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REACTION-DIFFUSION SYSTEMS FOR MULTIGROUP NEUTRON FISSION WITH TEMPERATURE FEEDBACK: POSITIVE STEADY-STATE AND STABILITY*

Anthony W. Leung
Department of Mathematical Sciences, University of Cincinnati, Cincinnati, Ohio 45221

Beatriz Villa
Department of Mathematics, Universidad Nacional, Bogotá, Colombia

(Submitted by: Glenn Webb)

Abstract. We consider a system of reaction-diffusion equations describing the dynamics of fission reactors with temperature feedback. There are m equations for the neutrons in m energy groups and a last temperature equation. We use the bifurcation method to find positive steady-states for the system which is not symmetric. We then analyze the linearized stability of the steady-state as a solution of the full system of $m+1$ parabolic equations. The asymptotic stability of the steady-state solution is proved by means of a stability theorem for sectorial operators.

1. Introduction and preliminaries. In the study of steady states and dynamics of nuclear fission reactors, it is crucial to understand the effect of temperature-dependent feedback, fission rates and reactor size on the behavior of the system. In [12], [13] and [15], various nonlinear models concerning multigroup neutron fission reaction-diffusion with temperature feedback are investigated. In these articles, it is always assumed that the scattering and reaction rates are in some sense larger than the principal eigenvalue of the domain representing the reactor core. Recently, in [14], the scattering and reaction rates are only assumed to be positive, and the emphasis is on the bifurcation of a positive steady-state at certain critical size of the reactor core. The stability of the bifurcating solution has also been investigated in [14]. The only important drawback in [14] is that the temperature is first expressed in terms of neutron-fluxes, and then substituted back into the first m equations for the m energy groups of neutron-fluxes. This model is implicitly assuming that the temperature is changing in a faster time-scale. The major difference in this article is in treating the $m+1$ equations simultaneously, without eliminating the last temperature equation as in [14]. The stability of the positive steady-state is then considered for the full system of $m+1$ equations. Thus the theory is more elegant and less restrictive.

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More precisely, we first consider the following elliptic system with Dirichlet boundary conditions on a domain with various sizes:

$$\begin{aligned} &\Delta \widehat{u}_i(x) + \sum_{j=1}^{m+1} \widehat{H}_{ij}(x, \widehat{u}_{m+1})\widehat{u}_j(x) = 0 \text{ for all } x \in k\Omega,\ i = 1, \dots, m+1, \\ &\widehat{u}_i(x) = 0 \qquad \text{for all } x \in \partial(k\Omega),\ i = 1, \dots, m+1, \end{aligned} \tag{1.1}$$

where $k\Omega = \{x = ky : y \in \Omega\}$, $k > 0$, Ω is a fixed domain in $\mathbb{R}^N$, and $m \geq 2$. The domain $k\Omega$ represents the reactor core; $\widehat{u}_j(x)$, $j = 1, \dots, m$ is the neutron-flux of the j^{th} energy group; and $\widehat{u}_{m+1}(x)$ denotes the temperature. $\widehat{H}_{ij}$, $i, j = 1, \dots, m$ represent the temperature-dependent fission and scattering rates of various energy groups; Δ is the Laplacian operator with x as independent variable. The function $\widehat{H}_{m+1,m+1}$ denotes the cooling coefficient, and $\widehat{H}_{m+1,j}$, $j = 1, \dots, m$, denotes the rate of temperature increase due to neutrons in group j. Consequently, we should have $\widehat{H}_{m+1,m+1} \leq 0$ and $\widehat{H}_{ij} \geq 0$ for all $(i, j) \neq (m+1, m+1)$. For more details concerning the physical meaning of these functions, see e.g. [2], [4], [6], [8], [9] and [11]. We will determine the parameter k when a positive steady-state will bifurcate from the trivial solution, and will thus find the critical size of the reactor core.

With the change of variable $x = ky$, problem (1.1) is transformed into

$$\begin{aligned} &\Delta_y u_i(y) + \lambda \sum_{j=1}^{m+1} H_{ij}(y, u_{m+1}(y))u_j(y) = 0 \text{ for } y \in \Omega, \quad i = 1, \dots, m+1, \\ &u_i(y) = 0 \qquad \text{for } y \in \partial\Omega, \quad i = 1, \dots, m+1, \end{aligned} \tag{1.2}$$

which is the Dirichlet problem for a fixed domain Ω, where $\lambda = k^2 > 0$. Here, $u_i(y) = \widehat{u}_i(x) = \widehat{u}_i(ky)$, and

$$H_{ij}(y, u_{m+1}(y)) = \widehat{H}_{ij}(x, \widehat{u}_{m+1}(x)) = \widehat{H}_{ij}(ky, \widehat{u}_{m+1}(ky));$$

Δ_y is the Laplacian on the y-variable; and for convenience, we will not display this variable in the following context. We will obtain a positive steady-state for (1.2) for certain values of λ; and consider the stability of this steady-state as a solution of the nonlinear parabolic system

$$\begin{aligned} &\frac{\partial u_i}{\partial t} = \Delta u_i + \lambda \sum_{j=1}^{m+1} H_{ij}(y, u_{m+1}(y,t))u_j(y,t) \text{ for } (y,t) \in \Omega \times (0, \infty), \\ &u_i(y,t) = 0 \quad \text{for} \quad (y,t) \in \partial\Omega \times (0,\infty), \quad i = 1, \dots, m+1, \\ &u_i(y, 0) = u_0^i(y) \quad \text{for } y \in \overline{\Omega},\ i = 1, \dots m+1. \end{aligned} \tag{1.3}$$

To fix ideas, we assume Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$ of class $C^{2+\mu}$ for some $\mu \in (0,1)$. For convenience, we denote

$$J = \{1, \dots, m+1\}, \quad B_{ij}(x) \stackrel{(\text{def})}{=} H_{ij}(x, 0) \quad \text{for} \quad x \in \overline{\Omega}, \quad i, j \in J.$$

We will assume

[H1] $H_{ij}(\cdot, \eta) \in C^\mu(\overline{\Omega})$ uniformly for η in bounded subsets of $\mathbb{R}^1$, $i, j \in J$; $B_{ij}(x) \geq 0$ in $\overline{\Omega}$, for all $(i,j) \neq (m+1, m+1)$;

[H2] $B_{ij} \not\equiv 0$ for $i \neq j$, $i, j = 1, \dots, m$; $B_{m+1,j} \not\equiv 0$, $j = 1, \dots, m$; $B_{i,m+1} \equiv 0$ for $i = 1, \dots, m$.

Note that in (1.3), the coefficients H_{ij} depend on the temperature u_{m+1}. Moreover, the last part in [H2], ($B_{i,m+1} \equiv 0$ $i = 1, \dots, m$) means that the effect of temperature on the system enters only through changes in the fission and scattering coefficients B_{ij}, $j < m+1$. Also, note that so far, the crucial assumptions are made only at temperature $u_{m+1} = 0$, which can be normalized as the exterior temperature. More hypotheses will be added later when stability is investigated.

For convenience, we will use the following notations:

$$\mathcal{E} = \{u = \text{col.}(u_1, \dots, u_{m+1}) : u_i \in C^{2+\mu}(\overline{\Omega}),\ u_i|_{\partial\Omega} = 0,\ i = 1, \dots, m+1\},$$
$$\mathcal{F} = \{u = \text{col.}(u_1, \dots, u_{m+1}) : u_i \in C^{\mu}(\overline{\Omega}),\ u_i|_{\partial\Omega} = 0,\ i = 1, \dots, m+1\},$$
$$\mathcal{F}_1 = \{u = \text{col.}(u_1, \dots, u_{m+1}) : u_i \in C^{1}(\overline{\Omega}),\ u_i|_{\partial\Omega} = 0,\ i = 1, \dots, m+1\},$$

$$\|\cdot\|_{\mathcal{E}} = \|\cdot\|_{C^{2+\mu}(\overline{\Omega})}, \qquad \|\cdot\|_{\mathcal{F}} = \|\cdot\|_{C^{\mu}(\overline{\Omega})}, \quad \|\cdot\|_{\mathcal{F}_1} = \|\cdot\|_{C^{1}(\overline{\Omega})};$$

$H(x,\eta) = [H_{ij}(x,\eta)]$ is an $(m+1) \times (m+1)$ matrix for $(x,\eta) \in \overline{\Omega} \times \mathbb{R}^1$, and $\mathcal{P} = \{u = \text{col.}(u_1, \dots, u_{m+1}) \in \mathcal{F}_1 : u_i \geq 0 \quad \text{on} \quad \overline{\Omega}\}$. Throughout this paper we will use the symbol $\Delta_D^{-1} f$ for $f \in \mathcal{F}$ or $\mathcal{F}_1$, to denote the function $w \in \mathcal{E}$ such that $\Delta w = f$ in Ω. Applying Δ_D^{-1} to (1.2), it can be written

$$u + \lambda \Delta_D^{-1}[H(\cdot, u_{m+1})u] = 0 \tag{1.4}$$

where u =col.$(u_1, \dots, u_{m+1})$. In section 2 we will use the bifurcation method to find a positive solution to (1.4). In section 3, we first analyze the linearized stability of this solution as a steady-state solution of (1.3). Finally, the asymptotic stability of the steady-state solution will be proved.

2. Positive eigenfunctions and bifurcating steady-state. In order to find a positive solution bifurcating from zero for equation (1.4), we will first consider the corresponding linearized eigenvalue problem. We will need the following comparison Lemma 2.1 in order to prove the main theorems in this section. For convenience we let $J = \{1, 2, \dots, m+1\}$, and define the operator

$$L_i = -\Delta + c_i(x)$$

for $i \in J$, where $c_i(x) \geq 0$ are functions in $C^\mu(\overline{\Omega})$, $0 < \mu < 1$.

Lemma 2.1 (Comparison). *Let $u, v \in [C^2(\Omega) \cap C^1(\overline{\Omega})]^{m+1}$, $u \not\equiv 0$, $v_i \geq 0, \not\equiv 0$ in Ω, for $i \in J$, satisfy*

$$L_i[u_i(x)] = \sum_{j=1}^{m+1} p_{ij}(x) u_j(x), \qquad \textit{for} \quad x \in \Omega,\ i \in J, \tag{2.1}$$

$$u|_{\partial\Omega} = 0, \qquad u = \textit{col.}\ (u_1, \dots, u_{m+1}),$$

$$L_i[v_i(x)] \geq \sum_{j=1}^{m+1} q_{ij}(x) v_j(x), \qquad \textit{for} \quad x \in \Omega,\ i \in J, \tag{2.2}$$

$$v = \textit{col.}\ (v_1, \dots, v_{m+1}),$$

where p_{ij} and q_{ij} are bounded functions in Ω. Suppose that

$$\begin{aligned} &q_{ij} \geq p_{ij} \qquad \textit{in} \quad \overline{\Omega} \quad \textit{for} \quad i, j \in J, \ \textit{and} \\ &q_{ij}, p_{ij} \geq 0 \qquad \textit{in} \quad \overline{\Omega} \quad \textit{for all} \quad i \neq j; \end{aligned} \tag{2.3}$$

then there exists $k \in J$ and a real number δ such that

$$\begin{aligned} &v_k \equiv \delta u_k\ ,\ p_{kj} \equiv q_{kj} \quad \textit{in} \quad \overline{\Omega} \quad \textit{for all} \quad j \in J, \\ &\textit{and} \quad v_j - \delta u_j \geq 0 \quad \textit{for all} \quad j \in J. \end{aligned}$$

Proof. Let $K > 0$ be a positive constant such that $p_{ii} + K, q_{ii} + K > 0$ in $\overline{\Omega}$, for $i \in J$. From (2.2) and (2.3), we have

$$L_i[v_i] + K v_i \geq (q_{ii} + K) v_i + \sum_{j=1, j \neq i}^{m+1} q_{ij} v_j \geq 0.$$

Thus the maximum principles (see [17]) imply that

$$v_i > 0 \ \text{ in } \ \Omega, \ \text{ and } \ (\partial v_i / \partial \eta)(\overline{x}) < 0 \ \text{ if } \ \overline{x} \in \partial\Omega \ \text{ and } \ v_i(\overline{x}) = 0. \tag{2.4}$$

Without loss of generality, we may assume that some component of u takes a positive value somewhere. Otherwise, replace u by $-u$. Since $u = 0$ on $\partial\Omega$, we can readily obtain from properties (2.4) that $v_i(x) - \epsilon u_i(x) > 0$ for some $\epsilon > 0$ and all $x \in \Omega$, $i \in J$. Let $\delta_i = \sup\{a : v_i - a u_i > 0 \text{ in } \Omega\}$ for those $i \in J$ such that δ_i can be finitely defined. Define δ to be the minimum of such δ_i's. Thus $\delta = \delta_k$ for some k, and $0 < \delta_k < \infty$, $v_i - \delta u_i \geq 0$ in Ω for $i \in J$. From (2.1) and (2.2), we find

$$\begin{aligned} (L_k + K)(v_k - \delta_k u_k) \geq (K + p_{kk})(v_k - \delta_k u_k) + \sum_{i=1, i \neq k}^{m+1} p_{ki}(v_i - \delta_k u_i) \\ + \sum_{i=1}^{m+1} (q_{ki} - p_{ki}) v_i \geq 0 \qquad \text{in} \quad \Omega. \end{aligned} \tag{2.5}$$

Consequently, the maximum principle implies that $v_k - \delta u_k \equiv 0$ in Ω. Then (2.5) further implies that we must have $q_{ki} \equiv p_{ki}$ for all $i \in J$.

For convenience, we define the following operators with $m+1$ components:

$$\widetilde{L}_q \equiv (-\Delta, \dots, -\Delta, -\Delta + q(x)), \quad \text{where } q(x) \geq 0 \text{ is a function in } C^\mu(\overline{\Omega});$$
$$T \equiv \widetilde{L}_q^{-1}(B): \ [C^1(\overline{\Omega})]^{m+1} \to [C^1(\overline{\Omega})]^{m+1}, \quad \text{so that for } u \in [C^1(\overline{\Omega})]^{m+1},$$
$$w = Tu \quad \text{is the function which satisfies } \widetilde{L}_q w = Bu \text{ and } w|_{\partial\Omega} = 0.$$

We first prove the existence of positive eigenfunctions for an appropriate linear system related to (1.4), under the restrictive condition $B_{m+1,m+1} \equiv 0$. This restriction will be modified later in Theorem 2.2.

Theorem 2.1. *Suppose that B satisfies* [H1], [H2] *and $B_{m+1,m+1} \equiv 0$ in $\overline{\Omega}$; then there exists $(\lambda_0, u^0) \in \mathbb{R} \times \mathcal{E}$, $\lambda_0 > 0$ such that*

$$\widetilde{L}_q[u^0] = \lambda_0 B u^0 \quad \text{in } \ \Omega, \ u^0 = 0 \text{ on } \ \partial\Omega, \tag{2.6}$$

with each component $u_i^0 > 0$ in Ω and $\partial u_i^0/\partial\eta < 0$ on $\partial\Omega$ for $i \in J$. Furthermore, $1/\lambda_0$ is a simple eigenvalue of the operator T (that is, the eigenfunction corresponding to this eigenvalue is unique up to a multiple). The number $\lambda = \lambda_0$ is the unique positive number so that the problem $u = \lambda T u$ has a nontrivial nonnegative solution for $u \in \mathcal{P}$.

Proof. The operator $T : [C^1(\overline{\Omega})]^{m+1} \to [C^1(\overline{\Omega})]^{m+1}$ is completely continuous and positive with respect to the cone $\mathcal{P}$. Let $z = col.(z_1, \dots, z_{m+1}) = col.((-\Delta_D)^{-1}(1), \dots, (-\Delta_D)^{-1}(1))$. The functions z_i satisfy $z_i(x) > 0$ in Ω, $z_i|_{\partial\Omega} = 0$ for $i \in J$. Define $v = T(z)$. Hypotheses [H1], [H2] and the maximum principles imply that the components $v_i > 0$ in Ω, $\partial v_i/\partial\eta < 0$ on $\partial\Omega$ for $i \in J$. Thus, there exists $\delta > 0$ such that $T(z) \geq \delta z$ with $z \in \mathcal{P}$. Theorem 2.5 in Krasnosel'skii [10] asserts that there exists a nontrivial $u^0 = \text{col.}(u_1^0, \dots, u_{m+1}^0) \in \mathcal{P}$ and $\rho_0 \geq \delta > 0$ such that $Tu^0 = \rho_0 u^0$ (i.e. (2.6) with $\lambda_0 = 1/\rho_0$). The last component of (2.6) implies that we cannot have $u_i^0 \equiv 0$ in $\overline{\Omega}$ for all $i = 1, \dots, m$. The maximum principle further implies that if $u_j^0 \not\equiv 0$, for $j \in J$, then $u_j^0(x) > 0$ for all $x \in \Omega$. We can obtain from hypotheses [H1], [H2] and the maximum principle that $u_i^0 > 0$ in Ω and $\partial u_i^0/\partial\eta < 0$ on $\partial\Omega$ for all $i \in J$.

Now, let $w = \text{col.}(w_1, \dots, w_{m+1}) \not\equiv 0$ be such that $\widetilde{L}_q[w] = \lambda_0 B w$. From Lemma 2.1, there must exist $\delta^* \in \mathbb{R}$ and some $k \in J$ such that

$$u_k^0 \equiv \delta^* w_k \quad \text{and} \quad u_j^0 - \delta^* w_j \geq 0 \ \text{ in } \ \overline{\Omega} \ \text{ for all } \ j \in J. \tag{2.7}$$

If there is an integer $r \in J$ such that

$$u_r^0(\overline{x}) - \delta^* w_r(\overline{x}) > 0 \quad \text{for some } \ \overline{x} \in \Omega, \tag{2.8}$$

then $-\Delta[u_r^0-\delta^* w_r] = \lambda_0 \sum_{j=1}^{m+1} B_{rj}(u_j^0 - \delta^* w_j) \geq 0$ implies that $u_r^0 - \delta^* w_r > 0$ in Ω for the case $r \neq m+1$. For the case $r = m+1$, we have $(-\Delta+q)(u_{m+1}^0 - \delta^* w_{m+1}) \geq 0$, which also implies that $u_r^0 - \delta^* w_r > 0$ in Ω. We then consider the i-th equation in (2.6), for $i \neq r$; the hypothesis [H2] implies that $-\Delta(u_i^0 - \delta^* w_i) \not\equiv 0$ if $i \neq m+1$ also, or $(-\Delta + q)(u_i^0 - \delta^* w_i) \not\equiv 0$ if $i = m+1$. Consequently, $u_i^0 - \delta^* w_i \not\equiv 0$ for each $i \neq r$. This contradicts the existence of an integer $k \in J$ such that (2.7) holds. This means that if (2.7) holds, there cannot exist an $r \in J$ such that (2.8) holds. That is, we have $u^0 \equiv \delta^* w$. Finally, suppose that there is another $\lambda_1 > 0$, $\lambda_1 \neq \lambda_0$ so that $\widetilde{u} = \lambda_1 T \widetilde{u}$ for some $\widetilde{u} \in \mathcal{P}$, $\widetilde{u} \not\equiv 0$. We can deduce as before that $\widetilde{u}_i > 0$ in Ω, $\partial \widetilde{u}_i / \partial \eta < 0$ on $\partial\Omega$ for $i \in J$. Then we can obtain from Lemma 2.1 that $\lambda_1 = \lambda_0$. This completes the proof of Theorem 2.1. $\square$

As described in section 1, the last component u_{m+1} denotes the temperature in the reactor, and the term $H_{m+1,m+1}(x,0) = B_{m+1,m+1}$ denotes the cooling coefficient. It is therefore reasonable to impose the hypothesis

[H3] $$H_{m+1,m+1}(x,0) = B_{m+1,m+1}(x) \leq 0 \quad \text{for all} \;\; x \in \overline{\Omega}.$$

To insure the existence of positive eigenfunctions, we further assume that

[H4] There exists some $k \in J$ such that $B_{kk}(x) > 0$ for some $x \in \Omega$.

Theorem 2.2. *Suppose B satisfies all the hypotheses* [H1] *to* [H4]; *then there exists* $(\widehat{\lambda}_0, v^0) \in \mathbb{R} \times \mathcal{E}$, $\widehat{\lambda}_0 > 0$, *such that*

$$-\Delta[v^0] = \widehat{\lambda}_0 B v^0 \quad \textit{in} \;\; \Omega, \quad v^0 = 0 \quad \textit{on} \;\; \partial\Omega, \tag{2.9}$$

with each component $v_i^0 > 0$ in Ω, $\partial v_i^0/\partial\eta < 0$ on $\partial\Omega$ for $i \in J$. Furthermore, $1/\widehat{\lambda}_0$ is a simple eigenvalue of the operator $(-\Delta_D)^{-1}B : [C^1(\overline{\Omega})]^{m+1} \to [C^1(\overline{\Omega})]^{m+1}$.

We will use Theorem 2.1 to prove this theorem. For convenience, define $\widetilde{B}$ to be the $(m+1) \times (m+1)$ matrix function on $\overline{\Omega}$ as follows

$$\begin{aligned} &\widetilde{B}_{ij}(x) = B_{ij}(x) \quad \text{for} \;\; i,j \in J, \; (i,j) \neq (m+1, m+1), \quad x \in \overline{\Omega}, \\ &\widetilde{B}_{m+1,m+1} \equiv 0. \end{aligned} \tag{2.10}$$

For each $\lambda \geq 0$, define the $m+1$ component vector operator

$$\widetilde{L}_\lambda \equiv (-\Delta, \ldots, -\Delta, -\Delta - \lambda B_{m+1,m+1}),$$

and consider the eigenvalue problem

$$\widetilde{L}_\lambda u = \rho \widetilde{B} u \quad \text{in} \;\; \Omega, \qquad u|_{\partial\Omega} = 0, \tag{2.11}$$

with eigenvalue ρ. Since $\widetilde{B}$ satisfies the conditions in Theorem 2.1, problem (2.11) has a unique positive eigenvalue $\rho = \widehat{\rho}(\lambda)$ with corresponding eigenfunction $u = u_\lambda = \text{col.}((u_\lambda)_1, \ldots, (u_\lambda)_{m+1})$, $(u_\lambda)_i > 0$ in Ω, $\partial(u_\lambda)_i/\partial\eta < 0$ on $\partial\Omega$, for $i \in J$. The proof of Theorem 2.2 will follow readily from the next two lemmas.

Lemma 2.2. *Under the hypotheses of Theorem* 2.2, *the function* $\widehat{\rho}(\lambda)$ *is bounded for all* $\lambda \in [0, \infty)$.

Proof. Let G be an open bounded set in Ω with its closure contained in Ω, and $B_{kk}(x) > 0$ for all $x \in G$. (Here, k is the integer described in [H4]). Let $\Phi \not\equiv 0$ be a C^∞ function with compact support contained in G. We clearly have $\int_G B_{kk}\Phi^2 dx > 0$. Let u_λ be as described above, and set $w_\lambda(x) = \ln(u_\lambda)_k(x)$ for $x \in \Omega$. Thus we have in G that

$$-\Delta w_\lambda - \sum_{i=1}^{N} \left(\frac{\partial(w_\lambda)}{\partial x_i} \right)^2 = \left[\frac{\widehat{\rho}(\lambda)}{(u_\lambda)_k} \right] \sum_{j=1}^{m} \widetilde{B}_{kj}(u_\lambda)_j \geq B_{kk}\widehat{\rho}(\lambda). \tag{2.12}$$

Multiplying by Φ^2 and integrating over G, we obtain

$$\int_G \left[-\Delta w_\lambda - \sum_{i=1}^{N} \left(\frac{\partial(w_\lambda)}{\partial x_i} \right)^2 \right] \Phi^2 dx \geq \widehat{\rho}(\lambda) \int_G B_{kk}(x)\Phi^2 dx. \tag{2.13}$$

Integrating by parts gives

$$\int_G \langle \Phi\nabla w_\lambda, 2\nabla\Phi - \Phi\nabla w_\lambda \rangle \, dx \geq \widehat{\rho}(\lambda) \int_G B_{kk}\Phi^2 dx. \tag{2.14}$$

Note that in G, we have

$$\begin{aligned} \langle \Phi\nabla w_\lambda, 2\nabla\Phi - \Phi\nabla w_\lambda \rangle &= -\langle \nabla\Phi - \Phi\nabla w_\lambda, \nabla\Phi - \Phi\nabla w_\lambda \rangle + \langle \nabla\Phi, \nabla\Phi \rangle \\ &\leq \langle \nabla\Phi, \nabla\Phi \rangle. \end{aligned}$$

Hence, (2.14) gives

$$0 < \widehat{\rho}(\lambda) \leq \frac{\int_G \langle \nabla\Phi, \nabla\Phi \rangle \, dx}{\int_G B_{kk}\Phi^2 \, dx} \tag{2.15}$$

for all $\lambda \in [0, \infty)$.

Lemma 2.3. *Under the hypotheses of Theorem* 2.2, *the function* $\widehat{\rho}(\lambda)$ *is continuous on* $\lambda \in [0, \infty)$.

Proof. Let $\lambda^* \geq 0$ and λ_i be a sequence with $\lambda_i \to \lambda^*$. By Lemma 2.2, we may assume without loss of generality that $\widehat{\rho}(\lambda_i) \to d$ for some $d \geq 0$. From Theorem 2.1, for each i there exists an eigenfunction $u_i \geq 0$, normalized to $\|u_i\|_\infty = 1$ satisfying

$$\widetilde{L}_{\lambda_i} u_i = \widehat{\rho}(\lambda_i)\widetilde{B}u_i \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{2.16}$$

Schauder's theory implies that $\{u_i\}$ is bounded in the $C^{1+\mu}(\overline{\Omega})$ norm. Using an embedding theorem we obtain without loss of generality that $u_i \to v$ for some $v \in [C^1(\overline{\Omega})]^{m+1}$ and $v \not\equiv 0, \geq 0$ in $\overline{\Omega}$. In the limit, we obtain

$$\widetilde{L}_{\lambda^*} v = d\widetilde{B}v \quad \text{in } \Omega, \qquad v|_{\partial\Omega} = 0. \tag{2.17}$$

From the maximum principle, we must have $v > 0$ in Ω. On the other hand, for $\lambda = \lambda^*$, Theorem 2.1 implies that there exists $v^* \in [C^{2+\mu}(\overline{\Omega})]^{m+1}$, $v^* > 0$ in Ω, and a number $\widehat{\rho}(\lambda^*)$ satisfying

$$\widetilde{L}_{\lambda^*} v^* = \widehat{\rho}(\lambda^*)\widetilde{B}v^* \quad \text{in } \Omega, \qquad v^*|_{\partial\Omega} = 0. \tag{2.18}$$

Using the comparison Lemma 2.1, (2.17) and (2.18), we can readily deduce by contradiction that $\widehat{\rho}(\lambda^*) = d$.

Proof of Theorem 2.2. By Lemma 2.3, the function $\omega(\lambda) \overset{\text{(def)}}{=} \widehat{\rho}(\lambda) - \lambda$ is continuous on $[0, \infty)$. From Theorem 2.1, we have $\omega(0) > 0$; and from Lemma 2.2, we have $\omega(\lambda) < 0$ for large $\lambda > 0$. Thus, there exists $\overline{\lambda}$ such that $\widehat{\rho}(\overline{\lambda}) = \overline{\lambda}$ and (2.11) becomes

$$\widetilde{L}_{\overline{\lambda}} v^0 = \widehat{\rho}(\overline{\lambda})\widetilde{B}v^0 \quad \text{in } \Omega, \qquad v^0|_{\partial\Omega} = 0 \tag{2.19}$$

for some v^0, with $v_i^0 > 0$ in Ω and $\partial v_i^0/\partial\eta < 0$ on $\partial\Omega$, $i \in J$. Comparing with (2.9), we clearly see that it is the same as (2.19) with $\widehat{\lambda}_0 = \widehat{\rho}(\overline{\lambda}) = \overline{\lambda}$. Finally, the simplicity of the eigenvalue $\widehat{\lambda}_0$ follows from (2.19) and Theorem 2.1. This completes the proof of Theorem 2.2. □

In order to apply Crandall-Rabinowitz's bifurcation theorem to the nonlinear problem (1.4) we will have to analyze the range of the operator $I - \widehat{\lambda}_0(-\Delta_D)^{-1}B$. This leads to the study of the adjoint problem

$$(-\Delta)v = \lambda B^T v \quad \text{in } \Omega, \qquad v|_{\partial\Omega} = 0. \tag{2.20}$$

Lemma 2.4. *Let B satisfy* [H1] *to* [H4]. *Then problem* (2.20) *has a nontrivial solution when $\lambda = \widehat{\lambda}_0$ with the corresponding eigenfunction $v(\widehat{\lambda}_0) \equiv \widehat{v}^0$ satisfying $\widehat{v}_i^0 > 0$ for $i = 1, \ldots, m$, and $\widehat{v}_{m+1}^0 \equiv 0$ in Ω. Moreover, the solution $v(\widehat{\lambda}_0)$ is unique up to a multiple.* (*Here, $\widehat{\lambda}_0$ is the same as that defined in Theorem* 2.2).

Proof. Let $\overline{k}$ be a positive constant such that $\overline{k} + H_{m+1,m+1} > 0$ in Ω. Let $D^* = B^T + \overline{k}I$, and consider the problem $(-\Delta + \widehat{\lambda}_0\overline{k})u = \lambda D^* u$ in Ω, $u|_{\partial\Omega} = 0$. As in the proof of Theorem 2.1, we can show that there exists $(\widetilde{\lambda}_0, u^*)$ such that

$$(-\Delta + \widehat{\lambda}_0\overline{k}I)u^* = \widetilde{\lambda}_0 D^* u^* = \widetilde{\lambda}_0(B^T + \overline{k}I)u^* \quad \text{in } \Omega,\ u^*|_{\partial\Omega} = 0, \tag{2.21}$$

with $u^* \geq 0, \not\equiv 0$ in Ω, $\widetilde{\lambda}_0 > 0$. Hypotheses [H1] and [H2] imply that $u_i^* > 0$ in Ω for $i = 1, \ldots, m$. Recall that from (2.9), v^0 satisfies

$$(-\Delta + \widehat{\lambda}_0\overline{k}I)v^0 = \widehat{\lambda}_0(B + \overline{k}I)v^0 \quad \text{in } \Omega, \qquad v^0|_{\partial\Omega} = 0. \tag{2.22}$$

Multiplying both sides of (2.22) by u^* and integrating over Ω, we obtain

$$\begin{aligned} \widehat{\lambda}_0\langle(B + \overline{k}I)v^0, u^*\rangle &= \langle(-\Delta + \widehat{\lambda}_0\overline{k}I)v^0, u^*\rangle \\ &= \langle v^0, (-\Delta + \widehat{\lambda}_0\overline{k}I)u^*\rangle = \widetilde{\lambda}_0\langle v^0, (B^T + \overline{k}I)u^*\rangle = \widetilde{\lambda}_0\langle(B + \overline{k}I)v^0, u^*\rangle. \end{aligned} \tag{2.23}$$

This implies that $\widetilde{\lambda}_0 = \widehat{\lambda}_0$, and (2.21) becomes

$$(-\Delta)\widehat{v}^0 = \widehat{\lambda}_0 B^T \widehat{v}^0 \quad \text{in } \Omega, \qquad \widehat{v}^0|_{\partial\Omega} = 0, \tag{2.24}$$

where we label $\widehat{v}^0 = u^*$. The $(m+1)$-th equation in (2.24) clearly implies that $\widehat{v}^0_{m+1} \equiv 0$ in Ω, since the last row of B^T is $\equiv 0$ except the diagonal entry, which is ≤ 0. Applying Lemma 2.1 to the first m equations of (2.24) (i.e., with m replacing $m+1$), we obtain the uniqueness of $u^* = \widehat{v}^0 = v(\widehat{\lambda}_0)$ up to a multiple. □

As described above, we can consider a solution of problem (1.2) as a solution of (1.4). For convenience, we define an operator $F : \mathbb{R}^+ \times \mathcal{F}_1 \to \mathcal{F}_1$ by

$$F(\lambda, u) = u - \lambda(-\Delta_D)^{-1} H(\cdot, u_{m+1})u \quad \text{for } (\lambda, u) \in \mathbb{R}^+ \times \mathcal{F}_1. \tag{2.25}$$

Problem (1.4) can be written in the form

$$F(\lambda, u) = 0. \tag{2.26}$$

Defining

$$\begin{aligned} &L_0 : \mathcal{F}_1 \to \mathcal{F}_1 \qquad \text{by } L_0 = I - \widehat{\lambda}_0(-\Delta_D)^{-1}B, \\ &L_1 : \mathcal{F}_1 \to \mathcal{F}_1 \qquad \text{by } L_1 = (\Delta_D)^{-1}B, \quad \text{and} \\ &G : \mathbb{R}^+ \times \mathcal{F}_1 \to \mathcal{F}_1 \qquad \text{by } G(\lambda, u) = -\lambda(-\Delta_D)^{-1}([H(\cdot, u_{m+1}) - B]u), \end{aligned} \tag{2.27}$$

equation (2.26) becomes

$$F(\lambda, u) \overset{\text{(def)}}{\equiv} L_0 u + (\lambda - \widehat{\lambda}_0)L_1 u + G(\lambda, u) = 0, \quad \text{for } (\lambda, u) \in \mathbb{R}^+ \times \mathcal{F}_1. \tag{2.28}$$

We clearly have $F(\lambda, 0) = 0$, for all $\lambda \in \mathbb{R}^+$. Let $N(L_0)$ and $R(L_0)$ respectively denote the null space and range of L_0. We will show that they have the following properties.

Lemma 2.5. *Assume that B satisfies hypotheses* [H1] *to* [H4]; *then*

(i) $N(L_0)$ *is one-dimensional, spanned by* v^0;
(ii) $\dim[\mathcal{F}_1/R(L_0)] = 1$;
(iii) $L_1 v^0 \notin R(L_0)$.

Proof. Part (i) was proved in Theorem 2.2. The operator L_0 can be extended naturally to $\widetilde{L}_0$ with the set $[L^2(\Omega)]^{m+1}$ as its domain. The range of $\widetilde{L}_0$ can be described by

$$\Big\{ z \in [L^2(\Omega)]^{m+1} : \int_\Omega g \cdot z \, dx = 0, \quad \text{for all } g \text{ satisfying} \\ \Delta g + \widehat{\lambda}_0 B^T g = 0 \quad \text{in } \Omega, \; g|_{\partial\Omega} = 0 \Big\}.$$

By Lemma 2.4, the g described above has to be a multiple of $\widehat{v}^0$. Thus, by means of the mapping $u \to \int_\Omega \widehat{v}^0 \cdot u\, dx$ from $\mathcal{F}_1$ onto $\mathbb{R}^1$, we conclude that (ii) must be true.

To prove (iii), we first assume the contrary that $L_1 v^0 \in R(L_0)$. Then there exists $w \in [\mathcal{F}_1]^{m+1}$ such that

$$[I - \widehat{\lambda}_0(-\Delta_D)^{-1}B]w = (\Delta_D)^{-1}Bv^0, \quad \text{i.e.,} \quad -\Delta w - \widehat{\lambda}_0 Bw = -Bv^0.$$

Multiplying both sides by $\widehat{v}^0$ (cf. equation (2.4)), and integrating over Ω, we obtain

$$\langle -\Delta w - \widehat{\lambda}_0 Bw, \widehat{v}^0 \rangle = \langle -Bv^0, \widehat{v}^0 \rangle,$$

and thus

$$0 = \langle w, 0 \rangle = \langle w, (-\Delta - \widehat{\lambda}_0 B^T)\widehat{v}^0 \rangle \; = \; \langle -Bv^0, \widehat{v}^0 \rangle.$$

However, the assumptions on B, and the fact that $v_i^0 > 0$ for $i = 1, \dots, m+1$, and $\widehat{v}_i^0 > 0$ for $i = 1, \dots, m$ imply that the expression on the right above is strictly negative. This contradiction implies that (iii) is valid. $\square$

In order to have enough smoothness for the function G given in (2.27), we now impose further hypotheses on the smoothness of $H(\cdot, \eta)$:

[H5] $(\partial H_{ij}/\partial \eta)(\cdot, \eta)$ and $(\partial^2 H_{ij}/\partial \eta^2)(\cdot, \eta)$ are in $C^\mu(\overline{\Omega})$ uniformly in η in bounded subsets of $\mathbb{R}^1$, for each $i, j \in J$.

Lemma 2.6.

(i) *Assume hypotheses* [H1] *to* [H5]*; then the Frechet derivatives D_2G, D_1G, $D_{12}G$ exist and are continuous on $\mathbb{R}^1 \times \mathcal{F}_1$. Moreover, we have*

$$D_2G(\lambda, u)w = \lambda(\Delta_D)^{-1}\big[H(\cdot, u_{m+1})w + \widetilde{H}(\cdot, u_{m+1})uw_{m+1} - Bw\big] \tag{2.29}$$

for all $(\lambda, u) \in \mathbb{R}^+ \times \mathcal{F}_1$, $w \in \mathcal{F}_1$ where $\widetilde{H}(\cdot, u_{m+1}) = (\partial H_{ij}/\partial \eta)(\cdot, u_{m+1})$ is an $(m+1) \times (m+1)$ matrix function;

(ii) $\|G(\lambda, u)\|_{\mathcal{F}_1}/\|u\|_{\mathcal{F}_1} \to 0$ *as* $\|u\|_{\mathcal{F}_1} \to 0$ *uniformly in λ near λ_0.*

Proof. Direct calculation and estimating the range of $(-\Delta)^{-1}$ gives

$$\begin{aligned}
&\|G(\lambda, w) - G(\lambda, u) - \lambda(\Delta_D)^{-1}[H(\cdot, u_{m+1})(w - u) \\
&\quad + \widetilde{H}(\cdot, u_{m+1})u(w_{m+1} - u_{m+1}) - B(w - u)]\|_{\mathcal{F}_1} \\
&\le k\lambda \|H(\cdot, w_{m+1})w - H(\cdot, u_{m+1})u - H(\cdot, u_{m+1})(w - u) \\
&\qquad - \widetilde{H}(\cdot, u_{m+1})u(w_{m+1} - u_{m+1})\|_\infty \\
&\le k\lambda \sum_{i,j=1}^{m+1} \|H_{ij}(\cdot, w_{m+1})w_j - H_{ij}(\cdot, u_{m+1})w_j \\
&\quad - \frac{\partial H_{ij}}{\partial \eta}(\cdot, u_{m+1})u_j(w_{m+1} - u_{m+1})\|_\infty
\end{aligned}$$

$$
\begin{aligned}
&\le k\lambda \sum_{i,j=1}^{m+1} \{\| \, [H_{ij}(\cdot, w_{m+1}) - H_{ij}(\cdot, u_{m+1}) - \frac{\partial H_{ij}}{\partial \eta}(\cdot, u_{m+1})(w_{m+1} - u_{m+1})] u_j \|_\infty \\
&\qquad\qquad + \| \, [H_{ij}(\cdot, w_{m+1}) - H_{ij}(\cdot, u_{m+1})][w_j - u_j] \|_\infty \} \\
&\le k\lambda \sum_{i,j=1}^{m+1} \{\| \frac{\partial H_{ij}}{\partial \eta}(\cdot, u_{m+1} + s_{ij}(w_{m+1} - u_{m+1})) - \frac{\partial H_{ij}}{\partial \eta}(\cdot, u_{m+1}) \|_\infty \| w_{m+1} \\
&\qquad\qquad - u_{m+1} \|_\infty \| u \|_\infty + \| H_{ij}(\cdot, w_{m+1}) - H_{ij}(\cdot, u_{m+1}) \|_\infty \| w - u \|_\infty \} \\
&\le \quad c(\lambda, \rho) \, \| w - u \|_\infty^2
\end{aligned}
$$

for $\|w\|_{\mathcal{F}_1}$, $\|u\|_{\mathcal{F}_1} \le \rho$, where s_{ij} above is a number between 0 and 1 and $c(\lambda, \rho)$ is a constant which depends on (λ, ρ). This inequality proves (2.29) and $G \in C^1(\mathbb{R}^+ \times \mathcal{F}_1, \mathcal{F}_1)$. One can then similarly show the existence of $D_{12}G$ and $D_1 G$.

For part (ii) we use Schauder's theory to obtain

$$
\begin{aligned}
\frac{\|G(\lambda, u)\|_{\mathcal{F}_1}}{\|u\|_{\mathcal{F}_1}} &\le \overline{k} \, \frac{\| [H(\cdot, u_{m+1}) - B] \, u \|_\infty}{\|u\|_\infty} \\
&\le \overline{k} \sum_{i,j=1}^{\infty} \| H_{ij}(\cdot, u_{m+1}) - H_{ij}(\cdot, 0) \|_\infty \to 0, \quad \text{as } \|u\|_{\mathcal{F}_1} \to 0.
\end{aligned}
$$

Continuing as in Lemma 2.6, one can show $F \in C^2(\mathbb{R}^+ \times \mathcal{F}_1, \mathcal{F}_1)$. Using (2.29) and Lemma 2.6 again, we obtain

$$
\begin{aligned}
&L_0 = D_2 F(\widehat{\lambda}_0, 0), \qquad L_1 = D_{12} F(\widehat{\lambda}_0, 0), \\
&G(\lambda, 0) \equiv 0, \qquad D_2 G(\widehat{\lambda}_0, 0) = D_{12} G(\widehat{\lambda}_0, 0) = 0.
\end{aligned}
$$

Consequently, by means of Lemma 2.5, we can apply a bifurcation theorem of Crandall-Rabinowitz [3] to $F(\lambda, u) = 0$ to obtain a C^1-curve $(\lambda(s), \theta(s))$ of solutions as described below (see also [5]).

Theorem 2.3. *Under the hypotheses* [H1] *to* [H5], *the point* $(\widehat{\lambda}_0, 0)$ *is a bifurcation point for problem* (2.28) (*or equivalently for problem* (1.2)). *Here,* $\widehat{\lambda}_0 > 0$ (*with corresponding eigenfunction* v^0), *is the eigenvalue described in Theorem* 2.2. *More precisely, there exists an interval* $[0, \delta)$, $\delta > 0$, *and a* C^1*-curve* $(\lambda(s), \theta(s)) : [0, \delta) \to \mathbb{R} \times \mathcal{F}_1$ *such that* $\lambda(0) = \widehat{\lambda}_0$, $\theta(0) = 0$, *and the solution* $\widehat{u}(x) = u(y) = u(x/k)$ *of* (2.28) *is of the form*

$$
u(x/k) = s(v^0 + \theta(s))(x/k) \quad \text{for } x \in k\Omega, \quad k = \sqrt{\lambda(s)}.
$$

The corresponding solution $\widehat{u}(x) = u(x/k)$ *of* (1.1) *is positive in* $k\Omega$ *and is in* $C^{2+\mu}(k\overline{\Omega})$.

Note that $\lambda(s)$ and $u(x/k)$ satisfy

$$F(\lambda(s), s(v^0 + \theta(s)) = 0 \quad \text{for} \quad s \in [0, \delta). \tag{2.30}$$

Remark. The above number $\widehat{\lambda}_0$, which is first defined in the statement of Theorem 2.2, is unique.

Proof of Remark. Suppose $\widehat{\lambda}_1$ is any number so that there exists $v^1 \in \mathcal{E}$ (with each component $v_i^1 > 0$ in Ω, $i \in J$) satisfying

$$(-\Delta)v^1 = \widehat{\lambda}_1 B v^1 \quad \text{in} \quad \Omega, \qquad v^1 = 0 \quad \text{on} \quad \partial\Omega, \tag{2.31}$$

where B satisfies hypotheses [H1] to [H4]. Let $\widehat{v}^0$ be as defined in Lemma 2.4. Taking inner products on both sides of (2.31) with $\widehat{v}^0$, we obtain

$$\widehat{\lambda}_1 \langle Bv^1, \widehat{v}^0 \rangle = \langle -\Delta v^1, \widehat{v}^0 \rangle = \langle v^1, -\Delta \widehat{v}^0 \rangle = \widehat{\lambda}_0 \langle v^1, B^T \widehat{v}^0 \rangle = \widehat{\lambda}_0 \langle Bv^1, \widehat{v}^0 \rangle.$$

Since $\widehat{v}^0_{m+1} \equiv 0$, the hypotheses [H1] to [H4] on B imply that $\langle Bv^1, \widehat{v}^0 \rangle \neq 0$. This shows that $\widehat{\lambda}_1 = \widehat{\lambda}_0$.

3. Linearized and asymptotic stability. For the rest of this article, we will always assume hypotheses [H1] to [H5]. We will investigate the linearized and asymptotic stability of the positive bifurcating solution found in Theorem 2.3. For this purpose, we will introduce more assumptions on the derivative of H_{ij} below. Applying the theory in Crandall and Rabinowitz [3] and the fact that

$$\int_\Omega \widehat{v}^0 \cdot \Delta^{-1} v^0 \, dx \neq 0 \tag{3.1}$$

(note that $\widehat{v}_i^0 > 0$ for $i = 1, \ldots, m$, $\widehat{v}^0_{m+1} = 0$, and $v_i^0 > 0$ for $i = 1, \ldots, m+1$), we can assert that there exists $\delta_1 \in (0, \delta)$ and two functions

$$\begin{aligned} &(\gamma(\cdot), z(\cdot)) \; : \; (\widehat{\lambda}_0 - \delta_1, \widehat{\lambda}_0 + \delta_1) \; \to \; \mathbb{R} \times \mathcal{F}_1, \\ &(\eta(\cdot), w(\cdot)) \; : \; [0, \delta_1) \; \to \; \mathbb{R} \times \mathcal{F}_1, \end{aligned}$$

with $(\gamma(\widehat{\lambda}_0), z(\widehat{\lambda}_0)) = (\eta(0), w(0)) = (0, v^0)$, such that

$$D_2 F(\lambda, 0) z(\lambda) = \gamma(\lambda) \Delta_D^{-1}(z(\lambda)), \quad \text{and} \tag{3.2}$$

$$D_2 F(\lambda(s), s(v^0 + \theta(s))) w(s) = \eta(s) \Delta_D^{-1}(w(s)). \tag{3.3}$$

Here, (3.1) and the theory in [3] imply that $\gamma(\lambda)$ and $\eta(s)$ are respectively Δ_D^{-1}-simple eigenvalues of $D_2F(\lambda, 0)$ and $D_2F(\lambda(s), s(v^0 + \theta(s))$, with eigenfunctions $z(\lambda)$ and $w(s)$. Moreover, the theory in [3] further leads to the following lemmas.

Lemma 3.1. *Assume the hypotheses* [H1] *to* [H5]. *There exists* $\rho > 0$ *such that for each* $s \in [0, \delta_1)$, *there is a unique (real) eigenvalue* $\eta(s)$ *for the linear operator*

$$F_s^* \overset{\text{(def)}}{\equiv} \Delta D_2 F(\lambda(s), s(v^0 + \theta(s))) \;:\; \mathcal{E} \to \mathcal{F} \tag{3.4}$$

satisfying $|\eta(s)| < \rho$ *with eigenfunction* $w(s) \in \mathcal{E}$. *That is,*

$$\begin{aligned} F_s^* w(s) &\overset{\text{(def)}}{\equiv} \Delta w(s) + \lambda(s) H(\cdot, (u_s)_{m+1}) w(s) + \lambda(s) \widetilde{H}(\cdot, (u_s)_{m+1}) u_s w_{m+1} \\ &= \eta(s) w(s), \end{aligned} \tag{3.5}$$

where $u_s \overset{\text{(def)}}{\equiv} s(v^0 + \theta(s))$.

The next few lemmas study the behavior of the eigenvalues $\lambda(s)$, $\eta(s)$ for small $s \geq 0$, and $\gamma(\lambda)$ near $\lambda = \lambda_0$. In order to obtain stability we will need the following additional hypotheses:

[H6] $$\Big(\frac{\partial H_{ij}}{\partial \eta}\Big)(\cdot, 0) \leq 0 \;\text{ in } \Omega, \text{ for all } i = 1, \ldots, m,\; j = 1, \ldots, m+1;$$

$$\Big(\frac{\partial H_{ij}}{\partial \eta}\Big)(\overline{x}, 0) < 0 \;\text{ for some } \overline{x} \in \Omega, \text{ some } i = 1, \ldots, m, \text{ and some } j = 1, \ldots, m+1.$$

For the remaining part of this article, we will always assume hypotheses [H1] to [H6].

Lemma 3.2. *The function* $\lambda(s)$ *satisfies* $\lambda'(0) > 0$.

Proof. Theorem 2.3 asserts that $\lambda'(0)$ exists. Equation (2.30) implies that $s(v^0 + \theta(s))$ is in $\mathcal{E}$; and for $s \in [0, \delta)$, we have

$$\Delta(s(v^0 + \theta(s)) + \lambda(s) H(\cdot, s((v^0)_{m+1} + \theta_{m+1}(s))) s(v^0 + \theta(s)) = 0.$$

Dividing by s, then differentiating with respect to s and setting $s = 0$, we obtain

$$\Delta(\theta'(0)) + \lambda'(0) H(\cdot, 0) v^0 + \widehat{\lambda}_0 H(\cdot, 0) \theta'(0) + \widehat{\lambda}_0 \widetilde{H}(\cdot, 0)(v^0)_{m+1} v^0 = 0.$$

Multiplying by $(\widehat{v}^0)^T$ and integrating over Ω we obtain

$$\begin{aligned} \int_\Omega \{ &(\widehat{v}^0)^T \Delta \theta'(0) + \lambda'(0)(\widehat{v}^0)^T B v^0 + \widehat{\lambda}_0 (\widehat{v}^0)^T B \theta'(0) \\ &+ \widehat{\lambda}_0 (\widehat{v}^0)^T \widetilde{H}(\cdot, 0)(v^0)_{m+1} v^0 \} \, dx = 0. \end{aligned}$$

Integrating by parts, we find

$$\int_\Omega \lambda'(0)(\widehat{v}^0)^T B v^0 \, dx = -\widehat{\lambda}_0 \int_\Omega (\widehat{v}^0)^T \widetilde{H}(\cdot, 0) v^0 (v^0)_{m+1} \, dx.$$

Consequently, we obtain

$$\lambda'(0) = \frac{-\widehat{\lambda}_0 \int_\Omega (\widehat{v}^0)^T \widetilde{H}(\cdot, 0) v^0 (v^0)_{m+1} \, dx}{\int_\Omega (\widehat{v}^0)^T B v^0 \, dx} > 0. \tag{3.6}$$

Note that the sign of the numerator above is determined by hypotheses [H6].

Lemma 3.3. *The function* $\gamma(\lambda)$ *in* (3.2) *satisfies* $\gamma'(\widehat{\lambda}_0) > 0$.

Proof. Note that $D_2F(\lambda, 0) = I + \lambda\Delta_D^{-1}B$. From (3.2), we have

$$(I + \lambda\Delta_D^{-1}B)z(\lambda) = \gamma(\lambda)\Delta_D^{-1}z(\lambda), \quad \text{for } \lambda \in (\widehat{\lambda}_0 - \delta_1, \widehat{\lambda}_0 + \delta_1).$$

We can thus readily obtain

$$\Delta z(\lambda) + \lambda Bz(\lambda) = \gamma(\lambda)z(\lambda),$$
$$\int_\Omega (\widehat{v}^0)^T\Delta z(\lambda) + \lambda(\widehat{v}^0)^T Bz(\lambda)\,dx = [\gamma(\lambda) - \gamma(\widehat{\lambda}_0)]\int_\Omega (\widehat{v}^0)^T z(\lambda)\,dx,$$

since $\gamma(\widehat{\lambda}_0) = 0$. Integrating the first term on the left above by parts, using the equation satisfied by $\widehat{v}^0$, factoring and cross multiplying, we deduce

$$\frac{\gamma(\lambda) - \gamma(\widehat{\lambda}_0)}{\lambda - \widehat{\lambda}_0} = \frac{\int_\Omega (\widehat{v}^0)^T Bz(\lambda)\,dx}{\int_\Omega (\widehat{v}^0)^T z(\lambda)\,dx}.$$

Taking the limit as λ tends to $\widehat{\lambda}_0$, we obtain

$$\gamma'(\widehat{\lambda}_0) = \frac{\int_\Omega (\widehat{v}^0)^T Bv^0\,dx}{\int_\Omega (\widehat{v}^0)^T v^0\,dx} > 0.$$

The strict inequality above is due to hypothesis [H4].

Lemma 3.4. *There exists* $\delta_2 \in (0, \delta_1)$, *such that* $\eta(s) < 0$ *and* $u_s \equiv s(v^0 + \theta(s)) > 0$ *in* Ω *for all* $s \in (0, \delta_2)$.

Proof. From Theorem 1.16 in [3], we find $-s\lambda'(s)\gamma'(\widehat{\lambda}_0)$ and $\eta(s)$ have the same sign for $s > 0$ near 0. Hence Lemmas 3.2 and 3.3 imply that $\eta(s) < 0$ for small positive s. Since v^0 is positive in Ω with negative outward normal derivative on the part of $\partial\Omega$ where it is zero, and $\theta(s) \to 0$ in C^1 as $s \to 0$, we must have $u_s \equiv s(v^0 + \theta(s)) > 0$ in Ω for $s > 0$ sufficiently small. □

It remains to investigate the other eigenvalues of F_s^*. Let $b > 0$ be a large enough constant such that

$$\sum_{j=1}^{m+1} \widehat{\lambda}_0 H_{ij}(x, 0) - b < 0 \quad \text{for all } x \in \Omega, \quad i = 1, \dots, m+1. \tag{3.7}$$

For convenience, let M_0 be the complex extension of the operator from $\mathcal{E}$ into $\mathcal{F}$ defined by

$$M_0 w = \Delta(w) + \widehat{\lambda}_0 B(w) - bI(w), \tag{3.8}$$

for $w \in \mathcal{E}$, where I is the identity operator.

Lemma 3.5.

(i) *The inverse of* M_0 *can be defined as* $M_0^{-1} \in \mathcal{L}(\mathcal{F})$, *i.e. a bounded linear operator* $\mathcal{F} \to \mathcal{F}$, *and it is compact.*

(ii) *If* $\lambda \neq 0$ *is an eigenvalue of* $M_0 + bI$ *(i.e.,* $\Delta + \widehat{\lambda}_0 B$*), then* $Re(\lambda) < -r$ *for some positive number* r.

The proof uses the sign of the off-diagonal terms of B, (3.7) and the maximum principle for the corresponding systems. It is essentially the same as the proof of Lemma 2.8 in [14]. The details will be omitted here.

For convenience, we let σ_s denote the point spectrum of F_s^*. The next linearized stability theorem follows from Lemmas 3.1 to 3.5.

Theorem 3.1. *Under hypotheses* [H1] *to* [H6], *there exists a number* $\delta^* \in (0, \delta)$ *where* δ *is described in Theorem* 2.3, *and a positive function* $\eta(s)$ *for* $s \in (0, \delta^*)$ *such that the point spectrum* σ_s *satisfies*

$$Re\ \sigma_s \subset \{w \in \mathbb{R}^1 : w \leq -\eta(s)\} \quad \text{for } \ s \in (0, \delta^*). \tag{3.9}$$

Here, Re σ_s *denotes the set of real numbers which are real parts of numbers in* σ_s.

The proof of Lemmas 3.2 to 3.4 are given in detail above, and they are the consequences of [H1] to [H6]. They are different from the model in [14]. However, the proof of Theorem 3.1, using assertions in Lemmas 3.1 to 3.5 are exactly the same as the proof of Theorem 2.2 in [14]. The details are thus omitted here.

For each fixed $\overline{s} \in (0, \delta^*)$, the function $u_{\overline{s}} = \overline{s}(v^0 + \theta(\overline{s}))$ is a steady-state solution of problem (1.3) with $\lambda = \lambda(\overline{s})$ given by Theorem 2.3. We now proceed to investigate the time asymptotic stability of this steady-state as a solution of the parabolic system (1.3). Let B_1 and B_2 be Banach spaces as follows:

$$\begin{aligned} &B_1 = \{u : \ u \in [C(\overline{\Omega})]^{m+1}, \ u = 0 \text{ on } \ \partial\Omega\}, \quad \text{and} \\ &B_2 = \{u : \ u \in [L_p(\Omega)]^{m+1}\} \text{ for } \ p \text{ large enough such that } \ \frac{N}{2p} < 1. \end{aligned} \tag{3.10}$$

Let A_1 be the Δ operator on B_1 with domain $D(A_1) = \{u : u \in [W^{2,p}(\Omega)]^{m+1}$ for all p, $\Delta u \in [C(\overline{\Omega})]^{m+1}$, $u = 0$ and $\Delta u = 0$ on $\partial\Omega\}$; and A_2 be the Δ operator on B_2 with domain $D(A_2) = \{u \in B_2 : \ u \in [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^{m+1}\}$. For $u =$col.$(u_1, \dots, u_{m+1})$, and $f(u) = \lambda H(\cdot, u_{m+1})u$, we can consider the following nonlinear initial-boundary value problem for each $i = 1, 2$ corresponding to (1.3):

$$\frac{du}{dt} - A_i u(t) = f(u(t)), \quad u(0) = u_0 \quad \text{ for } \ t \in (0, T] \tag{3.11}$$

with $u(t) \in D(A_i)$, $t > 0$, respectively for $i = 1, 2$. Here, we suppress writing the dependence of f on $\lambda(\overline{s})$, since it is fixed for some $\overline{s} \in (0, \delta^*)$.

Definition. A solution of (3.11) in B_i is a function

$$u \in C([0,T], B_i) \cap C^1((0,T], B_i),$$

with $u(0) = u_0$, $u(t) \in D(A_i)$ for all $t \in (0,T]$; and $u(t)$ satisfies (3.11) for $t \in (0,T]$.

The operator A_2 is an infinitesimal generator of an analytic semigroup $M(t)$, $t \geq 0$, on B_2. It is well known that for $\alpha > 0$

$$(-A_2)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \tau^{\alpha-1} M(\tau)\, d\tau$$

defines a bounded linear operator on B_2. Moreover, $[(-A_2)^{-\alpha}]^{-1} = (-A_2)^\alpha$ is a closed linear operator on B_2 with dense domain $D((-A_2)^\alpha) = (-A_2)^{-\alpha}(B_2)$. We denote by X_α the Banach space $(D(-A_2)^\alpha), \|\cdot\|_\alpha)$, where $\|u\|_\alpha = \|(-A_2)^\alpha u\|_{L^p}$ for all $u \in D((-A)^\alpha)$. Moreover, for $N/(2p) < \alpha < 1$, there exists a constant $C(\alpha) > 0$ such that $\|u\|_\infty \leq C(\alpha)\|u\|_\alpha$ for all $u \in X_\alpha$ (see [16] or [1]).

For further discussion of solutions of these spaces, we will make the following additional assumptions for $i, j = 1, \dots, m+1$:

[H7] $\quad H_{ij}(x,\eta)$ and $(\partial H_{ij}/\partial\eta)(x,\eta)$ are bounded for all $(x,\eta) \in \Omega \times \mathbb{R}^1$,

$$\Big|\frac{\partial H_{ij}}{\partial \eta}(x,\eta_1) - \frac{\partial H_{ij}}{\partial \eta}(x,\eta_2)\Big| \leq C\,|\eta_1 - \eta_2| \text{ for all } \eta_1, \eta_2 \in \mathbb{R}^1,$$
$$\text{for some constant } C > 0.$$

For the function f described above, the hypotheses [H1] and [H7] lead to the following Lipschitz properties in B_1 and B_2: There exists $\overline{C} > 0$ and $\rho > 0$, such that

$$\|f(u) - f(w)\|_\infty \leq \overline{C}\|u - w\|_\infty \text{ for all } u, w \in B_1 \text{ with } \|u\|_\infty, \|w\|_\infty \leq \rho, \tag{3.12}$$

and

$$\|f(u) - f(w)\|_{L^p} \leq \overline{C}\|u - w\|_\alpha \text{ for all } u, w \in X_\alpha \text{ with } \|u\|_\alpha, \|w\|_\alpha \leq \rho. \tag{3.13}$$

Here $\overline{C}$ depends on α and ρ. From the Lipschitz properties (3.12) and (3.13), we obtain local existence for solutions of (3.11) in B_1 and B_2 respectively (see e.g. [16]). Moreover, Corollary 3.3.5 in [7] implies that solutions of (3.11) are global. From hypotheses [H1] and [H7], we can further deduce

$$\|f(w) - f(u) - df_u(w-u)\|_\infty = o(\|w-u\|_\infty) \tag{3.14}$$

for all $u, w \in B_1$, $\|u\|_\infty \leq \rho$, as $\|w-u\|_\infty \to 0$, and

$$\|f(w) - f(u) - df_u(w-u)\|_{L^p} = o(\|w-u\|_\alpha) \tag{3.15}$$

for all $u, w \in X_\alpha,\ \|u\|_\alpha \le \rho,$ as $\|w-u\|_\alpha \to 0$. Here,

$$df_u z = \lambda H(\cdot, u_{m+1})z + \lambda\big[\frac{\partial H}{\partial \eta}(\cdot, u_{m+1})\big]u z_{m+1}.$$

Note that the operator $A_i + df_{\overline{u}}$, $\overline{u} \overset{(\mathrm{def})}{=} u_{\overline{s}}$, on B_i can be written as $F^*_{\overline{s}}$ as in (3.5). Thus the point spectrum of $A_i + df_{\overline{u}}$ lies in $\{\lambda \in C:\ Re\ \lambda < -\eta(\overline{s})\}$ by Theorem 3.1. Moreover, the spectrum of $F^*_{\overline{s}}$ is the set $\{1/\mu:\ \mu$ is in the spectrum of $(F^*_{\overline{s}})^{-1}\}$, and thus consists only of eigenvalues (see e.g. pp. 51 and 79 in [16]). By means of (3.14), (3.15) and Theorem 3.1, we can apply the stability Theorem 5.1.1 in [7] for sectorial operators to obtain the following theorem for the asymptotic stability of the steady-state solution.

Theorem 3.2. *Assume hypotheses* [H1] *to* [H7], *and let* $\overline{u} \overset{(\mathrm{def})}{=} u_{\overline{s}}$, $\lambda = \lambda(\overline{s})$ *for a fixed* $\overline{s} \in (0, \delta^*)$, $\alpha \in (N/(2p), 1)$. *Then, for each* $i = 1, 2$, *there exists* $\rho > 0$, $\beta > 0$ *and* $M > 1$ *such that equation* (3.11) *has a unique solution in* B_i *for all* $t > 0$ *if* $u_0 \in B_1$ *and* $\|u_0 - \overline{u}\|_\infty \le \rho/(2M)$ *for* $i = 1$ (*or* $u_0 \in X_\alpha$ *and* $\|u_0 - \overline{u}\|_\alpha \le \rho/(2M)$ *for* $i = 2$). *Moreover, the solution satisfies*

$$\|u(t) - \overline{u}\|_\infty \le 2Me^{-\beta t}\|u_0 - \overline{u}\|_\infty \quad \text{for all} \quad t \ge 0,\ i = 1, \quad \text{or} \tag{3.16}$$

$$\|u(t) - \overline{u}\|_\alpha \le 2Me^{-\beta t}\|u_0 - \overline{u}\|_\alpha \quad \text{for all} \quad t \ge 0,\ i = 2. \tag{3.17}$$

Note: *the condition on* α *is only assumed for solutions in* B_2.

The details of the proof are the same as those for Theorem 3.1 in [14], and will be omitted here. For (3.16), the theories in [18] and [19] are used.

Note that hypotheses [H1] to [H5] are sufficient to insure the existence of the steady-state $\overline{u}$, by Theorem 2.3.

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