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FREE ENERGY AND DOMAIN OF DEPENDENCE FOR WEAKLY COMPRESSIBLE VISCOELASTIC FLUIDS

Giorgio Gentili
Dipartimento di Matematica, P.zza di P.ta S.Donato 5, 40127 Bologna, Italy

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Abstract. We study the linear behavior of a weakly compressible viscoelastic fluid. Within an approximate theory of thermodynamics, compatible with the model, we construct the maximal free energy ψ_M that is shown to induce a "natural" norm $\|\cdot\|_M$ on the space state for such a fluid. Successively, since the stress $\mathbf{T}$ turns out to be continuous with respect to $\|\cdot\|_M$, we employ ψ_M to obtain some stability and domain of dependence results.

1. Introduction. Recent papers (e.g., [3], [4], [12]) proposed viscoelastic models in presence of weak compressibility, by using a pressure-density constitutive equation based upon a first-order Taylor expansion of the mass density ρ with respect to the pressure p, viz.,

$$\rho = \rho_0\big[1 + \frac{p - p_0}{\beta}\big]$$

and a linear approximation of the equation deriving from the balance of mass

$$\dot{p} + \beta\nabla\cdot\mathbf{v} = 0\ .$$

In this paper we describe the linear behavior of a weakly compressible viscoelastic fluid under isothermal conditions by using a linear expansion of the inverse of the mass density $r = \frac{1}{\rho}$, with respect to a dimensionless variable τ related to the pressure variation

$$r = \frac{1}{\rho_0}(1 - \tau) \tag{1.1}$$

so that the balance of mass is approximated by

$$\dot{\tau} + \nabla\cdot\mathbf{v} = 0.$$

Furthermore, following [1], [2], [5] and [8], the linear extra stress $\mathbf{V}$ is assumed to obey a hereditary law of Boltzmann's type

$$\mathbf{V} = \rho_0\int_0^\infty \lambda(s)\nabla\cdot\mathbf{v}^t(s)\,ds\mathbf{I} + 2\rho_0\int_0^\infty \mu(s)\nabla\mathbf{v}^t(s)\,ds\ , \tag{1.2}$$

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where $\lambda, \mu \in L^1(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+)$.

The main part of this paper is the construction and the study of a free energy ψ_M, consistent with such a model. By virtue of its properties, ψ_M is shown to induce a norm $\|\cdot\|_M$ on the space of the state variables. Moreover, since the constitutive parameters enter the definition of ψ_M, $\|\cdot\|_M$ turns out to be very appropriate since it is strictly related to the characteristic of the fluid.

For this sake, we first show approximated thermodynamic relations consistent with the approximate constitutive equations (1.1)–(1.2). In particular, we obtain the properties that a functional must satisfy in order to be considered a free energy for such a fluid and find again the thermodynamic restrictions on the Fourier transforms of λ and μ.

Then we exhibit an explicit formula for a free energy ψ_M, using an idea of [6]. ψ_M turns out to be the maximal one among the possible free energies, and, by virtue of its properties, it can be used to define a "natural" norm $\|\cdot\|_M$ on the state space. It is of interest to point out that, unlike the usual norms for materials with fading memory, this one is objective, since it does not depend on an arbitrary choice of an influence function, but the weight functions are the Fourier transforms of the memory kernels λ and μ. Furthermore, the stress $\mathbf{T}$ is continuous on the state space Σ with respect to such a norm.

In the remainder of the paper, we consider stability questions. Several authors studied stability problems for compressible and incompressible viscoelastic fluids obeying various constitutive equations (see e.g., [7], [10], [11], [13], [15], [14]). Our model leads to a purely hyperbolic evolution problem, as it occurs in [3] and [4], since the fluid is compressible and because of the regularity properties of λ and μ in (1.2). In such a context, we obtain a stability result to the evolution problem and derive a domain of dependence inequality by employing the "natural" norm $\|\cdot\|_M$ and requiring just the thermodynamic restrictions on the constitutive parameters and those conditions, on the initial data, that make $\psi_M(0)$ well defined.

It is interesting to observe that the domain of dependence inequality can be thought of as a further proof of the pure hyperbolicity of the problem. In fact, we prove that in the absence of body forces, the energy of the fluid within the ball B_R of radius R at time T is bounded above by the *initial* energy of the fluid within the ball B_{R+CT}, where C is a constant that is lower bounded by the speed of sound, by $\lambda(0)$ and by $\mu(0)$.

2. Weak compressibility and linear conditions. As is known ([7], [12]), the configuration of a compressible viscoelastic fluid can be described by means of the mass density $\rho(\mathbf{x}, t)$ and of the strain history $\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top)$ ($\mathbf{u}$ is the displacement) relative to its present value, i.e., by means of the couple $(\rho, \bar{\mathbf{E}}^t)$ where the relative strain history $\bar{\mathbf{E}}^t$ is defined by $\bar{\mathbf{E}}^t(\mathbf{x}, s) = \mathbf{E}(\mathbf{x}, t-s) - \mathbf{E}(\mathbf{x}, t)$.

For the purpose of this paper, we find more suitable the employment of $r = \frac{1}{\rho}$ rather than ρ, so that henceforth the material state of the fluid, at time t at the point $\mathbf{x}$, will be described by $\sigma(\mathbf{x},t) = (r(\mathbf{x},t), \bar{\mathbf{E}}^t(\mathbf{x},\cdot))$ whereas the set of all the admissible states σ will be denoted by Σ.

For simplicity we assume the material to be isotropic and homogeneous and omit the dependence on the spatial variable $\mathbf{x}$. The constitutive equation for the stress is given by

$$\mathbf{T} = -p(r)\mathbf{I} + \mathbf{V}(r, \bar{\mathbf{E}}^t), \tag{2.1}$$

where $\mathbf{I}$ is the identity tensor. Remember that, in correspondence of constant strain histories $\mathbf{E}^t = \mathbf{E}^\dagger$ ($\mathbf{E}^\dagger$ is such that $\mathbf{E}^\dagger(s) = \mathbf{E}\ \forall s > 0$), the relative strain history vanishes $\bar{\mathbf{E}}^t = \mathbf{0}^\dagger$ and so does the extra stress (see [2])

$$\mathbf{V}(r, \mathbf{0}^\dagger) = \mathbf{0}. \tag{2.2}$$

Moreover, in the linear approximation, the extra stress $\mathbf{V}$ can be expressed by (see [1], [2] and [8])

$$\mathbf{V}(r, \bar{\mathbf{E}}^t) = \int_0^\infty \lambda_1(r,s) tr(\bar{\mathbf{E}}^t(s))\, ds \mathbf{I} + 2\int_0^\infty \mu_1(r,s)\bar{\mathbf{E}}^t(s)\, ds. \tag{2.3}$$

We define the dimensionless variable

$$\tau = \frac{p - p_0}{\rho_0 c^2}, \tag{2.4}$$

where $p_0 = p(\rho_0)$ is the pressure at the given reference state σ_0 and $c^2 = \frac{\partial p}{\partial \rho}\big|_{\rho=\rho_0} > 0$ represents the square of the speed of the sound when the mass density of the fluid is ρ_0. Hence $\rho_0\tau$ denotes the ratio between the pressure variation from σ_0 and the speed of the sound at σ_0.

Thus the pressure takes the form

$$p = p_0 + \rho_0 c^2 \tau. \tag{2.5}$$

The weak compressibility assumption is that the mass density has very small variations as the pressure varies, and it is related to the fact that the speed of the sound is large but finite. Hence (2.4) implies that τ is assumed to be small since $\tau = O(c^{-2})$. Analogous to what [4] did for the density, we expand r into a first-order Taylor approximation

$$r = \frac{1}{\rho_0}(1 - \tau). \tag{2.6}$$

Observe that our method is equivalent to the one explained by [4] and [12]. In fact, by comparing equation (2.4) and the expression for the density $\rho = \rho_0(1+\tau)$ related to (2.6), with equations (5) of [4] and (29) of [12], viz.,

$$\rho = \rho_0[1 + (p - p_0)\varepsilon] = \rho_0\left[1 + \frac{p - p_0}{\beta}\right],$$

we obtain an equivalence, since the isothermal compressibility ε can be related to the speed of the sound by $\varepsilon^{-1} = \beta = \rho_0 c^2$. For simplicity, we assume λ_1 and μ_1 do not depend on the linear variation of r from $\frac{1}{\rho_0}$, so that, if we define λ' and μ' as

$$\lambda_1(\frac{1}{\rho_0}, s) = \rho_0 \lambda'(s) \quad \text{and} \quad \mu_1(\frac{1}{\rho_0}, s) = \rho_0 \mu'(s),$$

then (2.3) is approximated by $\mathbf{V}(r, \bar{\mathbf{E}}^t) = \rho_0 \mathbf{V}_l(\bar{\mathbf{E}}^t) + O(c^{-4})$ where

$$\mathbf{V}_l(\bar{\mathbf{E}}^t) = \int_0^\infty \lambda'(s) tr(\bar{\mathbf{E}}^t(s))\, ds \mathbf{I} + 2\int_0^\infty \mu'(s) \bar{\mathbf{E}}^t(s)\, ds$$

and the linearized stress tensor takes the form

$$\mathbf{T} = -\big(p_0 + \rho_0 c^2 \tau\big)\mathbf{I} + \rho_0 \int_0^\infty \lambda'(s)\ bI : \bar{\mathbf{E}}^t(s)\, ds \mathbf{I} + 2\rho_0 \int_0^\infty \mu'(s) \bar{\mathbf{E}}^t(s)\, ds. \tag{2.7}$$

We assume λ' and μ' belong to $L^1(\mathbf{R}^+)$ as well as the functions λ and μ defined by

$$\lambda(t) = -\int_t^\infty \lambda'(s)\, ds \qquad \mu(t) = -\int_t^\infty \mu'(s)\, ds\ ,$$

so that $\lambda, \mu \in L^1(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+)$. Under such hypotheses, integration by parts yields

$$\mathbf{T} = -\big(p_0 + \rho_0 c^2 \tau\big)\mathbf{I} + \rho_0 \int_0^\infty \lambda(s) \underline{\mathrm{I}} : \mathbf{D}^t(s)\, ds \mathbf{I} + 2\rho_0 \int_0^\infty \mu(s) \mathbf{D}^t(s)\, ds, \tag{2.8}$$

where $\mathbf{D} = \frac{1}{2}\big(\nabla \mathbf{v} + (\nabla \mathbf{v})^\top\big)$.

Since $\mathbf{T}$ must by symmetric, (1.2) and the extra stresses of (2.7) and (2.8) are equivalent expressions. Moreover, observe that (2.7) yields a hyperbolic problem not only because of the compressibility but also because the classical Lame' coefficients do not occur and are replaced by two integral kernels, so that the coefficients leading to a parabolic equation are replaced by convolution products with $L^1 \cap L^\infty$-functions.

We conclude the section giving some notations we use in the sequel.

Henceforth, given any second-order tensor $\mathbf{A}$, A denotes its trace and $\mathring{\mathbf{A}}$ denotes its trace-free part; viz.,

$$A = tr(\mathbf{A}) \quad , \quad \mathring{\mathbf{A}} = \mathbf{A} - \frac{1}{3} A \mathbf{I}.$$

In view of that (2.7) can be rewritten as

$$\mathbf{T} = -\rho_0 \big(\frac{p_0}{\rho_0} + c^2 \tau - \int_0^\infty \kappa'(s) \bar{E}^t(s)\, ds\big)\mathbf{I} + 2\rho_0 \int_0^\infty \mu'(s) \mathring{\bar{\mathbf{E}}}^t(s)\, ds, \tag{2.9}$$

where

$$\kappa(s) = \lambda(s) + \frac{2}{3}\mu(s)\ . \tag{2.10}$$

Furthermore, let $f(a(t), \mathbf{B}^t)$ be a continuous and differentiable functional, with $a(t) \in \mathbf{R}$ and $\mathbf{B}^t(s) \in Sym$ $\forall s \in \mathbf{R}$. We denote with $\mathrm{D}_a f$ and $\delta_B f(a, \mathbf{B}^t|h)$ the partial derivative of f with respect to a holding $\mathbf{B}^t$ fixed, and the partial (Fréchet) derivative of f with respect to $\mathbf{B}^t$, holding a fixed.

Thus we can define $J_B f(a, \mathbf{B}^t) \in \mathbf{R}^3$ such that

$$J_B f(a, \mathbf{B}^t) : \mathbf{D} = \delta_B f(a, \mathbf{B}^t|\mathbf{D}^\dagger) \qquad \forall \mathbf{D} \in Sym\ .$$

Finally, we introduce some notation about Fourier and Laplace transforms. For any function $f : \mathbf{R} \to \mathbf{R}^n$, $\hat{f}$ denotes the formal Fourier transform; viz.,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt.$$

If $f : [0, \infty) \to \mathbf{R}^n$ then f_c and f_s are respectively the formal half-range Fourier sine and cosine transforms

$$f_s(\omega) = \int_0^\infty f(t) \sin \omega t\, dt, \qquad f_c(\omega) = \int_0^\infty f(t) \cos \omega t\, dt.$$

Unless otherwise stated we identify functions f on $[0, \infty)$ with their causal extensions so that $\hat{f}(\omega) = f_c(\omega) - i f_s(\omega)$. Let $\mathbf{C}$ be the complex plane, z^* be the complex conjugate of $z \in \mathbf{C}$ and $\mathbf{C}^+ = \{z \in \mathbf{C} : \Re\{z\} > 0\}$. For each $z \in \mathbf{C}^+$ and $f : [0, \infty) \to \mathbf{R}^n$, $\tilde{f}$ denotes the Laplace transform of f; viz.,

$$\tilde{f}(\omega) = \int_0^\infty f(t) e^{-zt}\, dt,$$

and the Fourier transform $\hat{f} = f_c - i f_s$ of f can be thought as the limit of $\tilde{f}$ as $\Re\{z\} \to 0$.

3. Approximate thermodynamics. The balance of mass equation can be written in terms of r as follows:

$$\dot{r} = r \nabla \cdot \mathbf{v} \tag{3.1}$$

and the thermodynamic inequality, for sufficiently regular processes, is

$$\dot{\psi} \leq r\mathbf{T} : \mathbf{D}, \tag{3.2}$$

where ψ denotes the free energy and $\mathbf{D} = \dot{\mathbf{E}}$. When we consider a compressible viscoelastic fluid, by virtue of (2.1) and of (3.1), the inequality (3.2) becomes

$$\dot{\psi}(r, \bar{\mathbf{E}}^t) \leq -p(r)\dot{r} + r\mathbf{V}(r, \bar{\mathbf{E}}^t) : \mathbf{D}. \tag{3.3}$$

As a consequence, for a material of type (2.1), any history yields a greater free energy than the one corresponding to a constant strain history (i.e., a null relative strain history) with the same present value; that is,

$$\psi(r,\bar{\mathbf{0}}^{\dagger}) \leq \psi(r,\bar{\mathbf{E}}^{t}). \tag{3.4}$$

Moreover, (3.3) implies the following relations:

$$\frac{\partial\psi}{\partial r} = -p\ , \quad J_E\psi(r,\bar{\mathbf{E}}^t) = -r\mathbf{V}(r,\bar{\mathbf{E}}^t)\ , \quad \delta_E\psi(r,\bar{\mathbf{E}}^t|\mathbf{D}) \leq 0. \tag{3.5}$$

Therefore, an approximated theory, whose aim is linearizing these results about $\sigma_0 = (\frac{1}{\rho_0},\bar{\mathbf{0}}^{\dagger})$, should have to consider a free energy $\psi(\frac{1}{\rho_0}(1-\tau),\bar{\mathbf{E}}^t)$ such that

$$\frac{\partial\psi}{\partial\tau} = \frac{p}{\rho_0}\ , \quad J_E\psi = -\mathbf{V}_l\ , \quad \delta_E\psi(\frac{1}{\rho_0}(1-\tau),\bar{\mathbf{E}}^t|\mathbf{D}) \leq 0.$$

In particular, since we approximate the stress with the sum $\mathbf{T} = \mathbf{T}_0 + \mathbf{T}_1$ where

$$\mathbf{T}_0 = -\frac{p_0}{\rho_0}\mathbf{I} \quad \text{and} \quad \mathbf{T}_1 = -\rho_0 c^2\tau\mathbf{I} + \rho_0\mathbf{V}_l(\bar{\mathbf{E}}^t). \tag{3.6}$$

We consider a second-order approximation for the free energy $\psi = \psi_1 + \psi_2$ satisfying

$$\dot{\psi}_1 = \frac{p_0}{\rho_0}\dot{\tau}\ , \quad \dot{\psi}_2 \leq c^2\tau\dot{\tau} + \mathbf{V}_l : \mathbf{D} \tag{3.7}$$

so that $\psi_1 = \frac{p_0}{\rho_0}\tau$ and the quadratic part ψ_2 is of the type

$$\psi_2(\tau,\bar{\mathbf{E}}^t) = \frac{1}{2}c^2\tau^2 + \varphi(\bar{\mathbf{E}}^t), \tag{3.8}$$

where $\varphi(\bar{\mathbf{E}}^t)$ satisfies

$$J_E\varphi(\bar{\mathbf{E}}^t) = -\mathbf{V}_l(\bar{\mathbf{E}}^t) \quad , \quad \delta_E\varphi(\bar{\mathbf{E}}^t|\mathbf{D}^t) \leq 0 \tag{3.9}$$

and

$$\varphi(\bar{\mathbf{0}}^{\dagger}) \leq \varphi(\bar{\mathbf{E}}^t). \tag{3.10}$$

Note that the *linear* equation of the motion is given by

$$\dot{\mathbf{v}} = \frac{1}{\rho_0}\nabla\cdot\mathbf{T} + \mathbf{b} = \frac{1}{\rho_0}\nabla\cdot\mathbf{T}_1 + \mathbf{b}\ . \tag{3.11}$$

Hence the linear behavior of a compressible viscoelastic fluid near the state σ_0 can be described just by $\mathbf{T}_1$, so that it is not necessary to use the whole thermodynamic inequality (3.3), but just $(3.7)_2$.

Therefore, henceforth, we consider a viscoelastic fluid whose state is represented by $\sigma = (\tau, \bar{\mathbf{E}}^t)$, whose stress is $\mathbf{T} = \mathbf{T}_1$, namely,

$$\begin{aligned}\mathbf{T} &= \rho_0\big(-c^2\tau\mathbf{I} + \mathbf{V}_l(\bar{\mathbf{E}}^t)\big)\\ &= \rho_0\Big(-c^2\tau\mathbf{I} + \int_0^\infty \kappa'(s)\bar{E}^t(s)\,ds\mathbf{I} + 2\int_0^\infty \mu'(s)\mathring{\bar{\mathbf{E}}}^t(s)\,ds\Big)\end{aligned} \tag{3.12}$$

and whose free energy is $\psi = \psi_2$; i.e., it is defined as follows:

Definition 3.1. If a continuous differentiable functional $\psi : \Sigma \to \mathbf{R}$ is given by

$$\psi(\tau, \bar{\mathbf{E}}^t) = \frac{1}{2}c^2\tau^2 + \varphi(\bar{\mathbf{E}}^t),$$

where φ satisfies (3.9) and (3.10), then it is called a *free energy density* for a linear compressible viscoelastic fluid described by the constitutive equation (3.12). If $(3.9)_2$ is an equation, then ψ is the maximal free energy density.

It is well known that inequality $(3.7)_2$ implies also some restrictions to the parameters of (3.12). In fact, by integrating $(3.7)_2$ in the time during a cyclic process,we obtain the *Clausius property*:

$$\oint \frac{1}{\rho_0}\mathbf{T} : \mathbf{D}\,dt = \oint \mathbf{V}_l : \mathbf{D}\,dt \geq 0. \tag{3.13}$$

Following the method used in many papers (see [8]) we investigate such restrictions by calculating the left-hand side of (3.13) in correspondence of the following strain:

$$\mathbf{E}(t) = \mathbf{E}_1\cos\omega t + \mathbf{E}_2\sin\omega t$$

so that

$$\bar{\mathbf{E}}^t(s) = \mathbf{E}_1(\cos\omega(t-s) - \cos\omega t) + \mathbf{E}_2(\sin\omega(t-s) - \sin\omega t)$$

and

$$\mathbf{D} = \omega(\mathbf{E}_2\cos\omega t - \mathbf{E}_1\sin\omega t)\ .$$

Then (3.13) becomes

$$\begin{aligned}\oint \big[\omega\kappa_s'(\omega)\big(E_1^2\sin^2\omega t + E_2^2\cos^2\omega t\big)\\ + \omega\mu_s'(\omega)\big(|\mathring{\mathbf{E}}_1|^2\sin^2\omega t + |\mathring{\mathbf{E}}_2|^2\cos^2\omega t\big)\big]\,dt \leq 0,\end{aligned}$$

where $|\mathbf{E}|^2 = \mathbf{E} : \mathbf{E}$. The latter inequality implies

$$\omega\kappa_s'(\omega)(E_1^2 + E_2^2) + 2\omega\mu_s'(\omega)(|\mathring{\mathbf{E}}_1|^2 + |\mathring{\mathbf{E}}_2|^2) \leq 0. \tag{3.14}$$

Since E and $\overset{\circ}{\mathbf{E}}$ are independent, (3.14) implies

$$\omega\kappa_s'(\omega) \leq 0 \quad , \quad \omega\mu_s'(\omega) \leq 0 \quad \forall\omega \in \mathbf{R}. \tag{3.15}$$

Observe that inequalities (3.15) and the Fourier transform properties yield

$$\kappa_c(\omega) \geq 0 \quad , \quad \mu_c(\omega) \geq 0 \quad \forall\omega \in \mathbf{R}. \tag{3.16}$$

Moreover, we remark that the Clausius property is an equation only if the process is reversible. Actually reversible processes $\mathcal{P} = (\dot{\tau}, \mathbf{D})$ for a linear viscoelastic fluids are of the type $\mathcal{P}_r = (\dot{\tau}, \mathbf{0})$, so that inequalities (3.15) become

$$\omega\kappa_s'(\omega) < 0 \quad , \quad \omega\mu_s'(\omega) < 0 \quad \forall\omega \neq 0 \tag{3.17}$$

and

$$\kappa_c(\omega) > 0 \quad , \quad \mu_c(\omega) > 0 \qquad \forall\omega \neq 0. \tag{3.18}$$

By using the inverse half-range Fourier transform, from inequalities (3.15) it follows that

$$\mu_0 = \mu(0) = -\frac{2}{\pi}\int_0^\infty \mu_c(\omega)\, d\omega > 0 \tag{3.19}$$

and

$$\kappa_0 = \kappa(0) = -\frac{2}{\pi}\int_0^\infty \kappa_c(\omega)\, d\omega > 0\ . \tag{3.20}$$

Inequalities (3.16) imply some restrictions on the Laplace transform of κ and μ. In fact the Hilbert integral representation for the Laplace transform yields

$$\tilde{f}(z) = \frac{2}{\pi}\int_0^\infty \frac{z}{z^2+\omega^2} f_c(\omega)\, d\omega \quad z \in \mathbf{C}^+$$

so that, if $z = z_1 + iz_2$, we obtain

$$\Re\tilde{f}(z) = \frac{2}{\pi}\int_0^\infty \frac{z_1(z_1^2+z_2^2+\omega^2)}{(z_1^2-z_2^2+\omega^2)^2+4z_1^2z_2^2} f_c(\omega)\, d\omega$$

and thermodynamic restrictions (3.16) yield

$$\Re\{\tilde{\kappa}(z)\} > 0, \quad \Re\{\tilde{\mu}(z)\} > 0, \quad \forall z \in \mathbf{C}^+. \tag{3.21}$$

4. Maximal free energy. In this section we make use of the linearized stress tensor given by (3.12) as it is expressed in terms of Fourier transforms, by applying Plancherel's theorem:

$$\mathbf{T}(\tau, \bar{\mathbf{E}}^t) = -\rho_0 c^2 \tau \mathbf{I} + \frac{2\rho_0}{\pi}\int_0^\infty \kappa_s'(\omega)\bar{E}_s^t(\omega)\, d\omega \mathbf{I} + \frac{4\rho_0}{\pi}\int_0^\infty \mu_s'(\omega)\bar{\overset{\circ}{\mathbf{E}}}{}^t{}_s(\omega)\, d\omega. \tag{4.1}$$

Consider the functional $\psi_M(\tau, \bar{\mathbf{E}}^t) = \frac{1}{2}c^2\tau^2 + \varphi_M(\bar{\mathbf{E}}^t)$, where

$$\begin{aligned}\varphi_M(\bar{\mathbf{E}}^t) = &-\frac{1}{\pi}\int_0^\infty \omega\kappa_s'(\omega)\big[\bar{E}_s^t(\omega)^2 + \bar{E}_c^t(\omega)^2\big]\, d\omega \\ &-\frac{2}{\pi}\int_0^\infty \omega\mu_s'(\omega)\big[|\bar{\overset{\circ}{\mathbf{E}}}{}^t{}_s(\omega)|^2 + |\bar{\overset{\circ}{\mathbf{E}}}{}^t{}_c(\omega)|^2\big]\, d\omega.\end{aligned} \tag{4.2}$$

Theorem 4.1. *The functional* (4.2) *represents a free energy density in the sense of Definition* 3.1. *Moreover, it is the maximal free energy density.*

Proof. In fact,

$$J_E\varphi_M(\bar{\mathbf{E}}^t):\mathbf{B}=\lim_{s\to 0}\frac{d}{ds}\varphi_M(\bar{\mathbf{E}}^t+s\mathbf{B}^\dagger)=-\mathbf{V}_l(\bar{\mathbf{E}}^t):\mathbf{B},$$

where, in view of (3.12) and (4.1),

$$\mathbf{V}_l(\bar{\mathbf{E}}^t)=\frac{2}{\pi}\int_0^\infty \kappa_s'(\omega)\bar{E}^t_s(\omega)\,d\omega\mathbf{I}+\frac{4}{\pi}\int_0^\infty \mu_s'(\omega)\mathring{\bar{\mathbf{E}}}{}^t{}_s(\omega)\,d\omega\ .$$

Thus $(3.9)_1$ is satisfied. Moreover, the thermodynamic restrictions (3.17) on λ and μ ensure that $\varphi_M(\bar{\mathbf{E}}^t)>\varphi_M(\bar{\mathbf{0}}^\dagger)=0$ for all $\bar{\mathbf{E}}^t\neq\bar{\mathbf{0}}^\dagger$, so that also (3.10) is satisfied.

Finally, differentiating ψ_M by time and taking account of (4.1) we obtain

$$\begin{aligned}\dot{\psi}_M =&c^2\tau\dot{\tau}+\frac{2}{\pi}\int_0^\infty \omega\kappa_s'(\omega)\big[\frac{1}{\omega}\bar{E}^t_s(\omega)D(t)\big]\,d\omega\\ &+\frac{4}{\pi}\int_0^\infty \omega\mu_s'(\omega)\big[\mathring{\bar{\mathbf{E}}}{}^t{}_s(\omega):\big(\frac{1}{\omega}\mathring{\mathbf{D}}(t)\big)\big]\,d\omega=c^2\tau\dot{\tau}+\mathbf{V}_l:\mathbf{D}.\end{aligned}\tag{4.3}$$

Therefore, since $\dot{\psi}_M=c^2\tau\dot{\tau}-J_E\varphi_M:\mathbf{D}+\delta_E\varphi_M(\bar{\mathbf{E}}^t|\mathbf{D}^t)$, this is sufficient to show that $(3.9)_2$ is satisfied as an equation and ψ_M is a maximal free energy. □

Since $\varphi_M(\bar{\mathbf{E}}^t)\geq 0$ and it vanishes only if $\bar{\mathbf{E}}^t=\bar{\mathbf{0}}^\dagger$, φ_M can be thought of as a squared norm $\|\bar{\mathbf{E}}^t\|^2_M=\varphi_M(\bar{\mathbf{E}}^t)$ on the space

$$\mathcal{H}_M=\big\{\bar{\mathbf{E}}^t\ :\varphi_M(\bar{\mathbf{E}}^t)<\infty\big\}\ .$$

Thus we have constructed a functional space whose topology strictly depends on the characteristic of the fluid by means of the constitutive parameters entering the definition of φ_M.

$\mathcal{H}_M$ endowed with the norm $\|\cdot\|_M$ turns out to be a Banach space. Moreover, when endowed with the inner product

$$\begin{aligned}\langle\sigma_1,\sigma_2\rangle=&-\frac{1}{\pi}\int_0^\infty \omega\kappa_s'(\omega)\big[\bar{E}^t_{1s}(\omega)\bar{E}^t_{2s}(\omega)+\bar{E}^t_{1c}(\omega)\bar{E}^t_{2c}(\omega)\big]\,d\omega\\ &-\frac{2}{\pi}\int_0^\infty \omega\mu_s'(\omega)\big[\mathring{\bar{\mathbf{E}}}{}^t{}_{1s}(\omega):\mathring{\bar{\mathbf{E}}}{}^t{}_{2s}(\omega)+\mathring{\bar{\mathbf{E}}}{}^t{}_{1c}(\omega):\mathring{\bar{\mathbf{E}}}{}^t{}_{1c}(\omega)\big]\,d\omega\end{aligned}$$

it is a Hilbert space. Observe that $\mathcal{H}_M$ is complete, and in particular it contains L^2-functions.

A consequence of properties of φ_M is that $\psi_M(\tau, \bar{\mathbf{E}}^t) \geq 0$ and it vanishes only if $\tau = 0$ and $\bar{\mathbf{E}}^t = \bar{\mathbf{0}}^\dagger$, so that it defines a norm on the state space $\mathbf{R} \times \mathcal{H}_M$. In particular, such a norm $\|\sigma\|$ on $\mathbf{R} \times \mathcal{H}_M$ is

$$\|\sigma\|^2 = \frac{1}{2}c^2\tau^2 + \|\bar{\mathbf{E}}^t\|_M^2 = \frac{1}{2}c^2\tau^2 + \varphi_M(\bar{\mathbf{E}}^t) = \psi_M(\tau, \bar{\mathbf{E}}^t) > 0, \tag{4.4}$$

for all $\sigma = (\tau, \bar{\mathbf{E}}^t) \neq (0, \mathbf{0}^\dagger)$.

Unlike the usual topologies used in describing material with fading memory, this one has the following important continuity property, which is crucial for applications to stability and continuous dependence problems.

Theorem 4.2. *The stress tensor* $\mathbf{T}$ *is continuous in* $\mathbf{R} \times \mathcal{H}_M$ *with respect to the norm deriving from* ψ_M *by means of* (4.4)*; i.e., there exists a positive constant* C_0 *such that*

$$|\mathbf{T}(\tau, \bar{\mathbf{E}}^t)|^2 \leq 2C_0^2\rho_0^2\psi_M(\tau, \bar{\mathbf{E}}^t)\,. \tag{4.5}$$

Proof. In fact, if we consider $\mathbf{T}$ as expressed in (4.1) we obtain the following inequalities:

$$|\mathbf{T}(\tau, \bar{\mathbf{E}}^t)|^2 = \mathbf{T}(\tau, \bar{\mathbf{E}}^t) : \mathbf{T}(\tau, \bar{\mathbf{E}}^t) \leq 2\rho_0^2\big(3|c^2\tau(t)|^2 + |\mathbf{V}_l(\bar{\mathbf{E}}^t)|^2\big), \tag{4.6}$$

where

$$\begin{aligned}
|\mathbf{V}_l(\bar{\mathbf{E}}^t)|^2 &\leq 2\Big|\frac{2}{\pi}\int_0^\infty \kappa_s'(\omega)\bar{E}_s^t(\omega)\,d\omega\mathbf{I}\Big|^2 + 2\Big|\frac{4}{\pi}\int_0^\infty \mu_s'(\omega)\mathring{\bar{\mathbf{E}}}{}^t{}_s(\omega)\,d\omega\Big|^2 \\
&\leq 12\Big|\frac{2}{\pi}\int_0^\infty \frac{\kappa_s'(\omega)}{\omega}\,d\omega\Big|\Big|\frac{1}{\pi}\int_0^\infty \omega\kappa_s'(\omega)\big(\bar{E}_s^t(\omega)\big)^2\,d\omega\Big| \\
&\quad + 8\Big|\frac{2}{\pi}\int_0^\infty \frac{\mu_s'(\omega)}{\omega}\,d\omega\Big|\Big|\frac{2}{\pi}\int_0^\infty \omega\mu_s'(\omega)|\bar{\mathbf{E}}_s^t(\omega)|^2\,d\omega\Big| \\
&\leq -12\kappa_0\int_0^\infty \omega\kappa_s'(\omega)\big(\bar{E}_s^t(\omega)\big)^2\,d\omega - 8\mu_0\int_0^\infty \omega\mu_s'(\omega)|\bar{\mathbf{E}}_s^t(\omega)|^2\,d\omega.
\end{aligned}$$

Therefore, by virtue of (3.19) and (3.20) and recalling (4.2), we obtain (4.5) with

$$C_0 = \max\{\sqrt{3}\,c, 2\sqrt{3\kappa_0}, 2\sqrt{2\mu_0}\}. \tag{4.7}$$

As a consequence we obtain the following corollary:

Corollary. *If thermodynamic inequalities hold then we have*

$$|(\mathbf{T}\cdot\mathbf{v})\cdot\mathbf{a}| \leq \rho_0 C\big(\frac{1}{2}\mathbf{v}^2 + \psi\big)\,,$$

where $\mathbf{a}$ *is a vector field with* $|\mathbf{a}| = 1$*, for every* $C \geq C_0$*.*

Proof. In fact, using inequality (4.5) we have

$$|(\mathbf{T}\cdot\mathbf{v})\cdot\mathbf{a}| \leq \frac{1}{2}\rho_0 C\mathbf{v}^2 + \frac{1}{2\rho_0 C}|\mathbf{T}|^2 \leq \frac{1}{2}\rho_0 C\mathbf{v}^2 + \rho_0\frac{C_0^2}{C}\psi_M$$

and the thesis follows. □

We remark that Definition 3.1 allows many functionals to be free energies. This is well-known in the theories of materials with fading memory. As an illustration of this point, we show that, under suitable hypotheses, also the functional defined by $\psi_G(\tau, \bar{\mathbf{E}}^t) = \frac{1}{2}c^2\tau^2 + \varphi_G(\bar{\mathbf{E}}^t)$ where

$$\varphi_G(\bar{\mathbf{E}}^t) = -\frac{1}{2}\int_0^\infty \kappa'(s)|\bar{E}^t(s)|^2\,ds - \int_0^\infty \mu'(s)|\mathring{\bar{\mathbf{E}}}^t(s)|^2\,ds$$

is a free energy.

Proposition 4.1. *Suppose that* $\lambda', \mu' < 0$ *and* $\lambda'', \mu'' \geq 0$*. Then the functional* ψ_G *is a free energy in the sense of Definition* 3.1*.*

Proof. Firstly, note that $(3.9)_1$ is obviously satisfied, since

$$\begin{aligned}\lim_{h\to 0}\frac{d}{dh}\varphi_G(\bar{\mathbf{E}}^t + h\mathbf{B}^\dagger) = &-\int_0^\infty \kappa'(s)\bar{E}^t(s)\,ds\;\mathbf{I}:\mathbf{B}\\ &-2\int_0^\infty \mu'(s)\mathring{\bar{\mathbf{E}}}^t(s):\mathring{\mathbf{B}}\,ds = -\mathbf{V}_l:\mathbf{B}\,.\end{aligned}$$

It is also clear that (3.10) holds. Finally, we observe that

$$\begin{aligned}\dot{\varphi}_G + \mathbf{V}_l:\mathbf{D} = &+\int_0^\infty \kappa'(s)\bar{E}^t(s)D^t(s)\,ds + 2\int_0^\infty \mu'(s)\mathring{\bar{\mathbf{E}}}^t(s):\mathring{\mathbf{D}}^t(s)\,ds\\ &-\frac{1}{2}\int_0^\infty \kappa''(s)|\bar{E}^t(s)|^2\,ds - \int_0^\infty \mu''(s)|\mathring{\bar{\mathbf{E}}}^t(s)|^2\,ds.\end{aligned}$$

Our hypotheses on λ and μ ensure the latter expression to be nonpositive. □

Observe that φ_G is also a positive-definite quadratic form and can be considered as the square of a norm $\|\cdot\|_G$ defined on the space

$$\mathcal{H}_G = \left\{\bar{\mathbf{E}}^t\,:\varphi_G(\bar{\mathbf{E}}^t) < \infty\right\}.$$

Obviously, since φ_M corresponds to the *maximal* free energy, it induces the finest topology and the corresponding norm is such that $\|\bar{\mathbf{E}}^t\|_M \geq \|\bar{\mathbf{E}}^t\|_G$, and hence $\mathcal{H}_M \subseteq \mathcal{H}_G$.

The free energy ψ_G has all the properties of ψ_M; in particular, it satisfies the following

Proposition 4.2. *Under the hypotheses of Proposition* 4.1*, the stress tensor* $\mathbf{T}$ *is a continuous functional on* $\mathbf{R}\times\mathcal{H}_G$ *with respect to the norm* $\|\cdot\|$ *such that*

$$\|(\tau,\bar{\mathbf{E}}^t)\|^2=\psi_G(\tau,\bar{\mathbf{E}}^t)=\frac{1}{2}c^2\tau^2+\|\bar{\mathbf{E}}^t\|_G^2\;;$$

moreover, it satisfies

$$|\mathbf{T}(\tau,\bar{\mathbf{E}}^t)|^2\le 2C_0^2\rho_0^2\psi_G(\tau,\bar{\mathbf{E}}^t),\tag{4.8}$$

where C_0 *is given by* (4.7)*.*

Proof. In fact

$$\begin{aligned}
&|\mathbf{V}_l(\bar{\mathbf{E}}^t)|^2\\
&\le 6\int_0^\infty\kappa'(s)\,ds\int_0^\infty\kappa'(s)|\bar{E}^t(s)|^2\,ds+8\int_0^\infty\mu'(s)\,ds\int_0^\infty\mu'(s)|\mathring{\bar{\mathbf{E}}}^t(s)|^2\,ds\\
&=-6\kappa_0\int_0^\infty\kappa'(s)|\bar{E}^t(s)|^2\,ds-8\mu_0\int_0^\infty\mu'(s)|\mathring{\bar{\mathbf{E}}}^t(s)|^2\,ds\le 2C_1^2\varphi_G(\bar{\mathbf{E}}^t),
\end{aligned}$$

where $C_1=\max\{2\sqrt{3\kappa_0},2\sqrt{2\mu_0}\}$. Then recalling (4.7) and relation (4.6), we obtain (4.8). $\square$

In order to apply these results to stability problems it is suitable to write the free energy in terms of the history of the velocity gradient $\mathbf{D}^t$.

Since $\frac{d}{ds}\big(\bar{\mathbf{E}}^t(s)\big)=-\mathbf{D}^t(s)$, by the properties of the Fourier transforms of derivatives, (4.2) is equivalent to

$$\begin{aligned}
\psi_M(\tau,\mathbf{D}^t)=&\frac{1}{2}c^2\tau^2+\frac{1}{\pi}\int_0^\infty\kappa_c(\omega)\big[D_s^t(\omega)^2+D_c^t(\omega)^2\big]\,d\omega\\
&+\frac{2}{\pi}\int_0^\infty\mu_c(\omega)\big[|\mathring{\mathbf{D}}^t{}_s(\omega)|^2+|\mathring{\mathbf{D}}^t{}_c(\omega)|^2\big]\,d\omega;
\end{aligned}\tag{4.9}$$

as well, $\mathbf{T}$ can be rewritten as

$$\begin{aligned}
\mathbf{T}(\tau,\mathbf{D}^t)&=-\rho_0c^2\tau\mathbf{I}+\rho_0\int_0^\infty\kappa(s)D^t(s)\,ds\mathbf{I}+2\rho_0\int_0^\infty\mu(s)\mathring{\mathbf{D}}^t(s)\,ds\\
&=-\rho_0c^2\tau\mathbf{I}+\frac{2\rho_0}{\pi}\int_0^\infty\kappa_c(\omega)D_c^t(\omega)\,d\omega\mathbf{I}+\frac{4\rho_0}{\pi}\int_0^\infty\mu_c(\omega)\mathring{\mathbf{D}}^t{}_c(\omega)\,d\omega.
\end{aligned}\tag{4.10}$$

Therefore, in this context we redefine

$$\mathcal{H}_M=\big\{\mathbf{D}^t\,:\varphi_M(\mathbf{D}^t)<\infty\big\}\,.$$

Moreover, the property $\psi_M(\tau, \mathbf{D}^t) > \psi_M(\tau, \mathbf{0}^\dagger) = \frac{1}{2}c^2\tau^2$ for every $\mathbf{D}^t \neq \mathbf{0}^\dagger$ follows from (3.18) and it ensures the quantity

$$Y_M(t) = \int_\Omega \psi_M\big(\tau(\mathbf{x}, t), \mathbf{D}^t(\mathbf{x}, s)\big)\, d\mathbf{x}$$

induces a norm for $(\tau, \mathbf{D}^t)$ on the space $L^2(\Omega) \times \mathcal{H}_M(\mathbf{R}^+, L^2(\Omega))$ where

$$\mathcal{H}_M(\mathbf{R}^+, L^2(\Omega)) = \big\{\mathbf{D}^t : \Omega \to \mathcal{H}_M \ : \int_\Omega \varphi_M(\mathbf{D}^t)\, d\mathbf{x} < \infty\big\}.$$

Observe that , since κ_0 and μ_0 are finite, by virtue of (3.19) and (3.20), we have $\mu_c, \kappa_c \in L^1(\mathbf{R}^+)$ which implies $L^2(\mathbf{R}^+, L^2(\Omega)) \subseteq \mathcal{H}_M(\mathbf{R}^+, L^2(\Omega))$.

However, the usual norm for $L^2(\Omega) \times L^2(\mathbf{R}^+, L^2(\Omega))$ is stronger the one on $L^2(\Omega) \times \mathcal{H}_M(\mathbf{R}^+, L^2(\Omega))$ deriving from Y_M since $\mu_c(\omega), \kappa_c(\omega) \to 0$ as $\omega \to \infty$.

We complete the section by investigating the properties of the norm $\| \cdot \|_{\mathcal{H}_M}$ such that

$$\|\nabla \mathbf{v}^t\|_{\mathcal{H}_M(\mathbf{R}^+, L^2(\Omega))} = \big(\int_\Omega \varphi_M(\nabla \mathbf{v}^t) d\mathbf{x}\big)^{\frac{1}{2}} .$$

For this purpose, we compare it with the norms of two suitable weighted spaces.

Let V be a Hilbert space and let $\langle \cdot, \cdot \rangle_V$ denote its inner product. Henceforth, $\| \cdot \|_V$ denotes the norm on V, whereas $\| \cdot \|$ denotes the usual norm on $L^2(\Omega)$. For every $u : I \to V$, with $I \subseteq \mathbf{R}$, we put

$$u_I = \begin{cases} u(t) & \text{for } t \in I \\ 0 & \text{for } t \notin I \end{cases}$$

and define the space:

$$L^2_\mu(I, V) = \big\{u : I \to V : \int_{\mathbf{R}} \langle \mu * u_I, u_I \rangle_V \, dt < \infty\big\} , \tag{4.11}$$

where the symbol "$*$" denotes the usual Fourier convolution product, viz.,

$$a * b = \int_{-\infty}^{\infty} a(s) b(t - s)\, ds$$

and any function defined on a subset I of $\mathbf{R}$ is assumed to vanish for $t \in \mathbf{R} \setminus I$. Furthermore, we say that $u^t \in L^2_\mu(\mathbf{R}^+, V)$ if $u \in L^2_\mu(I, V)$ with $I = (-\infty, t]$. Observe that by virtue of the identity

$$\int_{\mathbf{R}} \langle (\mu * u_I)(t), u_I(t) \rangle_V \, dt = \frac{2}{\pi} \int_0^\infty \langle \mu_c(\omega) \hat{u}_I(\omega), \hat{u}_I^*(\omega) \rangle_V \, d\omega.$$

The usual norm of $\nabla \mathbf{v}^t$ on $L^2_\mu(\mathbf{R}^+, L^2(\Omega))$ is

$$\begin{aligned}\|\nabla \mathbf{v}^t\|^2_{L^2_\mu(R^+,L^2(\Omega))} &= \int_{\mathbf{R}} \langle (\mu * \nabla \mathbf{v}_h)(t'), \nabla \mathbf{v}_h(t')\rangle_{L^2(\Omega)}\, dt' \\ &= \frac{2}{\pi}\int_0^\infty \mu_c(\omega)\|\hat{\nabla \mathbf{v}^t}(\omega)\|^2\, d\omega\,. \end{aligned} \tag{4.12}$$

Analogously, we define $L^2_\kappa(R^+, L^2(\Omega))$ and its norm by replacing κ instead of μ in (4.11) and (4.12).

Since $\lambda_c(\omega) > -\frac{2}{3}\mu_c(\omega)$ by virtue of (3.18), and since $|\nabla\cdot\mathbf{v}|^2 \le |\nabla\mathbf{v}|^2$, we obtain:

$$\begin{aligned}&\int_{\mathbf{R}} \kappa_c(\omega)\|\nabla\cdot\mathbf{v}\|^2\, d\omega + 2\int_{\mathbf{R}} \mu_c(\omega)\|\nabla\overset{\circ}{\mathbf{v}}\|^2\, d\omega \\ &= \int_{\mathbf{R}} \lambda_c(\omega)\|\nabla\cdot\mathbf{v}\|^2\, d\omega + 2\int_{\mathbf{R}} \mu_c(\omega)\|\nabla\mathbf{v}\|^2\, d\omega > \frac{4}{3}\int_{\mathbf{R}} \mu_c(\omega)\|\nabla\mathbf{v}\|^2\, d\omega\end{aligned}$$

so that

$$L^2_\mu(R^+, L^2(\Omega)) \subset \mathcal{H}_M(\mathbf{R}^+, L^2(\Omega)) \subseteq L^2_{\mu+\kappa}(R^+, L^2(\Omega))$$

since

$$\frac{4}{3}\|\nabla\mathbf{v}^t\|^2_{L^2_\mu(R^+,L^2(\Omega))} < \|\nabla\mathbf{v}^t\|^2_{\mathcal{H}_M(R^+,L^2(\Omega))} \le \|\nabla\mathbf{v}^t\|^2_{L^2_{\mu+\kappa}(R^+,L^2(\Omega))}.$$

As a consequence, Y_M turns out to be bounded by the norms

$$\tfrac{1}{2}c^2\|\tau(t)\|^2 + \|\nabla\mathbf{v}^t\|^2_{L^2_\mu(R^+,L^2(\Omega))} < Y_M(t) \le \tfrac{1}{2}c^2\|\tau(t)\|^2 + \|\nabla\mathbf{v}^t\|^2_{L^2_{\mu+\kappa}(R^+,L^2(\Omega))}. \tag{4.13}$$

5. Stability of the evolution problem. We consider the following evolution problem for a compressible viscoelastic fluid occupying a bounded region Ω:

$$\begin{cases} \dot{r} = r\nabla\cdot\mathbf{v} & \text{on } \Omega\times\mathbf{R}^+ \\ \dot{\mathbf{v}} = r\nabla\cdot\mathbf{T} + \mathbf{b} & \text{on } \Omega\times\mathbf{R}^+ \\ \mathbf{v}(\mathbf{x},t) = 0 & \text{on } \partial\Omega\times\mathbf{R}^+ \\ \mathbf{v}(\mathbf{x},t) = \mathbf{v}^0(\mathbf{x},-t) & \text{on } \Omega\times\mathbf{R}^- \\ r(\mathbf{x},0) = \frac{1}{\rho_0}(1-\tau_0(\mathbf{x})) & \text{on } \Omega, \end{cases} \tag{5.1}$$

where

(i) $\mathbf{b} \in L^1(\mathbf{R}^+, L^2(\Omega))$
(ii) the initial conditions $\mathbf{v}^0$ and τ_0 are such that $Y_M(0)$ is finite.

Linearization of (5.1) and (3.12) yield

$$\begin{cases} \dot{\tau} = -\nabla \cdot \mathbf{v} & \text{on } \Omega \times \mathbf{R}^+ \\ \dot{\mathbf{v}} = -c^2 \nabla \tau + \nabla \cdot \mathbf{V}_l + \mathbf{b} & \text{on } \Omega \times \mathbf{R}^+ \\ \mathbf{v}(\mathbf{x}, t) = 0 & \text{on } \partial\Omega \times \mathbf{R}^+ \\ \mathbf{v}(\mathbf{x}, t) = \mathbf{v}^0(\mathbf{x}, -t) & \text{on } \Omega \times \mathbf{R}^- \\ \tau(\mathbf{x}, 0) = \tau_0(\mathbf{x}) & \text{on } \Omega, \end{cases} \tag{5.2}$$

where

$$\nabla \cdot \mathbf{V}_l = \int_0^\infty \lambda(s) \nabla(\nabla \cdot \mathbf{v}^t(s))\, ds + 2 \int_0^\infty \mu(s) \Delta \mathbf{v}^t(s)\, ds\ .$$

We can state a local-in-time existence and uniqueness result by rewriting problem (5.2) as follows:

$$\begin{cases} \dot{\tau} = -\nabla \cdot \mathbf{v} & \text{on } \Omega \times \mathbf{R}^+ \\ \dot{\mathbf{v}} = -c^2 \nabla \tau + \lambda *_{\mathcal{L}} \nabla(\nabla \cdot \mathbf{v}) + 2\mu *_{\mathcal{L}} \Delta \mathbf{v} + \mathbf{h} & \text{on } \Omega \times \mathbf{R}^+ \\ \mathbf{v}(\mathbf{x}, t) = 0 & \text{on } \partial\Omega \times \mathbf{R}^+ \\ \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) & \text{on } \Omega \\ \tau(\mathbf{x}, 0) = \tau_0(\mathbf{x}) & \text{on } \Omega, \end{cases} \tag{5.3}$$

where

$$\mathbf{h}(\mathbf{x}, t) = \mathbf{b}(\mathbf{x}, t) + \int_t^\infty \lambda(s) \nabla(\nabla \cdot \mathbf{v}^0(\mathbf{x}, s-t))\, ds + 2 \int_t^\infty \mu(s) \Delta \mathbf{v}^0(\mathbf{x}, s-t)\, ds\ ,$$

the symbol "$*_{\mathcal{L}}$" denotes the Laplace convolution product

$$a *_{\mathcal{L}} b = \int_0^t a(s) b(t-s)\, ds,$$

and $\mathbf{v}_0(\mathbf{x}) = \mathbf{v}^0(\mathbf{x}, 0)$. Observe that conditions (i) and (ii) yield $\mathbf{h} \in L^1_{loc}(\mathbf{R}^+, L^2(\Omega))$.

For each $z \in \mathbf{C}^+ = \{z \in \mathbf{C} : \Re z > 0\}$, we consider the Laplace transformed problem

$$\begin{cases} z\tilde{\tau} = -\nabla \cdot \tilde{\mathbf{v}} + \tau_0 & \text{on } \Omega \times \mathbf{R}^+ \\ z\tilde{\mathbf{v}} = -c^2 \tilde{\nabla} \tau + \tilde{\lambda}(z) \nabla(\nabla \cdot \tilde{\mathbf{v}}) + 2\tilde{\mu}(z) \Delta \tilde{\mathbf{v}} + \tilde{\mathbf{h}} + \mathbf{v}_0 & \text{on } \Omega \times \mathbf{R}^+ \\ \tilde{\mathbf{v}}(\mathbf{x}, z) = 0 & \text{on } \partial\Omega \times \mathbf{R}^+ \end{cases} \tag{5.4}$$

and the associated bilinear form $a\big((\tau_1, \mathbf{v}_1), (\tau_2, \mathbf{v}_2); z\big)$ given by

$$\begin{aligned} a\big((\tau_1, \mathbf{v}_1), (\tau_2, \mathbf{v}_2); z\big) = \int_\Omega \big[& c^2 (z\tau_1 + \nabla \cdot \mathbf{v}_1) \tau_2^* + z \mathbf{v}_1 \mathbf{v}_2^* - c^2 \tau_1 \nabla \cdot \mathbf{v}_2^* \\ & + \tilde{\lambda}(z) \nabla \cdot \mathbf{v}_1 \nabla \cdot \mathbf{v}_2^* + 2\tilde{\mu}(z) \nabla \mathbf{v}_1 \nabla \mathbf{v}_2^* \big] d\mathbf{x}. \end{aligned} \tag{5.5}$$

According to (5.5) we give the following definition of a weak solution for problem (5.4):

Definition 5.1. For each fixed $z \in \mathbf{C}^+$, the couple $(\tilde{\tau}, \tilde{\mathbf{v}})$ is a weak solution of (5.4) if $\tilde{\tau} \in L^2(\Omega)$, $\tilde{\mathbf{v}} \in H_0^1(\Omega)$ and the following holds:

$$a\big((\tilde{\tau}, \tilde{\mathbf{v}}), (\varphi, \nu); z\big) = \int_\Omega (c^2 \tau_0 \varphi + (\tilde{\mathbf{h}} + \mathbf{v}_0)\nu) d\mathbf{x}$$

for every $(\varphi, \nu) \in L^2(\Omega) \times H_0^1(\Omega)$.

Theorem 5.1. *If conditions* (i) *and* (ii) *are satisfied and if* λ *and* μ *satisfies thermodynamic restrictions* (3.18)*, then for each fixed* $z \in \mathbf{C}^+$*, problem* (5.4) *has one and only one weak solution* $(\tilde{\tau}(z), \tilde{\mathbf{v}}(z)) \in L^2(\Omega) \times H_0^1(\Omega)$.

Proof. In order to prove such a theorem it is sufficient to show that the bilinear form $a\big((\tau_1, \mathbf{v}_1), (\tau_2, \mathbf{v}_2; z\big)$ is coercive (cf. [16] and [9]); i.e., it satisfies the inequality

$$|a\big((\tilde{\tau}, \tilde{\mathbf{v}}), (\tilde{\tau}, \tilde{\mathbf{v}}); z\big)| \geq C_1(z)\|\tilde{\tau}\|^2 + C_2(z)\|\tilde{\mathbf{v}}\|_{,1}^2, \tag{5.6}$$

where $\|\cdot\|_{,1}$ denotes the norm on $H_0^1(\Omega)$, and $\|\cdot\|$ is the usual norm on $L^2(\Omega)$. We have

$$\begin{aligned}
|a\big((\tilde{\tau}, \tilde{\mathbf{v}}), (\tilde{\tau}, \tilde{\mathbf{v}}); z\big)| &\geq \big|\Re\big\{a\big((\tilde{\tau}, \tilde{\mathbf{v}}), (\tilde{\tau}, \tilde{\mathbf{v}}); z\big)\big\}\big| \\
&= \Big|\int_\Omega \big[c^2\Re\{z\}|\tilde{\tau}|^2 + \Re\{z\}|\tilde{\mathbf{v}}|^2 + \Re\{\tilde{\lambda}(z)\}|\nabla\cdot\tilde{\mathbf{v}}|^2 + 2\Re\{\tilde{\mu}(z)\}|\nabla\tilde{\mathbf{v}}|^2\big] d\mathbf{x}\Big| \\
&= \Big|\int_\Omega \big[c^2\Re\{z\}|\tilde{\tau}|^2 + \Re\{z\}|\tilde{\mathbf{v}}|^2 + \Re\{\tilde{\kappa}(z)\}|\nabla\cdot\tilde{\mathbf{v}}|^2 + 2\Re\{\tilde{\mu}(z)\}|\nabla\mathring{\tilde{\mathbf{v}}}|^2\big] d\mathbf{x}\Big|.
\end{aligned}$$

By virtue of inequalities (3.21), for each fixed $z \in \mathbf{C}^+$, we can find a positive constant $C_2'(z)$ such that

$$\Re\{\tilde{\kappa}(z)\}|\nabla\cdot\tilde{\mathbf{v}}|^2 + 2\Re\{\tilde{\mu}(z)\}|\nabla\mathring{\tilde{\mathbf{v}}}|^2 \geq C_2'(z)|\nabla\tilde{\mathbf{v}}|^2$$

so that inequality (5.6) holds with $C_1(z) = c^2\Re\{z\}$ and $C_2(z) = \min\{\Re\{z\}, C_2'(z)\}$.

Remark. By properties of Laplace transforms, for every $T < \infty$, problem (5.3) has a unique (weak) solution $(\tau, \mathbf{v}) \in L^1([0,T], L^2(\Omega)) \times L^1([0,T], H_0^1(\Omega))$ if and only if, for each fixed $z \in \mathbf{C}^+$, the couple $(\tilde{\tau}(z), \tilde{\mathbf{v}}(z)) \in L^2(\Omega) \times H_0^1(\Omega)$ is the unique (weak) solution of (5.4). Therefore, Theorem 5.1 constitutes also a local-in-time result for problem (5.3)

The next theorem will be proved by using the properties of the maximal free energy ψ_M and this is the main part of the section.

Theorem 5.2. *If* $(\tau, \mathbf{v})$ *is a solution for problem* (5.2) *with the data satisfying the hypotheses* (i) *and* (ii), *then it is stable; viz.,*

$$\tau \in L^\infty(\mathbf{R}^+, L^2(\Omega)) \quad \text{and} \quad \mathbf{v} \in L^\infty(\mathbf{R}^+, L^2(\Omega)) \cap L^2_\mu(\mathbf{R}^+, H^1_0(\Omega))\ ;$$

moreover, their norms in the respective spaces are bounded by the norms of the data.

Proof. In order show the stability of the solution by means of a priori estimates, we take the product of $(5.2)_2$ with $\mathbf{v}$, and integrate over $\Omega \times [0, T]$. Then integration by parts and substitution of $(5.2)_3$, and $(5.2)_1$ yield

$$\int_0^T \int_\Omega \big(\mathbf{v} \cdot \dot{\mathbf{v}} + c^2 \tau \dot{\tau} + \mathbf{V}_l \cdot \nabla \mathbf{v}\big)\, d\mathbf{x}\, dt = \int_0^T \int_\Omega \mathbf{b} \cdot \mathbf{v}\, d\mathbf{x}\, dt.$$

By virtue of the thermodynamic relation (4.3) we obtain

$$\int_0^T \frac{d}{dt}\big(\frac{1}{2}\|\mathbf{v}(t)\|^2 + Y_M(t)\big)\, dt \le \int_0^T \|\mathbf{b}(t)\|\, \|\mathbf{v}(t)\|\, dt$$

that implies

$$\frac{1}{2}\|\mathbf{v}(T)\|^2 + Y_M(T) \le \frac{1}{2}\|\mathbf{v}(0)\|^2 + Y_M(0) + \int_0^T \|\mathbf{b}(t)\|\, \|\mathbf{v}(t)\|\, dt.$$

Since $Y_M > 0$, applying Gronwall's generalized Lemma (cf. [17]) the following inequalities hold for any $T \in \mathbf{R}^+$:

$$\|\mathbf{v}(T)\| \le \big(\|\mathbf{v}(0)\|^2 + 2Y_M(0)\big)^{\frac{1}{2}} + \int_0^T \|\mathbf{b}(t)\|\, dt \tag{5.7}$$

$$Y_M(T) \le \frac{1}{2}\|\mathbf{v}(0)\|^2 + Y_M(0) + \sup_{t \in [O,T]} \|\mathbf{v}(t)\| \int_0^T \|\mathbf{b}(t)\|\, dt. \tag{5.8}$$

Observe that the Fourier transform of the history of the velocity satisfies

$$\begin{aligned}\hat{\mathbf{v}}^t(\omega) &= \int_0^t \mathbf{v}(t-s) e^{-i\omega s}\, ds\ + \int_t^\infty \mathbf{v}(t-s) e^{-i\omega s}\, ds \\ &= e^{-i\omega t}\big(\int_0^t \mathbf{v}(s') e^{-i\omega s'}\, ds'\ + \int_0^\infty \mathbf{v}^0(s') e^{i\omega s'}\, ds'\big) = e^{-i\omega t}\big(\hat{\mathbf{v}}_t^* + \hat{\mathbf{v}}^{0*}\big),\end{aligned}$$

where

$$\mathbf{v}_t(s) = \begin{cases} \mathbf{v}(t), & 0 \le s \le t\,, \\ 0, & s \notin [0,t]\,, \end{cases}$$

and $\mathbf{v}^0(s')$ is supposed to vanish when $s' < 0$. Hence the inequality $|\hat{\mathbf{v}}_t|^2 \le 2\big(|\hat{\mathbf{v}}^t|^2 + |\hat{\mathbf{v}}^0|^2\big)$ holds so that

$$\|\nabla \mathbf{v}^T\|^2_{L^2_\mu(R^+,L^2(\Omega))} \ge \frac{1}{2}\|\nabla \mathbf{v}_T\|^2_{L^2_\mu(R^+,L^2(\Omega))} - Y_M(0). \tag{5.9}$$

Using (4.13), (5.8), (5.9), and recalling Poincaré's inequality we obtain

$$\begin{aligned} c^2\|\tau(T)\|^2 &+ K\|\mathbf{v}_T\|^2_{L^2_\mu(R^+,H^1_0(\Omega))} \\ &\le \tfrac{1}{2}\|\mathbf{v}(0)\|^2 + 2Y_M(0) + \sup_{t\in[O,T]}\|\mathbf{v}(t)\| \int_0^T \|\mathbf{b}(t)\|\,dt, \end{aligned} \tag{5.10}$$

where $K > 0$. Taking the limit as $T \to \infty$ in (5.7) and (5.8) it follows that

$$\mathbf{v} \in L^\infty(\mathbf{R}^+, L^2(\Omega)) \cap L^2_\mu(R^+, H^1_0(\Omega)) \quad \text{and} \quad \tau \in L^\infty(\mathbf{R}^+, L^2(\Omega))\,.$$

Moreover, the solution is bounded by the data by means of the relations

$$\|\mathbf{v}\|_{L^\infty(R^+,L^2(\Omega))} \le \big(\|\mathbf{v}(0)\|^2 + 2Y_M(0)\big)^{\frac{1}{2}} + \|\mathbf{b}\|_{L^1(R^+,L^2(\Omega))}$$

$$c^2\|\tau\|^2_{L^\infty(R^+,L^2(\Omega))} + K\|\mathbf{v}\|^2_{L^2_\mu(R^+,H^1_0(\Omega))} \le \tfrac{3}{2}\|\mathbf{v}(0)\|^2 + 4Y_M(0) + \tfrac{3}{2}\|\mathbf{b}\|^2_{L^1(R^+,L^2(\Omega))}.$$

Finally, observe that Theorem 5.2 may be considered both a stability theorem in the usual sense when $Y_M(0)$ varies and $\mathbf{b}$ vanishes and a control theorem when $\mathbf{b}$ varies and $Y_M(0)$ vanishes.

6. Domain of dependence inequality. This section clearly shows the hyperbolicity of the model. In fact, the solution of the evolution problem (by means of its energy) is shown to propagate with finite speed. Such a speed is related to c, κ_0 and μ_0. As a consequence, it is apparent that hyberbolicity falls down when $c \to \infty$, or if λ or μ have an initial singularity.

We define the maximal energy of a subset U of $\mathbf{R}^3$ by

$$E(t;U) := \int_{\Omega\cap U} \rho_0\big(\frac{1}{2}\mathbf{v}^2 + \psi_M\big)\,d\mathbf{x}\,.$$

Also we let $B(\mathbf{x}_0, R)$ denote the closed ball

$$B(\mathbf{x}_0, R) = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| \le R\}.$$

Then the domain of dependence for the solution to (5.2) is stated by the following theorem.

Theorem 6.1. *If* $(\tau, \mathbf{v})$ *is a solution of the evolution problem* (5.2) *then, for every* $C \geq C_0$*, the energy* E *satisfies*

$$E(t; B(\mathbf{x}_0, R) \leq E(0; B(\mathbf{x}_0, R + CT)) + \int_0^T \int_{\Omega \cap B(\mathbf{x}_0, R+CT)} \rho_0 \mathbf{b} \cdot \mathbf{v} \, d\mathbf{x} \, dt.$$

Proof. For every smooth function ϕ on $\Omega \times \mathbf{R}$, we define a functional by

$$E_\phi(t) := \int_\Omega \rho_0 (\frac{1}{2}\mathbf{v}^2 + \psi_M) \phi \, d\mathbf{x}.$$

The derivative of $t \mapsto E_\phi(t)$ is given by

$$\dot{E}_\phi = \int_\Omega \rho_0 (\frac{1}{2}\mathbf{v}^2 + \psi_M)\dot{} \phi \, d\mathbf{x} + \int_\Omega \rho_0 (\frac{1}{2}\mathbf{v}^2 + \psi_M) \dot{\phi} \, d\mathbf{x}. \tag{6.1}$$

Remembering that ψ_M satisfies (4.3), the first integral of (6.1) is

$$\int_\Omega \rho_0 (\frac{1}{2}\mathbf{v}^2 + \psi_M)\dot{} \phi \, d\mathbf{x} = \int_\Omega (\rho_0 \mathbf{v} \cdot \dot{\mathbf{v}} + \mathbf{T} : \nabla \mathbf{v}) \phi \, d\mathbf{x}.$$

The evolution problem (5.2), the related boundary condition, and integration by parts yield

$$\int_\Omega \rho_0 (\frac{1}{2}\mathbf{v}^2 + \psi_M)\dot{} \phi \, d\mathbf{x} = \int_\Omega \big(\rho_0 \mathbf{b} \cdot \mathbf{v} \phi - (\mathbf{T} \cdot \mathbf{v}) \cdot \nabla \phi\big) \, d\mathbf{x} \,,$$

so that (6.1) becomes:

$$\dot{E}_\phi = \int_\Omega \rho_0 \mathbf{b} \cdot \mathbf{v} \, \phi \, d\mathbf{x} + \int_\Omega \big[\rho_0 \big(\frac{1}{2}\mathbf{v}^2 + \psi_M\big) \dot{\phi} - (\mathbf{T} \cdot \mathbf{v}) \cdot \nabla \phi\big] \, d\mathbf{x} \,. \tag{6.2}$$

For each $\mathbf{x}_0 \in \Omega$, $C > 0$, $R > 0$ and $T > 0$, we choose as a smooth function ϕ the following:

$$\phi(\mathbf{x}, t) = \phi_\delta(|\mathbf{x} - \mathbf{x}_0| - R - C(T - t)) \,,$$

where ϕ_δ is a nonnegative, decreasing C^∞ function on $\mathbf{R}$ with $\phi_\delta(z) = 1$ for $z \leq -\delta$, and $\phi_\delta(z) = 0$ for $z \geq \delta$. Thus the derivatives of ϕ are related to the ones of ϕ_δ by

$$\dot{\phi} = C\phi'_\delta \quad \text{and} \quad \nabla \phi = \mathbf{a}(\mathbf{x}) \phi'_{\delta,}$$

where $\mathbf{a}(\mathbf{x}) = \nabla_\mathbf{x} |\mathbf{x} - \mathbf{x}_0|$ is a unit vector. By substituting the latter into (6.2) we obtain

$$\dot{E}_\phi = \int_\Omega \rho_0 \mathbf{b} \cdot \mathbf{v} \, \phi \, d\mathbf{x} + \int_\Omega \big[\rho_0 C \big(\frac{1}{2}\mathbf{v}^2 + \psi_M\big) - (\mathbf{T} \cdot \mathbf{v}) \cdot \mathbf{a}\big] \phi'_\delta \, d\mathbf{x} \,.$$

By virtue of the Corollary inequality and by means of a Gronwall-type inequality (see [17], Theorem 5.3), integration in time yields

$$E_{\phi_\delta(|\mathbf{x}-\mathbf{x}_0|-R)} - E_{\phi_\delta(|\mathbf{x}-\mathbf{x}_0|-R-CT)} \leq \int_0^T \int_\Omega \rho_0 \mathbf{b}\cdot\mathbf{v}\,\phi\,d\mathbf{x}.$$

The result follows by letting $\delta \to 0$ in this inequality. □

Again, we note that an analogous result holds for the free energy ψ_G; moreover, since inequality (4.8) involves the same constant C_0 of inequality (4.5) (i.e., the one given by (4.7)) we obtain the same speed of the propagation for both the free energies.

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