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SPECTRAL ASYMPTOTICS OF NONLINEAR MULTIPARAMETER STURM-LIOUVILLE PROBLEMS

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Abstract. We consider the following nonlinear multiparameter Sturm-Liouville problem:

$$u''(x) + \sum_{k=1}^{n} \mu_k f_k(u(x)) = \lambda g(u(x)), \ \ u(x) > 0, \ \ x \in I := (0,1),$$
$$u(0) = u(1) = 0,$$

where $\mu = (\mu_1, \mu_2, \cdots, \mu_n) \in R_+^n$ ($R_+ := (0, \infty)$) and $\lambda \in R_+$ are parameters. By using Ljusternik-Schnirelman theory on general level set due to Zeidler, the variational eigenvalues $\lambda = \lambda(\mu, \alpha)$ are obtained. Here, $\alpha > 0$ is a parameter of general level sets. We shall establish an asymptotic formula of $\lambda(\mu, \alpha)$ as $\mu_1 \to \infty$.

1. Introduction. We consider the following nonlinear multiparameter Sturm-Liouville problem:

$$u''(x) + \sum_{k=1}^{n} \mu_k f_k(u(x)) = \lambda g(u(x)), \ \ u(x) > 0, \ \ x \in I := (0,1), \qquad u(0) = u(1) = 0, \tag{1.1}$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in R_+^n$ ($R_+ := (0, \infty)$) and $\lambda \in R_+$ are parameters. The purpose of this paper is to establish an asymptotic formula of the variational eigenvalues $\lambda = \lambda(\mu, \alpha)$ obtained by Ljusternik-Schnirelman (LS) theory on general level set

$$N_{\mu,\alpha} := \left\{ u \in W_0^{1,2}(I) : \frac{1}{2} \int_0^1 u'(x)^2 \, dx - \sum_{k=1}^{n} \mu_k \int_0^1 \left(\int_0^u f_k(s) \, ds \right) dx = -\alpha \right\}, \tag{1.2}$$

where $\alpha > 0$ is a parameter.

To motivate and clarify our intention, let us briefly recall some known facts concerning multiparameter problems. Multiparameter linear spectral theory began with the oscillation theory of Lame's differential equation and results obtained there have

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been extended to more general linear multiparameter Sturm-Liouville problems. In particular, there are many works concerning two-parameter problems of the form

$$u''(x) + \mu r(x)u(x) = \lambda s(x)u(x), \quad x \in I, \tag{1.3}$$

where $\mu, \lambda \in R$. One of the main interests in this problem is to study asymptotic directions for the j-th eigencurve $\lambda = \lambda_j(\mu)$ as $\mu \to \infty$. We refer to Faierman ([2]) and Turyn ([4]) and the references cited therein.

However, it seems that few results have been given for nonlinear multiparameter problems. Recently, Shibata ([5], [6]) studied simple nonlinear two-parameter problems:

$$\begin{gathered} u''(x) + \mu u = \lambda g(u(x)), \quad x \in I := (0,1), \\ u(0) = u(1) = 0, \end{gathered} \tag{1.4}$$

where $\mu \in R_+$, $g(u) = u + |u|^{p-1}u$ $(p > 1)$ (or $|u|^{p-1}u$) and obtained asymptotic formulas of the j-th variational eigenvalues $\lambda = \lambda_j(\mu, \alpha)$ as $\mu \to \infty$. The key ingredient seems to be the homogeneity of the left-hand side of (1.4) and scaling technique. In this paper, we treat more general nonlinear multiparameter problems by using LS theory on a general level set due to Zeidler ([8]), which seems suitable for our problem (1.1) and will play important roles, and shall establish an asymptotic formula of $\lambda(\mu, \alpha)$. We would like to emphasize that this formula is an intrinsic feature of LS theory on general level sets. The technical difficulty of this paper is as follows. Unfortunately, the *scaling technique*, which was useful in (1.4), cannot be applied directly to (1.1) any more. Hence, we begin our analysis by the estimate of the maximum norm of the eigenfunction $u(\mu, \alpha, x)$ associated with $\lambda(\mu, \alpha)$. By this procedure, our problem will be reduced to the form, to which the scaling is applicable. Hence, the greater part of this paper is occupied in this task.

2. Main Result. We explain notations before stating our result. Let C_j, δ_j $(j \in N)$ denote various positive constants. Let $X := W_0^{1,2}(I)$ be the usual real Sobolev space. For $u \in X$, let

$$\begin{gathered} \|u\|_X^2 := \int_0^1 u'(x)^2\,dx, \quad \|u\|_d^d = \int_0^1 |u(x)|^d\,dx \quad (d \geq 1), \\ (u,v)_2 := \int_0^1 u(x)v(x)\,dx, \\ \|u\|_\infty := \max_{x \in I} |u(x)|, \quad F_k(u) := \int_0^u f_k(s)\,ds, \quad G(u) := \int_0^u g(s)\,ds, \\ \Phi_k(u) := \int_0^1 F_k(u(x))\,dx, \quad \Psi(u) := \int_0^1 G(u(x))\,dx. \end{gathered}$$

$\lambda = \lambda(\mu, \alpha)$ is called the variational eigenvalue of (1.1) when the associated eigenfunction $u(\mu, \alpha, x) \in N_{\mu,\alpha}$ satisfies the following conditions (2.1)–(2.2):

$$(\mu, \lambda(\mu, \alpha), u(\mu, \alpha, x)) \in R_+^n \times R_+ \times N_{\mu,\alpha} \ \ satisfies \ \ (1.1). \tag{2.1}$$

$$\Psi(u(\mu, \alpha, \cdot)) = \beta(\mu, \alpha) := \inf_{u \in N_{\mu,\alpha}} \Psi(u). \tag{2.2}$$

We note that $\lambda(\mu, \alpha)$ is obtained as a Lagrange multiplier and explicitly represented as follows:

$$\lambda(\mu, \alpha) = \frac{2\alpha + \sum_{k=1}^n \mu_k \{(f_k(u(\mu, \alpha, \cdot)), u(\mu, \alpha, \cdot))_2 - 2\Phi_k(u(\mu, \alpha, \cdot))\}}{(g(u(\mu, \alpha, \cdot)), u(\mu, \alpha, \cdot))_2}. \tag{2.3}$$

In fact, multiplying (1.1) by $u(\mu, \alpha, x)$, we obtain by integration by parts that

$$\begin{aligned} &-\|u(\mu, \alpha, \cdot)\|_X^2 + \sum_{k=1}^n \mu_k (f_k(u(\mu, \alpha, \cdot)), u(\mu, \alpha, \cdot))_2 \\ &= \lambda(\mu, \alpha)(g(u(\mu, \alpha, \cdot)), u(\mu, \alpha, \cdot))_2; \end{aligned} \tag{2.4}$$

this along with the fact that $u(\mu, \alpha, \cdot) \in N_{\mu,\alpha}$ implies (2.3).

We assume the following conditions (A.1)–(A.3):

(A.1) $f_k(u), g(u)$ are locally Lipschitz continuous, odd in u and $g(u) > 0$, $f_1(u) > 0$, $f_k(u) \geq 0$ $(k = 2, \dots, n)$ for $u > 0$. Furthermore, $\frac{f_1(u)}{g(u)}$ is strictly increasing for $u \geq C_1 > 0$ and $\frac{f_1(u)}{g(u)} \to \infty$ as $u \to \infty$.

(A.2)

$$\lim_{u \to 0} \frac{f_k(u)}{|u|^{p_k}} = K_k, \quad \lim_{u \to 0} \frac{g(u)}{|u|^q} = K_0, \tag{2.5}$$

where $K_0, K_1 > 0, K_k \geq 0$ $(k = 2, \dots, n), 1 \leq q < p_1 < p_2 \leq \cdots \leq p_n < q + 2$ are constants. Furthermore, there exists $1 < j \leq n$ such that for $2 \leq k \leq n$

$$\frac{f_j(u)}{f_1(u)} \to \infty \ \text{ as } \ u \to \infty, \quad \frac{f_k(u)}{f_j(u)} \leq C_1 \ \text{ for } \ u \geq C_1. \tag{2.6}$$

(A.3) For $1 \leq k \leq n$, there exist constants $C_2, 1 \leq q_1 < p_0 < q_1 + 2$ and $\delta_1 > 0$ such that for $u \geq 0$

$$(2 + \delta_1) F_k(u) \leq f_k(u) u \leq C_2 F_k(u), \tag{2.7}$$

$$C_2^{-1} u^{q_1 + 1} \leq g(u) u \leq C_2 G(u), \tag{2.8}$$

$$F_k(u) \leq C_2 u^{p_0 + 1} \ \text{ for } \ u \geq C_3. \tag{2.9}$$

We say that a sequence $\{(\mu, \alpha)\} \subset R_+^{n+1}$ satisfies condition (B.1) if the following conditions are satisfied:

(B.1) Suppose that $1 < j \leq n$ satisfies (2.6). Then for $2 \leq k \leq n$ $(k \neq j)$

$$\mu_1 \to \infty, \ \ C_2^{-1} \leq \frac{\mu_j}{\mu_1} \leq C_2, \ \ \frac{\mu_k}{\mu_1} \to 0, \ \ \frac{\alpha^{2q+2}}{\mu_1^{2q - p_1 + 1}} \to 0, \ \ \alpha^{p_1 - 1} \mu_1^2 \to \infty.$$

Now we state our main result.

Theorem 2.1. *Assume* (A.1)–(A.3). *Furthermore, assume that* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies* (B.1). *Then the following asymptotic formula holds*:

$$\lambda(\mu,\alpha) = C_4 K_0^{-1} K_1^{\frac{q+3}{p_1+3}} \alpha^{\frac{2(p_1-q)}{p_1+3}} \mu_1^{\frac{q+3}{p_1+3}} + o(\alpha^{\frac{2(p_1-q)}{p_1+3}} \mu_1^{\frac{q+3}{p_1+3}}), \tag{2.10}$$

where

$$C_4 = \left\{ \left(\tfrac{q+1}{p_1+1}\right)^{\frac{q+3}{2(p_1-q)}} \tfrac{(p_1+3)(q+1)(p_1-q)}{2(2q-p_1+3)} \sqrt{\tfrac{2}{\pi(q+1)}} \frac{\Gamma\left(\frac{p_1+3}{2(p_1-q)}\right)}{\Gamma\left(\frac{q+3}{2(p_1-q)}\right)} \right\}^{\frac{2(p_1-q)}{p_1+3}}. \tag{2.11}$$

Examples 2.2. The typical examples of f_k, g which satisfy (A.1)–(A.3) are as follows. Let $1 \le q < p_1 \le p_2 \le \cdots \le p_n < q+2$.

$$f_k(u) = |u|^{p_k-1}u, \;\; g(u) = |u|^{q-1}u, \tag{2.12}$$

$$f_1(u) = |u|^{p_1-1}u, f_k(u) = |u|^{p_k-1}u + |u|^{p_n-1}u, g(u) = |u|^{q-1}u, \tag{2.13}$$

$$\bar{f}_k(u) = (p_k+1)u^{p_k}\log(1+u) + \frac{u^{p_k+1}}{1+u}, \quad u \ge 0, \;\; g(u) = |u|^{q-1}u,$$

$$f_k(u) = \begin{cases} \bar{f}_k(u), & u \ge 0, \\ -\bar{f}_k(|u|), & u < 0. \end{cases} \tag{2.14}$$

Finally, let us introduce the application of Theorem 2.1 to the following two-parameter problems:

$$\begin{aligned} u''(x) + \mu(u(x)^p + u(x)^r) = \lambda(u(x)^q + u(x)^s), \quad & x \in I, \\ u(x) > 0, \quad & x \in I, \\ u(0) = u(1) = 0, & \end{aligned} \tag{2.15}$$

where $\mu, \lambda \in R_+$ and $1 \le q < s < p < r < q+2$.

Corollary 2.3. *Assume that* $\{\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies* $\dfrac{\alpha^{2q+2}}{\mu^{2q-p+1}} \to 0, \alpha^{p-1}\mu^2 \to \infty$. *Let* $\lambda = \lambda(\mu,\alpha)$ *be the variational eigenvalue of* (2.15). *Then the asymptotic formula* (2.10) *holds for* $p = p_1$, $K_0 = K_1 = 1$.

Obviously, Corollary 2.3 follows from Theorem 2.1 by putting $n = 2$, $p_1 = p$, $p_2 = r$, $j = 2$, $f_1(u) = |u|^{p-1}u$, $f_2(u) = |u|^{r-1}u$, $\mu_1 = \mu_2 = \mu$. The remainder of this paper is organized as follows. In Section 3, the existence of $\lambda(\mu,\alpha)$ is formulated. In Section 4, we will prepare some fundamental tools. Finally, Section 5 is devoted the proof of Theorem 2.1.

3. Existence of variational eigenvalues. We shall apply the existence result of Zeidler ([8, Proposition 6a]) to obtain the existence of $\lambda(\mu,\alpha)$. To this end, we have to check the conditions imposed on this proposition.

Lemma 3.1. *Let* $(\mu, \alpha) \in R_+^{n+1}$ *be fixed. Then* $N_{\mu,\alpha} \neq \emptyset$.

Proof. We put $m(t) := \frac{1}{2}\|t\sin\pi x\|_X^2 - \sum_{k=1}^n \mu_k \Phi_k(t\sin\pi x)$ for $t \geq 0$. Then we see from (2.5) that $m(0) = 0$ and $m(t) > 0$ for $0 < t \ll 1$. By (A.1) and (2.8), we have $f_1(t) \geq C_5 t^{q_1}$ for $t \geq t_0 \gg 1$, where $C_5 \gg 1$. Then for a constant $C_6 > \dfrac{1}{2}\|\sin\pi x\|_X^2$ we have

$$\begin{aligned}\Phi_1(t\sin\pi x) &\geq \int_{1/4}^{1/2}\left(\int_0^{t\sin\pi x} f_1(s)\,ds\right)dx \geq \int_{1/4}^{1/2}\left(\int_{t_0}^{t/\sqrt{2}} f_1(s)\,ds\right)dx \\ &\geq \frac{1}{4}\int_{t_0}^{t/\sqrt{2}} C_5 s^{q_1}\,ds \geq C_6 t^{q_1+1}.\end{aligned} \tag{3.1}$$

Hence, by (A.3) and (3.1), we obtain that as $t \to \infty$

$$m(t) \leq \frac{1}{2}\|\sin\pi x\|_X^2 t^2 - C_6 t^{q_1+1} \to -\infty; \tag{3.2}$$

this implies that there exists $t_1 > 0$ such that $m(t_1) = -\alpha$; that is, $t_1 \sin\pi x \in N_{\mu,\alpha}$. Thus the proof is complete.

Lemma 3.2. *Let* $(\mu, \alpha) \in R_+^{n+1}$ *be fixed. Then, on all bounded subsets* $K \subset N_{\mu,\alpha}$

$$\inf_{u\in K} |\|u\|_X^2 - \sum_{k=1}^n \mu_k (f_k(u), u)_2| > 0. \tag{3.3}$$

Furthermore, for all constants $C_7 > 0$*, the set*

$$M_{\mu,\alpha} := \{u \in N_{\mu,\alpha} : \Psi(u) < C_7\} \subset X \tag{3.4}$$

is bounded.

Proof. Let $K \subset N_{\mu,\alpha}$ be a bounded subset. Then by (2.7)

$$(f_k(u), u)_2 - 2\Phi_k(u) \geq \delta_1 \Phi_k(u) > 0. \tag{3.5}$$

Hence, by (3.5)

$$\begin{aligned}|\|u\|_X^2 - \sum_{k=1}^n \mu_k (f_k(u), u)_2| &= 2\alpha + \sum_{k=1}^n \mu_k ((f_k(u), u)_2 - 2\Phi_k(u)) \\ &\geq 2\alpha + \delta_1 \sum_{k=1}^n \mu_k \Phi_k(u) > 2\alpha.\end{aligned}$$

Thus we obtain (3.3). Next, let $C_7 > 0$ be an arbitrary constant. For $u \in M_{\mu,\alpha}$, let

$$I_{1,u} := \{x \in I : |u(x)| \geq C_3\}, \quad I_{2,u} := I \setminus I_{1,u}.$$

Then by (2.9)

$$\begin{aligned}\Phi_k(u) &= \int_{I_{1,u}} F_k(u(x))\,dx + \int_{I_{2,u}} F_k(u(x))\,dx \\ &\leq C_2 \int_{I_{1,u}} |u(x)|^{p_0+1}\,dx + F_k(C_3) \leq C_2 \|u\|_{p_0+1}^{p_0+1} + C_8,\end{aligned} \tag{3.6}$$

where $C_8 = \max_{1\leq k\leq n} F_k(C_3)$. Then by (2.8) and (3.6), we obtain for $u \in M_{\mu,\alpha}$

$$\begin{aligned}\|u\|_X^2 &= 2\sum_{k=1}^n \mu_k \Phi_k(u) - 2\alpha \leq 2\sum_{k=1}^n \mu_k (C_2\|u\|_{p_0+1}^{p_0+1} + C_8) \\ &\leq 2\sum_{k=1}^n \mu_k (C_2 \|u\|_\infty^{p_0-q_1} \|u\|_{q_1+1}^{q_1+1} + C_8) \leq 2\sum_{k=1}^n \mu_k (C_2^3 \|u\|_X^{p_0-q_1} \Psi(u) + C_8) \\ &= 2\sum_{k=1}^n \mu_k (C_2^3 C_7 \|u\|_X^{p_0-q_1} + C_8).\end{aligned} \tag{3.7}$$

Since $0 < p_0 - q_1 < 2$, (3.4) follows from (3.7). Thus the proof is complete. □

By Lemmas 3.1 and 3.2, we see that we can apply [6, Proposition 6a] to (1.1) and obtain the following lemma:

Lemma 3.3. *For a fixed $(\mu, \alpha) \in R_+^{n+1}$, there exists $u_{\mu,\alpha} \in N_{\mu,\alpha}$ which satisfies* (2.2).

By Lagrange multiplier theory and (2.3), we see that there exists $\lambda(\mu, \alpha) > 0$ such that $(\mu, \lambda(\mu, \alpha), u_{\mu,\alpha})$ satisfies the equation in (1.1). So we shall show the positivity of $u_{\mu,\alpha}$.

Lemma 3.4. *Let $(\mu, \alpha) \in R_+^{n+1}$ be fixed. Then there exists $(\mu, \lambda(\mu, \alpha), u(\mu, \alpha, x))$ $\in R_+^n \times R_+ \times N_{\mu,\alpha}$ which satisfies* (2.1)–(2.2).

Proof. We put $u(\mu, \alpha, x) := |u_{\mu,\alpha}(x)|$, where $u_{\mu,\alpha} \in N_{\mu,\alpha}$ is a function obtained in Lemma 3.3. Since $u_{\mu,\alpha} \in X \subset C(\bar{I})$ satisfies the equation in (1.1), we see that $u_{\mu,\alpha} \in C^2(\bar{I})$ by a standard regularity argument. Then clearly, $\|u(\mu, \alpha, \cdot)\|_X = \|u_{\mu,\alpha}\|_X$. So we find that $u(\mu, \alpha, x) \in N_{\mu,\alpha}$. Furthermore, since $G(u)$ is even in u, we obtain that $\Psi(u(\mu, \alpha, \cdot)) = \Psi(u_{\mu,\alpha})$. By (A.1), we see that $f_k(u)u, F_k(u), g(u)u$ are also even in u. Hence, we find by (2.3) that for the same Lagrange multiplier $\lambda(\mu, \alpha)$ as that of $u_{\mu,\alpha}, (\mu, \lambda(\mu, \alpha), u(\mu, \alpha, x)) \in R_+^n \times R_+ \times N_{\mu,\alpha}$ satisfies the equation in (1.1) and (2.2). Finally, if there exists $x_0 \in I$ such that $u(\mu, \alpha, x_0) = 0$, then $u'(\mu, \alpha, x_0) = 0$, since $u(\mu, \alpha, x) \geq 0$ in I. Then by the uniqueness theorem of ODE, we obtain that $u(\mu,\alpha,x) \equiv 0$ in I. However, this is impossible, since $u(\mu,\alpha,x) \in N_{\mu,\alpha}$ and $0 \notin N_{\mu,\alpha}$. Thus, we obtain that $u(\mu,\alpha,x) > 0$ in I. Thus the proof is complete. □

4. Asymptotic behavior of $\|u(\mu,\alpha,\cdot)\|_\infty$. In what follows, for a subsequence, we use the same notation as that of original sequence for convenience. Let $\sigma_{\mu,\alpha} := \max_{x\in I} u(\mu,\alpha,x)$. By Gidas, Ni and Nirenberg ([3]), we know that $u(\mu,\alpha,x)$ satisfies the following properties:

$$u(\mu,\alpha,x) = u(\mu,\alpha,1-x), \;\; u'(\mu,\alpha,x) \geq 0, \;\; x \in (0,\frac{1}{2}), \tag{4.1}$$

$$u'(\mu,\alpha,\frac{1}{2}) = 0, \;\; \sigma_{\mu,\alpha} = u(\mu,\alpha,\frac{1}{2}). \tag{4.2}$$

The goal of this section is to prove the following Proposition 4.1:

Proposition 4.1. *Assume that* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies* (B.1). *Then* $\sigma_{\mu,\alpha} \to 0$ *as* $\mu_1 \to \infty$.

To prove Proposition 4.1, we will prepare several lemmas.

Lemma 4.2. *For a fixed* $(\mu,\alpha) \in R_+^{n+1}$, *the following equality holds for* $x \in \bar{I}$:

$$\frac{1}{2}u'(\mu,\alpha,x)^2 + J(\mu,\alpha,u(\mu,\alpha,x)) = J(\mu,\alpha,\sigma_{\mu,\alpha}) = \frac{1}{2}u'(\mu,\alpha,0)^2 > 0, \tag{4.3}$$

where

$$J(\mu,\alpha,u) := \sum_{k=1}^{n} \mu_k F_k(u) - \lambda(\mu,\alpha)G(u). \tag{4.4}$$

Proof. Multiplying (1.1) by $u'(\mu,\alpha,x)$, we obtain for $x \in \bar{I}$

$$u''(\mu,\alpha,x)u'(\mu,\alpha,x) + \sum_{k=1}^{n} \mu_k f_k(u(\mu,\alpha,x))u'(\mu,\alpha,x) \\ - \lambda(\mu,\alpha)g(u(\mu,\alpha,x))u'(\mu,\alpha,x) = 0;$$

namely,

$$\frac{d}{dx}\left\{\frac{1}{2}u'(\mu,\alpha,x)^2 + J(\mu,\alpha,u(\mu,\alpha,x))\right\} \equiv 0;$$

this implies that for $x \in \bar{I}$

$$\frac{1}{2}u'(\mu,\alpha,x)^2 + J(\mu,\alpha,u(\mu,\alpha,x)) \equiv \text{constant}. \tag{4.5}$$

Now put $x = 0, \frac{1}{2}$ in (4.5). Then (4.3) follows from (4.1) and (4.2). □

The following is a direct consequence of (4.3):

Corollary 4.3. *Let* $(\mu,\alpha) \in R_+^{n+1}$ *be fixed. Then*

$$\frac{\lambda(\mu,\alpha)}{\mu_1} < \sum_{k=1}^n \frac{\mu_k F_k(\sigma_{\mu,\alpha})}{\mu_1 G(\sigma_{\mu,\alpha})}, \quad \frac{\lambda(\mu,\alpha)}{\mu_j} < \sum_{k=1}^n \frac{\mu_k F_k(\sigma_{\mu,\alpha})}{\mu_j G(\sigma_{\mu,\alpha})}. \tag{4.6}$$

Lemma 4.4. *Assume that* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies* (B.1). *Then*

$$\|u(\mu,\alpha,\cdot)\|_{q_1+1}^{q_1+1} \le C_9 \Psi(u(\mu,\alpha,\cdot)) \le C_{10}\left(\frac{\alpha}{\mu_1}\right)^{\frac{q+1}{p_1+1}}. \tag{4.7}$$

Proof. Let $C_{11} \gg 1$ be fixed. We put for $0 \le t \le C_{11}$

$$L_{\mu,\alpha}(t) := \frac{1}{2}\|t(\alpha/\mu_1)^{\frac{1}{p_1+1}} \sin\pi x\|_X^2 - \sum_{k=1}^n \mu_k \Phi_k(t(\alpha/\mu_1)^{\frac{1}{p_1+1}} \sin\pi x). \tag{4.8}$$

Then by (A.1) and (B.1), we see that for $\mu_1 \gg 1$

$$\begin{aligned} L_{\mu,\alpha}(t) &= \frac{1}{2}K_0^2 t^2 (\alpha/\mu_1)^{\frac{2}{p_1+1}} \|\sin\pi x\|_X^2 \\ &\quad - \sum_{k=1}^n \frac{1}{p_k+1} K_k^{p_k+1} \mu_k (\alpha/\mu_1)^{\frac{p_k+1}{p_1+1}} t^{p_k+1}(1+o(1))\|\sin\pi x\|_{p_k+1}^{p_k+1}. \end{aligned} \tag{4.9}$$

Clearly, $L_{\mu,\alpha}(0) = 0$. Furthermore, by (B.1)

$$\frac{1}{\mu_1} = o(1)\alpha^{\frac{p_1-1}{2}}.$$

Hence, we obtain

$$\left(\frac{\alpha}{\mu_1}\right)^{\frac{2}{p_1+1}} = o(1)\alpha;$$

this along with (4.9) and the fact that $\mu_1(\alpha/\mu_1)^{\frac{p_1+1}{p_1+1}} = \alpha$ implies that $L_{\mu,\alpha}(C_{11}) < -\alpha$ for $C_{11} \gg 1$. Hence, there exists $0 < t_0 < C_{11}$ such that $L_{\mu,\alpha}(t_0) = -\alpha$; that is, $t_0(\alpha/\mu_1)^{\frac{1}{p_1+1}} \sin\pi x \in N_{\mu,\alpha}$. Then by (2.2) and (2.8) we obtain

$$\begin{aligned} C_2^{-1}\|u(\mu,\alpha,\cdot)\|_{q_1+1}^{q_1+1} &\le (g(u(\mu,\alpha,\cdot)), u(\mu,\alpha,\cdot))_2 \le C_2\Psi(u(\mu,\alpha,\cdot)) \\ &\le C_2\Psi(t_0(\alpha/\mu_1)^{\frac{1}{p_1+1}} \sin\pi x) \\ &= \frac{1}{q+1} C_2 K_0(1+o(1)) t_0^{q+1} (\alpha/\mu_1)^{\frac{q+1}{p_1+1}} \|\sin\pi x\|_{q+1}^{q+1} \\ &\le \frac{1}{q+1} C_2 K_0(1+o(1)) C_{11}^{q+1} (\alpha/\mu_1)^{\frac{q+1}{p_1+1}} \|\sin\pi x\|_{q+1}^{q+1} \\ &\le C_{10}(\alpha/\mu_1)^{\frac{q+1}{p_1+1}}. \end{aligned} \tag{4.10}$$

Thus the proof is complete. □

Lemma 4.5. *Assume that* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies* (B.1). *Furthermore, suppose that* $\frac{\lambda(\mu,\alpha)}{\mu_j} \to \infty$. *Then there exists a subsequence of* $\{(\mu,\alpha)\}$ *such that*

$$\frac{\lambda(\mu,\alpha)}{\mu_j} = \frac{F_j(\sigma_{\mu,\alpha})}{G(\sigma_{\mu,\alpha})} + o\left(\frac{F_j(\sigma_{\mu,\alpha})}{G(\sigma_{\mu,\alpha})}\right). \tag{4.11}$$

Proof. It follows from (4.6) that

$$\frac{\lambda(\mu,\alpha)}{\mu_j} < \sum_{k=1}^{n} \frac{\mu_k F_k(\sigma_{\mu,\alpha})}{\mu_j G(\sigma_{\mu,\alpha})} = \frac{F_j(\sigma_{\mu,\alpha})}{G(\sigma_{\mu,\alpha})} + \frac{\mu_1}{\mu_j}\frac{F_1(\sigma_{\mu,\alpha})}{G(\sigma_{\mu,\alpha})} + \sum_{k\neq 1,j}^{n} \frac{\mu_k}{\mu_j}\frac{F_k(\sigma_{\mu,\alpha})}{G(\sigma_{\mu,\alpha})}. \tag{4.12}$$

If $\{\sigma_{\mu,\alpha}\}$ is bounded, then we obtain by (4.12) and (B.1) that $\{\lambda(\mu,\alpha)/\mu_j\}$ is also bounded. Hence, there exists a subsequence of $\{(\mu,\alpha)\}$ such that $\sigma_{\mu,\alpha} \to \infty$. Since $\frac{F_k(u)}{F_j(u)} \le \dfrac{C_1 C_2}{2+\delta_1}$ for $u \gg 1$ and $\frac{F_1(u)}{F_j(u)} \to 0$ as $u \to \infty$ by (2.6) and (2.7), we obtain by (4.12) and (B.1) that

$$\limsup_{\mu_j\to\infty} \frac{\lambda(\mu,\alpha)}{\mu_j}\frac{G(\sigma_{\mu,\alpha})}{F_j(\sigma_{\mu,\alpha})} \le 1. \tag{4.13}$$

Next, we assume that there exists a subsequence of $\{(\mu,\alpha)\}$ and a constant $\delta_2 > 0$ such that

$$\frac{\lambda(\mu,\alpha)}{\mu_j}\frac{G(\sigma_{\mu,\alpha})}{F_j(\sigma_{\mu,\alpha})} \le \frac{1}{1+\delta_2} \tag{4.14}$$

and derive a contradiction. Then by (A.1), (4.3) and (4.14) we obtain

$$\begin{aligned} \frac{1}{2}u'(\mu,\alpha,0)^2 &= J(\mu,\alpha,\sigma_{\mu,\alpha}) = G(\sigma_{\mu,\alpha})\left(\sum_{k=1}^{n}\mu_k\frac{F_k(\sigma_{\mu,\alpha})}{G(\sigma_{\mu,\alpha})} - \lambda(\mu,\alpha)\right) \\ &\ge G(\sigma_{\mu,\alpha})\left\{\mu_j\frac{F_j(\sigma_{\mu,\alpha})}{G(\sigma_{\mu,\alpha})} - \frac{1}{1+\delta_2}\mu_j\frac{F_j(\sigma_{\mu,\alpha})}{G(\sigma_{\mu,\alpha})}\right\} = \frac{\delta_2}{1+\delta_2}\mu_j F_j(\sigma_{\mu,\alpha}). \end{aligned} \tag{4.15}$$

By (1.1) we have

$$u''(\mu,\alpha,x) = \mu_j g(u(\mu,\alpha,x))\left(\frac{\lambda(\mu,\alpha)}{\mu_j} - \sum_{k=1}^{n}\frac{\mu_k}{\mu_j}\frac{f_k(u(\mu,\alpha,x))}{g(u(\mu,\alpha,x))}\right). \tag{4.16}$$

Let $0 < \epsilon \ll 1$ be fixed. Then since $\sigma_{\mu,\alpha} \to \infty$, there exists $x_\epsilon := x_{\mu,\alpha,\epsilon} \in (0,\frac{1}{2})$ such that $u(\mu,\alpha,x_\epsilon) = \epsilon$. Then for $0 \le x \le x_\epsilon$ we obtain by (B.1) and (4.16) that

$$u''(\mu,\alpha,x) \ge \mu_j g(u(\mu,\alpha,x))\left(\frac{\lambda(\mu,\alpha)}{\mu_j} - \sum_{k=1}^{n}\frac{\mu_k}{\mu_j}\max_{0\le t\le\epsilon}\frac{f_k(t)}{g(t)}\right) > 0. \tag{4.17}$$

Hence, by (4.1), (4.15) and (4.17) we obtain

$$u'(\mu,\alpha,x) \ge u'(\mu,\alpha,0) \ge C_{12}\sqrt{\mu_j F_j(\sigma_{\mu,\alpha})}, \tag{4.18}$$

where $C_{12} = \sqrt{2\delta_2/(1+\delta_2)}$. Then by (4.17) and (4.18)

$$\frac{\epsilon}{x_\epsilon} = \frac{u(\mu,\alpha,x)-u(\mu,\alpha,0)}{x_\epsilon} \ge u'(\mu,\alpha,0) \ge C_{12}\sqrt{\mu_j F_j(\sigma_{\mu,\alpha})}. \tag{4.19}$$

This implies that $x_\epsilon = x_{\mu,\alpha,\epsilon} \to 0$ as $\mu_j \to \infty$. In particular, we see that $u(\mu,\alpha,x) \ge \epsilon$ for $1/4 \le x \le 1/2$. This contradicts Lemma 4.4. Thus the proof is complete. □

Lemma 4.6. *Assume that* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies* (B.1). *Furthermore, suppose that* $\dfrac{\lambda(\mu,\alpha)}{\mu_j} \to \infty$. *Then* $J(\mu,\alpha,\sigma_{\mu,\alpha}) = o(1)$.

Proof. By (4.3), it is sufficient to show that $u'(\mu,\alpha,0)^2 \to 0$. By (4.12), we see that $\sigma_{\mu,\alpha} \to \infty$. Hence, for any $0 < \epsilon \ll 1$, there exists $x_\epsilon := x_{\mu,\alpha,\epsilon} \in (0,1/2)$ such that $u(\mu,\alpha,x_\epsilon) = \epsilon$. Then by (2.5) and (B.1), we see that for $0 \le x \le x_\epsilon$

$$\begin{aligned}\sum_{k=1}^n \frac{\mu_k}{\mu_j}\frac{f_k(u(x))}{g(u(x))} &= \sum_{k=1}^n \frac{\mu_k}{\mu_j}(1+o(1))u(x)^{p_k-q} \\ &\le \sum_{k=1}^n \frac{\mu_k}{\mu_j}(1+o(1))\epsilon^{p_k-q} \le C_{13}\sum_{k=1}^n \epsilon^{p_k-q}.\end{aligned} \tag{4.20}$$

Hence, by (4.17), we see that $u''(\mu,\alpha,x) > 0$ for $0 \le x \le x_\epsilon$. If there exists a subsequence of $\{(\mu,\alpha)\}$ and $0 < \tau < \frac{1}{2}$ such that $0 \le x_\epsilon = x_{\mu,\alpha,\epsilon} \le \tau$, then $u(\mu,\alpha,x) \ge \epsilon$ for $\tau \le x \le 1-\tau$. Hence,

$$\|u(\mu,\alpha,\cdot)\|_{q_1+1}^{q_1+1} \ge \epsilon^{q_1+1}(1-2\tau) > 0. \tag{4.21}$$

This contradicts (4.10). Hence, we find that $x_\epsilon := x_{\mu,\alpha,\epsilon} \to \frac{1}{2}$. Then

$$u'(\mu,\alpha,0) \le \frac{u(\mu,\alpha,x_\epsilon)-u(\mu,\alpha,0)}{x_\epsilon} = \frac{\epsilon}{x_\epsilon} \le 3\epsilon. \tag{4.22}$$

Since $\epsilon > 0$ is arbitrary, we obtain our assertion. □

Lemma 4.7. *Assume that* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies* (B.1). *Then there exists a constant* $C_{14} > 0$ *such that* $\frac{\lambda(\mu,\alpha)}{\mu_j} \le C_{14}$.

Proof. We assume that there exists a subsequence of $\{(\mu,\alpha)\}$ such that $\frac{\lambda(\mu,\alpha)}{\mu_j} \to \infty$ and derive a contradiction. Then by Lemma 4.6 and (4.3)

$$\begin{aligned}\frac{1}{2}u'(\mu,\alpha,x)^2 &= J(\mu,\alpha,\sigma_{\mu,\alpha}) + \lambda(\mu,\alpha)G(u(\mu,\alpha,x)) - \sum_{k=1}^n \mu_k F_k(u(\mu,\alpha,x)) \\ &\le o(1) + \lambda(\mu,\alpha)G(\sigma_{\mu,\alpha}).\end{aligned} \tag{4.23}$$

Then by (B.1), (2.3), (2.4), (2.7), (2.8) and Lemma 4.4

$$\begin{aligned}\lambda(\mu,\alpha) &= \frac{2\alpha + \sum_{k=1}^{n} \mu_k \left\{(f_k(u(\mu,\alpha,\cdot)), u(\mu,\alpha,\cdot))_2 - 2\Phi_k(u(\mu,\alpha,\cdot))\right\}}{(g(u(\mu,\alpha,\cdot), u(\mu,\alpha,\cdot))_2} \\ &\geq \frac{2\alpha + \delta_1 \sum_{k=1}^{n} \mu_k \Phi_k(u(u,\alpha,\cdot))}{(g(u(\mu,\alpha,\cdot)), u(\mu,\alpha,\cdot))_2} \\ &\geq \frac{2\alpha}{(g(u(\mu,\alpha,\cdot)), u(\mu,\alpha,\cdot))_2} \geq \frac{2\alpha}{C_2 \Psi(u(\mu,\alpha,\cdot))} \geq \frac{2\alpha}{C_{10}(\alpha/\mu_1)^{\frac{q+1}{p_1+1}}} \qquad (4.24) \\ &= C_{15} \alpha^{\frac{p_1-q}{p_1+1}} \mu_1^{\frac{q+1}{p_1+1}} = (\alpha^{p_1-1}\mu_1^2)^{\frac{p_1-q}{(p_1+1)(p_1-1)}} \mu_1^{\frac{q-1}{p_1-1}} \to \infty, \\ G(\sigma_{\mu,\alpha}) &\geq C_2^{-2} \sigma_{\mu,\alpha}^{q_1+1} \to \infty.\end{aligned}$$

Then by (4.23), (4.24) and Lemma 4.5 we obtain that

$$u'(\mu,\alpha,x)^2 \leq C_{16}\lambda(\mu,\alpha)G(\sigma_{\mu,\alpha}) \leq C_{17}\mu_j F_j(\sigma_{\mu,\alpha}). \qquad (4.25)$$

We choose $x_{\mu,\alpha} \in (0, 1/2)$ such that $u(\mu,\alpha,x_{\mu,\alpha}) = \frac{1}{2}\sigma_{\mu,\alpha}$. Then by Lemma 4.4

$$(\frac{1}{2} - x_{\mu,\alpha})(\frac{1}{2}\sigma_{\mu,\alpha})^{q_1+1} \leq \|u(\mu,\alpha,\cdot)\|_{q_1+1}^{q_1+1} \leq C_{10}\left(\frac{\alpha}{\mu_1}\right)^{\frac{q+1}{p_1+1}}. \qquad (4.26)$$

Then by (2.9), (4.25), (4.26) and the mean value theorem, there exists $y_{\mu,\alpha} \in [x_{\mu,\alpha}, 1/2]$ such that

$$\begin{aligned}\frac{\left(\frac{1}{2}\sigma_{\mu,\alpha}\right)^{q_1+2}}{C_{10}\left(\frac{\alpha}{\mu_1}\right)^{\frac{q+1}{p_1+1}}} &\leq \frac{\frac{1}{2}\sigma_{\mu,\alpha}}{\frac{1}{2} - x_{\mu,\alpha}} = \frac{u(\mu,\alpha,\frac{1}{2}) - u(\mu,\alpha,x_{\mu,\alpha})}{\frac{1}{2} - x_{\mu,\alpha}} = u'(\mu,\alpha,y_{\mu,\alpha}) \qquad (4.27) \\ &\leq \sqrt{C_{18}\mu_j F_j(\sigma_{\mu,\alpha})} \leq C_{19}\mu_j^{\frac{1}{2}} \sigma_{\mu,\alpha}^{\frac{p_0+1}{2}};\end{aligned}$$

this implies that

$$\sigma_{\mu,\alpha}^{q_1+2} \mu_1^{\frac{q+1}{p_1+1}} \leq C_{20}\mu_j^{\frac{1}{2}} \alpha^{\frac{q+1}{p_1+1}} \sigma_{\mu,\alpha}^{\frac{p_0+1}{2}} \leq C_{21}\sigma_{\mu,\alpha}^{\frac{p_0+1}{2}} \mu_1^{\frac{1}{2}} \alpha^{\frac{q+1}{p_1+1}}. \qquad (4.28)$$

Hence, by (B.1)

$$\sigma_{\mu,\alpha}^{\frac{2q_1+3-p_0}{2}} \leq C_{21}\alpha^{\frac{q+1}{p_1+1}} \mu_1^{\frac{p_1-2q-1}{2(p_1+1)}} \leq C_{21}\left(\frac{\alpha^{2(q+1)}}{\mu_1^{2q+1-p_1}}\right)^{\frac{1}{2(p_1+1)}} \leq C_{21}. \qquad (4.29)$$

This is a contradiction, since $\sigma_{\mu,\alpha} \to \infty$ if $\frac{\lambda(\mu,\alpha)}{\mu_j} \to \infty$ by (4.6) and $2q_1 + 3 > p_0$. Thus the proof is complete. □

Lemma 4.8. *Assume that* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies* (B.1). *Then* $\dfrac{\lambda(\mu,\alpha)}{\mu_j} \to 0$.

Proof. We assume that there exists a subsequence of $\{(\mu,\alpha)\}$ and a constant $\delta_3 > 0$ such that $\frac{\lambda(\mu,\alpha)}{\mu_j} \ge \delta_3$ and derive a contradiction. If $\sigma_{\mu,\alpha} \to 0$, then by (2.5), (4.3) and (B.1)

$$\frac{\lambda(\mu,\alpha)}{\mu_j} < \sum_{k=1}^n \frac{\mu_k}{\mu_j}\frac{F_k(\sigma_{\mu,\alpha})}{G(\sigma_{\mu,\alpha})} = (1+o(1))\sum_{k=1}^n \frac{\mu_k}{\mu_j}\sigma_{\mu,\alpha}^{p_k-q} \to 0. \tag{4.30}$$

This contradicts the assumption. Hence, there exists a constant $\delta_4 > 0$ such that $\sigma_{\mu,\alpha} \ge \delta_4$. By (4.3),

$$\begin{aligned}\frac{1}{2}u'(\mu,\alpha,x)^2 &= J(\mu,\alpha,\sigma_{\mu,\alpha}) - J(\mu,\alpha,u(\mu,\alpha,x))\\ &\ge \lambda(\mu,\alpha)G(u(\mu,\alpha,x)) - \sum_{k=1}^n \mu_k F_k(u(\mu,\alpha,x))\\ &\ge \delta_3\mu_j G(u(\mu,\alpha,x)) - \sum_{k=1}^n \mu_k F_k(u(\mu,\alpha,x)).\end{aligned} \tag{4.31}$$

Let $0 < \epsilon \ll \delta_4$ be fixed. Furthermore, let $x_\epsilon := x_{\mu,\alpha,\epsilon} \in (0,1/2)$ satisfy $u(\mu,\alpha,x_\epsilon) = \epsilon$. Then by (B.1), (2.5) and (4.31)

$$\begin{aligned}\frac{1}{2}u'(\mu,\alpha,x_\epsilon)^2 &\ge \delta_3\mu_j\Big(\frac{1}{q+1}+o(1)\Big)\epsilon^{q+1} - \mu_j\sum_{k=1}^n\frac{\mu_k}{\mu_j}\left(\frac{1}{p_k+1}+o(1)\right)\epsilon^{p_k+1}\\ &\ge C_{23}\epsilon^{q+1}\mu_j \ge C_{23}C_2^{-1}\epsilon^{q+1}\mu_1.\end{aligned} \tag{4.32}$$

Since $u(\mu,\alpha,x) \in N_{\mu,\alpha}$, we obtain by (2.4), (2.7) and (4.7) that

$$\begin{aligned}&\delta_1\sum_{k=1}^n \mu_k\Phi_k(u(\mu,\alpha,\cdot)) \le \sum_{k=1}^n \mu_k\left\{f_k(u(\mu,\alpha,\cdot),u(\mu,\alpha,\cdot))_2 - 2\Phi_k(u(\mu,\alpha,\cdot))\right\}\\ &= \lambda(\mu,\alpha)(g(u(\mu,\alpha,\cdot)),u(\mu,\alpha,\cdot))_2 - 2\alpha\\ &\le \lambda(\mu,\alpha)(g(u(\mu,\alpha,\cdot)),u(\mu,\alpha,\cdot))_2\\ &\le C_2\lambda(\mu,\alpha)\int_0^1 G(u(\mu,\alpha,x))\,dx = C_2\lambda(\mu,\alpha)\Psi(u(\mu,\alpha,\cdot))\\ &\le C_2C_{10}\lambda(\mu,\alpha)\left(\frac{\alpha}{\mu_1}\right)^{\frac{q+1}{p_1+1}}.\end{aligned} \tag{4.33}$$

Furthermore, by (2.7), (2.8) and (4.7)

$$\int_{x_\epsilon}^{1/2} g(u(\mu,\alpha,x))\,dx \le C_2\int_{x_\epsilon}^{1/2}\frac{G(u(\mu,\alpha,x))}{u(\mu,\alpha,x)}\,dx \tag{4.34}$$

$$\le C_2\epsilon^{-1}\int_{x_\epsilon}^{1/2} G(u(\mu,\alpha,x))\,dx \le C_2C_{10}\epsilon^{-1}C_9^{-1}\left(\frac{\alpha}{\mu_1}\right)^{\frac{q+1}{p_1+1}},$$

$$\int_{x_\epsilon}^{1/2} f_k(u(\mu,\alpha,x))\,dx \le C_2 \int_{x_\epsilon}^{1/2} \frac{F_k(u(\mu,\alpha,x))}{u(\mu,\alpha,x)}\,dx \tag{4.35}$$
$$\le C_2\epsilon^{-1}\int_{x_\epsilon}^{1/2} F_k(u(\mu,\alpha,x))\,dx \le C_2\epsilon^{-1}\Phi_k(u(\mu,\alpha,\cdot)).$$

Hence, by using (1.1), (2.8), (4.7), (4.33)–(4.35) and Lemma 4.7 we obtain

$$\begin{aligned}
&u'(\mu,\alpha,x_\epsilon) = \left|\int_{x_\epsilon}^{1/2} u''(\mu,\alpha,x)\,dx\right| \\
&\le \lambda(\mu,\alpha)\int_{x_\epsilon}^{1/2} g(u(\mu,\alpha,x))\,dx + \sum_{k=1}^{n}\mu_k\int_{x_\epsilon}^{1/2} f_k(u(\mu,\alpha,x))\,dx \\
&\le C_{24}\epsilon^{-1}\lambda(\mu,\alpha)\left(\frac{\alpha}{\mu_1}\right)^{\frac{q+1}{p_1+1}} + C_2\epsilon^{-1}\sum_{k=1}^{n}\mu_k\Phi_k(u(\mu,\alpha,\cdot)) \\
&\le C_{24}\epsilon^{-1}\lambda(\mu,\alpha)\left(\frac{\alpha}{\mu_1}\right)^{\frac{q+1}{p_1+1}} + C_2\epsilon^{-1}\delta_1^{-1}\lambda(\mu,\alpha)(g(u(\mu,\alpha,\cdot)),u(\mu,\alpha,\cdot))_2 \\
&\le C_{14}C_{24}\epsilon^{-1}\left(\frac{\alpha}{\mu_1}\right)^{\frac{q+1}{p_1+1}}\mu_j + C_2^2\epsilon^{-1}\delta_1^{-1}\lambda(\mu,\alpha)\int_0^1 G(u(\mu,\alpha,x))\,dx \\
&= C_{14}C_{24}\epsilon^{-1}\left(\frac{\alpha}{\mu_1}\right)^{\frac{q+1}{p_1+1}}\mu_j + C_2^2\epsilon^{-1}\delta_1^{-1}\lambda(\mu,\alpha)\Psi(u(\mu,\alpha,\cdot)) \\
&= C_{14}C_{24}\epsilon^{-1}\left(\frac{\alpha}{\mu_1}\right)^{\frac{q+1}{p_1+1}}\mu_j + C_2^2C_9^{-1}C_{10}\epsilon^{-1}\delta_1^{-1}\lambda(\mu,\alpha)\left(\frac{\alpha}{\mu_1}\right)^{\frac{q+1}{p_1+1}} \qquad (4.36)\\
&\le C_{25}\epsilon^{-1}\left(\frac{\alpha}{\mu_1}\right)^{\frac{q+1}{p_1+1}}\mu_j \le C_{26}\epsilon^{-1}\left(\frac{\alpha}{\mu_1}\right)^{\frac{q+1}{p_1+1}}\mu_1.
\end{aligned}$$

Therefore, by (4.32) and (4.36) we obtain

$$\sqrt{2C_{23}C_2^{-1}\epsilon^{q+1}\mu_1} \le C_{26}\epsilon^{-1}\mu_1^{\frac{p_1-q}{p_1+1}}\alpha^{\frac{q+1}{p_1+1}}; \tag{4.37}$$

this implies that

$$\mu_1^{\frac{2q+1-p_1}{2(p_1+1)}} \le \frac{C_{26}}{\sqrt{2C_{23}C_2^{-1}}}\epsilon^{-\frac{q+3}{2}}\alpha^{\frac{q+1}{p_1+1}}. \tag{4.38}$$

Since $\epsilon > 0$ is a fixed constant, this contradicts (B.1). Thus the proof is complete. □

Lemma 4.9. *Assume that* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies* (B.1). *Then* $\sigma_{\mu,\alpha} \le C_{27}$.

Proof. We assume that there exists a subsequence of $\{(\mu,\alpha)\}$ such that $\sigma_{\mu,\alpha} \to \infty$ and derive a contradiction. Let $\epsilon > 0$ be fixed. Furthermore, let $x_\epsilon := x_{\mu,\alpha,\epsilon} \in (0, 1/2)$ satisfy $u(\mu,\alpha,x_\epsilon) = \epsilon$. By (A.1) and (2.7) and (2.8), we see that

$$\begin{aligned} F_k(\sigma_{\mu,\alpha}) &\ge F_k(\epsilon), \\ F_1(\sigma_{\mu,\alpha}) &\ge C_2^{-1}\sigma_{\mu,\alpha} f_1(\sigma_{\mu,\alpha}) = C_2^{-1}\sigma_{\mu,\alpha} g(\sigma_{\mu,\alpha}) \frac{f_1(\sigma_{\mu,\alpha})}{g(\sigma_{\mu,\alpha})} \\ &\ge C_2^{-2}\sigma_{\mu,\alpha}^{q_1+1} \frac{f_1(\sigma_{\mu,\alpha})}{g(\sigma_{\mu,\alpha})} \to \infty. \end{aligned} \tag{4.39}$$

Furthermore, we see from (A.1) that there exists a constant $C_{28} > 0$ such that $g(u) \le \epsilon f_1(u)$ for $u \ge C_{28}$. Then

$$\begin{aligned} G(\sigma_{\mu,\alpha}) &= \int_0^{\sigma_{\mu,\alpha}} g(s)\,ds = \int_0^{C_{28}} g(s)\,ds + \int_{C_{28}}^{\sigma_{\mu,\alpha}} g(s)\,ds \\ &= C_{29} + \epsilon \int_{C_{28}}^{\sigma_{\mu,\alpha}} f_1(s)\,ds \le C_{29} + \epsilon F_1(\sigma_{\mu,\alpha}); \end{aligned} \tag{4.40}$$

this along with (4.3), (4.39) and Lemma 4.8 implies that

$$\begin{aligned} \frac{1}{2}u'(\mu,\alpha,x_\epsilon)^2 &= \sum_{k=1}^n \mu_k(F_k(\sigma_{\mu,\alpha}) - F_k(\epsilon)) - \lambda(\mu,\alpha)(G(\sigma_{\mu,\alpha}) - G(\epsilon)) \\ &\ge \mu_1(F_1(\sigma_{\mu,\alpha}) - F_1(\epsilon)) - o(\mu_1)(\epsilon F_1(\sigma_{\mu,\alpha}) + C_{29}) \\ &\ge C_{30}\mu_1 F_1(\sigma_{\mu,\alpha}) \ge C_{30}\mu_1. \end{aligned} \tag{4.41}$$

Thus, we obtain the same inequality as (4.32). Then by repeating the same arguments as those used in the proof of Lemma 4.8, we can derive a contradiction. Thus the proof is complete. $\square$

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. We assume that there exists a subsequence of $\{(\mu,\alpha)\}$ and a constant $\epsilon > 0$ such that $\sigma_{\mu,\alpha} \ge 2\epsilon$ and derive a contradiction. Let $x_\epsilon := x_{\mu,\alpha,\epsilon} \in (0, 1/2)$ satisfy $u(\mu,\alpha,x_\epsilon) = \epsilon$. Then by Lemma 4.9, we have $G(\sigma_{\mu,\alpha}) \le G(C_{27}) \le C_{31}$. Furthermore, by (A.1), we have $F_1(\epsilon) < F_1(2\epsilon)$. By (A.1), (4.3) and Lemma 4.8

$$\begin{aligned} \frac{1}{2}u'(\mu,\alpha,x_\epsilon)^2 &= \sum_{k=1}^n \mu_k(F_k(\sigma_{\mu,\alpha}) - F_k(\epsilon)) - \lambda(\mu,\alpha)(G(\sigma_{\mu,\alpha}) - G(\epsilon)) \\ &\ge \mu_1(F_1(2\epsilon) - F_1(\epsilon)) - o(\mu_1)C_{31} \ge C_{32}\mu_1. \end{aligned} \tag{4.42}$$

Thus, we obtain the same inequality as (4.32). Then by repeating the same arguments as those used in the proof of Lemma 4.8, we can derive a contradiction. Thus the proof is complete. $\square$

5. Proof of Theorem 2.1. At first, we consider the case $K_0 = K_1 = \cdots = K_n = 1$. We explain the idea of the proof of Theorem 2.1 briefly. Let

$$\xi := \xi_{\mu,\alpha} = (\lambda(\mu,\alpha)/\mu_1)^{\frac{1}{p_1-q}}, \;\; v(\mu,\alpha,x) := \xi^{-1}u(\mu,\alpha,x),$$
$$\nu := \nu_{\mu,\alpha} = \mu_1^{\frac{1-q}{2(p_1-q)}}\lambda(\mu,\alpha)^{\frac{p_1-1}{2(p_1-q)}}.$$

Furthermore, let

$$h_0(u) := g(u) - u^q, \qquad h_k(u) := f_k(u) - u^{p_k},$$
$$H_0(u) := \int_0^u h_0(s)\,ds, \quad H_k(u) := \int_0^u h_k(s)\,ds.$$

Then by (1.1), v satisfies the following equation:

$$\begin{aligned}
&v''(\mu,\alpha,x) + \nu^2(v(\mu,\alpha,x)^{p_1} - v(\mu,\alpha,x)^q) + \sum_{k=2}^n \mu_k\xi^{p_k-1}v(\mu,\alpha,x)^{p_k}\\
&+\sum_{k=1}^n \mu_k\xi^{-1}h_k(\xi v(\mu,\alpha,x)) - \lambda(\mu,\alpha)\xi^{-1}h_0(\xi v(\mu,\alpha,x)) = 0, \;\; x\in I,\\
&v(\mu,\alpha,x) > 0, \;\; x \in I,\\
&v(\mu,\alpha,0) = v(\mu,\alpha,1) = 0.
\end{aligned} \tag{5.1}$$

Furthermore, put $t := \nu(x - \dfrac{1}{2}), w(\mu,\alpha,t) = v(\mu,\alpha,x)$. Then by (5.1), $w(\mu,\alpha,t)$ satisfies

$$\begin{aligned}
&w''(\mu,\alpha,t) + w(\mu,\alpha,t)^{p_1} - w(\mu,\alpha,t)^q + \sum_{k=2}^n \nu^{-2}\mu_k\xi^{p_k-1}w(\mu,\alpha,t)^{p_k}\\
&+\sum_{k=1}^n \nu^{-2}\mu_k\xi^{-1}h_k(\xi w(\mu,\alpha,t)) - \lambda(\mu,\alpha)\nu^{-2}\xi^{-1}h_0(\xi w(\mu,\alpha,t))\\
&= 0, \;\; t \in I_{\mu,\alpha} := (-\frac{1}{2}\nu, \frac{1}{2}\nu),\\
&w(\mu,\alpha,t) > 0, \;\; t\in I_{\mu,\alpha},\\
&w(\mu,\alpha,\pm\frac{1}{2}\nu) = 0.
\end{aligned} \tag{5.2}$$

In connection with (5.2), the following nonlinear scalar field equation plays important roles:

$$\begin{aligned} w''(t) + w(t)^{p_1} - w(t)^q = 0,\ \ t \in R, \\ w(t) > 0,\ \ t \in I, \\ \lim_{|t|\to\infty} w(t) = 0. \end{aligned} \tag{5.3}$$

We see by Berestycki and Lions ([1]) that there uniquely exists a solution w of (5.3), which is called the *ground state*, and satisfies the following properties:

$$w(0) = \zeta := \left(\frac{p_1+1}{q+1}\right)^{\frac{1}{p_1-q}},\ \ w(t) = w(-t),\ \ t \in R,\ \ w'(t) \le 0,\ \ t \ge 0, \tag{5.4}$$

$$\frac{1}{2}w'(t)^2 + \frac{1}{p_1+1}w(t)^{p_1+1} - \frac{1}{q+1}w(t)^{q+1} = 0,\ \ t \in R, \tag{5.5}$$

$$w(t) \le C_{32}e^{-\delta_5|t|},\ \ t \in R. \tag{5.6}$$

Furthermore, if $w(t) > 0$ satisfies the equation in (5.3) with $w(0) = \zeta$, then w is the ground state of (5.3).

We shall show that $w(\mu, \alpha, t) \to w(t)$ in $L^{q+1}(R)$. To this end, we prepare several lemmas. Let $\zeta_{\mu,\alpha} := \max_{t\in I_{\mu,\alpha}} w(\mu, \alpha, t)$.

Lemma 5.1. *Assume that* $\{(\mu, \alpha)\} \subset R_+^{n+1}$ *satisfies* (B.1). *Then* $\zeta_{\mu,\alpha} \le C_{33}$.

Proof. Since $\zeta_{\mu,\alpha} = \xi^{-1}\sigma_{\mu,\alpha}$, it is sufficient to show that $\sigma_{\mu,\alpha} \le C_{33}\xi$. Let $z(\mu, \alpha, x) := \sigma_{\mu,\alpha}^{-1}u(\mu, \alpha, x)$. Then by (2.3), (2.5), (2.7) and Proposition 4.1 we obtain

$$\begin{aligned} \lambda(\mu, \alpha) &= \frac{2\alpha + \sum_{k=1}^n \mu_k \left\{(f_k(u(\mu,\alpha,\cdot)), u(\mu,\alpha,\cdot))_2 - 2\Phi_k(u(\mu,\alpha,\cdot))\right\}}{(g(u(\mu,\alpha,\cdot)), u(\mu,\alpha,\cdot))_2} \\ &\ge \frac{2\alpha + \delta_1\sum_{k=1}^n \mu_k\Phi_k(u(\mu,\alpha,\cdot))}{(g(u(\mu,\alpha,\cdot)), u(\mu,\alpha,\cdot))_2} \\ &\ge \frac{2\alpha + \sum_{k=1}^n \dfrac{\delta_1}{p_k+1}\mu_k\sigma_{\mu,\alpha}^{p_k+1}(1+o(1))\|z(\mu,\alpha,\cdot)\|_{p_k+1}^{p_k+1}}{\sigma_{\mu,\alpha}^{q+1}(1+o(1))\|z(\mu,\alpha,\cdot)\|_{q+1}^{q+1}}. \end{aligned} \tag{5.7}$$

Integrate (4.3) over I. Then we obtain by (2.5) and Proposition 4.1 that

$$\begin{aligned} &\frac{1}{2}\|u(\mu,\alpha,\cdot)\|_X^2 + \sum_{k=1}^n \frac{1}{p_k+1}\mu_k\sigma_{\mu,\alpha}^{p_k+1}(1+o(1))\|z(\mu,\alpha,\cdot)\|_{p_k+1}^{p_k+1} \\ &- \frac{1}{q+1}\lambda(\mu,\alpha)\sigma_{\mu,\alpha}^{q+1}(1+o(1))\|z(\mu,\alpha,\cdot)\|_{q+1}^{q+1} = J(\mu,\alpha,\sigma_{\mu,\alpha}) \\ &= \sum_{k=1}^n \frac{1}{p_k+1}\mu_k(1+o(1))\sigma_{\mu,\alpha}^{p_k+1} - \frac{1}{q+1}\lambda(\mu,\alpha)(1+o(1))\sigma_{\mu,\alpha}^{q+1}; \end{aligned}$$

this along with (2.4) implies that

$$\begin{aligned} &\sum_{k=1}^{n} \mu_k \sigma_{\mu,\alpha}^{p_k+1} \left\{ \frac{p_k+3}{2(p_k+1)}(1+o(1))\|z(\mu,\alpha,\cdot)\|_{p_k+1}^{p_k+1} - \frac{1}{p_k+1}(1+o(1)) \right\} \\ &= \lambda(\mu,\alpha)\sigma_{\mu,\alpha}^{q+1} \left\{ \frac{q+3}{2(q+1)}(1+o(1))\|z(\mu,\alpha,\cdot)\|_{q+1}^{q+1} - \frac{1}{q+1}(1+o(1)) \right\}. \end{aligned} \tag{5.8}$$

There are two cases to consider:

Case 1: Assume that $\|z(\mu,\alpha,\cdot)\|_{q+1}^{q+1} \to 0$. Then clearly, $\|z(\mu,\alpha,\cdot)\|_{p_k+1}^{p_k+1} \to 0$. Then we obtain by (B.1), Proposition 4.1 and (5.8) that

$$\left(\frac{\xi}{\sigma_{\mu,\alpha}}\right)^{p_1-q} = \frac{\lambda(\mu,\alpha)}{\mu_1 \sigma_{\mu,\alpha}^{p_1-q}} \to \frac{q+1}{p_1+1}. \tag{5.9}$$

This implies our conclusion.

Case 2: Assume that there exists a subsequence of $\{(\mu,\alpha)\}$ such that

$$\|z(\mu,\alpha,\cdot)\|_{q+1}^{q+1} > \delta_6 > 0.$$

Then there exists $x_0 \in I$ such that $z(\mu,\alpha,x_0) \ge \delta_7 > 0$ for $\mu_1 \gg 1$. Hence, $\|z(\mu,\alpha,\cdot)\|_{p_k+1}^{p_k+1} \ge \delta_8$ follows. Then by (5.7) we obtain

$$\lambda(\mu,\alpha) \ge \frac{\dfrac{\delta_1\delta_8}{p_1+1}\mu_1\sigma_{\mu,\alpha}^{p_1+1}(1+o(1))}{\sigma_{\mu,\alpha}^{q+1}(1+o(1))} \ge C_{34}\mu_1\sigma_{\mu,\alpha}^{p_1-q}. \tag{5.10}$$

Now our conclusion follows from (5.10). Thus the proof is complete. □

Lemma 5.2. *Assume that* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies* (B.1). *Then* $\nu := \nu_{\mu,\alpha} \to \infty$.

Proof. By (4.24), (5.10) and (B.1)

$$\nu^2 = \lambda(\mu,\alpha)^{\frac{p_1-1}{p_1-q}} \mu_1^{\frac{1-q}{p_1-q}} \ge \left(C_{15}\alpha^{\frac{p_1-q}{p_1+1}}\mu_1^{\frac{q+1}{p_1+1}}\right)^{\frac{p_1-1}{p_1-q}} \mu_1^{\frac{1-q}{p_1-q}} = C_{35}\alpha^{\frac{p_1-1}{p_1+1}}\mu_1^{\frac{2}{p_1+1}} \to \infty. \tag{5.11}$$

Thus the proof is complete. □

Lemma 5.3. *Assume that* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies* (B.1). *Then the following equality holds*:

$$\frac{1}{2}w'(\mu,\alpha,t)^2 + R(\mu,\alpha,w(\mu,\alpha,t)) = R(\mu,\alpha,\zeta_{\mu,\alpha}), \tag{5.12}$$

where

$$R(\mu,\alpha,w) := \frac{1}{p_1+1}w^{p_1+1} - \frac{1}{q+1}w^{q+1} + \frac{1}{p_k+1}\sum_{k=2}^{n}\nu^{-2}\mu_k\xi^{p_k-1}w^{p_k+1} + \sum_{k=1}^{n}\mu_k\nu^{-2}\xi^{-2}H_k(\xi w) - \lambda(\mu,\alpha)\nu^{-2}\xi^{-2}H_0(\xi w). \tag{5.13}$$

By using (5.2) and the same arguments as those used in the proof of Lemma 4.2, we can obtain Lemma 5.3 easily. Hence, we omit the proof.

We note that if $\{(\mu,\alpha)\} \subset R_+^{n+1}$ satisfies (B.1), then by (B.1) and Lemma 4.8

$$\lambda(\mu,\alpha)\nu^{-2}\xi^{q-1} = 1, \tag{5.14}$$

$$\nu^{-2}\mu_k\xi^{p_k-1} = \frac{\mu_k}{\mu_1}\xi^{p_k-p_1} \to 0 \quad (2 \le k \le n). \tag{5.15}$$

Lemma 5.4. *Assume that* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies* (B.1). *Then* $\zeta_{\mu,\alpha} \to \zeta$.

Proof. By (B.1), (4.6) and Proposition 4.1 we have

$$\frac{\lambda(\mu,\alpha)}{\mu_1} < \sum_{k=1}^{n}\frac{\mu_k}{\mu_1}\frac{F_k(\sigma_{\mu,\alpha})}{G(\sigma_{\mu,\alpha})} = \frac{q+1}{p_1+1}(1+o(1))\sigma_{\mu,\alpha}^{p_1-q} + \sum_{k=2}^{n}\frac{\mu_k}{\mu_1}(1+o(1))\sigma_{\mu,\alpha}^{p_k-q};$$

this implies that

$$\zeta^{p_1-q} \le \liminf \zeta_{\mu,\alpha}^{p_1-q}. \tag{5.16}$$

We assume that there exists a subsequence of $\{(\mu,\alpha)\}$ and a constant $\delta_8 > 0$ such that

$$\zeta_{\mu,\alpha}^{p_1-q} > \zeta^{p_1-q} + \delta_8 \tag{5.17}$$

and derive a contradiction. We put $y(\mu,\alpha,t) := \zeta_{\mu,\alpha}^{-1}w(\mu,\alpha,t)$. Then by (5.12) we obtain

$$\begin{aligned}
&\frac{1}{2}y'(\mu,\alpha,t)^2 \\
&= \frac{1}{p_1+1}\zeta_{\mu,\alpha}^{p_1-1}(1-y(\mu,\alpha,t)^{p_1+1}) - \frac{1}{q+1}\zeta_{\mu,\alpha}^{q-1}(1-y(\mu,\alpha,t)^{q+1}) \\
&+ \sum_{k=2}^{n}\frac{1}{p_k+1}\nu^{-2}\mu_k\xi^{p_k-1}\zeta_{\mu,\alpha}^{p_k-1}(1-y(\mu,\alpha,t)^{p_k+1}) \\
&+ \sum_{k=1}^{n}\mu_k\zeta_{\mu,\alpha}^{-2}\nu^{-2}\xi^{-2}(H_k(\xi\zeta_{\mu,\alpha}) - H_k(\xi\zeta_{\mu,\alpha}y(\mu,\alpha,t))) \\
&- \lambda(\mu,\alpha)\zeta_{\mu,\alpha}^{-2}\nu^{-2}\xi^{-2}(H_0(\xi\zeta_{\mu,\alpha}) - H_0(\xi\zeta_{\mu,\alpha}y(\mu,\alpha,t))).
\end{aligned} \tag{5.18}$$

Then by (B.1), (2.5), (5.14), (5.15), (5.17), Lemma 4.8 and Lemma 5.1

$$\begin{aligned}&\mu_k\zeta_{\mu,\alpha}^{-2}\nu^{-2}\xi^{-2}|H_k(\xi\zeta_{\mu,\alpha}) - H_k(\xi\zeta_{\mu,\alpha}y(\mu,\alpha,t))|\\ &\le \mu_k\zeta_{\mu,\alpha}^{-2}\nu^{-2}\xi^{-2}\max_{0\le y\le 1}|h_k(\xi\zeta_{\mu,\alpha}y)|\xi\zeta_{\mu,\alpha}(1-y(\mu,\alpha,t))\\ &\le o(1)\zeta_{\mu,\alpha}^{p_k-1}\nu^{-2}\mu_k\xi^{p_k-1}(1-y(\mu,\alpha,t)) = o(1)(1-y(\mu,\alpha,t)),\end{aligned} \tag{5.19}$$

$$\begin{aligned}&\zeta_{\mu,\alpha}^{-2}\lambda(\mu,\alpha)\nu^{-2}\xi^{-2}|H_0(\xi\zeta_{\mu,\alpha}) - H_0(\xi\zeta_{\mu,\alpha}y(\mu,\alpha,t))|\\ &\le \zeta_{\mu,\alpha}^{-2}\lambda(\mu,\alpha)\nu^{-2}\xi^{-2}\max_{0\le y\le 1}|h_0(\xi\zeta_{\mu,\alpha}y)|\xi\zeta_{\mu,\alpha}(1-y(\mu,\alpha,t))\\ &= o(1)\zeta_{\mu,\alpha}^{q-1}\lambda(\mu,\alpha)\nu^{-2}\xi^{q-1}(1-y(\mu,\alpha,t)) = o(1)(1-y(\mu,\alpha,t)).\end{aligned} \tag{5.20}$$

Hence, by (5.17)–(5.20) we obtain for $-\frac{1}{2}\nu \le t \le 0$

$$\begin{aligned}y'(\mu,\alpha,t) &\ge \sqrt{S(\mu,\alpha,y(\mu,\alpha,t))}\\ &\ge \sqrt{\frac{2}{p_1+1}\zeta_{\mu,\alpha}^{q-1}(\zeta_{\mu,\alpha}^{p_1-q} - \zeta^{p_1-q})(1-y(\mu,\alpha,t)^{q+1}) - o(1)(1-y(\mu,\alpha,t))}\\ &\ge \sqrt{\left(\frac{2}{p_1+1}\zeta_{\mu,\alpha}^{q-1}\delta_8 - o(1)\right)(1-y(\mu,\alpha,t))} \ge \delta_9\sqrt{1-y(\mu,\alpha,t)},\end{aligned} \tag{5.21}$$

where

$$S(\mu,\alpha,y) := \frac{2}{p_1+1}\zeta_{\mu,\alpha}^{p_1-1}(1-y^{p_1+1}) - \frac{2}{q+1}\zeta_{\mu,\alpha}^{q-1}(1-y^{q+1}) - o(1)(1-y).$$

Then by (5.21)

$$\begin{aligned}1 &= \int_{-\nu/2}^{0} y'(\mu,\alpha,t)dt \ge \int_{-\nu/2}^{0}\delta_9\sqrt{1-y(\mu,\alpha,t)}dt\\ &= \nu\delta_9\int_0^{1/2}\sqrt{1-z(\mu,\alpha,x)}\,dx,\end{aligned} \tag{5.22}$$

where $z(\mu,\alpha,x) = \sigma_{\mu,\alpha}^{-1}u(\mu,\alpha,x)$. Hence, by Lemma 5.2, $z(\mu,\alpha,x_0) \to 1$ for any fixed $x_0 \in I$. Then it follows from (5.8) that $\zeta_{\mu,\alpha}^{p_1-q} \to 1$. This contradicts (5.16). Thus the proof is complete. □

We identify $w(\mu,\alpha,x)$ with a function defined on R by 0-extension.

Lemma 5.5. *Assume that* $\{(\mu,\alpha)\}$ *satisfies* (B.1). *Then* $w(\mu,\alpha,t) \to w(t)$ *uniformly on any compact subsets on* R.

Proof. We see that $\{w(\mu,\alpha,t)\}, \{w'(\mu,\alpha,t)\}$ and $\{w''(\mu,\alpha,t)\}$ are bounded by (5.2), (5.12), (5.14), (5.15) and Lemma 5.4. Hence, we apply Ascoli-Arzela's theorem to extract a subsequence such that

$$w(\mu,\alpha,t) \to w_1(t),\;\; w'(\mu,\alpha,t) \to w_2(t) \tag{5.23}$$

uniformly on any compact subsets on R. Then for a fixed $t \in R$

$$w(\mu,\alpha,t) - w(\mu,\alpha,0) = \int_0^1 w'(\mu,\alpha,s)\,ds;$$

by letting $\mu_1 \to \infty$, we obtain by Lemma 5.4 and (5.23) that

$$w_1(t) - \zeta = \int_0^t w_2(s)\,ds.$$

Hence, $w_2(t) = w_1'(t)$. For fixed $\psi(t) \in C_0^\infty(R)$ and $\mu_1 \gg 1$, we obtain by (5.2) and integration by parts that

$$\begin{aligned}
&-\int_R w'(\mu,\alpha,t)\psi'(t)dt + \int_R (w(\mu,\alpha,t)^{p_1} - w(\mu,\alpha,t)^q)\psi(t)dt \\
&+ \sum_{k=2}^n \nu^{-2}\mu_k\xi^{p_k-1}\int_R w(\mu,\alpha,t)^{p_k}\psi(t)dt \\
&+ \sum_{k=1}^n \nu^{-2}\mu_k\xi^{-1}\int_R h_k(\xi w(\mu,\alpha,t))\psi(t)dt \\
&- \lambda(\mu,\alpha)\nu^{-2}\xi^{-1}\int_R h_0(\xi w(\mu,\alpha,t))\psi(t)dt = 0.
\end{aligned} \tag{5.24}$$

Then by (B.1), (2.5), (5.14), (5.15) and Lemma 4.8

$$\nu^{-2}\mu_k\xi^{p_k-1}\left|\int_R w(\mu,\alpha,t)^{p_k}\psi(t)dt\right| = o(1)\left|\int_R w(\mu,\alpha,t)^{p_k}\psi(t)dt\right| \to 0, \tag{5.25}$$

$$\begin{aligned}
&\nu^{-2}\mu_k\xi^{-1}\left|\int_R h_k(\xi w(\mu,\alpha,t))\psi(t)dt\right| \\
&\quad = o(1)\nu^{-2}\mu_k\xi^{p_k-1}\left|\int_R w(\mu,\alpha,t)^{p_k}\psi(t)dt\right| \to 0,
\end{aligned}$$

$$\begin{aligned}
&\lambda(\mu,\alpha)\nu^{-2}\xi^{-1}\left|\int_R h_0(\xi w(\mu,\alpha,t))\psi(t)dt\right| \\
&\quad = o(1)\lambda(\mu,\alpha)\nu^{-2}\xi^{q-1}\left|\int_R w(\mu,\alpha,t)^q\psi(t)dt\right| \to 0.
\end{aligned}$$

Hence, by letting $\mu_1 \to \infty$ in (5.24), we obtain by (5.23) and (5.25) that

$$-\int_R w_1'(t)\psi'(t)dt + \int_R (w_1(t)^{p_1} - w_1(t)^q)\psi(t)dt = 0.$$

That is, $w_1(t) \in C^1(R)$ is a weak solution (and consequently, a classical solution by a standard regularity argument) of (5.3). Finally, since $\zeta_{\mu,\alpha} = w(\mu,\alpha,0)$, we obtain by Lemma 5.4 and (5.23) that

$$w_1(0) = \lim_{\mu_1\to\infty} w(\mu,\alpha,0) = \lim_{\mu_1\to\infty} \zeta_{\mu,\alpha} = \zeta. \tag{5.26}$$

Thus, $w_1 \equiv w$, the ground state of (5.3). Now our assertion follows from a standard compactness argument. Thus the proof is complete. $\square$

Lemma 5.6. *Assume that* $\{(\mu,\alpha)\}$ *satisfies* (B.1). *Then there exists* $y(t) \in L^{q+1}(R) \cap L^{p_k+1}(R)$ *such that* $w(\mu,\alpha,t) \le y(t)$ *for* $t \in R$.

Proof. Let $q < r < 2q+3$ be fixed. Furthermore, let $y_1(t) := (t+1)^{-2/(r-1)}$. Then $y_1(t)$ satisfies

$$\begin{aligned} &y_1'(t) = -\sqrt{T_0(y_1(t))}, \quad t > 0, \\ &y_1(0) = 1, \end{aligned} \tag{5.27}$$

where $T_0(y) := 4(r-1)^{-2}y^{r+1}$. Moreover, since $r < 2q+3 < 2p_k+3$, it is clear that $y_1(t) \in L^{q+1}(R_+) \bigcap L^{p_k+1}(R_+)$. Let $y(\mu,\alpha,t) := \zeta_{\mu,\alpha}^{-1} w(\mu,\alpha,t)$. Then by (5.12), $y(\mu,\alpha,t)$ satisfies

$$\begin{aligned} &y'(\mu,\alpha,t) = -\sqrt{T_1(y(\mu,\alpha,t))}, \quad 0 < t < \frac{1}{2}\nu, \\ &y(\mu,\alpha,0) = 1, \end{aligned} \tag{5.28}$$

where

$$T_1(y) = 2\zeta_{\mu,\alpha}^{-2}(R(\mu,\alpha,\zeta_{\mu,\alpha}) - R(\mu,\alpha,\zeta_{\mu,\alpha} y(\mu,\alpha,t))).$$

We note that $R(\mu,\alpha,\zeta_{\mu,\alpha}) = \frac{1}{2}w'(\mu,\alpha,0)^2 > 0$ by (5.12). Let $0 < \epsilon \ll 1$ be fixed. We shall show that $T_1(y) - T_0(y) > 0$ for $0 \le y \le \epsilon$. To this end, we estimate the terms in $R(\mu,\alpha,\zeta_{\mu,\alpha} y)$. By (2.5), (5.14), (5.15) and Lemma 5.4

$$\begin{aligned} \nu^{-2}\mu_k \xi^{p_k-1}\zeta_{\mu,\alpha}^{p_k+1} y^{p_k+1} &= o(1) y^{p_k+1}, \\ \mu_k \nu^{-2}\xi^{-2}|H_k(\xi\zeta_{\mu,\alpha} y)| &= o(1)\mu_k\nu^{-2}\xi^{p_k-1}\zeta_{\mu,\alpha}^{p_k+1} y^{p_k+1} = o(1) y^{p_k+1}, \\ \lambda(\mu,\alpha)\nu^{-2}\xi^{-2}|H_0(\xi\zeta_{\mu,\alpha} y)| &= o(1)\lambda(\mu,\alpha)\nu^{-2}\xi^{q-1}\zeta_{\mu,\alpha}^{q+1} y^{q+1} = o(1) y^{q+1}. \end{aligned} \tag{5.29}$$

Hence, by (5.29) we obtain that for $0 \le y \le \epsilon$

$$\begin{aligned} T_1(y) - T_0(y) \ge{}& 2\zeta_{\mu,\alpha}^{-2} R(\mu,\alpha,\zeta_{\mu,\alpha}) + \frac{2}{q+1}\zeta_{\mu,\alpha}^{q-1} y^{q+1} - \frac{2}{p_1+1}\zeta_{\mu,\alpha}^{p_1-1} y^{p_1+1} \\ &- o(1)\sum_{k=1}^{n} y^{p_k+1} - o(1) y^{q+1} - \frac{4}{(r-1)^2} y^{r+1} > 0. \end{aligned} \tag{5.30}$$

We fix $t_0 \gg 1$ satisfying $y_1(t_0) < \epsilon$. Then by (5.6) and Lemma 5.5, we see that $y(\mu, \alpha, t_0) < y_1(t_0)$ for $\mu_1 \gg 1$. Then we apply the comparison theorem of ODE to obtain that $y(\mu, \alpha, t) \le y_1(t)$ for $t > t_0$. This along with Lemma 5.4 implies that $w(\mu, \alpha, t) = \zeta_{\mu,\alpha} y(\mu, \alpha, t) \le 2\zeta y_1(t)$ for $t > t_0$. Now define $y(t)$ by

$$y(t) = \begin{cases} 2\zeta, & |t| \le t_0, \\ 2\zeta y_1(|t|), & |t| > t_0. \end{cases} \tag{5.31}$$

This is the desired function. □

Now we obtain the following lemma by Lemma 5.5, 5.6 and Lebesgue's convergence theorem:

Lemma 5.7. *Assume that* $\{(\mu, \alpha)\}$ *satisfies* (B.1). *Then* $w(\mu, \alpha, t) \to w(t)$ *as* $\mu_1 \to \infty$ *in* $L^{q+1}(R) \bigcap L^{p_k+1}(R)$.

Lemma 5.8 ([7, Lemma 4.6])**.** *Let* $w(t)$ *be the ground state of* (5.3). *Then*

$$\int_R w(t)^{q+1} dt = \frac{2}{p_1 - q}\sqrt{\frac{\pi(q+1)}{2}}\zeta^{\frac{q+3}{2}} \frac{\Gamma\left(\dfrac{q+3}{2(p_1-q)}\right)}{\Gamma\left(\dfrac{p_1+3}{2(p_1-q)}\right)}. \tag{5.32}$$

Proof. By (5.4) and (5.5), we have for $t \ge 0$

$$w(t) = -\frac{w'(t)}{w(t)^{\frac{q-1}{2}}\sqrt{\dfrac{2}{q+1} - \dfrac{2}{p_1+1} w(t)^{p_1-q}}}. \tag{5.33}$$

Then by (5.33), putting $s = w(t), t = \zeta^{-1}s$ and $t = \sin^{\frac{2}{p_1-q}}\theta$, we obtain

$$\begin{aligned} \int_R w(t)^{q+1} dt &= 2\int_0^\infty w(t)^q \cdot w(t) dt \\ &= 2\int_0^\infty w(t)^q \cdot \frac{-w'(t)}{w(t)^{\frac{q-1}{2}}\sqrt{\dfrac{2}{q+1} - \dfrac{2}{p_1+1} w(t)^{p_1-q}}} dt \\ &= 2\int_0^\zeta \frac{s^{\frac{q+1}{2}}}{\sqrt{\dfrac{2}{q+1} - \dfrac{2}{p_1+1} s^{p_1-q}}} ds \\ &= 2\sqrt{\frac{p_1+1}{2}}\zeta^{\frac{2q+3-p_1}{2}} \int_0^1 \frac{t^{\frac{q+1}{2}}}{\sqrt{1 - t^{p_1-q}}} dt \\ &= \frac{4}{p_1 - q}\sqrt{\frac{q+1}{2}}\zeta^{\frac{q+3}{2}} \int_0^{\pi/2} \sin^{\frac{2q+3-p_1}{p_1-q}} \theta d\theta \\ &= \frac{2}{p_1 - q}\sqrt{\frac{\pi(q+1)}{2}}\zeta^{\frac{q+3}{2}} \frac{\Gamma\left(\dfrac{q+3}{2(p_1-q)}\right)}{\Gamma\left(\dfrac{p_1+3}{2(p_1-q)}\right)}. \end{aligned} \tag{5.34}$$

We used here the following formula: for $r > -1$

$$\int_0^{\pi/2} \sin^r \theta d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{r}{2}+1\right)}. \tag{5.35}$$

Thus the proof is complete. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Multiply (5.3) by $w(t)$ and integrate it over R to obtain

$$\|w'\|^2_{L^2(R)} = \|w\|^{p_1+1}_{L^{p_1+1}(R)} - \|w\|^{q+1}_{L^{q+1}(R)}. \tag{5.36}$$

Integrate (5.5) over R to obtain

$$\frac{1}{2}\|w'\|^2_{L^2(R)} + \frac{1}{p+1}\|w\|^{p_1+1}_{L^{p_1+1}(R)} - \frac{1}{q+1}\|w\|^{q+1}_{L^{q+1}(R)} = 0. \tag{5.37}$$

It follows from (5.36) and (5.37) that

$$\|w\|^{p_1+1}_{L^{p_1+1}(R)} = \frac{(p_1+1)(q+3)}{(p_1+3)(q+1)}\|w\|^{q+1}_{L^{q+1}(R)}. \tag{5.38}$$

Then by Lemma 5.7 we obtain

$$\|w(\mu,\alpha,\cdot)\|^{q+1}_{L^{q+1}(R)} \longrightarrow \|w\|^{q+1}_{L^{q+1}(R)}, \;\; \|w(\mu,\alpha,\cdot)\|^{p_k+1}_{L^{p_k+1}(R)} \longrightarrow \|w\|^{p_k+1}_{L^{p_k+1}(R)}. \tag{5.39}$$

By definition of $w(\mu,\alpha,t)$, we have

$$\begin{aligned}
&\mu_1\|u(\mu,\alpha,\cdot)\|^{p_1+1}_{p_1+1} = \lambda(\mu,\alpha)^{\frac{p_1+3}{2(p_1-q)}}\mu_1^{-\frac{q+3}{2(p_1-q)}}\|w(\mu,\alpha,\cdot)\|^{p_1+1}_{L^{p_1+1}(R)},\\
&\|u(\mu,\alpha,\cdot)\|^{q+1}_{q+1} = \lambda(\mu,\alpha)^{\frac{2q+3-p_1}{2(p_1-q)}}\mu^{-\frac{q+3}{2(p_1-q)}}\|w(\mu,\alpha,\cdot)\|^{q+1}_{L^{q+1}(R)},\\
&\mu_k\|u(\mu,\alpha,\cdot)\|^{p_k+1}_{p_k+1} = \xi^{p_k-p_1}\frac{\mu_k}{\mu_1}\lambda(\mu,\alpha)^{\frac{p_1+3}{2(p_1-q)}}\mu_1^{-\frac{q+3}{2(p_1-q)}}\|w(\mu,\alpha,\cdot)\|^{p_k+1}_{L^{p_k+1}(R)}\\
&= o(1)\lambda(\mu,\alpha)^{\frac{p_1+3}{2(p_1-q)}}\mu_1^{-\frac{q+3}{2(p_1-q)}}\|w(\mu,\alpha,\cdot)\|^{p_k+1}_{L^{p_k+1}(R)}.
\end{aligned} \tag{5.40}$$

Now, by (2.5) and Proposition 4.1

$$\begin{aligned}
(f_k(u(\mu,\alpha,\cdot)),u(\mu,\alpha,\cdot))_2 &= (1+o(1))\|u(\mu,\alpha,\cdot)\|^{p_k+1}_{p_k+1},\\
\Phi_k(u(\mu,\alpha,\cdot)) &= (1+o(1))\frac{1}{p_k+1}\|u(\mu,\alpha,\cdot)\|^{p_k+1}_{p_k+1},\\
(g(u(\mu,\alpha,\cdot)),u(\mu,\alpha,\cdot))_2 &= (1+o(1))\|u(\mu,\alpha,\cdot)\|^{q+1}_{q+1}.
\end{aligned} \tag{5.41}$$

Then by (2.3), (5.40) and (5.41) we obtain

$$\lambda(\mu,\alpha)=\frac{2\alpha+\sum_{k=1}^{n}\mu_k\dfrac{p_k-1}{p_k+1}(1+o(1))\|u(\mu,\alpha,\cdot)\|_{p_k+1}^{p_k+1}}{(1+o(1))\|u(\mu,\alpha,\cdot)\|_{q+1}^{q+1}}$$
$$=\frac{2\alpha+\dfrac{p_1-1}{p_1+1}\lambda(\mu,\alpha)^{\frac{p_1+3}{2(p_1-q)}}\mu_1^{-\frac{q+3}{2(p_1-q)}}\{(1+o(1))\|w(\mu,\alpha,\cdot)\|_{L^{p_1+1}(R)}^{p_1+1}+o(1)\}}{(1+o(1))\lambda(\mu,\alpha)^{\frac{2q+3-p_1}{2(p_1-q)}}\mu_1^{-\frac{q+3}{2(p_1-q)}}\|w(\mu,\alpha,\cdot)\|_{L^{q+1}(R)}^{q+1}}; \tag{5.42}$$

this along with (5.38) implies that

$$\frac{\lambda(\mu,\alpha)}{\alpha^{\frac{2(p_1-q)}{p_1+3}}\mu_1^{\frac{q+3}{p_1+3}}}=\left(\frac{2}{S_{\mu,\alpha}}\right)^{\frac{2(p_1-q)}{p_1+3}}\to\left(\frac{2}{S}\right)^{\frac{2(p_1-q)}{p_1+3}}, \tag{5.43}$$

where

$$\begin{aligned}S_{\mu,\alpha}&=(1+o(1))\|w(\mu,\alpha,\cdot)\|_{L^{q+1}(R)}^{q+1}\\&\quad-\frac{p_1-1}{p_1+1}((1+o(1))\|w(\mu,\alpha,\cdot)\|_{L^{p_1+1}(R)}^{p_1+1}+o(1)),\\S&=\|w\|_{L^{q+1}(R)}^{q+1}-\frac{p_1-1}{p_1+1}\|w\|_{L^{p_1+1}(R)}^{p_1+1}=\frac{2(2q+3-p_1)}{(p_1+3)(q+1)}\|w\|_{L^{q+1}(R)}^{q+1}.\end{aligned} \tag{5.44}$$

Then Theorem 2.1 follows from (5.43), (5.44) and Lemma 5.8. Thus we obtain the formula (2.10) for the case $K_0=K_1=\cdots=K_n=1$. For general K_j, we replace μ_1 and $\lambda(\mu,\alpha)$ by $K_1\mu_1$ and $K_0\lambda(\mu,\alpha)$, respectively. Then by the same arguments as those used above, we obtain the formula (2.10). Thus the proof is complete. □

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