac{\partial v}{\partial x}\big|^2(x,t')\,dt'\,dx\big)^{\frac{1}{2}} \\
&\le C\|g\|_\infty\big(\int_{-\infty}^\infty \int_{-T}^T |v|^4\big|\frac{\partial v}{\partial x}\big|^2(x,t)\,dt\,dx\big)^{\frac{1}{2}} \\
&\le C\|g\|_\infty\big(\int_{-\infty}^\infty \big(\sup_{t\in[-T,T]}|v|\big)^4\big\|\frac{\partial v}{\partial x}\big\|^2_{L^2_T}\,dx\big)^{\frac{1}{2}} \\
&= C\|g\|_\infty\|v\|^2_{L^4_x L^\infty_T}\big\|\frac{\partial v}{\partial x}\big\|_{L^\infty_x L^2_T} = c\|g\|_\infty(\mu_5^T(v))^2(\mu_2^T(v)),
\end{aligned}$$

so that

$$\big\|v^2 g(v)\frac{\partial v}{\partial x}\big\|_{L^2_T L^2_x} \le C\|g\|_\infty (\Omega^T(v))^3. \tag{45}$$

In order to estimate (43) we write $g(u) = g(0) + f(u)u$ where $f(u) = \int_0^1 g'(\theta u)\, d\theta$. Then (43) becomes

$$\int_{-T}^{T} \big\|D^{1/4}\big([v^2 g(0) + v^3 f(v)]\frac{\partial v}{\partial x}\big)\big\|_2(t')\, dt' \le CT^{\frac12}\|g\|_\infty \big\|D^{1/4}\big(v^2\frac{\partial v}{\partial x}\big)\big\|_{L^2_x L^2_T} + C\int_{-T}^{T}\big\|D_x^{1/4}\big(v^3 f(v)\frac{\partial v}{\partial x}\big)\big\|_2(t')\, dt'.$$

To estimate the first term of the right-hand side we use the argument in the proof of Theorem 2.3 in [14]. By the Leibeniz rule stated in Theorem A.9 of [14], with $f = v^2$, $g = \partial_x v$, $s = s_1 = 1/4$, $p = q = 2$, we obtain the following string of inequalities:

$$\begin{aligned}
&\big\|D^{1/4}\big(v^2\frac{\partial v}{\partial x}\big)\big\|_{L^2_x L^2_T}\\
&\le C\big\|D_x^{1/4}(v^2)\big\|_{L_x^{\frac{20}{9}} L_T^{10}}\big\|\frac{\partial v}{\partial x}\big\|_{L_x^{20} L_T^{\frac52}} + C\big\|v^2\big\|_{L^2_x L^\infty_T}\big\|D_x^{1/4}\frac{\partial v}{\partial x}\big\|_{L^\infty_x L^2_T}\\
&\le C\|v\|_{L^4_x L^\infty_T}\big\|D_x^{1/4}v\big\|_{L^5_x L^{10}_T}\mu_5^T(v) + C(\mu_3^T(v))^2\mu_2^T(v) \le C(\Omega^T(v))^3;
\end{aligned}$$

that is,

$$\big\|D^{1/4}\big(v^2\frac{\partial v}{\partial x}\big)\big\|_{L^2_x L^2_T} \le C(\Omega^T(v))^3. \tag{46}$$

To estimate the second term we follow instead the idea used to prove Theorem 2.15 in [14]. Using again the Leibniz rule we can write

$$\begin{aligned}
&\big\|D^{1/4}\big(v^3 f(v)\frac{\partial v}{\partial x}\big)\big\|_{L^2_x}\\
&\le C\|vf(v)\|_{L^\infty_x}\big\|D_x^{1/4}\big(v^2\frac{\partial v}{\partial x}\big)\big\|_{L^2_x} + C\big\|D_x^{1/4}(vf(v))v^2\frac{\partial v}{\partial x}\big\|_{L^2_x} = A_0 + A_1.
\end{aligned}$$

First we consider the term A_0. Using the fact that $\|f\|_\infty \le C\|g'\|_\infty$ we can write

$$\begin{aligned}
\int_{-T}^{T} A_0(t)\, dt &\le \|g'\|_\infty \int_{-T}^{T}\|v\|_{L^\infty_x}\big\|D^{1/4}\big(v^2\frac{\partial v}{\partial x}\big)\big\|_{L^2_x}\, dt\\
\text{and by Hölder's inequality } &\le CT^\alpha\|g'\|_\infty\|v\|_{L^5_T L^\infty_x}\big\|D_x^{1/4}\big(v^2\frac{\partial v}{\partial x}\big)\big\|_{L^2_x L^2_T}\\
\text{and by Lemma 4.7 and (46) } &\le CT^\alpha\|g'\|_\infty\mu_4^T(v)(\Omega^T(v))^3.
\end{aligned}$$

Now we consider A_1. Using the chain rule for fractional derivatives as stated in Lemma 4.6, with $F(v) = vf(v)$, $h = v^2\partial_x v$ and $1 < r < 5/4$, it is easy to obtain the following string of inequalities:

$$\int_{-T}^{T} A_1(t)\,dt = C\int_{-T}^{T} \|f(v) + vf'(v)\|_{L_x^\infty}\|v\|_{L_x^\infty}\big\|D^{1/4}(v)\big(M\big(v\frac{\partial v}{\partial x}\big)^{2r}\big)^{1/2r}\big\|_{L_x^2}\,dt$$

and again by Hölder's inequality

$$\leq CT^\beta[\|g'\|_\infty + \|g''\|_\infty\|v\|_{L_T^5 L_x^\infty}]\|v\|_{L_T^5 L_x^\infty}\big\|D_x^{1/4}(v)\big(M\big(v\frac{\partial v}{\partial x}\big)^{2r}\big)^{1/2r}\big\|_{L_x^2 L_T^2}$$

and by Lemma 4.7 and Hölder's inequality once more

$$\leq CT^\beta[\|g'\|_\infty + \|g''\|_\infty\mu_4^T(v)](\mu_4^T(v))^2\big\|\big(M\big(v\frac{\partial v}{\partial x}\big)^{2r}\big)^{1/2r}\big\|_{L_x^{\frac{10}{3}} L_T^{\frac{5}{2}}}$$

and by Lemma A.3 in [14]

$$\leq CT^\beta[\|g'\|_\infty + \|g''\|_\infty\mu_4^T(v)](\mu_4^T(v))^2\mu_3^T(v)\mu_5^T(v).$$

The collection of the above estimates leads to

$$\int_{-T}^{T}\big\|D^{\frac14}\big(v^2g(v)\frac{\partial v}{\partial x}\big)\big\|_{L_x^2}\,dt' \leq T^\gamma C(g, g', g'', \Omega^T(v))(\Omega^T(v))^3, \tag{47}$$

where $\gamma > 0$ and

$$\begin{aligned} &C(g, g', g'', \Omega^T(v)) \\ &= C[\|g\|_\infty + \|g\|_\infty\Omega^T(v) + \|g'\|_\infty(\Omega^T(v))^2 + \|g''\|_\infty(\Omega^T(v))^3]. \end{aligned} \tag{48}$$

Then from (40), (42), (45) and (47) it follows

$$\Omega^T(\Phi(v)) \leq C_0\|u_0\|_{H^{1/4}} + C_1T^\gamma C(g, g', g'', \Omega^T(v))(\Omega^T(v))^3. \tag{49}$$

With a similar argument we can also prove

$$\Omega^T(\Phi(v) - \Phi(\tilde v)) \leq CT^\gamma C(g, g', g'', \Omega^T(v) + \Omega^T(\tilde v))\Omega^T(v - \tilde v). \tag{50}$$

Once the estimates (49) and (50) are established the rest of the proof follows by the same arguments presented in the proof of Theorem 2.3 and 2.4 in [14].

Proof of Theorem 3.2. We prove first part (ii). Here we assume that $F(u) \simeq u^k$ for $k \geq 1$. As in the proof of Theorem 3.1 we define the operator

$$\Phi(v(t)) = W(t)u_0 - \int_0^t W(t-t')F(v)(t')\frac{\partial v}{\partial x}(t')\,dt' \tag{51}$$

acting on the space

$$Y_T^s = \{v \in C([-T,T] : H^s(\mathbb{R}))/\Omega_s^T(v) < \infty\},$$

where

$$\Omega_s^T(v) = \max_{i=1,\dots,5} \mu_{i,s}^T(v), \;\; s > 1/2.$$

We recall that by the Sobolev embedding theorem, if $s > 1/2$,

$$\|v\|_{L_T^\infty L_x^\infty} \leq C\mu_1^T(v). \tag{52}$$

Hence from (49) and (52) it follows

$$\Omega_s^T(\Phi(v)) \leq C_0\|u_0\|_{H^s} + C_1 T^\gamma (\Omega_s^T(v))^k, \tag{53}$$

and from (50) and (52)

$$\Omega^T(\Phi(v) - \Phi(\tilde{v})) \leq CT^\gamma C(\Omega^T(v), \Omega^T(\tilde{v}))\Omega^T(v-\tilde{v}). \tag{54}$$

It is now easy to see that part (ii) of Theorem 3.2 follows from (53) and (54) in the same way as Theorem 3.1 follows from (49) and (50).

We begin the proof of part (i) of Theorem 3.2 by making the extra assumption $F'(0) = 0$. We write $F(u) = u^2 g(u)$, where

$$g(u) = \int_0^1 \int_0^1 \rho F''(\sigma\rho u)\,d\sigma\,d\rho.$$

Again our goal is to establish the key inequalities of type (49) and (50). We assume that $\Omega^T(v) \leq 1$. Then (52) guarantees that the function F lives on a closed interval $[-C_0, C_0]$, and therefore it is bounded with all its derivatives. We can then repeat the argument we presented for Theorem 3.1.

We now consider the more general case when $F'(0) \neq 0$. For simplicity we assume $F'(0) = 1$, so that $F(u) = u + u^2 g(u)$. The operator Φ becomes

$$\Phi(v(t)) = W(t)u_0 + \int_0^t W(t-t')v(t')\frac{\partial v}{\partial x}(t')\,dt' + \int_0^t W(t-t')\big(v^2 g(v)\frac{\partial v}{\partial x}\big)(t')\,dt',$$

and Φ acts on the space of functions $\tilde{Y}_T^s = \{v \in C([-T,T] : H^s(\mathbb{R}))/\tilde{\Omega}_s^T(v) < \infty\}$, with $s > 1/2$ and $\tilde{\Omega}_s^T(v) = \max_{i=0,\dots,5} \mu_{i,s}^T(v)$, with $\mu_0^T(v) = \|\phi(T^{-1}t)v\|_{X_{s,b}}$, $1/2 < b < 3/4$ and $\mu_j^T(v)$, $j = 1, \dots, 5$, as defined in (16)–(20). Again we need to estimate $\tilde{\Omega}_s^T(\Phi(v))$ when $v \in \tilde{Y}_T^s$. Theorem 4.1 gives

$$\tilde{\Omega}_s^T(W(t)u_0) \le C(1 + T^{\frac{(1-2b)}{2}})\|u_0\|_{H^s}. \tag{55}$$

To control the nonlinear term with $v\partial_x v$ we first use Lemma 4.3 to obtain

$$\max_{j=1,\dots,5} \mu_j^T(\int_0^t W(t-t')v(t')\frac{\partial v}{\partial x}(t')\,dt') \le C\mu_0^T(\int_0^t W(t-t')v(t')\frac{\partial v}{\partial x}(t')\,dt'), \tag{56}$$

then Lemmas 4.3 and 4.5 to get

$$\mu_0^T(\int_0^t W(t-t')v(t')\frac{\partial v}{\partial x}(t')\,dt') \le CT^\beta[\mu_0^T(v)]^2, \tag{57}$$

where $\beta = \theta_0 + (1-2b)/2 > 0$. We now estimate the term with $v^2 g(v)\partial_x v$. As we did above we assume that $\tilde{\Omega}_s^T(v) \le 1$, which together with (52) guarantees that the function g lives on a bounded interval $[-C_0, C_0]$. We first observe that using (52), the fact that $\|v\|_{L_T^\infty L_x^\infty} \le C\tilde{\Omega}_s^T(v)$ and an argument similar to the one we presented to prove (47), we can show that

$$\|D_x^s(v^2 g(v)\partial_x v)\|_{L_T^2 L_x^2} \le C(g, g'g'', \tilde{\Omega}_s^T(v))(\tilde{\Omega}_s^T(v))^3. \tag{58}$$

Then we use Lemma 4.3, Lemma 4.4, Lemma 4.2, (45) and (58) to obtain

$$\max_{i=1,\dots,5} \mu_i^T\big(\int_0^t W(t-t')\big(v^2 g(v)\frac{\partial v}{\partial x}\big)(t')\,dt'\big) \le CT^\delta C(g, g'g'', \tilde{\Omega}_s^T(v))(\tilde{\Omega}_s^T(v))^3,$$

for some $\delta > 0$. Similarly by Lemma 4.4, Lemma 4.2, (45) and (58)

$$\begin{aligned}
&\mu_0^T\big(\int_0^t W(t-t')\big(v^2 g(v)\frac{\partial v}{\partial x}\big)(t')\,dt'\big) \le cT^{\frac{1-2b}{2}}\|\phi(T^{-1}\cdot)v^2 g(v)\frac{\partial v}{\partial x}\|_{Xs,b-1} \\
&\quad \le cT^\rho\big[\|v^2 g(v)\frac{\partial v}{\partial x}\|_{L_T^2 L_x^2} + \|D_x^s v^2 g(v)\frac{\partial v}{\partial x}\|_{L_T^2 L_x^2}\big] \\
&\quad \le T^\delta C(g, g', g'', \tilde{\Omega}_s^T(v))(\tilde{\Omega}_s^T(v))^3.
\end{aligned}$$

Collecting the estimates above we finally obtain

$$\begin{aligned}
&\tilde{\Omega}^T(\Phi(v)) \\
&\le C_0 T^{\frac{(1-2b)}{2}}\|u_0\|_{H^s} + C_1 T^\beta(\tilde{\Omega}_s^T(v))^2 + C_2 T^\delta C(g, g', g'', \tilde{\Omega}_s^T(v))(\tilde{\Omega}_s^T(v))^3,
\end{aligned}$$

where $\beta, \delta > 0$. With similar arguments we can also prove an inequality of type (50).

Proof of Theorem 3.3. We restrict ourselves to the most interesting case $s = 1/4$. The proof of the general case $s \geq 1/4$ follows by combining the result for $s = 1/4$ and the fact that in all the estimates needed in the proof the highest derivatives (after interpolation) appear linearly.

If $u_0 \in H^{1/4}(\mathbb{R})$, we denote by $u = \Phi(v) = \Phi_{u_o}(v)$ the solution of the linear inhomogeneous IVP

$$\begin{aligned} &iu_t + \alpha u_{xx} + \beta u_{xxx} + \gamma|v|^2 v + i\delta|v|^2 v_x + i\epsilon v^2 \bar{v}_x = 0, \\ &u(x,0) = u_0(x) \quad x, t \in \mathbb{R}, \end{aligned} \tag{59}$$

where $v \in Y_T^a = \{w \in Y_T / \Omega^T(v) \leq a\}$ as defined in (38). We prove that there exist $T = T(\|u_0\|_{H^{1/4}}) > 0$ and $a = a(\|u_0\|_{H^{1/4}}) > 0$ such that if $v \in Y_T^a$, then $u = \Phi(v) \in Y_T^a$ and $\Phi : Y_T^a \to Y_T^a$ is a contraction. To achieve this goal we proceed as in the proof of Theorem 3.1. We recall that $\Phi(v)(t) = U(t)u_0 - \int_0^t U(t-t')\omega(v)\,dt'$, where $\omega(v) = \gamma|v|^2 v + i\delta|v|^2 v_x + i\epsilon v^2 \bar{v}_x$.

We begin the estimate of $\Omega^T(\Phi(v))$ by considering first the linear term, which by Theorem 4.1 satisfies

$$\Omega^T(U(t)u_0) \leq C\|u_0\|_{H^{1/4}}. \tag{60}$$

To estimate the nonlinear term we follow the same argument presented in the proof of Theorem 3.1. We can easily see that the only fact we need to prove is that

$$\|\omega(v)\|_{L_x^2 L_T^2} + \|D_x^{1/4}\omega(v)\|_{L_x^2 L_T^2} \leq C(\Omega^T(v))^3. \tag{61}$$

We need to consider three terms:

$$\|v^2\bar{v}\|_{L_x^2 L_T^2} + \|D_x^{1/4} v^2 \bar{v}\|_{L_x^2 L_T^2} \tag{62}$$

$$\|v\bar{v}v_x\|_{L_x^2 L_T^2} + \|D_x^{1/4} v\bar{v}v_x\|_{L_x^2 L_T^2} \tag{63}$$

$$\|v^2\bar{v}_x\|_{L_x^2 L_T^2} + \|D_x^{1/4} v^2\bar{v}_x\|_{L_x^2 L_T^2}. \tag{64}$$

For (63) and (64) we repeat the proof given for Theorem 2.3 in [14]

$$\begin{aligned} \|v\bar{v}v_x\|^2 &= \int_{-\infty}^{\infty}\int_{-T}^{T} |v|^4 |v_x|^2\,dt\,dx \leq \int_{-\infty}^{\infty} \big(\max_{[-T,T]} |v|\big)^4 \|v_x\|_{L_T^2}^2\,dx \\ &\leq \|v_x\|_{L_x^\infty L_T^2}^2 \|v\|_{L_x^4 L_T^\infty}^4 \leq (\Omega^T(v))^6. \end{aligned}$$

On the other hand, using Theorem A.8 and A.6 in [14], we deduce the following string of inequalities:

$$\begin{aligned}\|D_x^{1/4}(v\bar{v}v_x)\|_{L^2_xL^2_T} &\le \|D_x^{1/4}(v\bar{v})\|_{L_x^{\frac{20}{9}}L_T^{10}}\|v_x\|_{L_x^{20}L_T^{\frac{5}{2}}} + \|v\bar{v}\|_{L^2_xL^\infty_T}\|D_x^{1/4}v_x\|_{L^\infty_xL^2_T}\\ &\le C\|v\|_{L^4_xL^\infty_T}\|D_x^{1/4}v\|_{L^5_xL^{10}_T}\|v_x\|_{L_x^{20}L_T^{\frac{5}{2}}} + C\|v\|^2_{L^4_xL^\infty_T}\|D_x^{1/4}v_x\|_{L^\infty_xL^2_T}\\ &= C\mu_5^T(v)\mu_4^T(v)\mu_3^T(v) + C(\mu_4^T(v))^2\mu_2^T(v) \le C(\Omega^T(v))^3.\end{aligned}$$

For (64) the same argument works. Let us consider (62). We claim that

$$\|v^2\bar{v}\|_{L^2_xL^2_T} \le CT^\beta(\Omega^T(v))^3. \tag{65}$$

In fact

$$\begin{aligned}\|v^2\bar{v}\|_{L^2_xL^2_T} &= \Big(\int_{-\infty}^{\infty}\int_{-T}^{T}|v|^6\,dt\,dx\Big)^{\frac{1}{2}} = \Big(\int_{-T}^{T}\big(\|v\|^4_{L^\infty_x}\int_{-\infty}^{\infty}|v|^2\,dx\big)\,dt\Big)^{\frac{1}{2}}\\ &\le \|v\|_{L^\infty_TL^2_x}\Big(\int_{-T}^{T}\|v\|^4_{L^\infty_x}\,dt\Big)^{\frac{1}{2}} \le \|v\|_{L^\infty_TL^2_x}T^\alpha\Big(\int_{-T}^{T}\|v\|^5_{L^\infty_x}\,dt\Big)^{\frac{2}{5}}\\ \text{and by Lemma 4.7 } &\le T^\beta\mu_1^T(v)(\mu_4^T(v))^2 \le T^\beta C(\Omega^T(v))^3.\end{aligned}$$

On the other hand, using the chain rule as in Lemma 4.6 with $h = 1$,

$$\begin{aligned}\|D_x^{1/4}(|v|^2v)\|_{L^2_TL^2_x} &\le C\Big(\int_{-T}^{T}\big[\|F'(v)\|_{L^\infty_x}\|D_x^{1/4}v\|_{L^2_x}\big]^2\,dt\Big)^{\frac{1}{2}}\\ &\le C\Big(\int_{-T}^{T}\big[\|v\|^2_{L^\infty_x}\|D_x^{1/4}v\|_{L^2_x}\big]^2\,dt\Big)^{\frac{1}{2}}\\ \text{and by Hölder's inequality } &\le CT^\alpha\|v\|^2_{L^5_TL^\infty_x}\|D_x^{1/4}v\|_{L^\infty_TL^2_x}\\ \text{and by Lemma 4.7 } &\le CT^\alpha\mu_1^T(v)(\mu_4^T(v))^2 = CT^\alpha(\Omega^T(v))^3,\end{aligned}$$

where $F(v) = |v|^2v$ and $|F'(v)| \le c|v|^2$.

Proof of Theorem 3.4. This proof is very much like the one we presented for Theorem 3.2. Below we describe briefly the steps in which we need to give some extra arguments.

We first define the space $\tilde{X}_{s,b}$, the analogy of the space $X_{s,b}$ in Definition 2.1. The space $\tilde{X}_{s,b}$ is the closure of the set of Schwartz functions under the norm

$$\|f\|_{X_{s,b}} = \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|\tau-\beta\xi^3-\alpha\xi^2|)^{2b}(1+|\xi|)^{2s}|\hat{f}(\xi,\tau)|^2\,d\xi\,d\tau\Big)^{\frac{1}{2}}. \tag{66}$$

It is not hard to show that (21) in Theorem 4.1, Lemmas 4.2, 4.3 and 4.4 hold with the space $X_{s,b}$ replaced with $\tilde{X}_{s,b}$. If we repeat the argument presented for the proof of Theorem 3.2 we find that the only new inequalities we need to obtain are

$$\int_{-T}^{T} [\|\gamma(v)v^2\bar{v}\|_{L_x^2} + \|D_x^{1/4}\gamma(v)v^2\bar{v}\|_{L_x^2}]\,dt \le CT^{\alpha}(\|\gamma\|_\infty, \|\gamma'\|_\infty)(\Omega^T(v))^4 \tag{67}$$

and for $(\tilde{\Omega}_s^T(v)) < 1$, $s > 1/2$

$$\|\gamma(v)v^2\bar{v}\|_{L_x^2L_T^2} + \|D_x^s\gamma(v)v^2\bar{v}\|_{L_x^2L_T^2} \le C(\|\gamma\|_\infty, \|\gamma'\|_\infty)\tilde{\Omega}_s^T(v)^k. \tag{68}$$

To prove (67) we use first Cauchy-Schwarz and (62) to obtain

$$\int_{-T}^{T} \|\gamma(v)v^2\bar{v}\|_{L_x^2}\,dt \le CT^{\frac{1}{2}+\beta}\|\gamma\|_\infty(\Omega^T(v))^3. \tag{69}$$

Then we set $G(v) = \gamma(v)|v|^2v$ so that

$$|G'(v)| \le C(\|\gamma\|_\infty + \|\gamma'\|_\infty|v|)|v|^2. \tag{70}$$

Using the chain rule stated in Lemma 4.6 we have

$$\int_{-T}^{T} \|D_x^{1/4}(\gamma(v)v^2\bar{v})\|_{L_x^2}\,dt \le C\int_{-T}^{T} \|G'(v)\|_{L_x^\infty}\|D_x^{1/4}v\|_{L_x^2}\,dt$$

$$\le C\int_{-T}^{T} ||\gamma||_\infty\|v\|_{L_x^\infty}^2\|D_x^{1/4}v\|_{L_x^2}\,dt + C\int_{-T}^{T} ||\gamma'||_\infty\|v\|_{L_x^\infty}^3\|D_x^{1/4}v\|_{L_x^2}\,dt$$

and by Hölder's inequality

$$\le CT^{\alpha}||\gamma||_\infty\|v\|_{L_T^5L_x^\infty}^2\|D_x^{1/4}v\|_{L_T^\infty L_x^2} + CT^{\rho}\|\gamma'\|_\infty\|v\|_{L_T^5L_x^\infty}^3\|D_x^{1/4}v\|_{L_T^\infty L_x^2}$$

$$\le CT^{\beta}\mu_1^T(v)(\mu_4^T(v))^3 = CT^{\beta}(\Omega^T(v))^4 \quad \text{by Lemma 4.7 .}$$

To prove (68) we use a similar argument and the fact that for $s > 1/2$

$$\|u\|_{L_x^\infty L_T^\infty} \le C\tilde{\Omega}_s^T(v). \tag{71}$$

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