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MULTIPLE COMPLETENESS OF ROOT VECTORS OF A SYSTEM OF OPERATOR PENCILS AND ITS APPLICATION

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Abstract. The role and importance of the Fourier method for investigation of mathematical physics problems is well known. By this method we can investigate problems for which the corresponding spectral problem is self-adjoint. By the Fourier method a solution $u(t,x)$ to the problem is written in the form of the series $u(t,x) = \sum_{k=1}^{\infty} C_k u_k(t,x)$, where $u_k(t,x)$ are elementary solutions of the considered problem. In the self-adjoint case the Hilbert theory of self-adjoint operators with a compact resolvent gives us information about the existence of elementary solutions and whether enough elementary solutions exist to write $u(t,x)$ in this series form. In the case when the principal part of the corresponding spectral problem is non-self-adjoint the existence of elementary solutions and convergence of the series are already problems. And this, in turn, leads to the problem of completeness of root functions. The completeness problem of the root vectors of a system of operator pencils and the root functions of elliptic boundary value problems is a subject of this paper.

1. Statement of problems and reference notes.

1.1. A system of operator pencils. The results of M. V. Keldysh ([13]) about the completeness of root vectors of an operator pencil and N. Dunford and J. T. Schwartz ([6, Corollary 11.9.31]) about the completeness of root vectors of an operator were essentially improved and generalized in works of the author ([23–26]).

Let H and H^ν, $\nu = 1, \ldots, m$, be Hilbert spaces. Consider a problem for a system of polynomial operator pencils in H:

$$\begin{aligned} L(\lambda)u &\equiv \lambda^n u + \lambda^{n-1} A_1 u + \cdots + A_n u = 0, \\ L_\nu(\lambda)u &\equiv \lambda^{n_\nu} A_{\nu 0} u + \lambda^{n_\nu - 1} A_{\nu 1} u + \cdots + A_{\nu n_\nu} u = 0, \quad \nu = 1, \ldots, m, \end{aligned} \tag{1}$$

where $n \geq 1$, $0 \leq n_\nu \leq n-1$, $m \geq 0$, and A_k are, generally speaking, unbounded operators in H and $A_{\nu k}$, $k = 0, \ldots, n_\nu$ are, generally speaking, unbounded operators from H into H^ν.

System (1) is an abstract interpretation of boundary value problems for elliptic equations. The definition of root vectors of problem (1) is given and conditions that guarantee n-fold completeness of root vectors of problem (1) are found.

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The case of N. Dunford and J. T. Schwartz ([6, Corollary 11.9.31]) is $n = 1, m = 0$; and the case of M.V. Keldysh ([13]) is n is arbitrary, $m = 0, A_n = S^n, S = S^*$, operators $A_k S^{-k}, k = 1, \dots, n-1$, are compact in H. Our results in [26] contain both the case of N. Dunford and J.T. Schwartz ([6, Corollary 11.9.31]) and the case of M.V. Keldysh ([13]).

In the literature we often have different formulations of the same theorem. But if the theorem will be applied in the theory of differential equations it is useful to formulate the theorem in such a way that its conditions can be easily checked in the application. For example, in the theory of the completeness of root vectors N. Dunford and J.T. Schwartz ([6, Corollary 11.9.31]) claim that $R(\lambda, A) \in \sigma_p(H)$; I. Gohberg and M.G. Krein ([9]) claim that singular numbers $s_j(R(\lambda, A)) \le Cj^{-q}$ for some q. In this paper we claim that $s_j(J; H(A), H) \le Cj^{-q}$ for some q. Since estimates of singular numbers of the embedding operator of one functional space into another one is a subject of many papers and monographs, our condition is more checkable in the application.

We prove the completeness of root vectors of an unbounded polynomial operator pencils system. These results, if even one operator pencil is available, strengthen the analogous results given in the books by I. Gohberg and M.G. Krein ([9]), A.S. Markus ([18]), and S. Yakubov ([26]) and in the paper by G.V. Radzievskii ([19]). S. Yakubov ([26]) in contrast to [9], [18] and [19] does not assume that the operator pencil

$$L(\lambda) = \lambda^n I + \lambda^{n-1} A_1 + \dots + A_n$$

has a weight; that is, the estimate $\|A_k u\| \le C\|A^k u\|$, $u \in D(A^k)$, where $A \ge \gamma^2 I$ in H, is not assumed. In spite of this fact, the results in [26] can not be applied to noncoercive elliptic boundary value problems for $n \ge 2$. In this paper we derive the completeness theory of a system of polynomial operator pencils in such a way that it can be applied to noncoercive elliptic boundary value problems.

1.2. Regular elliptic boundary value problems. Let G be a bounded domain in the Euclidean space $\mathbb{R}^r$ with an $(r-1)$-dimensional boundary ∂G. Consider the spectral problem in G for principally regular elliptic boundary value problems with a polynomial parameter

$$\begin{aligned} L(\lambda)u &= \lambda^n u(x) + \sum_{k=1}^{n} \lambda^{n-k} \Big(\sum_{|\alpha|=dk} a_{k\alpha}(x) D^\alpha u(x) + B_k u|_x \Big) = 0, \quad x \in G, \\ L_\nu(\lambda)u &= \sum_{k=0}^{n_\nu} \lambda^k \Big(\sum_{|\alpha|=m_\nu - dk} b_{\nu k\alpha}(x') D^\alpha u(x') + T_{\nu k} u|_{x'} \Big) = 0, \quad x' \in \partial G, \end{aligned} \tag{2}$$

$\nu = 1, \dots, m$, where the problem's weight $d = \frac{2m}{n}$ is, in general, a noninteger, and $a_{k\alpha}(x) = b_{\nu k\alpha}(x) = 0$ if dk is a noninteger, $0 \le n_\nu \le n-1$, $D^\alpha = D_1^{\alpha_1} \cdots D_r^{\alpha_r}$, $D_j = -i\frac{\partial}{\partial x_j}$, $j = 1, \dots, r$, $\alpha = (\alpha_1, \dots, \alpha_r)$ is a multi-index, $|\alpha| = \sum_{j=1}^r \alpha_j$, $\lambda \in \mathbb{C}$, $x = (x_1, \dots, x_r)$, $a_{k\alpha} \in C^{\ell-2m}(\overline{G})$, $b_{\nu k\alpha} \in C^{\ell-m_\nu}(\overline{G})$, $\partial G \in C^\ell$, $\ell \ge \max\{2m, m_\nu +$

1}. Here operators B_k from $W_q^{dk}(G)$ into $L_q(G)$ are compact, operators $T_{\nu k}$ from $W_q^{m_\nu - dk + \frac{1}{q}}(G)$ into $L_q(\partial G)$ are compact, $q \in (1, \infty)$, $B_k u|_x = (B_k u)(x)$.

The definition of root functions of problem (2) is given and conditions that guarantee n-fold completeness of root functions of problem (2) are found. In the papers by M.V. Keldysh ([13]) and F.E. Browder ([5]) the completeness of root functions of elliptic boundary value problems with a self-adjoint principal part is proved. In papers by S. Agmon ([1]), G. Geymonat and P. Grisvard ([7]), A.N. Kozhevnikov ([15]), Z.A. Kotko and S.G. Krein ([14]), S. Ya. Yakubov ([23–26]), M.S. Agranovich ([4]), A. Kozhevnikov and S. Yakubov ([16]) the completeness of root functions is proved when the principal part of the problem is non-self-adjoint. In the paper by S. Agmon ([1]) a spectral parameter enters linearly into the equation and does not appear in the boundary value conditions. Except in the paper by S. Ya. Yakubov ([25]) in other works it is assumed that $m_\nu \leq 2m - 1$. A condition of the form $\max\{m_\nu\} - \min\{m_\nu - dn_\nu\} \leq 2m - 1$ appears for the first time in the author's work ([25]). In Agmon's case ($n = 1, n_p = 0$) ([1]) this condition is transformed into the following: $\max\{m_\nu\} - \min\{m_\nu\} \leq 2m - 1$, i.e., the case $m_\nu \leq 2m - 1$, as it relates to the work of S. Agmon and others, is covered in [25]. The case when $n = n_\nu = 1$ was investigated by A. Kozhevnikov and S. Yakubov ([16]). The main goal of this paper is the improvement of the corresponding results in the author's book ([26]).

2. Operator pencils.

2.1. Introduction. In the theory of differential equations with constant operator coefficients, polynomial operator pencils play approximately the same role as characteristic polynomials play in the theory of differential equations with constant coefficients.

Consider, in a Banach space E, the following differential-operator equation:

$$L(D)u \equiv u^{(n)}(t) + A_1 u^{(n-1)}(t) + \cdots + A_n u(t) = 0, \tag{1}$$

where A_k, $k = 1, \ldots, n$ are given, generally speaking, unbounded operators in E, $u(t)$ is an unknown function with values in E, $D = D_t = \frac{\mathrm{d}}{\mathrm{dt}}$, and the characteristic operator pencil

$$L(\lambda) \equiv \lambda^n I + \lambda^{n-1} A_1 + \cdots + A_n. \tag{2}$$

The main connection between equation (1) and pencil (2) is shown by the following lemma.

Lemma 1. *The function $u(t)$ of the form*

$$u(t) \equiv \mathrm{e}^{\lambda_0 t}\big(\frac{t^k}{k!}u_0 + \frac{t^{k-1}}{(k-1)!}u_1 + \cdots + u_k\big), \tag{3}$$

where $u_j \in E$, $u_0 \neq 0$, is a solution to equation (1) *if and only if the following correlations hold:*

$$L(\lambda_0)u_p + \frac{1}{1!}L'(\lambda_0)u_{p-1} + \cdots + \frac{1}{p!}L^{(p)}(\lambda_0)u_0 = 0, \quad p = 0, \ldots, k. \tag{4}$$

Proof. If function (3) is a solution to equation (1), then

$$L(D)\sum_{q=0}^{k} e^{\lambda_0 t}\frac{t^q}{q!}u_{k-q} \tag{5}$$
$$=\sum_{q=0}^{k} e^{\lambda_0 t}\big(L(\lambda_0)\frac{t^q}{q!}+\frac{L'(\lambda_0)}{1!}\frac{t^{q-1}}{(q-1)!}+\cdots+\frac{L^{(q)}(\lambda_0)}{q!}\big)u_{k-q}=0.$$

Here the Leibniz formula is used

$$L(D)(\varphi_1\varphi_2)=\sum_s \frac{1}{s!}(L^{(s)}(D)\varphi_1)D^s\varphi_2.$$

Hence

$$L(\lambda_0)u_k+\frac{1}{1!}L'(\lambda_0)u_{k-1}+\cdots+\frac{1}{k!}L^{(k)}(\lambda_0)u_0=0, \quad \ldots\ldots, \quad L(\lambda_0)u_0=0;$$

i.e., (4) is valid. Conversely, if (4) holds, then (5) is true; i.e., function (3) is a solution to (1).

Corollary 2. *If function* (3) *is a solution to equation* (1)*, then functions*

$$u_q(t)\equiv e^{\lambda_0 t}\big(\frac{t^q}{q!}u_0+\frac{t^{q-1}}{(q-1)!}u_1+\cdots+u_q\big), \quad q=0,\ldots,k,$$

are also solutions to equation (1)*.*

2.2. Completeness of root vectors of an unbounded operator.

2.2.1. Notations and definitions. E denotes a Banach space, and H denotes a Hilbert space. The domain of definition of a linear operator A in E is denoted by $D(A)$, and the range of its values is denoted by $R(A)$. The point λ of the complex plane is called a *regular point* of the operator A, if the operator $A-\lambda I$ in E is invertible. The set $\rho(A)$ of all regular points of the operator A is called a *resolvent set* of the operator A, and the operator $R(\lambda, A)\equiv(\lambda I-A)^{-1}$ is called the *resolvent* of the operator A. The complement of the set $\rho(A)$ in the whole complex plane is called the *spectrum* $\sigma(A)$ of the operator A. Thus, the spectrum of an operator is a closed set. The infinite point $\lambda=\infty$ is always attached to the resolvent set of a bounded operator A and to the spectrum of an unbounded operator A. All *eigenvalues* of the operator A belong to the spectrum $\sigma(A)$, i.e., those numbers λ for which the equation $Au=\lambda u$ has at least one nonzero solution $u\in D(A)$. The element $u_0\neq 0$ that satisfies the equation $Au_0=\lambda_0 u_0$ is called an *eigenvector* of the operator A, corresponding to the eigenvalue λ_0. If the elements $u_0, u_1, \ldots, u_k$ correlate with

$$Au_p=\lambda_0 u_p+u_{p-1}, \qquad p=1,\ldots,k, \tag{1}$$

then the element u_k is called an *associated vector of the k-th rank* to the eigenvector u_0. The number $k+1$ is called the *length of the chain* $u_0, u_1, \ldots, u_k$. The element u_0 is called an *eigenvector of the r-th rank*, if the longest chain corresponding to u_0 has length equal to r. The chain $u_0, u_1, \ldots, u_{r-1}$ is called the *Jordan chain*. The elements $u_0, u_1, \ldots, u_{r-1}$ are linearly independent. The linear manifold

$$N^k = N^k_{\lambda_0} = \{u | u \in D(A^{k+1}),\ (A - \lambda_0 I)^{k+1} u = 0\}, \quad k = 0, \ldots, \infty,$$

is called a *root lineal of the k-th rank*. Obviously, $N^0 \subset N^1 \subset N^2 \subset \cdots$. The linear manifold

$$N = N_{\lambda_0} = \bigcup_{k=0}^{\infty} N^k$$

is called a *root lineal*. The dimension $\gamma = \gamma(\lambda_0)$ of the lineal N is called an *algebraic multiplicity of the eigenvalue* λ_0. If $\gamma < \infty$, then the lineals N^k and N are closed, and in this case we call them a *root subspace of the k-th rank* and a *root subspace* respectively.

The eigenvectors and associated vectors are joined under the common name of *root vectors*. In the case of a linear pencil $L(\lambda) = \lambda I - A$ correlations 2.1/4 and (1) coincide. So Lemma 2.1/1 yields the following:

Lemma. *The function $u(t)$ of the form $u(t) \equiv e^{\lambda_0 t}\big(\frac{t^k}{k!}u_0 + \frac{t^{k-1}}{(k-1)!}u_1 + \cdots + u_k\big)$, where $u_j \in E$, $u_0 \neq 0$, is the solution to the equation*

$$u'(t) - Au(t) = 0$$

if and only if the chain $u_0, u_1, \ldots, u_k$ is a chain of root vectors of the operator A, corresponding to the eigenvalue λ_0.

2.2.2. Completeness of root vectors of unbounded operators.

Theorem 1. *Let the following conditions be satisfied:*

(1) *an unbounded closed operator A in a Hilbert space H has a dense domain $D(A)$;*
(2) *the embedding $H(A) \subset H$ is compact and for some $p > 0$ $s_j(J; H(A), H) \leq Cj^{-p}$, $j = 1, \ldots, \infty$;*
(3) *there exist rays ℓ_k with angles between the neighboring rays less than $p\pi$ and $\eta \geq -1$ such that*[2] $\|R(\lambda, A)\| \leq C|\lambda|^{\eta}, \qquad \lambda \in \ell_k, \quad |\lambda| \to \infty.$

Then the spectrum of the operator A is discrete and a system of root vectors of the operator A is complete in the space $H(A^k), k = 0, \ldots, \infty$.

Proof. By [26, Lemma 2.3.1] $s_j(J; H(A), H) = s_j(R(\lambda_0, A); H, H)$ (for any λ_0 in the resolvent set of the operator A); i.e., the operator A satisfies conditions of Corollary [6, 11.9.31]. So, Corollary [6, 11.9.31] is applicable to the operator A, from which the statement of the theorem follows.

[2]For $p > 2$ the existence of one such ray is enough.

Theorem 2. *Let the following conditions be satisfied:*

(1) *an unbounded closed operator A in a Hilbert space H has a dense domain $D(A)$;*
(2) *the embedding $H(A) \subset H$ is compact and for some $p > 0$ $s_j(J; H(A), H) \leq Cj^{-p}$, $j = 1, \ldots, \infty$;*
(3) *there exist rays ℓ_k with angles between the neighboring rays less than $p\pi$ and $\eta \in (0, 1]$ such that*[3] $\|R(\lambda, A)\| \leq C|\lambda|^{-\eta}, \qquad \lambda \in \ell_k, \quad |\lambda| \to \infty;$
(4) *B is an operator in H, $D(B) \supset D(A)$ and for any $\varepsilon > 0$*

$$\|Bu\| \leq \varepsilon\|Au\|^{\eta}\|u\|^{1-\eta} + C(\varepsilon)\|u\|, \qquad u \in D(A).$$

Then the spectrum of the operator $A + B$ is discrete and a system of root vectors of the operator $A + B$ is complete in the spaces H and $H(A)$.

Proof. Since $H(A + B) = H(A)$, then $s_j(J; H(A + B), H) \leq Cj^{-p}$, $j = 1, \ldots, \infty$. By [26, Lemma 2.3.1] $s_j(J; H(A + B), H) = s_j(R(\lambda_0, A + B); H, H)$ (for any λ_0 in the resolvent set of the operator $A + B$); i.e., the operator $A + B$ satisfies conditions 1 and 2 of Theorem 1. From conditions 3 and 4 and by [26, Lemma 2.2.8] it follows that the operator $A + B$ satisfies condition 3 of Theorem 1. So, Theorem 1 is applicable to the operator $A + B$.

2.3. n-fold completeness of root vectors of a system of unbounded polynomial operator pencils. The results of this paragraph are main and have many applications in the theory of differential equations.

2.3.1. Notations and definitions. Let H and H^{ν}, $\nu = 1, \ldots, m$, be Hilbert spaces. Consider a problem for a system of polynomial operator pencils in H

$$\begin{aligned} &L(\lambda)u \equiv \lambda^n u + \lambda^{n-1}A_1 u + \cdots + A_n u = 0, \\ &L_{\nu}(\lambda)u \equiv \lambda^{n_{\nu}}A_{\nu 0}u + \lambda^{n_{\nu}-1}A_{\nu 1}u + \cdots + A_{\nu n_{\nu}}u = 0, \quad \nu = 1, \ldots, m, \end{aligned} \tag{1}$$

where $n \geq 1$, $0 \leq n_{\nu} \leq n - 1$, $m \geq 0$, and A_k are, generally speaking, unbounded operators in H and $A_{\nu k}$, $k = 0, \ldots, n_{\nu}$, are, generally speaking, unbounded operators from H into H^{ν}. It is obvious that for $\lambda \neq 0$ $D(L(\lambda)) = \bigcap\limits_{k=1}^{n} D(A_k)$. A number λ_0 is called an *eigenvalue* of problem (1) if the problem

$$L(\lambda_0)u = 0, \qquad L_{\nu}(\lambda_0)u = 0, \quad \nu = 1, \ldots, m$$

has a nontrivial solution. The nontrivial solution u_0 is called an *eigenvector* of problem (1) corresponding to the eigenvalue λ_0. A solution of the problem

$$\begin{aligned} &L(\lambda_0)u_p + \frac{1}{1!}L'(\lambda_0)u_{p-1} + \cdots + \frac{1}{p!}L^{(p)}(\lambda_0)u_0 = 0, \\ &L_{\nu}(\lambda_0)u_p + \frac{1}{1!}L'_{\nu}(\lambda_0)u_{p-1} + \cdots + \frac{1}{p!}L_{\nu}^{(p)}(\lambda_0)u_0 = 0, \qquad \nu = 1, \ldots, m, \end{aligned}$$

[3] See the footnote of Theorem 1.

u_p, is called an *associated vector* of the p-th rank to the eigenvector u_0 of problem (1).

Eigenvectors and associated vectors of problem (1) are combined under the general name *root vectors* of problem (1). A complex number λ is called a *regular point* of problem (1) or of the pencil $\mathbb{L}(\lambda): \ u \to \mathbb{L}(\lambda)u = (L(\lambda)u, L_1(\lambda)u, \dots, L_m(\lambda)u)$ acting from H into $H \oplus H^1 \oplus \cdots \oplus H^m$, if the problem

$$L(\lambda)u = f, \qquad L_\nu(\lambda)u = f_\nu, \quad \nu = 1, \dots, m$$

for any $f \in H,\ f_\nu \in H^\nu$, has a unique solution

$$u \in D(\mathbb{L}(\lambda)) \equiv D(L(\lambda)) \bigcap_{\nu=1}^{m} D(L_\nu(\lambda))$$

and the estimate

$$\|u\| \le C(\lambda)\big(\|f\| + \sum_{\nu=1}^{m} \|f_\nu\|_{H^\nu}\big)$$

is satisfied. The complement of the regular point set in the complex plane is called the *spectrum* of problem (1) or of the pencil $\mathbb{L}(\lambda)$. The spectrum of problem (1) is called *discrete*, if:

a) all points λ, not coinciding with the eigenvalues of problem (1), are regular points of problem (1);
b) the eigenvalues are isolated and have finite algebraic multiplicities;
c) infinity is the only limit point of the set of the eigenvalues of problem (1).

Consider a system of differential equations

$$\begin{aligned} &L(D)u \equiv u^{(n)}(t) + A_1 u^{(n-1)}(t) + \cdots + A_n u(t) = 0, \\ &L_\nu(D)u \equiv A_{\nu 0} u^{(n_\nu)}(t) + \cdots + A_{\nu n_\nu} u(t) = 0, \quad \nu = 1, \dots, m, \end{aligned} \tag{2}$$

$$u^{(k)}(0) = v_{k+1}, \qquad k = 0, \dots, n-1, \tag{3}$$

where v_{k+1} are given elements of H, $D = \frac{d}{dt}$, $t \ge 0$. By virtue of Lemma 2.1/1 a function of the form

$$u(t) \equiv e^{\lambda_0 t}\big(\frac{t^k}{k!}u_0 + \frac{t^{k-1}}{(k-1)!}u_1 + \cdots + u_k\big) \tag{4}$$

is a solution to system (2), if and only if the system of vectors $u_0, u_1, \dots, u_k$ is a chain of root vectors of problem (1), corresponding to the eigenvalue λ_0. A solution of the form (4) is called an *elementary solution* to system (2). The inclination to approximate a solution to the Cauchy problem (2)–(3), by linear combinations of the elementary solutions, suggests that the vector $(v_1, v_2, \dots, v_n)$ should be approximated by linear combinations of vectors of the form

$$(u(0), u'(0), \dots, u^{(n-1)}(0)), \tag{5}$$

where $u(t)$ is an elementary solution of the form (4). Let $\mathcal{H}$ be a Hilbert space, continuously embedded into $\overset{n}{\bigoplus} H$. A system of the root vectors of problem (1) is called *n-fold complete* in $\mathcal{H}$ if the system of vectors (5) is complete in the space $\mathcal{H}$.

2.3.2. n-fold completeness of root vectors of a system of operator pencils.

Theorem 1. *Let the following conditions be satisfied:*

(1) *there exist Hilbert spaces H_k, $k = 0, \dots, n$, for which the compact embeddings $H_n \subset H_{n-1} \subset \cdots \subset H_0 = H$ take place;*
(2) *for some $p > 0$ $\quad s_j(J_k; H_k, H_{k-1}) \le Cj^{-p}$, $j = 1, \dots, \infty$, $k = 1, \dots, n$;*
(3) *the operators A_k, $k = 1, \dots, n$, from H_k into H, act boundedly;*
(4) *the operators $A_{\nu k}$, $k = 0, \dots, n_\nu$, $\nu = 1, \dots, m$, from $H_{n-n_\nu+k}$ into H^ν, act boundedly;*
(5) *there exist Hilbert spaces H_0^ν such that continuous embeddings $H^\nu \subset H_0^\nu$, $\nu = 1, \dots, m$, hold, and the linear manifold*

$$\mathcal{H}_1 = \{v|\ v = (v_1, \dots, v_n) \in \bigoplus_{k=0}^{n-1} H_{n-k}, \quad \sum_{k=0}^{n_\nu} A_{\nu k} v_{n_\nu - k + s} = 0,$$
$$\text{for such integers } \nu \in [1, m] \text{ and } s \in [1, n - n_\nu] \text{ for which}$$
$$A_{\nu k} \text{ (for all } k = 0, \dots, n_\nu) \text{ from } H_{n+1-n_\nu+k-s} \text{ into } H_0^\nu \text{ are bounded}\}$$

is dense in the Hilbert space

$$\mathcal{H} = \{v|\ v = (v_1, \dots, v_n) \in \bigoplus_{k=0}^{n-1} H_{n-k-1}, \quad \sum_{k=0}^{n_\nu} A_{\nu k} v_{n_\nu - k + s} = 0,$$
$$\text{for such integers } \nu \in [1, m] \text{ and } s \in [1, n - n_\nu - 1] \text{ for which}$$
$$A_{\nu k} \text{ (for all } k = 0, \dots, n_\nu) \text{ from } H_{n-n_\nu+k-s} \text{ into } H_0^\nu \text{ are bounded}\};$$

(6) *there exist[4] rays ℓ_k with the angles between the neighboring rays less than $p\pi$ and a number η such that all numbers λ on ℓ_k and with sufficiently large moduli are regular points of the operator pencil $\mathbb{L}(\lambda):\ u \to \mathbb{L}(\lambda)u = (L(\lambda)u, L_1(\lambda)u, \dots, L_m(\lambda)u)$, which acts boundedly from H_n into $H \oplus H^1 \oplus \cdots \oplus H^m$ and $\|\mathbb{L}(\lambda)^{-1}\|_{B(H\oplus H^1 \oplus \cdots \oplus H^m, H_{n-1})} \le C|\lambda|^\eta$, $\lambda \in \ell_k$, $|\lambda| \to \infty$.*

Then the spectrum of problem 2.3.1/1 *is discrete and a system of root vectors of problem* 2.3.1/1 *is n-fold complete in the spaces $\mathcal{H}$ and $\mathcal{H}_1$.*

Proof. By the substitution $v_k = \lambda^{k-1} u$, $k = 1, \dots, n$, the system

$$L(\lambda)u = 0, \qquad L_\nu(\lambda)u = 0, \quad \nu = 1, \dots, m$$

[4]For $p > 2$ the existence of one such ray is enough.

is reduced to the equivalent system

$$\lambda v = \mathbb{A}v,$$

where $\mathbb{A}$ is an operator in the Hilbert space $\mathcal{H}$ (see condition 5) given by the equalities

$$D(\mathbb{A}) \equiv \mathcal{H}_1, \qquad \mathbb{A}(v_1, \dots, v_n) \equiv (v_2, \dots, v_n, -A_n v_1 - \dots - A_1 v_n).$$

If $u(t)$ is a solution to system 2.3.1/2, then $v(t) \equiv (u(t), \dots, u^{(n-1)}(t))$ is a solution to the system

$$v'(t) = \mathbb{A}v(t). \tag{1}$$

Conversely, if $v(t) = (v_1(t), \dots, v_n(t))$ is a solution to system (1), then $u(t) \equiv v_1(t)$ is a solution to system 2.3.1/2. Since the set of root vectors coincides with the set of values of elementary solutions in zero, then the set of root vectors of the operator $\mathbb{A}$ coincides with the set $\{v(0)\} = \{(u(0), \dots, u^{(n-1)}(0))\}$, where $u(t)$ are elementary solutions (4) to system 2.3.1/2.

Let us apply Theorem 2.2.2/1 to the operator $\mathbb{A}$. Find the resolvent of the operator $\mathbb{A}$. Instead of the equation

$$(\lambda I - \mathbb{A})v = F$$

we solve the system

$$\begin{aligned} &\lambda v_k - v_{k+1} = f_k, \quad k = 1, \dots, n-1, \\ &\lambda v_n + \sum_{k=1}^{n} A_{n-k+1} v_k = f_n, \\ &A_{\nu 0} v_{n_\nu + 1} + A_{\nu 1} v_{n_\nu} + \dots + A_{\nu n_\nu} v_1 = 0, \qquad \nu = 1, \dots, m, \end{aligned} \tag{2}$$

which is equivalent to it in the space $\mathcal{H}$. Let us show that if $v = (v_1, \dots, v_n)$ is a solution to problem (2) and $F = (f_1, \dots, f_n) \in \mathcal{H}$, then v satisfies all conditions of connection in $\mathcal{H}_1$. Let operators $A_{\nu k}$, $k = 0, \dots, n_\nu$, from $H_{n+1-n_\nu+k-s}$ into H_0^ν be bounded for some $s = r$, $r = 2, \dots, n - n_\nu$. Then, by virtue of the continuity of the embeddings $H_{k+1} \subset H_k$, $k = 1, \dots, n-1$, they are bounded for all $s = 2, \dots, r-1$. Further, if under some s

$$\sum_{k=0}^{n_\nu} A_{\nu k} v_{n_\nu - k + s - 1} = 0 \tag{3}$$

then, since $F \in \mathcal{H}$, i.e.,

$$\sum_{k=0}^{n_\nu} A_{\nu k} f_{n_\nu - k + s - 1} = 0, \qquad s = 2, \dots, r-1$$

from the first $n-1$ equations of the system (2), (3) and the last equality we get

$$\sum_{k=0}^{n_\nu} A_{\nu k} v_{n_\nu - k + s} = \lambda \sum_{k=0}^{n_\nu} A_{\nu k} v_{n_\nu - k + s - 1} + \sum_{k=0}^{n_\nu} A_{\nu k} f_{n_\nu - k + s - 1} = 0. \tag{4}$$

In turn, if $s = 2$ then (3) holds. It follows from the last m equations of system (2). So (4) is true for $s = 2$. This means that (3) is true for $s = 3$. Continuing these considerations we find that (4) is also true for $s = r$; i.e., v satisfies all conditions of connections in $\mathcal{H}_1$.

From the first $n-1$ equations of system (2) we successively find

$$v_k = \lambda^{k-1} v_1 - \sum_{j=1}^{k-1} \lambda^{k-1-j} f_j, \qquad k = 2, \dots, n. \tag{5}$$

Substituting these values of v_k into other equations of system (2), we obtain

$$\begin{aligned} &L(\lambda) v_1 - \sum_{k=2}^{n} A_{n-k+1} \sum_{j=1}^{k-1} \lambda^{k-1-j} f_j - \sum_{j=1}^{n-1} \lambda^{n-j} f_j = f_n, \\ &L_\nu(\lambda) v_1 + \sum_{k=0}^{n_\nu - 1} A_{\nu k} \sum_{j=1}^{n_\nu - k} \lambda^{n_\nu - k - j} f_j = 0, \qquad \nu = 1, \dots, m. \end{aligned} \tag{6}$$

If λ is a regular point of the pencil $\mathbb{L}(\lambda) = (L(\lambda), L_1(\lambda), \dots, L_m(\lambda))$ acting from H into $H \oplus H^1 \oplus \cdots \oplus H^m$ then by virtue of conditions 1, 3, 4 and 6 for any $(f_1, \dots, f_n) \in H_{n-1} \oplus \cdots \oplus H_0$ problem (6) has a unique solution

$$\begin{aligned} v_1 = \mathbb{L}(\lambda)^{-1} \Big(&\sum_{k=2}^{n} A_{n-k+1} \sum_{j=1}^{k-1} \lambda^{k-1-j} f_j + \sum_{j=1}^{n} \lambda^{n-j} f_j, \\ &- \sum_{k=0}^{n_1 - 1} A_{1k} \sum_{j=1}^{n_1 - k} \lambda^{n_1 - k - j} f_j, \dots, - \sum_{k=0}^{n_m - 1} A_{mk} \sum_{j=1}^{n_m - k} \lambda^{n_m - k - j} f_j \Big). \end{aligned} \tag{7}$$

From conditions 3, 4, 6 and formulas (7) and (5) it follows that $v_k \in H_{n-k+1}$, $k = 1, \dots, n$. From condition 6 it follows that the operator $\mathbb{L}(\lambda)^{-1}$ from $H \oplus H^1 \oplus \cdots \oplus H^m$ into H_{n-1} acts boundedly; consequently, the operator from $H \oplus H^1 \oplus \cdots \oplus H^m$ into H acts also boundedly. From conditions 3, 4, 6 and equalities (7) and (5) it follows that there exists a number r such that

$$\|R(\lambda, \mathbb{A}) F\|_{\mathcal{H}} \le C(\|v_1\|_{H_{n-1}} + \cdots + \|v_n\|_H) \le C|\lambda|^r \|F\|_{\mathcal{H}}, \quad \lambda \in \ell_k, \ |\lambda| \to \infty;$$

i.e., condition 3 of Theorem 2.2.2/1 is satisfied. Consequently the operator $\mathbb{A}$ is closed and, by virtue of condition 5, condition 1 of Theorem 2.2.2/1 is satisfied, too.

It is clear that the operator $\mathbb{J} : v \to \mathbb{J}v \equiv (J_n v_1, \ldots, J_1 v_n)$ is the embedding operator from $\bigoplus_{k=0}^{n-1} H_{n-k}$ into $\bigoplus_{k=0}^{n-1} H_{n-k-1}$ and

$$s_j(\mathbb{J}; \bigoplus_{k=0}^{n-1} H_{n-k}, \bigoplus_{k=0}^{n-1} H_{n-k-1}) \leq Cj^{-p}, \; j = 1, \ldots, \infty. \tag{8}$$

By virtue of condition 5, $\mathcal{H}_1$ is a subspace of $\bigoplus_{k=0}^{n-1} H_{n-k}$. Then, by virtue of Lemma 2.3.3 ([26]),

$$s_j(\mathbb{J}; \mathcal{H}_1, \bigoplus_{k=0}^{n-1} H_{n-k-1}) \leq Cs_j(\mathbb{J}; \bigoplus_{k=0}^{n-1} H_{n-k}, \bigoplus_{k=0}^{n-1} H_{n-k-1}), \; j = 1, \ldots, \infty. \tag{9}$$

Since $\mathcal{H}$ is a subspace of $\bigoplus_{k=0}^{n-1} H_{n-k-1}$, then

$$s_j(\mathbb{J}; \mathcal{H}_1, \mathcal{H}) = s_j(\mathbb{J}; \mathcal{H}_1, \bigoplus_{k=0}^{n-1} H_{n-k-1}), \; j = 1, \ldots, \infty. \tag{10}$$

From (8)–(10) follows that $s_j(\mathbb{J}; \mathcal{H}_1, \mathcal{H}) \leq Cj^{-p}, \; j = 1, \ldots, \infty$, i.e., condition 2 of Theorem 2.2.2/1 is satisfied, too.

Remark 2. The condition $R(\lambda, A) \in \sigma_q(H)$ of the Corollary [6, 11.9.31], we replaced in Theorem 2.2.2/1 by the condition $s_j(J; H(A), H) \leq Cj^{-p}, \; j = 1, \ldots, \infty$. This fact gives us the essential simplification not only in application but also in the proof of Theorem 1 and other theorems (cp. with Theorem 2.3.4 ([26])).

Remark 3. One can apply Theorem 1 in the case when an elliptic boundary value problem with a spectral parameter is coercive with respect to the space variable. It is better to use Theorem 2.3.3 for problems without a spectral parameter in boundary conditions. In this case one can investigate both coercive and noncoercive elliptic boundary value problems.

2.3.3. n-fold completeness of root vectors of an operator pencil. Here we will give the proof of the theorem for differential equations when boundary conditions do not depend on the spectral parameter.

Let H be a Hilbert space. Consider in H the operator pencil

$$L(\lambda)u = \lambda^n u + \lambda^{n-1} A_1 u + \cdots + A_n u. \tag{1}$$

Theorem. *Let the following conditions be satisfied:*

(1) *there exist Hilbert spaces H_k, $k=0,\dots,n$, for which the compact embeddings $H_n \subset H_{n-1} \subset \cdots \subset H_0 = H$ take place and $\overline{H_k}|_{H_{k-1}} = H_{k-1}$, $k=1,\dots,n$;*

(2) *for some $p>0$ $s_j(J_k; H_k, H_{k-1}) \le j^{-p}$, $j=1,\dots,\infty$, $k=1,\dots,n$;*

(3) *the operators A_k, $k=1,\dots,n$, from H_k into H, act boundedly;*

(4) *there exist*[5] *rays ℓ_k with the angles between the neighboring rays less than $p\pi$ and a number η such that all numbers λ on ℓ_k and with sufficiently large moduli are regular points of the operator pencil $L(\lambda)$, which acts boundedly from H_n into H and $\|L(\lambda)^{-1}\|_{B(H,H_{n-1})} \le C|\lambda|^\eta$, $\lambda \in \ell_k$, $|\lambda| \to \infty$.*

Then the spectrum of operator pencil (1) *is discrete and a system of root vectors of operator pencil* (1) *is n-fold complete in the spaces* $\bigoplus_{k=0}^{n-1} H_{n-k-1}$ *and* $\bigoplus_{k=0}^{n-1} H_{n-k}$.

Proof. As in Theorem 2.3.2/1 by the substitution $v_k = \lambda^{k-1}u$, $k=1,\dots,n$, the equation $L(\lambda)u=0$ is reduced to the equivalent system $\lambda v = \mathbb{A}v$, where $\mathbb{A}$ is an operator in the Hilbert space $\mathcal{H} \equiv \bigoplus_{k=0}^{n-1} H_{n-k-1}$ which is given by the equalities

$$D(\mathbb{A}) \equiv \mathcal{H}_1 \equiv \bigoplus_{k=0}^{n-1} H_{n-k}, \quad \mathbb{A}(v_1,\dots,v_n) \equiv (v_2,\dots,v_n,-A_n v_1 - \cdots - A_1 v_n).$$

Now we repeat the proof of Theorem 2.3.2/1. Let us mention that in this case the proof is more easy because system 2.3.2/2 has a more simple form

$$\lambda v_k - v_{k+1} = f_k, \quad k=1,\dots,n-1, \quad \lambda v_n + \sum_{k=1}^{n} A_{n-k+1} v_k = f_n.$$

The operator $\mathbb{J} : v \to \mathbb{J}v \equiv (J_n v_1,\dots,J_1 v_n)$ is the embedding operator from $\bigoplus_{k=0}^{n-1} H_{n-k}$ into $\bigoplus_{k=0}^{n-1} H_{n-k-1}$ and

$$s_j(\mathbb{J}; \mathcal{H}(\mathbb{A}), \mathcal{H}) = s_j(\mathbb{J}; \bigoplus_{k=0}^{n-1} H_{n-k}, \bigoplus_{k=0}^{n-1} H_{n-k-1}) \le Cj^{-p}, \ j=1,\dots,\infty.$$

Then we apply Theorem 2.2.2/1 to the operator $\mathbb{A}$.

2.3.4. n-fold completeness of root vectors of a perturbed operator pencil, coercive with a defect. In the theory of regular boundary value problems for ordinary differential equations and elliptic partial differential equations, *coercive*

[5]For $p>2$ the existence of one such ray is enough.

operator pencils 2.3.3/1 appear, i.e., operator pencils for which on some rays ℓ_k the estimate

$$\sum_{k=0}^{n} |\lambda|^{n-k} \|L(\lambda)^{-1}\|_{B(E,E_k)} \leq C \tag{1}$$

holds. However, in the theory of irregular boundary value problems *coercive operator pencils with a defect* appear, i.e., pencils for which the estimate (1) holds with some loss $d \geq 0$ in λ. Let us show that for such pencils it is possible to prove the perturbation theorem on the completeness of root vectors.

Consider, in a Banach space E, the following unbounded operator pencil:

$$L(\lambda) = \lambda^n I + \lambda^{n-1}(A_1 + B_1) + \cdots + (A_n + B_n). \tag{2}$$

Denote

$$L_0(\lambda) = \lambda^n I + \lambda^{n-1} A_1 + \cdots + A_n, \quad L_1(\lambda) = \lambda^{n-1} B_1 + \cdots + B_n. \tag{3}$$

Lemma 1. *Let the following conditions be satisfied:*

(1) *A_k are operators in E; there exist Banach spaces E_k, $k = 1, \ldots, n$, continuously embedded into $E_0 = E$, and a number $d \geq 0$ such that $D(L_0(\lambda)) = D(L(\lambda)) = \bigcap_{k=1}^{n} E_k$ and*

$$\sum_{k=0}^{n} |\lambda|^{n-k-d} \|L_0(\lambda)^{-1}\|_{B(E,E_k)} \leq C, \qquad \lambda \in S,\ |\lambda| \to \infty,$$

where S is an unbounded set of the complex plane;

(2) *B_k are operators in E; under $k = 1, \ldots, [d]$, where $d = [d] + \{d\}$, $0 \leq \{d\} < 1$, the operators $B_k = 0$ and under $k = [d] + 1, \ldots, n$ the domain $D(B_k) \supset \bigcap_{k=1}^{n} E_k$ and for any $\varepsilon > 0$*

$$\|B_k u\| \leq \varepsilon \|u\|_{E_k}^{1-\frac{d}{k}} \|u\|^{\frac{d}{k}} + C(\varepsilon)\|u\|, \qquad u \in \bigcap_{k=1}^{n} E_k.$$

Then there exists $R > 0$ such that all complex numbers $\lambda \in S$ for which $|\lambda| > R$ are regular points of pencil (2) *and the estimate*

$$\sum_{k=0}^{n} |\lambda|^{n-k-d} \|L(\lambda)^{-1}\|_{B(E,E_k)} \leq C, \qquad \lambda \in S,\ |\lambda| \to \infty$$

holds.

Proof. Use the identity

$$L(\lambda) = [I + L_1(\lambda)L_0(\lambda)^{-1}]L_0(\lambda), \qquad \lambda \in S,\ |\lambda| \to \infty, \tag{4}$$

where $L_0(\lambda)$, $L_1(\lambda)$ are defined by formulas (3). By virtue of conditions 1 and 2 for $\lambda \in S$, $|\lambda| \to \infty$ and for any $\varepsilon > 0$ we have

$$\sum_{k=[d]+1}^{n} |\lambda|^{n-k}\|B_k L_0(\lambda)^{-1}\| \le \sum_{k=[d]+1}^{n} |\lambda|^{n-k}(\varepsilon\|L_0(\lambda)^{-1}\|_{B(E,E_k)}^{1-\frac{d}{k}}\|L_0(\lambda)^{-1}\|^{\frac{d}{k}} + C(\varepsilon)\|L_0(\lambda)^{-1}\|) \le C\varepsilon + C(\varepsilon)|\lambda|^{\{d\}-1}.$$

Thus, for $\lambda \in S$, $|\lambda| \to \infty$,

$$\|L_1(\lambda)L_0(\lambda)^{-1}\| \le \sum_{k=[d]+1}^{n} |\lambda|^{n-k}\|B_k L_0(\lambda)^{-1}\| \le q < 1. \tag{5}$$

Then from (4) it follows that for $\lambda \in S$, $|\lambda| \to \infty$, the operator $L(\lambda)$ is invertible in E and

$$L(\lambda)^{-1} = L_0(\lambda)^{-1}[I + L_1(\lambda)L_0(\lambda)^{-1}]^{-1}.$$

Hence, by virtue of condition 1 and (5) we have, for $\lambda \in S$, $|\lambda| \to \infty$,

$$\sum_{k=0}^{n} |\lambda|^{n-k-d}\|L(\lambda)^{-1}\|_{B(E,E_k)} \le \sum_{k=0}^{n} |\lambda|^{n-k-d}\|L_0(\lambda)^{-1}\|_{B(E,E_k)}\|[I + L(\lambda)L_0(\lambda)^{-1}]^{-1}\| \le C.$$

For ideal pencils, i.e., for pencils in which the action law of subordinated terms has interpolation character, the following becomes useful.

Lemma 2. *Let the following conditions be satisfied:*

(1) *A_k are operators in E; there exist Banach spaces E_k, $k = 1, \dots, n$, continuously embedded into $E_0 = E$ and a number $d \ge 0$ such that $D(L_0(\lambda)) = D(L(\lambda)) = \bigcap_{k=1}^{n} E_k$ and*

$$\sum_{k=0}^{n} |\lambda|^{n-k-d}\|L_0(\lambda)^{-1}\|_{B(E,E_k)} \le C, \qquad \lambda \in S,\ |\lambda| \to \infty,$$

where S is an unbounded set of the complex plane;

(2) *B_k are operators in E; under $k = 1, \dots, [d]$ the operators $B_k = 0$ and under $k = [d]+1, \dots, n$ the domain $D(B_k) \supset \bigcap_{k=1}^{n} E_k$ and for any $\varepsilon > 0$*

$$\|B_k u\| \le \varepsilon\|u\|_{E_n}^{\frac{k-d}{n}}\|u\|^{1-\frac{k-d}{n}} + C(\varepsilon)\|u\|, \qquad u \in \bigcap_{k=1}^{n} E_k.$$

Then there exists $R > 0$ such that all complex numbers $\lambda \in S$ for which $|\lambda| > R$ are regular points of pencil (2) *and the estimate*

$$\sum_{k=0}^{n} |\lambda|^{n-k-d} \|L(\lambda)^{-1}\|_{B(E,E_k)} \le C, \quad \lambda \in S, \ |\lambda| \to \infty$$

holds.

Proof. Under $\lambda \in S$, $|\lambda| \to \infty$ and $\varepsilon > 0$ we have

$$\sum_{k=[d]+1}^{n} |\lambda|^{n-k} \|B_k L_0(\lambda)^{-1}\| \le \sum_{k=[d]+1}^{n} |\lambda|^{n-k} \big(\varepsilon \|L_0(\lambda)^{-1}\|_{B(E,E_n)}^{\frac{k-d}{n}} \|L_0(\lambda)^{-1}\|^{1-\frac{k-d}{n}} + C(\varepsilon)\|L_0(\lambda)^{-1}\|\big) \le C\varepsilon + C(\varepsilon)|\lambda|^{\{d\}-1}.$$

The rest of the proof coincides with the proof of the above Lemma 1.

Theorem 3. *Let the following conditions be satisfied:*

(1) *there exist Hilbert spaces H_k, $k = 0, \dots, n$, for which the compact embeddings $H_n \subset H_{n-1} \subset \cdots \subset H_0 = H$ take place and $\overline{H_k}|_{H_{k-1}} = H_{k-1}$, $k = 1, \dots, n$;*
(2) *for some $p > 0$ $s_j(J_k; H_k, H_{k-1}) \le j^{-p}$, $j = 1, \dots, \infty$, $k = 1, \dots, n$;*
(3) *the operators A_k, $k = 1, \dots, n$, from H_k into H, act boundedly;*
(4) *there exist*[6] *rays ℓ_k with the angles between the neighboring rays less than $p\pi$ and a number $d \ge 0$ such that all numbers λ on ℓ_k and with sufficiently large moduli are regular points of the operator pencil $L_0(\lambda) = \lambda^n + \lambda^{n-1}A_1 + \cdots + A_n$, which acts boundedly from H_n into H and*

$$\sum_{k=0}^{n} |\lambda|^{n-k-d} \|L_0(\lambda)^{-1}\|_{B(H,H_k)} \le C, \qquad \lambda \in \ell_k, \ |\lambda| \to \infty;$$

(5) *B_k are operators in H; under $k = 1, \dots, [d]$ the operators $B_k = 0$ and under $k = [d]+1, \dots, n$ the domain $D(B_k) \supset H_k$ and for any $\varepsilon > 0$*

$$\|B_k u\| \le \varepsilon \|u\|_{H_k}^{1-\frac{d}{k}} \|u\|^{\frac{d}{k}} + C(\varepsilon)\|u\|, \quad u \in H_k.$$

Then the spectrum of operator pencil (2) *is discrete and a system of root vectors of pencil* (2) *is n-fold complete in the spaces $H_n \oplus \cdots \oplus H_1$ and $H_{n-1} \oplus \cdots \oplus H_0$.*

Proof. By virtue of Lemma 1, Theorem 2.3.3 is applicable to pencil (2), from which follows the statement of Theorem 3.

[6]For $p > 2$ the existence of one such ray is enough.

Remark 4. When applying Theorems 2.3.2/1, 2.3.3 and 2.3.4/3 to differential pencils, the spaces H_k, $k = 1, \dots, n$, become optimal if they are chosen in accordance with the following principle: let A_k be a differential operator of the order m_k, then

$$H_k = W_2^{\frac{m_n k}{n}}, \qquad m_k \leq \frac{m_n k}{n},$$
$$H_k = W_2^{m_k}, \qquad m_k > \frac{m_n k}{n}.$$

Let us call the lower bound of numbers d, which satisfies the 5-th condition of Theorem 3, the *defect of coerciveness* or the *order of noncoerciveness* of the pencil $L_0(\lambda)$. From Theorem 3 it follows that all terms of the coercive pencil, i.e., the pencil $L_0(\lambda)$ that satisfies condition 5 of Theorem 3 when $d = 0$, admit the perturbation (the most possible) since, by virtue of Lemma 2.2.6 ([26]), condition 5 is transformed into the condition of compactness of the operator B_k acting from H_k into H. The greater the defect of coerciveness d, the fewer terms that admit the perturbation, and the order of the perturbation operators becomes less. When the defect of the coerciveness is too high, i.e., $d > n$, then no term admits the perturbation. Also note that the order of the perturbation operator may be more than the order of the perturbed operator. This happens when $m_k < \frac{m_n k}{n}$. The differential pencils of normal type have $m_1 < m_2 < \cdots < m_n$.

2.3.5. n-fold completeness of root vectors of an operator pencil with a weight. Consider in a Hilbert space H the unbounded operator pencil

$$L(\lambda) = \lambda^n I + \lambda^{n-1} A_1 + \cdots + A_n \tag{1}$$

with a weight A, i.e., the pencil, for which there exists an unbounded invertible operator A in H such that operators $A_k A^{-k}$, $k = 1, \dots, n$, in H act boundedly. The class of pencils with a weight is a subclass of the normal pencil class and does not coincide with the latter. However, as a rule, polynomial operator pencils with a weight appear in the theory of differential equations.

Lemma 1. *Let an operator A in a Banach space E be invertible and let A have a dense domain $D(A)$. Then, $\overline{E(A^k)}|_{E(A^{k-1})} = E(A^{k-1})$, $k = 1, \dots, \infty$.*

Proof. Let $u \in E(A^{k-1})$. Denote $v = A^{k-1}u$. Since $\overline{D(A)} = E$ then there exists a sequence $v_n \in D(A)$ such that $\lim_{n\to\infty} \|v_n - v\| = 0$. It is easy to see that $u_n = A^{-k+1} v_n \in D(A^k)$ and

$$\lim_{n\to\infty} \|A^{k-1}u_n - A^{k-1}u\| = \lim_{n\to\infty} \|v_n - v\| = 0.$$

Corollary 2. *Let the condition of Lemma 1 hold. Then $\overline{E(A^k)}|_E = E$, $k = 1, \dots, \infty$.*

The simple corollary of Theorem 2.3.3 is the following

Theorem 3. *Let the following conditions be satisfied:*

(1) *an operator A in H is invertible and has a dense domain $D(A)$;*
(2) *the embedding $H(A) \subset H$ is compact and $s_j(J; H(A), H) \le Cj^{-p}$, $j = 1, \ldots, \infty$, for some $p > 0$;*
(3) *A_k are operators in H, $D(A_k) \supset D(A^k)$, $D(A_n) = D(A^n)$ and operators $A_k A^{-k}$ in H act boundedly;*
(4) *there exist*[7] *rays ℓ_k with the angles between the neighboring rays less than $p\pi$ and a number η such that all numbers λ on ℓ_k and with sufficiently large moduli are regular points of the operator pencil $L(\lambda)$, which acts boundedly from $H(A^n)$ into H and $\|A^{n-1}L(\lambda)^{-1}\| \le C|\lambda|^\eta$, $\lambda \in \ell_k$, $|\lambda| \to \infty$.*

Then the spectrum of operator pencil (1) *is discrete and a system of root vectors of pencil* (1) *is n-fold complete in the spaces $H(A^n) \oplus \cdots \oplus H(A)$ and $H(A^{n-1}) \oplus \cdots \oplus H$.*

Proof. It is obvious from Lemma 1 that Hilbert spaces $H_k = H(A^k)$, $k = 1, \ldots, n$, satisfy condition 1 of Theorem 2.3.3. By virtue of [26, Lemma 2.3.1] the condition $s_j(J; H(A^k), H(A^{k-1})) \le Cj^{-p}$, is equivalent to the condition $s_j(J; H(A), H) \le Cj^{-p}$, $j = 1, \ldots, \infty$. Hence, from condition 2 follows condition 2 of Theorem 2.3.3. Conditions 3 and 4 of Theorem 2.3.3 coincide with conditions 3 and 4 of Theorem 3. □

Similarly, from Theorem 2.3.4/3 follows the completeness theorem of root vectors of a perturbed operator pencil with a weight.

Theorem 4. *Let the following conditions be satisfied:*

(1) *an operator A in H is invertible and has a dense domain $D(A)$;*
(2) *the embedding $H(A) \subset H$ is compact and $s_j(J; H(A), H) \le Cj^{-p}$, $j = 1, \ldots, \infty$, for some $p > 0$;*
(3) *A_k are operators in H, $D(A_k) \supset D(A^k)$, $D(A_n) = D(A^n)$ and operators $A_k A^{-k}$ in H act boundedly;*
(4) *there exist*[8] *rays ℓ_k with the angles between the neighboring rays less than $p\pi$ and a number $d \ge 0$ such that all numbers λ on ℓ_k and with sufficiently large moduli are regular points of the operator pencil $L_0(\lambda) = \lambda^n + \lambda^{n-1}A_1 + \cdots + A_n$, which acts boundedly from $H(A^n)$ into H and*

$$\sum_{k=0}^{n} |\lambda|^{n-k-d} \|A^k L_0(\lambda)^{-1}\| \le C, \qquad \lambda \in \ell_k,\ |\lambda| \to \infty;$$

(5) *B_k are operators in H; when $k = 1, \ldots, [d]$ the operators $B_k = 0$ and when $k = [d]+1, \ldots, n$ the domain $D(B_k) \supset D(A^k)$ and for any $\varepsilon > 0$*

$$\|B_k u\| \le \varepsilon \|A^k u\|^{1-\frac{d}{k}} \|u\|^{\frac{d}{k}} + C(\varepsilon)\|u\|, \qquad u \in D(A^k).$$

[7]For $p > 2$ the existence of one such ray is enough.
[8]For $p > 2$ the existence of one such ray is enough.

Then the spectrum of the operator pencil

$$L(\lambda) = \lambda^n I + \lambda^{n-1}(A_1 + B_1) + \cdots + (A_n + B_n) \tag{2}$$

is discrete and a system of root vectors of pencil (2) *is n-fold complete in the spaces* $H(A^n) \oplus \cdots \oplus H(A)$ *and* $H(A^{n-1}) \oplus \cdots \oplus H$.

2.3.6. n-fold completeness of root vectors of the Keldysh operator pencil. Consider, in a Hilbert space H, the operator pencil

$$L(\lambda) = \lambda^n I + \lambda^{n-1}(a_1 A + B_1) + \cdots + (a_n A^n + B_n), \tag{1}$$

where A, B_k, $k = 1, \dots, n$, are operators in H, a_k are complex numbers, $a_n \neq 0$. Let us show that for such pencils the formulation of completeness theorems of root vectors becomes simpler. Denote by ω_k, $k = 1, \dots, n$, roots of the characteristic equation $\omega^n + a_1\omega^{n-1} + \cdots + a_n = 0$, $\ell(a, \varphi) = \{z | \ z \in \mathbb{C}, \ z = a + re^{i\varphi}, r \geq 0\}$ and $\ell(\varphi) = \ell(0, \varphi)$.

Lemma 1. *Let for some* $d \geq 0$ $|\lambda|^{1-d}\|R(\lambda, A)\| \leq C$, $\lambda \in \ell(a\omega_k^{-1}, \varphi - \arg \omega_k)$, $|\lambda| \to \infty$. *Then for the operator pencil*

$$L_0(\lambda) = \lambda^n I + \lambda^{n-1} a_1 A + \cdots + a_n A^n$$

the estimate

$$\sum_{k=0}^{n} |\lambda|^{n-k-nd} \|A^k L_0(\lambda)^{-1}\| \leq C, \qquad \lambda \in \ell(a, \varphi), \ |\lambda| \to \infty$$

is valid.

Proof. Let us use the relation

$$L_0(\lambda) = \prod_{k=1}^{n} (\lambda I - \omega_k A) = \prod_{k=1}^{n} \omega_k (\lambda \omega_k^{-1} I - A) \tag{2}$$

which is true for any $\lambda \in \mathbb{C}$. Since for $k = 1, \dots, n$

$$|\lambda|^{1-d}\|R(\lambda\omega_k^{-1}, A)\| \leq C, \qquad \lambda \in \ell(a, \varphi), \ |\lambda| \to \infty, \tag{3}$$

then from $(\lambda\omega_k^{-1} I - A)R(\lambda\omega_k^{-1}, A) = I$ it follows that for $k = 1, \dots, n$, $\lambda \in \ell(a, \varphi)$, $|\lambda| \to \infty$,

$$|\lambda|^{-d}\|AR(\lambda\omega_k^{-1}, A)\| \leq C\big(|\lambda|^{-d} + |\lambda|^{1-d}\|R(\lambda\omega_k^{-1}, A)\|\big) \leq C. \tag{4}$$

From (2)–(4) it follows that for $k = 0, \dots, n$, $\lambda \in \ell(a, \varphi)$, $|\lambda| \to \infty$ the estimate

$$|\lambda|^{n-k-nd}\|A^k L_0(\lambda)^{-1}\| \leq |\lambda|^{-kd} \prod_{j=1}^{k} |\omega_j| \ \|A(\lambda\omega_j^{-1} I - A)^{-1}\|$$
$$\times |\lambda|^{(n-k)(1-d)} \prod_{j=k+1}^{n} |\omega_j| \ \|(\lambda\omega_j^{-1} I - A)^{-1}\| \leq C$$

is valid.

Theorem 2. *Let the following conditions be satisfied:*

(1) *an operator A in H is invertible and has a dense domain $D(A)$;*

(2) *the embedding $H(A) \subset H$ is compact and $s_j(J; H(A), H) \leq Cj^{-p}$, $j = 1, \dots, \infty$, for some $p > 0$;*

(3) *there exist*[9] *rays $\ell_k(a, \varphi_k)$ with the angles between the neighboring rays less than πp and a number $d \geq 0$ such that*

$$|\lambda|^{1-d}\|R(\lambda, A)\| \leq C, \qquad \lambda \in \ell_{kj}(a\omega_j^{-1}, \varphi_k - \arg\omega_j),\ |\lambda| \to \infty;$$

(4) *B_k are operators in H; when $k = 1, \dots, [nd]$ the operators $B_k = 0$ and when $k = [nd] + 1, \dots, n$ the domain $D(B_k) \supset D(A^k)$ and for any $\varepsilon > 0$*

$$\|B_k u\| \leq \varepsilon\|A^k u\|^{1-\frac{nd}{k}}\|u\|^{\frac{nd}{k}} + C(\varepsilon)\|u\|, \qquad u \in D(A^k).$$

Then the spectrum of operator pencil (1) *is discrete and a system of root vectors of pencil* (1) *is n-fold complete in the spaces $H(A^n) \oplus \cdots \oplus H(A)$ and $H(A^{n-1}) \oplus \cdots \oplus H$.*

Proof. By virtue of Lemma 1, Theorem 2 is a special case of Theorem 2.3.5/4.

Theorem 3. *Let the following conditions be satisfied:*

(1) *an operator A in H is invertible and self-adjoint;*

(2) *the embedding $H(A) \subset H$ is compact and $s_j(J; H(A), H) \leq Cj^{-p}$, $j = 1, \dots, \infty$, for some $p > 0$;*

(3) *B_k are operators in H and operators $B_k A^{-k}$ in H are compact.*

Then the spectrum of operator pencil (1) *is discrete, under any $\varepsilon > 0$ outside the angles $|\arg\lambda - \arg\omega_k| < \varepsilon$, $|\arg\lambda - \arg\omega_k - \pi| < \varepsilon$, $k = 1, \dots, n$, there exists a finite number of eigenvalues, and a system of root vectors of pencil* (1) *is n-fold complete in the spaces $H(A^n) \oplus \cdots \oplus H(A)$ and $H(A^{n-1}) \oplus \cdots \oplus H$.*

Proof. Since for any $\lambda \neq \lambda_k(A), k = 1, \dots, \infty$, the expansion

$$R(\lambda, A) = \sum_{k=1}^{\infty} \frac{1}{\lambda - \lambda_k}(\cdot, u_k)u_k$$

holds, where $\{u_k\}_1^\infty$ is a complete orthonormal system of the eigenvectors of the operator A then

$$\|R(\lambda, A)\| \leq \frac{1}{|\mathrm{Im}\lambda|} \leq \frac{C(\varphi)}{|\lambda|}, \qquad \lambda \in \ell(0, \varphi),\ \varphi \neq 0,\ \varphi \neq \pi.$$

Hence the operator A satisfies condition 3 of Theorem 2 when $d = 0$. Then, from Lemmas 2.2.6 and 2.2.7 ([26]) it follows that condition 3 of our Theorem and condition 4 of Theorem 2 are equivalent. So, Theorem 2 is applicable to pencil (1).

[9] For $p > 2$ the existence of one such ray is enough.

The fact that outside the angles $|\arg\lambda - \arg\omega_k| < \varepsilon$, $|\arg\lambda - \arg\omega_k - \pi| < \varepsilon$, $k = 1, \ldots, n$, there is a finite number of eigenvalues follows from Lemmas 1 and 2.3.4/1.

2.4. Completeness of root vectors of a system of unbounded operator pencils with a linear parameter. Let us now discuss the case of a system of linear operator pencils. Let H and H^ν, $\nu = 1, \ldots, m$, be Hilbert spaces. Consider, in H, the problem for a system of linear operator pencils

$$\begin{aligned} &L(\lambda)u \equiv \lambda u + Au = 0, \\ &L_\nu(\lambda)u \equiv \lambda A_{\nu 0}u + A_{\nu 1}u = 0, \quad \nu = 1, \ldots, s, \\ &L_\nu u = 0, \quad \nu = s+1, \ldots, m, \end{aligned} \tag{1}$$

where λ is a complex number ($\lambda \in \mathbb{C}$), A is an unbounded operator in H, and $A_{\nu k}$ and L_ν are, generally speaking, unbounded operators from H into H^ν, s is a fixed natural number such that $0 \leq s \leq m$. If $s = 0$ then the pencils $L_\nu(\lambda)$ are absent in (1) and if $s = m$ then the pencils L_ν are absent in (1).

Elements $u_0, u_1, \ldots, u_k$ of H, satisfying the relations

$$\begin{aligned} &(\lambda_0 I + A)u_q + u_{q-1} = 0, \\ &(\lambda_0 A_{\nu 0} + A_{\nu 1})u_q + A_{\nu 0}u_{q-1} = 0, \qquad \nu = 1, \ldots, s, \\ &L_\nu u_q = 0, \quad \nu = s+1, \ldots, m, \end{aligned} \tag{2}$$

where $q = 0, \ldots, k$ and $u_{-1} = 0$, are called *root vectors* of problem (1), corresponding to the eigenvalue λ_0. The vector $u_0 \neq 0$ is called an *eigenvector* of problem (1).

Let for all $\lambda \in \mathbb{C}$ the operator $B(\lambda)$ from a Banach space E into a Banach space F act boundedly. Let F_0 be a subspace of F. A number $\lambda_0 \in \mathbb{C}$ is called a F_0-*regular point* of the operator $B(\lambda)$ if there exists the bounded inverse operator $B(\lambda)^{-1}$ from F_0 into E. A F-regular point of the operator $B(\lambda)$ is called a *regular point* of the operator $B(\lambda)$. The complement of all regular points of the operator $B(\lambda)$ in the complex plane is called the *spectrum* of the operator $B(\lambda)$.

Theorem. *Let the following conditions be satisfied:*

(1) *there exist Hilbert spaces H_1 and H_1^ν, $\nu = 1, \ldots, s$, such that the embeddings $H_1 \subset H$ and $H_1^\nu \subset H^\nu$, $\nu = 1, \ldots, s$, are compact and dense;*
(2) *for some $p > 0$ $s_j(J; H_1, H) \leq Cj^{-p}$ and $s_j(J_\nu; H_1^\nu, H^\nu) \leq Cj^{-p}$, $\nu = 1, \ldots, s$;*
(3) *the operator A from H_1 into H acts boundedly;*
(4) *the operators $A_{\nu 0}$, $\nu = 1, \ldots, s$, from H_1 into H_1^ν act boundedly;*
(5) *the operators $A_{\nu 1}$, $\nu = 1, \ldots, s$, and operators L_ν, $\nu = s+1, \ldots, m$, from H_1 into H^ν act boundedly;*
(6) *the linear manifold*

$$\{v \mid v = (u, A_{10}u, \ldots, A_{s0}u),\ u \in H_1,\ L_\nu u = 0,\ \nu = s+1, \ldots, m\},$$

is dense in the Hilbert space $H \oplus H^1 \oplus \cdots \oplus H^s$;
(7) *there exist*[10] *rays ℓ_k with the angles between the neighboring rays less than*

[10] For $p > 2$ the existence of one such ray is enough.

$p\pi$ and numbers η such that all numbers λ on ℓ_k and with sufficiently large moduli are $H \bigoplus_{\nu=1}^{s} H^\nu \bigoplus_{s+1}^{m}\{0\}$–regular for the operator pencil $\mathbb{L}(\lambda)$: $u \to \mathbb{L}(\lambda)u = (L(\lambda)u, L_1(\lambda)u, \dots, L_s(\lambda)u, L_{s+1}u, \dots, L_m u)$, which acts boundedly from H_1 into $H \oplus H^1 \oplus \cdots \oplus H^m$ and

$$\|\mathbb{L}(\lambda)^{-1}\|_{B(H \bigoplus_{\nu=1}^{s} H^\nu \bigoplus_{s+1}^{m}\{0\}, H_1)} \le C|\lambda|^\eta, \quad \lambda \in \ell_k, \ |\lambda| \to \infty.$$

Then the spectrum of (1) *is discrete and a system of vectors $(u_k, A_{10}u_k, \dots, A_{s0}u_k)$, where u_k are root vectors of problem* (1)*, complete in the space $H \oplus H^1 \oplus \cdots \oplus H^s$.*

Proof. Consider, in the Hilbert space $\mathcal{H} = H \oplus H^1 \oplus \cdots \oplus H^s$, the operator $\mathbb{A}$, given by the equalities

$$D(\mathbb{A}) = \{v \mid v = (u, A_{10}u, \dots, A_{s0}u), \ u \in H_1, \ L_\nu u = 0, \ \nu = s+1, \dots, m\},$$

$$\mathbb{A}(u, A_{10}u, \dots, A_{s0}u) = (-Au, -A_{11}u, \dots, -A_{s1}u).$$

It is easy to see that if $u_0, u_1, \dots, u_k$ are root vectors of problem (1), i.e., they satisfy relations (2), then the vectors $v_p = (u_p, A_{10}u_p, \dots, A_{s0}u_p)$ are root vectors of the operator $\mathbb{A}$; i.e., they satisfy the relations $(\lambda I - \mathbb{A})v_p + v_{p-1} = 0$, $p = 0, \dots, k$, where $v_{-1} = 0$, and conversely.

Let us apply Theorem 2.2.2/1 to the operator $\mathbb{A}$. Let us check condition 3 of Theorem 2.2.2/1 for the operator $\mathbb{A}$. For this instead of the equation

$$(\lambda I - \mathbb{A})v = F \tag{3}$$

we solve the system

$$\begin{aligned} &\lambda u + Au = f, \\ &\lambda A_{\nu 0}u + A_{\nu 1}u = f_\nu, \qquad \nu = 1, \dots, s, \\ &L_\nu u = 0, \qquad \nu = s+1, \dots, m, \end{aligned} \tag{4}$$

which is equivalent to (3). By virtue of condition 7 problem (4) has a unique solution $u = \mathbb{L}(\lambda)^{-1}(f, f_1, \dots, f_s, 0, \dots, 0)$ for numbers λ with sufficiently large moduli and $\lambda \in \ell_k$. So, a solution to (3) has the following form:

$$\begin{aligned} v = (&\mathbb{L}(\lambda)^{-1}(f, f_1, \dots, f_s, 0, \dots, 0), A_{10}\mathbb{L}(\lambda)^{-1}(f, f_1, \dots, f_s, 0, \dots, 0), \\ &\dots, A_{s0}\mathbb{L}(\lambda)^{-1}(f, f_1, \dots, f_s, 0, \dots, 0)), \end{aligned} \tag{5}$$

for the same λ. From conditions 4 and 7 follows the estimate

$$\|A_{\nu 0}\mathbb{L}(\lambda)^{-1}\|_{B(H \bigoplus_{\nu=1}^{s} H^\nu \bigoplus_{s+1}^{m}\{0\}, H^\nu)} \le C|\lambda|^r, \qquad \lambda \in \ell_k, \ |\lambda| \to \infty,$$

for $\nu = 1, \dots, s$ and some $r \in \mathbb{R}$. It implies that

$$\|R(\lambda, \mathbb{A})\| \leq C|\lambda|^r, \qquad \lambda \in \ell_k,\ |\lambda| \to \infty; \tag{6}$$

i.e., condition 3 of Theorem 2.2.2/1 is satisfied too. Consequently the operator $\mathbb{A}$ is closed and, by virtue of condition 6, the operator $\mathbb{A}$ satisfies condition 1 of Theorem 2.2.2/1.

From (5) and condition 4 it follows that the operator $R(\lambda, \mathbb{A})$ from $\mathcal{H}$ into $H_1 \oplus H_1^1 \oplus \cdots \oplus H_1^s$ acts boundedly. Then, by virtue of Lemma 2.3.1 ([26]) and condition 2, for some $\lambda_0 \in \rho(\mathbb{A})$ we have

$$\begin{aligned} s_j(J; \mathcal{H}(\mathbb{A}), \mathcal{H}) &= s_j(J; \mathcal{H}(\lambda_0 I - \mathbb{A}), \mathcal{H}) = s_j(JR(\lambda_0, \mathbb{A}), \mathcal{H}, \mathcal{H}) \\ &\leq \|R(\lambda_0, \mathbb{A})\|_{B(\mathcal{H}, H_1 \bigoplus_{\nu=1}^{s} H_1^\nu)} s_j(J; H_1 \bigoplus_{\nu=1}^{s} H_1^\nu, \mathcal{H}) \leq Cj^{-p},\ j = 1, \dots, \infty; \end{aligned}$$

i.e., the operator $\mathbb{A}$ satisfies condition 2 of Theorem 2.2.1/1.

3. Completeness of root functions of principally regular elliptic boundary.

3.1. Introduction. It is known that the ellipticity condition of a boundary value problem (ellipticity of the equation and Lopatinskii condition for the problem) guarantee that the problem is Noether. As was shown by R.T. Seely ([20]), the ellipticity does not guarantee the existence of a regular point of the problem. The condition of ellipticity with a parameter which was found by S. Agmon ([1]) ensures that the problem is Fredholm and the completeness of root functions of the problem. In the papers by M.V. Keldysh ([13]) and F.E. Browder ([5]) the completeness of root functions of elliptic boundary value problems with a self-adjoint principal part is proved. In papers by S. Agmon ([1]), G. Geymonat and P. Grisvard ([7]), A.N. Kozhevnikov ([15]), Z.A. Kotko and S.G. Krein ([14]), S. Ya. Yakubov ([23–26]), M.S. Agranovich ([4]) and A. Kozhevnikov and S. Yakubov ([16]) the completeness of root functions is proved when the principal part of the problem is non-self-adjoint.

3.2. Coerciveness of principally regular elliptic boundary value problems.

3.2.1. Ellipticity with a parameter. Let G be a bounded domain in the Euclidean space $\mathbb{R}^r$ with an $(r-1)$-dimensional boundary ∂G. Consider the spectral problem in G for in principally regular elliptic boundary value problems with a polynomial parameter

$$L(\lambda)u = \lambda^n u(x) + \sum_{k=1}^{n} \lambda^{n-k} \Big(\sum_{|\alpha|=dk} a_{k\alpha}(x) D^\alpha u(x) + B_k u|_x \Big) = f(x), \quad x \in G, \tag{1}$$

$$L_\nu(\lambda)u = \sum_{k=0}^{n_\nu} \lambda^k \Big(\sum_{|\alpha|=m_\nu - dk} b_{\nu k\alpha}(x') D^\alpha u(x') + T_{\nu k} u|_{x'} \Big) = f_\nu(x'),\ x' \in \partial G, \tag{2}$$

$\nu = 1, \dots, m$, where the problem's weight $d = \frac{2m}{n}$ is, in general, a noninteger, and $a_{k\alpha}(x) = b_{\nu k\alpha}(x) = 0$ if dk is a noninteger, $0 \le n_\nu \le n-1$, $D^\alpha = D_1^{\alpha_1} \cdots D_r^{\alpha_r}$, $D_j = -i\frac{\partial}{\partial x_j}$, $j = 1, \dots, r$, $\alpha = (\alpha_1, \dots, \alpha_r)$ is a multi-index, $|\alpha| = \sum_{j=1}^r \alpha_j$, $\lambda \in \mathbb{C}$, $x = (x_1, \dots, x_r)$, $a_{k\alpha} \in C^{\ell-2m}(\overline{G})$, $b_{\nu k\alpha} \in C^{\ell - m_\nu}(\overline{G})$, $\partial G \in C^\ell$, $\ell \ge \max\{2m, m_\nu + 1\}$. Here operators B_k from $W_q^{dk}(G)$ into $L_q(G)$ are compact, operators $T_{\nu k}$ from $W_q^{m_\nu - dk + \frac{1}{q}}(G)$ into $L_q(\partial G)$ are compact, $q \in (1, \infty)$, $B_k u|_x = (B_k u)(x)$. Let us denote

$$L_0(x, \sigma, \lambda) = \lambda^n + \sum_{k=1}^n \lambda^{n-k} \sum_{|\alpha|=dk} a_{k\alpha}(x)\sigma^\alpha,$$

$$L_{\nu 0}(x', \sigma, \lambda) = \sum_{k=0}^{n_\nu} \lambda^k \sum_{|\alpha| = m_\nu - dk} b_{\nu k\alpha}(x')\sigma^\alpha, \quad \nu = 1, \dots, m,$$

where $\sigma^\alpha = \sigma_1^{\alpha_1} \cdots \sigma_r^{\alpha_r}$, $\sigma \in \mathbb{R}^r$. Let S be a set in the complex plane $\mathbb{C}$.

Condition I. *Let for $x \in \overline{G}$, $\sigma \in \mathbb{R}^r$, $\lambda \in S$, $|\sigma| + |\lambda| \ne 0$, $L_0(x, \sigma, \lambda) \ne 0$. If $r = 1$, in addition, suppose that roots of the equation $L_0(x, \sigma, \lambda) = 0$, with respect to σ for $x \in \overline{G}$, $\lambda \in S$, are equally distributed between upper and lower half-planes.*

Condition II. *Let x' be any point on ∂G. Let the vector σ' be tangent and σ be a normal vector to ∂G at the point $x' \in \partial G$. Consider the following ordinary differential problem on a half line $y > 0$:*

$$L_0(x', \sigma' - i\sigma\frac{\mathrm{d}}{\mathrm{dy}}, \lambda)u(y) = 0, \quad y > 0,$$

$$L_{\nu 0}(x', \sigma' - i\sigma\frac{\mathrm{d}}{\mathrm{dy}}, \lambda)u(y)|_{y=0} = h_\nu, \quad \nu = 1, \dots, m.$$

It is required that for $\lambda \in S$ this problem should have one and only one solution including all its derivatives, tending to 0 as $y \to \infty$ for any numbers $h_\nu \in \mathbb{C}$.

We will say that conditions I and II are fulfilled on the set S. Condition II has many equivalent formulations ([17, 2.4.1]). Let us show one of them. Let x', σ' and σ be the same as in condition II. Consider τ-roots of the polynomial

$$L_0(x', \sigma' + \tau\sigma, \lambda) = \lambda^n + \sum_{k=1}^n \lambda^{n-k} \sum_{|\alpha|=dk} a_{k\alpha}(x')(\sigma' + \tau\sigma)^\alpha = 0.$$

By virtue of condition I τ-roots are nonreal and

$$\mathrm{Im}\tau_k^+(x', \sigma', \sigma, \lambda) > 0, \quad k = 1, \dots, m,$$

$$\mathrm{Im}\tau_k^-(x', \sigma', \sigma, \lambda) < 0, \quad k = 1, \dots, m,$$

Condition II′. *For any tangent vector σ' and normal vector σ to ∂G at the point $x' \in \partial G$, $\lambda \in S$ polynomials on τ*

$$L_{\nu 0}(x', \sigma' + \tau\sigma, \lambda) = \sum_{k=1}^{n_\nu} \lambda^k \sum_{|\alpha|=m_\nu - dk} b_{\nu k\alpha}(x')(\sigma' + \tau\sigma)^\alpha, \quad \nu = 1, \dots, m,$$

are linearly independent on modulo a polynomial

$$M^+(x', \sigma', \sigma, \lambda, \tau) = \prod_{k=1}^{m} (\tau - \tau_k^+(x', \sigma', \sigma, \lambda)).$$

In other words, if the polynomial

$$L'_{\nu 0}(x', \sigma', \sigma, \lambda, \tau) = \sum_{k=0}^{m-1} b'_{\nu k}(x', \sigma', \sigma, \lambda)\tau^k$$

is the residue of the division of the polynomial $L_{\nu 0}$ on M^+, then

$$\det |b'_{\nu k}(x', \sigma', \sigma, \lambda)|_{\substack{\nu=1,\dots,m \\ k=0,\dots,m-1}} \neq 0.$$

A system

$$L_\nu u = \sum_{|\alpha|=m_\nu} b_{\nu\alpha}(x')D^\alpha u(x') + T_\nu u|_{x'}, \quad x' \in \partial G, \ \ \nu = 1, \dots, m,$$

is called *normal*, if $m_j \neq m_k$ for $j \neq k$ and for any vector σ normal to the boundary ∂G at the point $x' \in \partial G$ the following condition is fulfilled

$$L_{\nu 0}(x', \sigma) = \sum_{|\alpha|=m_\nu} b_{\nu\alpha}(x')\sigma^\alpha \neq 0, \quad \nu = 1, \dots, m,$$

and T_ν from $W_q^{m_\nu + \frac{1}{q}}(G)$ into $L_q(\partial G)$ are compact. Let us denote

$$L_{\nu, n_\nu - k} u = \sum_{|\alpha|=m_\nu - dk} b_{\nu k\alpha}(x')D^\alpha u(x') + T_{\nu k} u|_{x'}, \tag{3}$$

$k = 0, \dots, n_\nu$, $\nu = 1, \dots, m$.

Theorem 1. *Let the following conditions be satisfied:*

(1) $n \geq 1$, $m \geq 1$, $d = \frac{2m}{n}$, $m_\nu \geq dn_\nu$;
(2) $a_{k\alpha} \in C^{\ell-2m}(\overline{G})$, $b_{\nu k\alpha} \in C^{\ell-m_\nu}(\overline{G})$, $\partial G \in C^\ell$, *where* $\ell \geq \max\{2m, m_\nu + 1\}$;
(3) *operators* B_k *from* $W_q^{dk}(G)$ *into* $L_q(G)$ *and from* $W_q^{\ell-2m+dk}(G)$ *into* $W_q^{\ell-2m}(G)$ *are compact, where* $q \in (1, \infty)$;
(4) *operators* $T_{\nu k}$ *from* $W_q^{m_\nu - dk + \frac{1}{q}}(G)$ *into* $L_q(\partial G)$ *and from* $W_q^{\ell-dk}(G)$ *into* $W_q^{\ell-m_\nu-\frac{1}{q}}(\partial G)$ *are compact;* $T_{\nu n_\nu} = 0$, *if* $m_\nu - dn_\nu = 0$;
(5) *the system* $L_{\nu n_\nu} = \sum_{|\alpha|=m_\nu} b_{\nu 0\alpha}(x')D^\alpha + T_{\nu 0}$, $\nu = 1, \dots, m$, *is normal;*
(6) *conditions I and II are fulfilled on the rays* $\ell(\varphi)$.

Then the operator $\mathbb{L}(\lambda): \ u \to \mathbb{L}(\lambda)u = (L(\lambda)u, L_1(\lambda)u, \ldots, L_m(\lambda)u)$, *for* $\lambda \in \ell(\varphi)$, $|\lambda| \to \infty$, *from* $W_q^{\ell}(G)$ *onto* $W_q^{\ell-2m}(G) \dot{+}_{\nu=1}^{m} W_q^{\ell-m_\nu-\frac{1}{q}}(\partial G)$ *is an isomorphism, and for a solution to problem* (1)–(2) *the estimate*

$$\sum_{k=0}^{\ell} |\lambda|^{d^{-1}(\ell-k)} \|u\|_{k,q,G} \leq C[\|f\|_{\ell-2m,q,G} + |\lambda|^{d^{-1}(\ell-2m)} \|f\|_{0,q,G} + \sum_{\nu=1}^{m} (|\lambda|^{d^{-1}(\ell-m_\nu-\frac{1}{q})} \|f_\nu\|_{0,q,\partial G} + \|f_\nu\|_{\ell-m_\nu-\frac{1}{q},q,\partial G})] \tag{4}$$

is valid.

Proof. Using the substitution $\lambda = \mu^d$ problem (1)–(2) is reduced to

$$\mu^{2m}u(x) + \sum_{k=1}^{n} \mu^{2m-dk}\big(\sum_{|\alpha|=dk} a_{k\alpha}(x)D^\alpha u(x) + B_k u|_x\big) = f(x), \quad x \in G, \tag{5}$$

$$\sum_{k=0}^{n_\nu} \mu^{dk}(\sum_{|\alpha|=m_\nu-dk} b_{\nu k\alpha}(x')D^\alpha u(x') + T_{\nu k}u|_{x'}) = f_\nu(x'), \quad x' \in \partial G, \ \nu = 1, \ldots, m. \tag{6}$$

Let $u \in W_q^{\ell}(G)$ be a solution to problem (1)–(2), i.e., (5)–(6). Then by virtue of [2] and [3], for any $\mu \in \ell(d^{-1}\varphi)$, $|\mu| \to \infty$, we have

$$\begin{aligned}
\sum_{k=0}^{\ell} |\mu|^{\ell-k} \|u\|_{k,q,G} &\leq C[\|f - \sum_{k=1}^{n} \mu^{2m-dk} B_k u\|_{\ell-2m,q,G} \\
&\quad + |\mu|^{\ell-2m} \|f - \sum_{k=1}^{n} \mu^{2m-dk} B_k u\|_{0,q,G} + \sum_{\nu=1}^{m} (|\mu|^{\ell-m_\nu-\frac{1}{q}} \|f_\nu \\
&\quad - \sum_{k=0}^{n_\nu} \mu^{dk} T_{\nu k} u\|_{0,q,\partial G} + \|f_\nu - \sum_{k=0}^{n_\nu} \mu^{dk} T_{\nu k} u\|_{\ell-m_\nu-\frac{1}{q},q,\partial G})] \\
&\leq C[\|f\|_{\ell-2m,q,G} + |\mu|^{\ell-2m} \|f\|_{0,q,G} \\
&\quad + \sum_{\nu=1}^{m} (|\mu|^{\ell-m_\nu-\frac{1}{q}} \|f_\nu\|_{0,q,\partial G} + \|f_\nu\|_{\ell-m_\nu-\frac{1}{q},q,\partial G}) \\
&\quad + \sum_{k=1}^{n} |\mu|^{2m-dk} (\|B_k u\|_{\ell-2m,q,G} + |\mu|^{\ell-2m} \|B_k u\|_{0,q,G}) \\
&\quad + \sum_{\nu=1}^{m} \sum_{k=0}^{n_\nu} |\mu|^{dk} (|\mu|^{\ell-m_\nu-\frac{1}{q}} \|T_{\nu k} u\|_{0,q,\partial G} + \|T_{\nu k} u\|_{\ell-m_\nu-\frac{1}{q},q,\partial G})].
\end{aligned} \tag{7}$$

From (7), by virtue of conditions 3–4 and [26, Lemma 3.1.6], we obtain

$$\begin{aligned}\sum_{k=0}^{\ell}|\mu|^{\ell-k}\|u\|_{k,q,G} \le C\{&\|f\|_{\ell-2m,q,G}+|\mu|^{\ell-2m}\|f\|_{0,q,G}\\ &+\sum_{\nu=1}^{m}(|\mu|^{\ell-m_\nu-\frac{1}{q}}\|f_\nu\|_{0,q,\partial G}+\|f_\nu\|_{\ell-m_\nu-\frac{1}{q},q,\partial G})\\ &+\delta\sum_{k=1}^{n}(|\mu|^{2m-dk}\|u\|_{\ell-2m+dk,q,G}+|\mu|^{\ell-dk}\|u\|_{dk,q,G})\\ &+C(\delta)\sum_{k=1}^{n}(|\mu|^{2m-dk}\|u\|_{\ell-2m,q,G}+|\mu|^{\ell-dk}\|u\|_{0,q,G})\\ &+\sum_{\nu=1}^{m}\sum_{k=0}^{n_\nu-1}[|\mu|^{\ell-(m_\nu-dk)-\frac{1}{q}}(\delta\|u\|_{m_\nu-dk+\frac{1}{q},q,G}\\ &+C(\delta)\|u\|_{\frac{1}{q},q,G})+|\mu|^{dk}(\delta\|u\|_{\ell-dk,q,G}+C(\delta)\|u\|_{\ell-m_\nu,q,G})]\},\end{aligned}$$

for any $\delta > 0$. Using [26, Lemma 1.2.4], i.e.,

$$|\mu|^{s}\|u\|_{\ell-s,q,G}\le C(\|u\|_{\ell,q,G}+|\mu|^{\ell}\|u\|_{0,q,G}),$$

we have

$$\begin{aligned}\sum_{k=0}^{\ell}|\mu|^{\ell-k}\|u\|_{k,q,G} \le C[&\|f\|_{\ell-2m,q,G}+|\mu|^{\ell-2m}\|f\|_{0,q,G}\\ &+\sum_{\nu=1}^{m}(|\mu|^{\ell-m_\nu-\frac{1}{q}}\|f_\nu\|_{0,q,\partial G}+\|f_\nu\|_{\ell-m_\nu-\frac{1}{q},q,\partial G})]\\ &+(C\delta+C(\delta)|\mu|^{-\theta})(\|u\|_{\ell,q,G}+|\mu|^{\ell}\|u\|_{0,q,G}),\end{aligned}$$

where $\theta = \min_{m_\nu-dn_\nu\neq 0}\{d, m_\nu-dn_\nu\}$. It is clear that one can choose $\delta>0$ so small and $|\mu|$ so large that $C\delta+C(\delta)|\mu|^{-\theta}<1$. Hence, for $\mu\in\ell(d^{-1}\varphi)$, $|\mu|\to\infty$ we obtain an a priori estimate

$$\begin{aligned}\sum_{k=0}^{\ell}|\mu|^{\ell-k}\|u\|_{k,q,G} \le C[&\|f\|_{\ell-2m,q,G}+|\mu|^{\ell-2m}\|f\|_{0,q,G}\\ &+\sum_{\nu=1}^{m}(|\mu|^{\ell-m_\nu-\frac{1}{q}}\|f_\nu\|_{0,q,\partial G}+\|f_\nu\|_{\ell-m_\nu-\frac{1}{q},q,\partial G})].\end{aligned}\tag{8}$$

From (8) it follows that, for $\lambda\in\ell(\varphi)$, $|\lambda|\to\infty$, estimate (4) is valid. Hence, for the indicated λ, a solution to problem (1)–(2) in $W_q^{\ell}(G)$ is unique.

Let us represent the operator $\mathbb{L}(\lambda)$ in the form $\mathbb{L}(\lambda) = \mathbb{L}_0(\lambda) + \mathbb{L}_1(\lambda)$, where

$$\mathbb{L}_0(\lambda)u \equiv (L_0(x, D, \lambda)u, L_{10}(x', D, \lambda)u, \dots, L_{m0}(x', D, \lambda)u),$$

$$L_0(x, D, \lambda) = \lambda^n + \sum_{k=1}^{n} \lambda^{n-k} \sum_{|\alpha|=dk} a_{k\alpha}(x) D^\alpha,$$

$$L_{\nu 0}(x', D, \lambda) = \sum_{k=0}^{n_\nu} \lambda^k \sum_{|\alpha|=m_\nu - dk} b_{\nu k\alpha}(x') D^\alpha, \quad \nu = 1, \dots, m,$$

$$\mathbb{L}_1(\lambda)u = \Big(\sum_{k=1}^{n} \lambda^{n-k} B_k u|_x, \sum_{k=0}^{n_1} \lambda^k T_{1k} u|_{x'}, \dots, \sum_{k=0}^{n_m} \lambda^k T_{mk} u|_{x'}\Big).$$

From [2] and [3] it follows that the operator $\mathbb{L}_0(\lambda)$, for $\lambda \in \ell(\varphi)$, $|\lambda| \to \infty$, from $W_q^\ell(G)$ onto $W_q^{\ell-2m}(G) \dot{+}_{\nu=1}^{m} W_q^{\ell-m_\nu-\frac{1}{q}}(\partial G)$ is an isomorphism. By virtue of conditions 3 and 4 the operator $\mathbb{L}_1(\lambda)$ from $W_q^\ell(G)$ into $W_q^{\ell-2m}(G) \dot{+}_{\nu=1}^{m} W_q^{\ell-m_\nu-\frac{1}{q}}(\partial G)$ is compact. Then it is possible to apply the perturbation theorem of Fredholm operators of T. Kato ([12, page 238]) to the operator $\mathbb{L}(\lambda) = \mathbb{L}_0(\lambda) + \mathbb{L}_1(\lambda)$, from which the statement of Theorem 1 follows. □

If problem (1)–(2) is a regular elliptic problem with a parameter, i.e.,

$$B_k u = \sum_{|\alpha| \le dk-1} a_{k\alpha}(x) D^\alpha u(x), \quad T_{\nu k} u = \sum_{|\alpha| \le m_\nu - dk - 1} b_{\nu k\alpha}(x') D^\alpha u(x'),$$

then conditions of coercive solvability of problem (1)–(2) in the space $W_q^\ell(G)$ have been found in the works of S. Agmon and L. Nirenberg ([2]), M.S. Agranovich and M.I. Vishik ([3]), and in the space $C(\overline{G})$, in the paper by H.B. Stewart ([21]) (for $n = 1$).

Corollary 2. *If, in condition 6 of Theorem 1, Conditions I and II are fulfilled in some angle S, then the statement of Theorem 1 is also true for $\lambda \in S$, $|\lambda| \to \infty$.*

Theorem 3. *Let the following conditions be satisfied:*

(1) *$b_{\nu\alpha} \in C^{\ell-m_\nu}(\overline{G})$, operators $K_{\nu p}$ from $W_q^{\ell-p-\frac{1}{q}}(\partial G)$ into $W_q^{\ell-m_\nu-\frac{1}{q}}(\partial G)$ are compact, where $\ell \ge \max\{m_\nu\} + 1$, $q \in (1, \infty)$;*

(2) *the system*

$$L_\nu u \equiv \sum_{|\alpha|=m_\nu} b_{\nu\alpha}(x') D^\alpha u(x') + \sum_{p=0}^{m_\nu - 1} K_{\nu p} \frac{\partial^p u}{\partial n^p}\Big|_{x'}, \quad \nu = 1, \dots, m$$

is normal, where n is a normal vector to the boundary ∂G at the point $x' \in \partial G$.

Then the operator $u \to (L_1 u, \dots, L_m u)$ *from* $W_q^\ell(G)$ *onto* $\dot{+}_{\nu=1}^{m} W_q^{\ell-m_\nu-\frac{1}{q}}(\partial G)$ *has a continuous right-inverse; in other words, there exists such an operator* $R(f_1, \dots, f_m) = u$ *continuous from* $\dot{+}_{\nu=1}^{m} W_q^{\ell-m_\nu-\frac{1}{q}}(\partial G)$ *into* $W_q^\ell(G)$*, where* u *is a solution of the system*

$$L_\nu u = \sum_{|\alpha|=m_\nu} b_{\nu\alpha}(x')D^\alpha u(x') + \sum_{p=0}^{m_\nu-1} K_{\nu p}\frac{\partial^p u}{\partial n^p}\Big|_{x'} = f_\nu(x'), \quad \nu = 1, \dots, m. \tag{9}$$

Moreover, the inverse operator does not depend on $\ell \in [\max\{m_\nu\}+1, \ell_0]$.

Proof. We now apply to (9) the scheme of the proof of Lemma 5.4.4 from [22].

3.2.2. Dense sets in Sobolev spaces. Before we pass to the main results of this item, we prove a theorem of dense sets in direct sums of Sobolev spaces, which is also of independent mathematical interest. In what follows this theorem is essential.

Theorem 1. *Let the following conditions be satisfied:*

(1) $b_{\nu\alpha} \in C^{\ell-m_\nu}(\overline{G})$*; operators* $K_{\nu p}$ *from* $W_q^{\ell-p-\frac{1}{q}}(\partial G)$ *into* $W_q^{\ell-m_\nu-\frac{1}{q}}(\partial G)$ *are compact, where* $\ell \geq \max\{m_\nu\}+1$, $q \in (1,\infty)$*;*

(2) *the system*

$$L_\nu u = \sum_{|\alpha|=m_\nu} b_{\nu\alpha}(x')D^\alpha u(x') + \sum_{p=0}^{m_\nu-1} K_{\nu p}\frac{\partial^p u}{\partial n^p}\Big|_{x'}, \quad \nu = 1, \dots, m,$$

is normal, where n *is a normal vector to the boundary* ∂G *at the point* $x' \in \partial G$.

Then for any integer $k \in [0, \ell]$

$$\overline{W_q^\ell(G; L_\nu u = 0, \nu = 1, \dots, m)}\big|_{W_q^k(G)} = W_q^k(G; L_\nu u = 0, m_\nu \leq k-1). \tag{1}$$

Proof. (1) follows from the known equality $\overline{C_0^\infty(G)}|_{W_q^k(G)} = W_q^k(G; L_\nu u = 0, m_\nu \leq k-1)$.

Consider operators

$$L_\nu(\lambda)u = \sum_{k=1}^{n_\nu} \lambda^k \Big(\sum_{|\alpha|=m_\nu-dk} b_{\nu k\alpha}(x')D^\alpha u(x') + T_{\nu k}u|_{x'} \Big)$$
$$+ \sum_{|\alpha|=m_\nu} b_{\nu 0\alpha}(x')D^\alpha u(x') + \sum_{p=0}^{m_\nu-1} K_{\nu p}\frac{\partial^p u}{\partial n^p}\Big|_{x'} = f_\nu(x'), \quad x' \in \partial G,$$

$\nu = 1, \ldots, m$. Operators $L_\nu(\lambda)$ have the form

$$L_\nu(\lambda)u = \sum_{k=0}^{n_\nu} \lambda^k L_{\nu, n_\nu - k} u,$$

where

$$L_{\nu, n_\nu - k} u = \sum_{|\alpha| = m_\nu - dk} b_{\nu k \alpha}(x') D^\alpha u(x') + T_{\nu k} u|_{x'}, \quad k = 1, \ldots, n_\nu,$$
$$L_{\nu n_\nu} u = \sum_{|\alpha| = m_\nu} b_{\nu 0 \alpha}(x') D^\alpha u(x') + \sum_{p=0}^{m_\nu - 1} K_{\nu p} \frac{\partial^p u}{\partial n^p}\Big|_{x'}.$$

Theorem 2. *Let the following conditions be satisfied:*

(1) $n \geq 1$, $m \geq 1$, $0 \leq n_\nu \leq n - 1$, $d = \frac{2m}{n}$ *is an integer,* $m_\nu \geq dn_\nu$, $\max\{m_\nu\} - \min\{m_\nu - dn_\nu\} \leq 2m - 1$;

(2) $b_{\nu k \alpha} \in C^{\theta - m_\nu}(\overline{G})$, *operators* $T_{\nu k}$ *from* $W_q^{m_\nu - dk + \frac{1}{q}}(G)$ *into* $L_q(\partial G)$ *and from* $W_q^{\theta - dk}(G)$ *into* $W_q^{\theta - m_\nu - \frac{1}{q}}(\partial G)$ *are compact, where* $\theta \geq \max\{2m, m_\nu + 1\}$, $q \in (1, \infty)$; $T_{\nu n_\nu} = 0$, *if* $m_\nu - dn_\nu = 0$;

(3) *operators* $K_{\nu p}$ *from* $W_q^{m_\nu - p}(\partial G)$ *into* $L_q(\partial G)$ *and from* $W_q^{\theta - p - \frac{1}{q}}(\partial G)$ *into* $W_q^{\theta - m_\nu - \frac{1}{q}}(\partial G)$ *are compact;*

(4) *the system* $L_{\nu, n_\nu}, \nu = 1, \ldots, m$, *is normal.*

Then the set

$$\mathcal{H}_d = \{v \mid v = (v_1, \ldots, v_n) \in \dot{+}_{k=0}^{n-1} W_q^{\ell + d(n-k)}(G),\ \sum_{k=0}^{n_\nu} L_{\nu, n_\nu - k} v_{k+s} = 0,$$
$$m_\nu \leq \ell + d(n - s + 1) - 1,\ s = 1, \ldots, n - n_\nu,\ \nu = 1, \ldots, m\}$$

is dense in the space

$$\mathcal{H} = \{v \mid v = (v_1, \ldots, v_n) \in \dot{+}_{k=0}^{n-1} W_q^{\ell + d(n-k-1)}(G),\ \sum_{k=0}^{n_\nu} L_{\nu, n_\nu - k} v_{k+s} = 0,$$
$$m_\nu \leq \ell + d(n - s) - 1,\ s = 1, \ldots, n - n_\nu - 1,\ \nu = 1, \ldots, m\}$$

for an integer $\ell \in [\max\{0, m_\nu - (2m - 1)\}, \min\{m_\nu - dn_\nu\}]$.

Proof. Let $\varepsilon > 0$ and $v = (v_1, \ldots, v_n) \in \mathcal{H}$. Set $t = \min\{n_\nu\}$. Construct functions $\varphi_s \in C^\infty(\overline{G})$, $s = n - t + 1, \ldots, n$, such that

$$\|\varphi_s - v_s\|_{\ell + d(n-s), q, G} < \varepsilon, \quad s = n - t + 1, \ldots, n. \tag{2}$$

Since every subsystem of a normal system is also normal and $\varphi_s \in C^\infty(\overline{G})$, $s = n-t+1, \dots, n$, then, by virtue of Theorem 3.2.1/3, there exists a function $\varphi^0_{n-t} \in W_q^{\ell+d(t+1)}(G)$ such that

$$L_{\nu n_\nu}\varphi^0_{n-t} = -\sum_{k=1}^{n_\nu} L_{\nu,n_\nu-k}\varphi_{k+n-t}, \quad m_\nu \le \ell + d(t+1) - 1, \; n_\nu = t. \tag{3}$$

Since $\ell \le m_\nu - dn_\nu$, then for $n_\nu = t$ we have $m_\nu \ge \ell + dt$. Then, by virtue of Theorem 1, there exists a function $g_{n-t} \in W_q^{\ell+d(t+1)}(G)$ such that

$$\begin{gathered} L_{\nu n_\nu} g_{n-t} = 0, \quad m_\nu \le \ell + d(t+1) - 1, \; n_\nu = t, \\ \|g_{n-t} - (v_{n-t} - \varphi^0_{n-t})\|_{\ell+dt,q,G} < \varepsilon. \end{gathered} \tag{4}$$

By virtue of (3)–(4), the function $\varphi_{n-t} = g_{n-t} + \varphi^0_{n-t} \in W_q^{\ell+d(t+1)}(G)$ satisfies

$$L_{\nu n_\nu}\varphi_{n-t} = -\sum_{k=1}^{n_\nu} L_{\nu,n_\nu-k}\varphi_{k+n-t}, \quad m_\nu \le \ell + d(t+1) - 1, \; n_\nu = t. \tag{5}$$

Obviously,

$$\|\varphi_{n-t} - v_{n-t}\|_{\ell+dt,q,G} < \varepsilon. \tag{6}$$

From the conditions of connection in $\mathcal{H}_d$, for $n_\nu \le t+i$, $s = n-t-i$, $m_\nu \le \ell + d(t+i+1) - 1$, $i = 0, \dots, n-t-1$, one can obtain

$$L_{\nu n_\nu}\varphi_{n-t-i} = -\sum_{k=1}^{n_\nu} L_{\nu,n_\nu-k}\varphi_{k+n-t-i}, \tag{7}$$

for $\varphi \in \mathcal{H}_d$. Let us show by induction on i that there exist functions

$$\varphi_{n-t-i} \in W_q^{\ell+d(t+i+1)}(G), \quad i = 0, \dots, n-t-1,$$

that satisfy relations (7) and

$$\|\varphi_{n-t-i} - v_{n-t-i}\|_{\ell+d(t+i),q,G} < C\varepsilon. \tag{8}$$

Relations (7) and (8) for $i = 0$ are true since in this case they transform into relations (5) and (6). Suppose that there exist functions $\varphi_{n-t-i} \in W_q^{\ell+d(t+i+1)}(G)$ that satisfy relations (7) and (8) for $i = 0, \dots, s-1$, where $s = 1, \dots, n-t-1$. We will now show that there exists a function $\varphi_{n-t-s} \in W_q^{\ell+d(t+s+1)}(G)$ which satisfies (7) and (8) for $i = s$. Obviously, if $m_\nu \le \ell + d(t+s+1) - 1$, $n_\nu \le t+s$, then from

$$\|\sum_{k=1}^{n_\nu} L_{\nu,n_\nu-k}\varphi_{k+n-t-s}\|_{\ell+d(t+s+1)-m_\nu-\frac{1}{q},q,\partial G} \le C\sum_{k=1}^{n_\nu} \|\varphi_{k+n-t-s}\|_{\ell+d(t+s-k+1),q,G}$$

it follows that $\sum_{k=1}^{n_\nu} L_{\nu,n_\nu-k}\varphi_{k+n-t-s} \in W_q^{\ell+d(t+s+1)-m_\nu-\frac{1}{q}}(\partial G)$. Then, by virtue of Theorem 3.2.1/3, there exist functions $\varphi^0_{n-t-s} \in W_q^{\ell+d(t+s+1)}(G)$ such that

$$L_{\nu n_\nu}\varphi^0_{n-t-s} = -\sum_{k=1}^{n_\nu} L_{\nu,n_\nu-k}\varphi_{k+n-t-s} \tag{9}$$

for $m_\nu \le \ell + d(t+s+1) - 1$, $n_\nu \le t+s$.

Since $\ell \le m_\nu - dn_\nu$, then for $n_\nu = t+s$ we have $m_\nu \ge \ell + d(t+s)$. This means that boundary value conditions of the type $L_{\nu n_\nu}$ with data $n_\nu = t+s$ and $m_\nu \le \ell + d(t+s) - 1$ do not exist. Consequently,

$$v_{n-t-s} - \varphi^0_{n-t-s} \in W_q^{\ell+d(t+s)}(G; L_{\nu n_\nu}u = 0, \quad m_\nu \le \ell + d(t+s) - 1, n_\nu = t+s).$$

Then, by virtue of Theorem 1, there exist functions $g_{n-t-s} \in W_q^{\ell+d(t+s+1)}(G)$, such that

$$L_{\nu n_\nu} g_{n-t-s} = 0, \quad m_\nu \le \ell + d(t+s+1) - 1,\ n_\nu = t+s, \tag{10}$$

$$\|g_{n-t-s} - (v_{n-t-s} - \varphi^0_{n-t-s})\|_{\ell+d(t+s),q,G} < \varepsilon. \tag{11}$$

Now it is enough to note that the functions

$$\varphi_{n-t-s} = g_{n-t-s} + \varphi^0_{n-t-s} \in W_q^{\ell+d(t+s+1)}(G)$$

satisfy both (7) and (8) for $i = s$. Relation (7) for $i = s$ follows from (10) and (9). Relation (8) for $i = s$ follows from (11). From (7) follows that the constructed function $\varphi = (\varphi_1, \dots, \varphi_n) \in \mathcal{H}_d$. From (2) and (8) follows that $\|\varphi - v\|_{\mathcal{H}} < C\varepsilon$.

3.2.3. Fold completeness of root functions of principally regular elliptic boundary value problems. Let us formulate and prove the main theorem on the completeness of root functions of regular elliptic boundary value problems. Let G be a bounded domain in the Euclidean space $\mathbb{R}^r$ with an $(r-1)$-dimensional boundary ∂G. Consider the spectral problem in G for principally regular elliptic boundary value problems with a polynomial parameter

$$L(\lambda)u = \lambda^n u(x) + \sum_{k=1}^{n} \lambda^{n-k}\big(\sum_{|\alpha|=dk} a_{k\alpha}(x)D^\alpha u(x) + B_k u|_x\big) = 0, \quad x \in G, \tag{1}$$

$$L_\nu(\lambda)u = \sum_{k=1}^{n_\nu} \lambda^k\big(\sum_{|\alpha|=m_\nu-dk} b_{\nu k\alpha}(x')D^\alpha u(x') + T_{\nu k}u|_{x'}\big)$$

$$+ \sum_{|\alpha|=m_\nu} b_{\nu 0\alpha}(x')D^\alpha u(x') + \sum_{p=0}^{m_\nu-1} K_{\nu p}\frac{\partial^p u}{\partial n^p}\big|_{x'} = 0, \quad x' \in \partial G, \quad \nu = 1, \dots, m, \tag{2}$$

where the problem's weight $d = \frac{2m}{n}$ is an integer, $0 \le n_\nu \le n-1$.

A number λ_0 is called an *eigenvalue* of problem (1)–(2) if the problem

$$L(\lambda_0)u = 0, \qquad L_\nu(\lambda_0)u = 0, \quad \nu = 1, \dots, m, \tag{3}$$

has a nontrivial solution that belongs to $W_2^{2m}(G)$. The nontrivial solution $u_0(x)$ of problem (3) that belongs to $W_2^{2m}(G)$ is called an *eigenfunction* of problem (1)–(2), and corresponds to the eigenvalue λ_0. A solution $u_p(x)$ to the problem

$$\begin{aligned} &L(\lambda_0)u_p + \frac{1}{1!}L'(\lambda_0)u_{p-1} + \cdots + \frac{1}{p!}L^{(p)}(\lambda_0)u_0 = 0,\\ &L_\nu(\lambda_0)u_p + \frac{1}{1!}L'_\nu(\lambda_0)u_{p-1} + \cdots + \frac{1}{p!}L_\nu^{(p)}(\lambda_0)u_0 = 0, \quad \nu = 1, \dots, m, \end{aligned}$$

that belongs to $W_2^{2m}(G)$ is called an *associated function* of the p-th rank to the eigenfunction $u_0(x)$ of problem (1)–(2).

Eigenfunctions and associated functions of problem (1)–(2) are combined under the general name *root functions* of problem (1)–(2).

A complex number λ is called a *regular point* of problem (1)–(2), if the problem

$$L(\lambda)u = f, \qquad L_\nu(\lambda)u = f_\nu, \quad \nu = 1, \dots, m, \tag{4}$$

for any $f \in L_2(G)$, $f_\nu \in W_2^{2m-m_\nu-\frac{1}{2}}(\partial G)$, has a unique solution belonging to $W_2^{2m}(G)$ and the estimate

$$\|u\|_{2m,2,G} \le C(\lambda)\big(\|f\|_{0,2,G} + \sum_{\nu=1}^{m} \|f_\nu\|_{2m-m_\nu-\frac{1}{2},2,\partial G}\big)$$

is satisfied.[11]

The complement of the set of regular points in the complex plane is called the *spectrum* of problem (1)–(2). The spectrum of problem (1)–(2) is called *discrete*, if

a) all points λ, not coinciding with eigenvalues of problem (1)–(2), are regular points of problem (1)–(2);
b) the eigenvalues are isolated and have finite algebraic multiplicities;
c) infinity is the only limit point of the set of eigenvalues of the problem (1)–(2).

[11]In the general case, i.e., when a boundary value problem is irregular, it should be required that a solution belongs to $L_2(G)$ and the estimate

$$\|u\|_{0,2,G} \le C(\lambda)(\|f\|_{0,2,G} + \sum_{\nu=1}^{m} \|f_\nu\|_{2m-m_\nu-\frac{1}{2},2,\partial G})$$

holds.

Consider a system of differential equations

$$L(D_t)u(t,x) = 0, \quad t > 0,\ x \in G, \tag{5}$$

$$L_\nu(D_t)u(t,x') = 0, \quad t > 0,\ x' \in \partial G,\ \nu = 1,\dots,m, \tag{6}$$

where $D_t = \frac{\partial}{\partial t}$. By virtue of Lemma 2.1/1, a function of the form

$$u(t,x) = \mathrm{e}^{\lambda_0 t}\big(\frac{t^k}{k!}u_0(x) + \frac{t^{k-1}}{(k-1)!}u_1(x) + \cdots + u_k(x)\big) \tag{7}$$

is a solution to system (5)–(6) if and only if the system of functions $u_0(x), u_1(x), \dots, u_k(x)$ is a chain of root functions of problem (1)–(2), that corresponds to the eigenvalue λ_0. A solution of the form (7) is called an *elementary solution* of system (5)–(6).

Let $\mathcal{H}$ be a Hilbert space, continuously embedded into $[L_2(G)]^n = \bigoplus^n L_2(G)$. A system of root functions of problem (1)–(2) is called *n-fold complete* in $\mathcal{H}$ if the system of functions $(u(0,x)u'_t(0,x),\dots,u_t^{(n-1)}(0,x))$ is complete in the space $\mathcal{H}$. Conditions (2) can be rewritten in the form

$$L_\nu(\lambda)u = \sum_{k=0}^{n_\nu} \lambda^k L_{\nu,n_\nu-k}u.$$

Theorem 1. *Let the following conditions be satisfied:*

(1) $n \geq 1$, $m \geq 1$, $n_\nu \leq n-1$, $m_\nu \geq dn_\nu$, $\max\{m_\nu\} - \min\{m_\nu - dn_\nu\} \leq 2m-1$;
(2) *the weight* $d = \frac{2m}{n}$ *of problem (1)–(2) is an integer;*
(3) $a_{k\alpha} \in C^\ell(\overline{G})$, *operators* B_k *from* $W_2^{dk}(G)$ *into* $L_2(G)$ *and from* $W_2^{\ell+dk}(G)$ *into* $W_2^\ell(G)$ *are compact, where* $\ell \in [\max\{0, m_\nu-(2m-1)\}, \min\{m_\nu-dn_\nu\}]$;
(4) $b_{\nu k\alpha} \in C^{\ell+2m-m_\nu}(\overline{G})$, *operators* $T_{\nu k}$ *from* $W_2^{m_\nu-dk+\frac{1}{2}}(G)$ *into* $L_2(\partial G)$ *and from* $W_2^{\ell+2m-dk}(G)$ *into* $W_2^{\ell+2m-m_\nu-\frac{1}{2}}(\partial G)$ *are compact;* $T_{\nu n_\nu} = 0$, *if* $m_\nu - dn_\nu = 0$;
(5) *operators* $K_{\nu p}$ *from* $W_2^{m_\nu-p}(\partial G)$ *into* $L_2(\partial G)$ *and from* $W_2^{\ell+2m-p-\frac{1}{2}}(\partial G)$ *into* $W_2^{\ell+2m-m_\nu-\frac{1}{2}}(\partial G)$ *are compact;*
(6) *the system*

$$L_{\nu,n_\nu}u = \sum_{|\alpha|=m_\nu} b_{\nu 0\alpha}(x')D^\alpha u(x') + \sum_{p=0}^{m_\nu-1} K_{\nu p}\frac{\partial^p u}{\partial n^p}\Big|_{x'}, \quad \nu = 1,\dots,m,$$

is normal, where n *is a normal vector to the boundary* ∂G *at the point* $x' \in \partial G$

(7) *there exist rays* $\ell_k(\varphi_k)$ *with the angles between the neighboring rays less than* $\frac{2m\pi}{rn}$, *such that conditions I and II are fulfilled on the rays* ℓ_k.

Then the spectrum of problem (1)–(2) *is discrete and the system of root functions of problem* (1)–(2) *is n-fold complete in the space*

$$\mathcal{H} = \{u \mid u = (u_1, \dots, u_n) \in \bigoplus_{k=0}^{n-1} W_2^{\ell+d(n-k-1)}(G),\ \sum_{k=0}^{n_\nu} L_{\nu,n_\nu-k} u_{k+s} = 0,$$
$$m_\nu \leq \ell + d(n-s) - 1,\ s = 1, \dots, n - n_\nu - 1,\ \nu = 1, \dots, m\}.$$

Proof. Let us denote $H_k = W_2^{\ell+dk}(G)$, $k = 0, \dots, n$, $H^\nu = W_2^{\ell+2m-m_\nu-\frac{1}{2}}(\partial G)$, $H_0^\nu = W_2^{\frac{1}{2}}(\partial G)$, $\nu = 1, \dots, m$. Consider the operators A_k and $A_{\nu k}$

$$A_k u = \sum_{|\alpha|=dk} a_{k\alpha}(x) D^\alpha u(x) + B_k u|_x, \quad k = 1, \dots, n,$$

$$A_{\nu,n_\nu-k} u = \sum_{|\alpha|=m_\nu-dk} b_{\nu k\alpha}(x') D^\alpha u(x') + T_{\nu k} u|_{x'}, \quad k = 1, \dots, n_\nu,\ \nu = 1, \dots, m,$$

$$A_{\nu,n_\nu} u = \sum_{|\alpha|=m_\nu} b_{\nu 0\alpha}(x') D^\alpha u(x') + \sum_{p=0}^{m_\nu-1} K_{\nu p} \frac{\partial^p u}{\partial n^p}\Big|_{x'}, \quad \nu = 1, \dots, m, \tag{8}$$

Then problem (1)–(2) can be rewritten in the form of a system of operator pencil equations

$$\begin{aligned} &L(\lambda)u = \lambda^n u + \lambda^{n-1} A_1 u + \dots + A_n u = 0, \\ &L_\nu(\lambda)u = \lambda^{n_\nu} A_{\nu 0} u + \lambda^{n_\nu-1} A_{\nu 1} u + \dots + A_{\nu n_\nu} u = 0, \quad \nu = 1, \dots, m, \end{aligned} \tag{9}$$

where $u \in H_n = W_2^{\ell+2m}(G)$. Let us apply Theorem 2.3.2/1 to (9). By virtue of [22, page 258], the compact embeddings $W_2^{\ell+2m}(G) \subset W_2^{\ell+d(n-1)}(G) \subset \dots \subset W_2^\ell(G)$ hold, and by virtue of [22, page 350],

$$s_j(J, W_2^{\ell+dk}(G), W_2^{\ell+d(k-1)}(G)) \sim j^{-\frac{d}{r}}, \quad k = 1, \dots, n. \tag{10}$$

So, conditions 1 and 2 of Theorem 2.3.2/1 have been checked. Condition 3 of Theorem 2.3.2/1 is obvious. For operators $A_{\nu k}$, defined by equalities (8), we have $\mathrm{ord} A_{\nu k} = m_\nu - d(n_\nu - k)$. Since $\ell + d(n - n_\nu + k) - [m_\nu - d(n_\nu - k)] = \ell + dn - m_\nu = \ell + 2m - m_\nu \geq 1$, then, by virtue of [22, page 330], operators $A_{\nu k}$ act boundedly from $H_{n-n_\nu+k} = W_2^{\ell+d(n-n_\nu+k)}(G)$ into $H^\nu = W_2^{\ell+2m-m_\nu-\frac{1}{2}}(\partial G)$; i.e., condition 4 of Theorem 2.3.2/1 holds. By virtue of Theorem 3.2.2/2, condition 5 of Theorem 2.3.2/1 also holds. Condition 6 of Theorem 2.3.2/1 follows from Theorem 3.2.1/1. Indeed, from Theorem 3.2.1/1 it follows that all complex numbers $\lambda \in \ell_k$ and with large enough moduli are regular points of the pencil $\mathbb{L}(\lambda) = (L(\lambda), L_1(\lambda), \ldots, L_m(\lambda))$, which acts boundedly from $W_2^{\ell+2m}(G)$ into $W_2^\ell(G) \oplus W_2^{\ell+2m-m_\nu-\frac{1}{2}}(\partial G)$ and for a solution to the problem

$$L(\lambda)u = f, \qquad L_\nu(\lambda)u = f_\nu, \quad \nu = 1, \ldots, m,$$

the following estimate (for the above indicated λ)

$$\sum_{k=0}^{\ell+2m} |\lambda|^{d^{-1}(\ell+2m-k)} \|u\|_{k,2,G} \leq C\{\|f\|_{\ell,2,G} + |\lambda|^{d^{-1}\ell} \|f\|_{0,2,G}$$
$$+ \sum_{\nu=1}^{m} (|\lambda|^{d^{-1}(\ell+2m-m_\nu-\frac{1}{2})} \|f_\nu\|_{0,2,\partial G} + \|f_\nu\|_{\ell+2m-m_\nu-\frac{1}{2},2,\partial G})\}$$

holds. From here, in particular, it follows that for the pencil $\mathbb{L}(\lambda)$ the estimate

$$\|\mathbb{L}(\lambda)^{-1}\|_{B(W_2^\ell(G) \oplus W_2^{\ell+2m-m_\nu-\frac{1}{2}}(\partial G), W_2^{\ell+2m-d}(G))} \leq C|\lambda|^h, \quad \lambda \in \ell_k, \ |\lambda| \to \infty$$

is fulfilled for some $h \in \mathbb{R}$. By virtue of (10), condition 2 of Theorem 2.3.2/1 is satisfied for $p = \frac{d}{r}$ and angles between the neighboring rays less than $p\pi = \frac{d\pi}{r} = \frac{2m\pi}{nr}$. So, Theorem 2.3.2/1 is applicable to the system of pencils (9).

Corollary 2. *Let the conditions of Theorem 1 be satisfied. Then a system of root functions of problem* (1)–(2) *is n-fold complete in the space $\mathcal{H}_d$ (see Theorem* 3.2.2/2*).*

The next two theorems are particular cases of Theorem 1. Consider the problem

$$L(\lambda)u = \lambda^n u(x) + \sum_{k=1}^{n} \lambda^{n-k} \Big(\sum_{|\alpha|=dk} a_{k\alpha}(x) D^\alpha u(x) + B_k u|_x \Big) = 0, \ x \in G, \quad (11)$$

$$L_\nu u = \sum_{|\alpha|=m_\nu} b_{\nu\alpha}(x') D^\alpha u(x') + \sum_{p=0}^{m_\nu-1} K_{\nu p} \frac{\partial^p u}{\partial n^p}\Big|_{x'} = 0, \quad x' \in \partial G, \ \nu = 1, \ldots, m, \quad (12)$$

where n is a normal vector to the boundary ∂G at the point $x' \in \partial G$.

Theorem 3. *Let the following conditions be satisfied:*

(1) $n \geq 1$, $m \geq 1$, $n \leq 2m$, $m_\nu \geq 0$, $\max\{m_\nu\} - \min\{m_\nu\} \leq 2m - 1$;
(2) *the weight* $d = \frac{2m}{n}$ *of problem* (11)–(12) *is an integer;*
(3) $a_{k\alpha} \in C^\ell(\overline{G})$, *operators* B_k *from* $W_2^{dk}(G)$ *into* $L_2(G)$ *and from* $W_2^{\ell+dk}(G)$ *into* $W_2^\ell(G)$ *are compact, where* $\ell \in [\max\{0, m_\nu - (2m-1)\}, \min\{m_\nu\}]$;
(4) $b_{\nu\alpha} \in C^{\ell+2m-m_\nu}(\overline{G})$; *operators* $K_{\nu p}$ *from* $W_2^{m_\nu-p}(\partial G)$ *into* $L_2(\partial G)$ *and from* $W_2^{\ell+2m-p-\frac{1}{2}}(\partial G)$ *into* $W_2^{\ell+2m-m_\nu-\frac{1}{2}}(\partial G)$ *are compact;*
(5) *system* (12) *is normal;*
(6) *there exist rays* ℓ_k *with the angles between the neighboring rays less than* $\frac{2m\pi}{rn}$, *such that conditions I and II with* $n_\nu = 0$ *are fulfilled on the rays* ℓ_k.

Then the spectrum of problem (11)–(12) *is discrete and a system of root functions of problem* (11)–(12) *is n-fold complete in the space*

$$\mathcal{H} = \bigoplus_{k=0}^{n-1}\{u \mid u \in W_2^{\ell+d(n-k-1)}(G),\ \sum_{|\alpha|=m_\nu} b_{\nu\alpha}(x')D^\alpha u(x') + \sum_{p=0}^{m_\nu-1} K_{\nu p}\frac{\partial^p u}{\partial n^p}\big|_{x'} = 0,$$
$$x' \in \partial G,\ m_\nu \le \ell + d(n-k-1) - 1,\ \nu = 1,\dots,m\}.$$

Consider problem (11)–(12) in the case of $n = 1$:

$$L(\lambda)u = \lambda u(x) + \sum_{|\alpha|=2m} a_\alpha(x)D^\alpha u(x) + Bu|_x = 0, \quad x \in G, \tag{13}$$

$$L_\nu u = \sum_{|\alpha|=m_\nu} b_{\nu\alpha}(x')D^\alpha u(x') + \sum_{p=0}^{m_\nu-1} K_{\nu p}\frac{\partial^p u}{\partial n^p}\big|_{x'} = 0, \quad x' \in \partial G,\ \nu = 1,\dots,m, \tag{14}$$

where n is a normal vector to the boundary ∂G at the point $x' \in \partial G$.

Theorem 4. *Let the following conditions be satisfied:*

(1) $m_\nu \ge 0$, $\max\{m_\nu\} - \min\{m_\nu\} \le 2m - 1$;
(2) $a_\alpha \in C^\ell(\overline{G})$, *operator* B *from* $W_2^{2m}(G)$ *into* $L_2(G)$ *and from* $W_2^{\ell+2m}(G)$ *into* $W_2^\ell(G)$ *is compact, where* $\ell \in [\max\{0, m_\nu - (2m-1)\}, \min\{m_\nu\}]$;
(3) $b_{\nu\alpha} \in C^{\ell+2m-m_\nu}(\overline{G})$; *operators* $K_{\nu p}$ *from* $W_2^{m_\nu-p}(\partial G)$ *into* $L_2(\partial G)$ *and from* $W_2^{\ell+2m-p-\frac{1}{2}}(\partial G)$ *into* $W_2^{\ell+2m-m_\nu-\frac{1}{2}}(\partial G)$ *are compact;*
(4) *system* (14) *is normal;*
(5) *there exist rays* ℓ_k *with the angles between the neighboring rays less than* $\frac{2m\pi}{r}$, *such that for* $x \in \overline{G}$, $\sigma \in \mathbb{R}^r$, $\lambda \in \ell_k$, $|\sigma| + |\lambda| \ne 0$,

$$\lambda + \sum_{|\alpha|=2m} a_\alpha(x)\sigma^\alpha \ne 0;$$

(6) x' *is any point on* ∂G, *the vector* σ' *is tangent and* σ *is a normal vector to* ∂G *at the point* $x' \in \partial G$. *Consider the following ordinary differential problem:*

$$[\lambda + \sum_{|\alpha|=2m} a_\alpha(x')(\sigma' - i\sigma\frac{\mathrm{d}}{\mathrm{dy}})^\alpha]u(y) = 0, \quad y \ge 0,\ \lambda \in \ell_k, \tag{15}$$

$$\sum_{|\alpha|=m_k} b_{k\alpha}(x')(\sigma' - i\sigma\frac{\mathrm{d}}{\mathrm{dy}})^\alpha u(y)|_{y=0} = h_k, \quad k = 1,\dots,m; \tag{16}$$

it is required that problem (15)–(16) *has one and only one solution, including all its derivatives, tending to zero as* $y \to \infty$ *for any numbers* $h_k \in \mathbb{C}$.

Then the spectrum of problem (13)–(14) *is discrete, and a system of root functions of problem* (13)–(14) *is complete in the spaces* $W_2^\ell(G)$ *and* $W_2^{\ell+2m}(G, L_\nu u = 0, \nu = 1, \ldots, m)$.[12]

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