Differential and Integral Equations Volume 10, Number 4, July 1997, pp. 609 – 624.

TWO POSITIVE SOLUTIONS FOR A CLASS OF NONHOMOGENEOUS ELLIPTIC EQUATIONS

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Abstract. In this paper we develop an original variational approach which allows us to prove the existence of two positive solutions for a class of semilinear nonhomogeneous elliptic equations set on $\mathbb{R}^N$. In addition to a lack of compactness the main difficulty to overcome is the degenerated structure of the set of possible critical points.

1. Introduction. This paper studies the existence of positive solutions $u \in H^1(\mathbb{R}^N)$ for the equation

$$-\Delta u(x) + u(x) = f(x, u(x)) + h(x),\ x \in \mathbb{R}^N,\ N \geq 3, \tag{1.1}$$

where we assume that:

(H0) $h \in H^{-1}(\mathbb{R}^N)$ and $0 \not\equiv h$;
(H1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory conditions;
(H2) $\exists a > 0,\ b \in [0,1)$ and $2 < p < \frac{2N}{N-2}$ such that

$$|f(x,t)| \leq a|t|^{p-1} + b|t|,\ a.e.\ x \in \mathbb{R}^N,\ \forall t \in \mathbb{R};$$

(H3) $\exists \mu > 2,\ \bar{b} \in [0,1)$ such that, setting $\bar{f}(x,t) = f(x,t) - \bar{b}t$, we have for almost every $x \in \mathbb{R}^N$ and $\forall t \neq 0$

$$0 < \mu \bar{F}(x,t) \leq t\bar{f}(x,t) \ \text{ with } \bar{F}(x,t) = \int_0^t \bar{f}(x,s)\,ds;$$

(H4) There is $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that
(i) $\lim_{|x|\to\infty} f(x,t) = \tilde{f}(t)$,
(ii) $f(x,t) \geq \tilde{f}(t)$, for almost every $x \in \mathbb{R}^N,\ \forall t \geq 0$.

Received for publication May 1996.
AMS Subject Classifications: 35A15, 35J20, 35J65.

The existence of positive solutions for equation (1.1) has already been investigated in a series of works ([3, 9, 10]) (see also [4, 8] for closely related problems). To our knowledge the more complete results are contained in [3]. In addition to (H0)–(H4) Cao-Zhou require:

(H5) $f(x,.) \in C^2(0,+\infty)$ and $\frac{\partial^2}{\partial^2 t} f(x,t) \geq 0,\ \forall x \in \mathbb{R}^N,\ \forall t \geq 0$;
(H6) For some constants $k > 0,\ 0 < q < \frac{4}{N-2}$ and uniformly, for $x \in \mathbb{R}^N$,

$$\lim_{t\to+\infty} \{t^{1-q}\ \frac{\partial^2}{\partial^2 t} f(x,t)\} \leq k \ \text{ and } \lim_{t\to 0^+} \{t\ \frac{\partial^2}{\partial^2 t} f(x,t)\} = 0.$$

Then, under the assumptions (H0)–(H6), they show that (1.1) has two positive solutions if $h \geq 0$ and satisfies the following bound:

$$||h||_{H^{-1}} \equiv \sup\Big\{\int_{\mathbb{R}^N} hu \ :\ \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 = 1\Big\} < \tilde{c}\ (1-b)^{\frac{p-1}{p-2}}\ S_0^{\frac{p}{2(p-2)}}, \qquad (1.2)$$

where

$$S_0 = \inf\{\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 : \int_{\mathbb{R}^N} |u|^p = 1\} \ \text{ and } \ \tilde{c} = a^{-\frac{1}{p-2}}(p-1)^{-\frac{p-1}{p-2}}(p-2).$$

Moreover, under the assumptions (H0)–(H2) and for $h \geq 0$, if $||h||_{H^{-1}} \leq \tilde{c}\ (1-b)^{\frac{p-1}{p-2}}\ S_0^{\frac{p}{2(p-2)}}$, they prove that (1.1) has one positive solution.

Weak solutions of (1.1) are critical points of the functional I defined on $H^1(\mathbb{R}^N)$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 - \int_{\mathbb{R}^N} F(x,u) - \int_{\mathbb{R}^N} hu,$$

where $F(x,t) = \int_0^t f(x,s)\,ds$. When we look for positive solutions we can assume without restriction $f(x,t) = 0$, for almost every $x \in \mathbb{R}^N$ for $t \leq 0$. When $h \geq 0$ it readily implies that any critical point u of I satisfies $u \geq 0$ and using the maximum principle of weak solution it follows that $u > 0$ for almost every $x \in \mathbb{R}^N$. Thus the hard work is to get distinct critical points.

To this end Cao-Zhou first study the limit case $f(x,t) = a|t|^{p-2}t + bt$ (here $\bar{b} = b$ in (H3)). For this nonlinearity, using a constrained minimization method inspired by [8], they obtain a positive solution of (1.1) and hence, since in the general case $f(x,t) \leq a|t|^{p-2}t + bt,\ \forall t \geq 0$, a supersolution of (1.1). This enables them, using a sub- and supersolutions method ($u \equiv 0$ is the subsolution), to derive a minimal positive solution u_0 of (1.1). Then they prove that u_0 is a local minimum for I (at this step (H5)–(H6) are used) and using a Mountain Pass lemma around u_0 they obtain a second solution. We refer the reader to [3] for more details.

In this paper we develop an original approach, purely variational, which allows us to strongly enlarge the results of Cao-Zhou. Let us introduce some notation:

$$||h||_b = \sup\{\int_{\mathbb{R}^N} hu \ : \ \int_{\mathbb{R}^N} |\nabla u|^2 + (1-b)u^2 = 1\}$$
$$S_b = \inf\{\int_{\mathbb{R}^N} |\nabla u|^2 + (1-b)u^2 : u \in H \text{ and } \int_{\mathbb{R}^N} |u|^p = 1\}$$
$$F_b = \{u \in H : \int_{\mathbb{R}^N} |u|^p = 1 \text{ and } \int_{\mathbb{R}^N} |\nabla u|^2 + (1-b)u^2 = S_b\}.$$

Our main results are the following:

Theorem 1.1. *Assume that* (H0)–(H4) *hold,* $h \geq 0$ *and*

$$\inf\Big\{ \ \tilde{c} \ \Big[\int_{\mathbb{R}^N} |\nabla u|^2 + (1-b)u^2\Big]^{\frac{p-1}{p-2}} - \int_{\mathbb{R}^N} hu \ : \ \int_{\mathbb{R}^N} |u|^p = 1\Big\} > 0, \tag{1.3}$$

where $\tilde{c}$ *is defined as before. Then* (1.1) *has two positive solutions.*

Theorem 1.2. *Assume that* (H0)–(H4) *hold and* $h \geq 0$. *Then equation* (1.1) *has two positive solutions if either*

$$||h||_b < \tilde{c} \ S_b^{\frac{p}{2(p-2)}} \ \text{or} \ \Big\{||h||_b = \tilde{c} \ S_b^{\frac{p}{2(p-2)}} \ \text{and} \ \int_{\mathbb{R}^N} hu < ||h||_b \ S_b \ \text{when} \ u \in F_b\Big\}. \tag{1.4}$$

Theorem 1.2 is a weak version of Theorem 1.1 with hypotheses easier to check. Indeed, as we shall see, (1.4) implies (1.3). We also derive an existence result for one solution where the assumption $h \geq 0$ is no longer required.

Theorem 1.3. *Assume that* (H0)–(H2) *hold. Then* (1.1) *has one solution if*

$$\inf\Big\{ \ \tilde{c} \ \Big[\int_{\mathbb{R}^N} |\nabla u|^2 + (1-b)u^2\Big]^{\frac{p-1}{p-2}} - \int_{\mathbb{R}^N} hu \ : \ \int_{\mathbb{R}^N} |u|^p = 1\Big\} \geq 0. \tag{1.5}$$

If $h \geq 0$ *this solution is positive. In particular* (1.5) *holds when* $||h||_b \leq \tilde{c} \ S_b^{\frac{p}{2(p-2)}}$.

Our results compare to [3] in the following way. With respect to [3] we do not require any more hypotheses (H5)–(H6) to get two positive solutions. This is our main improvement. We also extend the class of admissible h since already condition (1.4) is implied by condition (1.2) but the converse is not true. In fact following Tarantello ([8]) we suspect that in the case where $f(x,t) = a|t|^{p-2}t + bt$, assumption (1.3) is not only sufficient but necessary to guarantee the statement of Theorem 1.1. In that direction we mention that in [3] it is shown that for this nonlinearity equation (1.1) has no positive solution when $h \geq 0$ and $||h||_{H^{-1}}$ is large. Finally in [3] it is always required that $h \geq 0$ and thus there is no equivalent to Theorem 1.3.

In view of Theorems 1.1 and 1.3 one may wonder if it is possible to find two solutions for equation (1.1) without requiring that $h \geq 0$. Closely related to this question is the work of Tarantello ([8]). She considers the existence of solutions to the Dirichlet problem

$$-\Delta u = |u|^{p-2}u + h$$

set on a bounded regular domain $\Omega \subset \mathbb{R}^N, N \geq 3$ and with $p = \frac{2N}{N-2}$. Here the noncompactness of the problem stems from the precise choice of p, giving rise to a so-called critical Sobolev exponent problem. If $h \in H^{-1}(\mathbb{R}^N)$ with $0 \not\equiv h$ satisfies condition (1.3) (here $a = 1, b = 0$) Tarantello shows that the Dirichlet problem has two distinct solutions which proves to be positive if $h \geq 0$. In view of this result we conjecture a positive answer also for equation (1.1).

Let us indicate the main lines of the proof of Theorem 1.1. Our first solution is obtained directly as a local minimum of I. We make explicit a manifold Λ, homeomorphic to the unit sphere in H, which disconnects H into exactly two connected components U_1 and U_2: $H \setminus \Lambda = U_1 \cup U_2$ and $0 \in U_1$, and for which, if condition (1.3) holds,

$$-\infty < c_0 \equiv \inf_{u \in U_1} I(u) < \inf_{u \in \Lambda} I(u) \equiv c_1. \tag{1.6}$$

We prove that a minimum u_0 for I on U_1 exists. Indeed the strict inequality in (1.6) and the Ekeland's variational principle insure the existence of a Palais-Smale sequence $\{u_n\} \subset U_1$ for I at level c_0. Since we work on all $\mathbb{R}^N$ it is not to be expected that the Palais-Smale conditions hold at all levels. However, using the geometrical structure of the functional I, we manage to show that, up to a subsequence, $\{u_n\}$ strongly converges to a $u_0 \in U_1$ which is thus a minimum.

We obtain a second critical point using a Mountain Pass-type procedure. There exists a $v \in U_2$ such that $I(v) \leq I(u_0)$ and setting $\gamma \equiv \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t))$, where

$$\Gamma = \{g \in C([0,1], H), g(0) = u_0, g(1) = v\},$$

we have $\gamma > I(u_0)$ since $\inf_\Lambda I = c_1 > c_0 = I(u_0)$. Then, still by Ekeland's principle, we get a Palais-Smale sequence $\{v_n\}$ for I at level γ. Some sharp estimates on the value of γ allow us to show that $\{v_n\}$ has a weak limit v_0 which is a critical point different from u_0. It is at this step that we need to assume that our first solution u_0 is positive.

The article is organized as follows. In Section 2, we introduce our special partition of H which, when (1.3) holds, satisfies (1.6). In Section 3 we prove that I admits a critical point $u_0 \in \bar{U}_1$. Moreover, if (1.3) is satisfied, $u_0 \in U_1$ and corresponds to a minimum. This gives us a first solution which proves to be positive if $h \geq 0$. The Mountain Pass procedure is developed in Section 4 to get a second positive solution when (1.3) and $h \geq 0$ hold. In Section 5 we give the proofs of Theorems 1.1, 1.2 and 1.3. In particular we show that (1.4) implies (1.3). This is the content of Theorem 1.2.

Notation. The spaces $L^p(\mathbb{R}^N)$ are equipped with their usual norms denoted by $||.||_p$. We shall often work on $H = H^1(\mathbb{R}^N)$ with the norm $||u||_b = \{||\nabla u||_2^2 + (1-b)||u||_2^2\}^{\frac{1}{2}}$ which by (H2) is equivalent to the standard norm $||.||$.

2. A special partition of H. In this section we introduce our special partition of H. Namely we prove that H can be disconnected into two connected components $U_1 \ni 0$ and U_2 by a manifold Λ, homeomorphic to the unit sphere, and that, when (1.3) holds, this partition satisfies (1.6). Namely we require that

$$c_0 \equiv \inf_{u \in U_1} I(u) < \inf_{u \in \Lambda} I(u) \equiv c_1.$$

Let $g : H \to \mathbb{R}$ be defined by $g(u) = ||u||_b^2 - a(p-1)||u||_p^p$.

Lemma 2.1.

(i) *For any* $u \in H, ||u|| = 1$, *there is a unique* $t = t(u) \in \mathbb{R}^+$ *such that* $g(tu) = 0$;
(ii) *Let* $u \in H \setminus \{0\}$ *be such that* $g(u) = 0$; *then*

$$g(tu) > 0, \ \forall t \in (0,1) \quad \textit{and} \quad g(tu) < 0, \ \forall t \in \ (1, +\infty).$$

Proof. For all $s \in \mathbb{R}^+$ we have $g(su) = s^2||u||_b^2 - s^p a(p-1)||u||_p^p$. Thus Point (i) holds since $g(0) = 0$ and $s \to g(su)$ is a strictly concave function. Now using the fact that $g(u) = 0$ we get $g(su) = (s^2 - s^p)||u||_b^2$ and Point (ii) follows. $\square$

We shall prove that a partition we are looking for is given by

$$U_1 \equiv \{u \in H : \ u = 0 \text{ or } g(u) > 0\}, \quad U_2 \equiv \{u \in H : \ g(u) < 0\}$$

and

$$\Lambda \equiv \{u \in H \setminus \{0\} : \ g(u) = 0\}.$$

Lemma 2.2. *For all* $u \in \Lambda$, $||u||$, $||u||_b$ *and* $||u||_p$ *are bounded away from zero.*

Proof. For all $u \in \Lambda$ we get using the Sobolev embedding

$$(1-b)||u||^2 \leq a(p-1)||u||_p^p \leq a(p-1)S_0^{-\frac{p}{2}}||u||^p.$$

Since $p > 2$ it shows there is $k > 0$ such that $||u|| \geq k$, $||u||_b \geq k$ and $||u||_p \geq \ k$. $\square$

Now let $J : H \to \mathbb{R}$ be defined by

$$J(u) = \frac{1}{2}||u||_b^2 - \frac{a}{p}||u||_p^p - \int_{\mathbb{R}^N} hu.$$

Lemma 2.3. *Assume that* (1.3) *holds. Then there is* $\mu > 0$ *such that, for all* $u \in \Lambda$

$$\frac{d}{dt}J(tu)_{|t=1} \geq \mu.$$

Proof. We have

$$\frac{d}{dt}J(tu) = t||u||_b^2 - at^{p-1}||u||_p^p - \int_{\mathbb{R}^N} hu, \ \forall u \in H, \ \forall t \geq 0.$$

Thus

$$\frac{d}{dt}J(tu)_{|t=1} = ||u||_b^2 - a||u||_p^p - \int_{\mathbb{R}^N} hu$$

and, using the fact that $u \in \Lambda$, we easily get

$$\frac{d}{dt}J(tu)_{|t=1} = \frac{p-2}{p-1}||u||_b^2 - \int_{\mathbb{R}^N} hu = \tilde{c}\ \frac{||u||_b^{\frac{2(p-1)}{p-2}}}{||u||_p^{\frac{p}{p-2}}} - \int_{\mathbb{R}^N} hu.$$

Condition (1.3) says there exists $d > 0$ such that

$$\inf_{||u||_p=1}\{\tilde{c}\ ||u||_b^{\frac{2(p-1)}{p-2}} - \int_{\mathbb{R}^N} hu\} \geq d. \tag{2.1}$$

But

$$\begin{aligned}(2.1) \iff \tilde{c}\ \frac{||u||_b^{\frac{2(p-1)}{p-2}}}{||u||_p^{\frac{p}{p-2}}} - \int_{\mathbb{R}^N} hu \geq d, \ ||u||_p = 1 \\ \iff \tilde{c}\ \frac{||u||_b^{\frac{2(p-1)}{p-2}}}{||u||_p^{\frac{p}{p-2}}} - \int_{\mathbb{R}^N} hu \geq d\ ||u||_p, \ u \in H \setminus \{0\}.\end{aligned}$$

Thus, since by Lemma 2.2, $||u||_p \geq k > 0, \ \forall u \in \Lambda$, our claim is proved. □

Now let $\{u_n\}$ be a minimizing sequence for I on Λ. By (H2), $I(u) \geq J(u), \ \forall u \in H$. Thus there exists $M > 0$ such that $J(u_n) \leq M, \ \forall n \in \mathbb{N}$ and since $u_n \in \Lambda$, we have

$$J(u_n) = \{\frac{1}{2} - \frac{1}{p(p-1)}\}||u_n||_b^2 - \int_{\mathbb{R}^N} hu_n \geq \{\frac{1}{2} - \frac{1}{p(p-1)}\}||u_n||_b^2 - ||h||_b||u_n||_b.$$

Since $p > 2$ this shows that $||u_n||_b$ and thus $||u_n||_p$ are bounded. We claim that, for $n \in \mathbb{N}$ large, $\int_{\mathbb{R}^N} hu_n > 0$. To see this first notice that by the geometrical structure of I, $c_0 < 0$ and thus we must have $I(u_n) \leq 0$ for $n \in \mathbb{N}$ large. On the other hand when $\int_{\mathbb{R}^N} hu_n \leq 0$ we get, using the fact that $u_n \in \Lambda$,

$$I(u_n) \geq J(u_n) \geq \frac{1}{2}||u_n||_b^2 - \frac{a}{p}||u_n||_p^p \geq \frac{a}{2}(p-2)||u_n||_p^p > 0.$$

This proves our claim, and from now on we assume without restriction that

$$\int_{\mathbb{R}^N} hu_n > 0 \qquad \forall n \in \mathbb{N}.$$

Since $\int_{\mathbb{R}^N} hu_n > 0$ we have $\frac{d}{dt}J(tu_n) < 0$ for $t > 0$ small enough and by Lemma 2.3 there exists a $t_n \in (0,1)$ such that $\frac{d}{dt}J(t_n u_n) = 0$. Moreover, t_n is unique since

$$\frac{d^2}{d^2t}J(tu) = ||u||_b^2 - a(p-1)t^{p-2}||u||_p^p > 0, \ \forall u \in \Lambda, \ \forall t \in [0,1).$$

Let us show that

$$\liminf_{n\to\infty}\big\{J(u_n) - J(t_n u_n)\big\} > 0. \tag{2.2}$$

We have

$$J(u_n) - J(t_n u_n) = \int_{t_n}^1 \frac{d}{dt}\{J(tu_n)\}\,dt$$

and, for all $n \in \mathbb{N}$, there is $\xi_n > 0$ such that

$$t_n \in (0, 1-2\xi_n) \ \text{ and } \ \frac{d}{dt}J(tu_n) \geq \frac{\mu}{2} \ \text{ for } t \in [1-\xi_n, 1].$$

Thus to establish (2.2) it is enough to show that $\xi_n > 0$ can be chosen independently of $n \in \mathbb{N}$. But this is the case since $\frac{d}{dt}J(tu_n)_{|t=1} \geq \mu$ and by the boundedness of $\{u_n\}$,

$$\Big|\frac{d^2}{d^2t}J(tu_n)\Big| = \Big|||u_n||_b^2 - a(p-1)t^{p-2}||u_n||_p^p\Big| \leq k, \ \forall n \in \mathbb{N}, \ \forall t \in [0,1],$$

for some $k > 0$. Now, still by (H2), we have $\frac{d}{dt}I(tu) \geq \frac{d}{dt}J(tu)$, $\forall u \in H, \forall t \geq 0$ and thus

$$I(u_n) - I(t_n u_n) = \int_{t_n}^1 \frac{d}{dt}\{I(tu_n)\}\,dt \geq \int_{t_n}^1 \frac{d}{dt}\{J(tu_n)\}\,dt = J(u_n) - J(t_n u_n).$$

Since $\{u_n\} \subset \Lambda$ is a minimizing sequence for I and $t_n u_n \in U_1$ we conclude using (2.2) that $c_0 < c_1$. Thus the partition U_1, Λ, U_2 is of the required type.

3. Existence of a minimum for I on $\bar{U}_1$. In this section, under assumptions (H0)–(H4) and (1.5), we obtain a first critical point u_0 of I. It is characterized as a minimum of I on $\bar{U}_1$ and when (1.3) holds it lies in U_1. We also prove that u_0 is positive if $h \geq 0$.

We recall that $\{u_n\} \subset H$ is a Palais-Smale sequence for I at level $c \in \mathbb{R}$ if and only if $I(u_n) \to c$ and $I'(u_n) \to 0$ in H^{-1}.

Lemma 3.1. *Assume that* (H0)–(H2) *hold. Then* $c_0 > -\infty$ *and any Palais-Smale sequence in* U_1 *for* I *at level* c_0 *is bounded. If in addition* (1.3) *is satisfied there does exist a bounded Palais-Smale sequence* $\{u_n\} \subset U_1$ *for* I *at level* c_0*.*

Proof. Let us show that J is bounded from below on U_1. Since by (H2), $I(u) \geq J(u), \forall u \in H$ this is clearly sufficient to prove that $c_0 > -\infty$. By definition of U_1, we have

$$||u||_b^2 \geq a(p-1)||u||_p^p,\ \forall u \in U_1,$$

or equivalently

$$\frac{a}{p}||u||_p^p \leq \frac{1}{p(p-1)}||u||_b^2,\ \forall u \in U_1.$$

Thus

$$J(u) \geq \big[\frac{1}{2} - \frac{1}{p(p-1)}\big]||u||_b^2 - ||h||_b||u||_b,\ \forall u \in U_1, \tag{3.1}$$

and this proves our claim. Another consequence of (3.1) is that all Palais-Smale sequences in U_1 for I at level c_0 are bounded. Now let $\{v_n\} \subset \bar{U_1} = U_1 \cup \Lambda$ be such that $I(v_n) \to c_0$. Since $c_0 < c_1$ we can assume without restriction that $\{v_n\}$ lies in the interior of $\bar{U_1}$, namely in U_1. Thus the Ekeland's variational principle (see [7] for example) guarantees the existence of a Palais-Smale sequence $\{u_n\} \subset U_1$ for I at level c_0. □

Let us now introduce the problem at infinity associated to (1.1),

$$-\Delta u + u = \tilde{f}(u), \tag{$\widetilde{1.1}$}$$

and the corresponding functional $\tilde{I} : H \to \mathbb{R}$ defined by

$$\tilde{I}(u) = \frac{1}{2}||u||^2 - \int_{\mathbb{R}^N} \tilde{F}(u) \ \text{ where } \tilde{F}(t) = \int_0^t \tilde{f}(s)\, ds.$$

We recall (see [2] for such results) that

$$S^\infty = \inf\{\tilde{I}(u), u \in H, u \not\equiv 0, \tilde{I}'(u) = 0\} > 0. \tag{3.2}$$

For future reference note also that a minimum exists and is realized by a positive function w. This function satisfies (as all classical solutions) the Pohozaev's identity (see [2]). In our case this means that:

$$\frac{N-2}{2}||\nabla w||_2^2 = N\big\{-\frac{1}{2}||w||_2^2 + \int_{\mathbb{R}^N} \tilde{F}(w)\big\}. \tag{3.3}$$

Proposition 3.1. *Assume that* (H0)–(H4) *hold and let* $\{v_n\} \subset H$ *be a bounded Palais-Smale sequence for* I. *Then there exists a subsequence* (*still denoted* $\{v_n\}$) *for which the following holds: there exist an integer* $m \geq 0$, *sequences* $\{x_n^i\} \subset \mathbb{R}^N$ *for* $1 \leq i \leq m$, *a solution* v_0 *of* (1.1) *and solutions* v^i, *for* $1 \leq i \leq m$, *of* $\widetilde{(1.1)}$ *such that:*

$v_n \rightharpoonup v_0$ *weakly in* H,
$I(v_n) \to I(v_0) + \sum_{i=1}^m \tilde{I}(v^i)$,
$v_n - v_0 - \sum_{i=1}^m v^i(x - x_n^i) \to 0$ *strongly in* H,
$|x_n^i| \to \infty$, $|x_n^i - x_n^j| \to \infty$, *for* $1 \leq i \neq j \leq m$,

where we agree that in the case $m = 0$ *the above holds without* v^i, x_n^i.

This is a standard result that we give here without proof (see [1] for analogous statements).

Lemma 3.2. *Assume that* (H0)–(H4) *and* (1.5) *hold. Then any Palais-Smale sequence* $\{u_n\} \subset U_1$ *for* I *at level* c_0 *converges strongly to a critical point* $u_0 \in \bar{U}_1$. *In particular, if* (1.3) *holds,* $u_0 \in U_1$ *and is a minimum for* I *on* U_1.

Proof. By Proposition 3.1 we know that $\{u_n\}$ satisfies

$$u_n - u_0 - \sum_{i=1}^{m} u^i(x - x_n^i) \to 0 \text{ strongly in } H, \tag{3.4}$$

for some u_0 with $I'(u_0) = 0$ and some appropriate $u^i, \{x_n^i\}$. Let us prove there is no term in the summation, namely, that $m = 0$. If this is the case then $u_n \to u_0$ strongly in H and $u_0 \in \bar{U}_1$ is a critical point. Moreover if (1.3) holds, then $c_0 < c_1$ and since $I(u_0) = c_0$ it follows that $u_0 \in U_1$ and is a minimum for I on U_1.

We start by the observation that if $u^i \neq 0$ is such that $\tilde{I}'(u^i) = 0$ then $g(u^i) < 0$. Indeed,

$$\tilde{I}'(u) = 0 \Longrightarrow ||u||^2 - \int_{\mathbb{R}^N} \tilde{f}(u)u = 0. \tag{3.5}$$

On the other hand, from (H2) and (H4), we get

$$\int_{\mathbb{R}^N} \tilde{f}(u)u \leq \int_{\mathbb{R}^N} f(x,u)u \leq a||u||_p^p + b||u||_2^2. \tag{3.6}$$

Combining (3.5) and (3.6) we obtain

$$\begin{aligned} g(u^i) &= ||u^i||^2 - b||u^i||_2^2 - a(p-1)||u^i||_p^p \\ &= \int_{\mathbb{R}^N} \tilde{f}(u^i)u^i - b||u^i||_2^2 - a(p-1)||u^i||_p^p \leq -a(p-2)||u^i||_p^p < 0. \end{aligned}$$

Moreover, since

$$I(u_n) \to I(u_0) + \sum_{i=1}^{m} \tilde{I}(u^i),$$

and by (3.2), $\tilde{I}(u^i) > 0$, for all $1 \le i \le m$, if we assume for the sake of contradiction that $m \ge 1$ we get

$$I(u_0) < \liminf_{n\to\infty} I(u_n) = c_0.$$

This implies that $u_0 \notin U_1$ and consequently that $g(u_0) < 0$.

Let us now evaluate $g(u_0 + \sum_{i=1}^m u^i(x - x_n^i))$. Since $u_n \in U_1$ we have $g(u_n) > 0$. Thus because of (3.4) the uniform continuity of g implies that

$$0 \le \liminf_{n\to\infty} g(u_n) = \liminf_{n\to\infty} g(u_0 + \sum_{i=1}^{m} u^i(x - x_n^i)). \tag{3.7}$$

On the other hand, since $|x_n^i| \to \infty$, $|x_n^i - x_n^j| \to \infty$, for $1 \le i \ne j \le m$, the supports of $u_0(\cdot)$ and $u^i(\cdot - x_n^i)$ are going increasingly far away as $n \to +\infty$ and we get

$$\lim_{n\to\infty} g(u_0 + \sum_{i=1}^{m} u^i(x - x_n^i)) = g(u_0) + \lim_{n\to\infty} \sum_{i=1}^{m} g(u^i(x - x_n^i)) = g(u_0) + \sum_{i=1}^{m} g(u^i)$$

with the last equality obtained using the fact that g is invariant under translation in $\mathbb{R}^N$. Now since $g(u_0) < 0$ and $g(u^i) < 0$, for $1 \le i \ne j \le m$, this contradicts (3.7).

Lemma 3.3. *Assume that* (H0)–(H4) *and* (1.5) *hold. Then I has a critical point $u_0 \in \bar{U}_1$ with $u_0 \in U_1$ and $I(u_0) = c_0$ if* (1.3) *holds.*

Proof. If (1.3) holds we know by Lemma 3.1 that there exists a Palais-Smale sequence for I at level c_0. Thus in this case the claim of the lemma follows directly from Lemma 3.2. When (1.5) is satisfied but not (1.3) we use an approximation argument. If h satisfies (1.5) then $h_n = (1 - \frac{1}{n})h$ satisfies (1.3) for all $n \in \mathbb{N}$. Thus by Lemma 3.2, for all $n \in \mathbb{N}$, there is $u_n \in U_1$ such that $I_n'(u_n) = 0$ with

$$I_n(u) = \frac{1}{2}||u||^2 - \int_{\mathbb{R}^N} F(x,u) - (1 - \frac{1}{n}) \int_{\mathbb{R}^N} hu.$$

Clearly $\{u_n\} \subset U_1$ is a Palais-Smale sequence for I at level c_0. Thus, still by Lemma 3.2, it strongly converges to a critical point $u_0 \in \bar{U}_1$.

Lemma 3.4. *The critical point $u_0 \in \bar{U}_1$ obtained in Lemma* 3.3 *is a weak solution of* (1.1) *which is positive if $h \geq 0$.*

Proof. Clearly u_0 is a weak solution of (1.1). Moreover if $h \geq 0$ we can choose $u_0 \geq 0$. Indeed, when (1.3) holds we have $I(|u_0|) \leq I(u_0)$ with $|u_0| \in U_1$ and thus $|u_0|$ is also a minimum for I on U_1. If (1.5) holds u_0 may not belong to U_1 but to Λ. However the approximation procedure (used with $u_n \geq 0$) guarantees that we still have $u_0 \geq 0$. Now by the maximum principle of weak solution (see, for example, Theorem 8.19 in [5]) we can show that $u_0 > 0$ for almost every $x \in \mathbb{R}^N$.

4. Existence of a second positive solution. In this section, under assumptions (H0)–(H4), (1.3) and $h \geq 0$, we prove by a Mountain Pass-type argument, the existence of a second positive solution. Since we look for positive solution we assume that $f(x,t) = \tilde{f}(t) = 0$ for almost every $x \in \mathbb{R}^N$ for $t \leq 0$.

We start with some preliminaries. Let $w > 0$ be a fixed positive function which realizes the infimum in (3.2) and for each $t > 0$, let $w_t \in H$ be defined by $w_t(x) = w(\frac{x}{t})$.

Lemma 4.1. *Assume that* (H0)–(H4), (1.3) *and $h \geq 0$ hold and let $u_0 \in H$ be the critical point obtained in Lemma* 3.3. *Then*

(i) $I(u_0 + w_t) < I(u_0) + \tilde{I}(w_t),\ \forall t \in \mathbb{R}^+$,

(ii) $\tilde{I}(w_t)$ *is maximal for $t = 1$ and thus* $I(u_0 + w_t) < I(u_0) + S^\infty,\ \forall t \in \mathbb{R}^+$,

(iii) $\lim_{t\to 0} ||w_t|| = 0$ *and* $\lim_{t\to\infty} ||w_t|| = +\infty$,

(iv) $\lim_{t\to\infty} \tilde{I}(w_t) = -\infty$.

Proof. Using the fact that u_0 is a critical point for I we get:

$$\begin{aligned} I(u_0 + w_t) &= \frac{1}{2}||u_0 + w_t||^2 - \int_{\mathbb{R}^N} F(x, u_0 + w_t) - \int_{\mathbb{R}^N} h(u_0 + w_t) \\ &= \frac{1}{2}||u_0||^2 + \frac{1}{2}||w_t||^2 + \int_{\mathbb{R}^N} \nabla u_0 \nabla w_t + u_0 w_t - \int_{\mathbb{R}^N} F(x, u_0 + w_t) \\ &\quad - \int_{\mathbb{R}^N} hu_0 - \int_{\mathbb{R}^N} hw_t \\ &= I(u_0) + \int_{\mathbb{R}^N} F(x, u_0) + \tilde{I}(w_t) + \int_{\mathbb{R}^N} \tilde{F}(w_t) \\ &\quad - \int_{\mathbb{R}^N} F(x, u_0 + w_t) + \int_{\mathbb{R}^N} f(x, u_0) w_t. \end{aligned}$$

Thus, using (H4), it follows that

$$\begin{aligned} &I(u_0 + w_t) \\ &= I(u_0) + \tilde{I}(w_t) - \int_{\mathbb{R}^N} F(x, u_0 + w_t) - F(x, u_0) - \tilde{F}(w_t) - f(x, u_0) w_t \\ &\leq I(u_0) + \tilde{I}(w_t) - \int_{\mathbb{R}^N} \int_0^{w_t(x)} \big\{ f(x, u_0 + s) - f(x, u_0) - f(x, s) \big\}\, ds. \end{aligned}$$

Clearly the growth condition (H3) implies that

$$f(x, t_1 + t_2) > f(x, t_1) + f(x, t_2), \text{ a.e. } x \in \mathbb{R}^N, \quad \forall t_1, t_2 > 0,$$

and since both $u_0 > 0$ (by Lemma 3.4) and $w > 0$ we get Point (i), namely that, for all $t > 0$,

$$I(u_0 + w_t) < I(u_0) + \tilde{I}(w_t).$$

Let us now prove the remaining points. From standard calculations we get

$$||w_t||_2 = t^{\frac{N}{2}}||w||_2, \ ||\nabla w_t||_2 = t^{(\frac{N}{2}-1)}||\nabla w||_2 \text{ and } \int_{\mathbb{R}^N} \tilde{F}(w_t) = t^N \int_{\mathbb{R}^N} \tilde{F}(w), \tag{4.1}$$

and Point (iii) is established. Using (4.1) we also get

$$\tilde{I}(w_t) = \frac{1}{2}t^{(N-2)}||\nabla w||_2^2 - t^N\{\int_{\mathbb{R}^N} \tilde{F}(w)\, dx - \frac{1}{2}||w||_2^2\}$$

and taking into account (3.3)

$$\tilde{I}(w_t) = \frac{1}{2}t^{(N-2)}||\nabla w||_2^2 - t^N\{\frac{N-2}{2N}||\nabla w||_2^2\}.$$

This shows in particular that Point (iv) holds. To complete the proof of the lemma we now check Point (ii), namely that $\sup_{t\in\mathbb{R}^+} \tilde{I}(w_t) = \tilde{I}(w)$. But this holds since for $t > 0$

$$0 = \frac{d}{dt}\tilde{I}(w_t) = \frac{1}{2}(N-2)t^{(N-3)}||\nabla w||_2^2 - Nt^{(N-1)}\{\frac{N-2}{2N}||\nabla w||_2^2\} \Longleftrightarrow t = 1.$$

Lemma 4.2. *Assume that* (H0)–(H3) *hold. Then, for any $c \in \mathbb{R}$, all Palais-Smale sequences for I at level c are bounded.*

Proof. Let $\{u_n\} \subset H$ be a Palais-Smale sequence for I at level $c \in \mathbb{R}$. Using (H3) we have, for $n \in \mathbb{N}$ sufficiently large,

$$\begin{aligned}
1 + c + ||u_n|| &\geq I(u_n) - \frac{1}{\mu}I^{'}(u_n)u_n \\
&= (\frac{1}{2} - \frac{1}{\mu})||u_n||^2 + \int_{\mathbb{R}^N} \{\frac{1}{\mu}f(x, u_n)u_n - F(x, u_n)\} + (\frac{1}{\mu} - 1)\int_{\mathbb{R}^N} hu_n \\
&= (\frac{1}{2} - \frac{1}{\mu})||u_n||_{\bar{b}}^2 + \int_{\mathbb{R}^N} \{\frac{1}{\mu}\bar{f}(x, u_n)u_n - \bar{F}(x, u_n)\} + (\frac{1}{\mu} - 1)\int_{\mathbb{R}^N} hu_n \\
&\geq (\frac{1}{2} - \frac{1}{\mu})||u_n||_{\bar{b}}^2 - \frac{\mu - 1}{\mu}||h||_{H^{-1}}||u_n||,
\end{aligned}$$

where we have set $||u||_{\bar{b}}^2 = ||\nabla u||_2^2 + (1 - \bar{b})||u||_2^2$. This proves that $\{u_n\}$ is bounded.

Lemma 4.3. *Assume that* (H0)–(H4), (1.3) *and* $h \geq 0$ *hold. Then equation* (1.1) *has a second critical point* $v_0 \neq u_0$.

Proof. In view of Lemma 4.1, for $t > 0$ large, $I(u_0 + w_t) \leq I(u_0)$ and $u_0 + w_t \in U_2$. Let $t_0 > 0$ be one of these values. We set $\gamma = \inf_{g\in\Gamma} \max_{t\in[0,1]} I(g(t))$, where

$$\Gamma = \{g \in C([0,1], H), g(0) = u_0, g(1) = u_0 + w_{t_0}\}.$$

From Lemma 4.1, Point (ii) and since $\inf_\Lambda I(u) > I(u_0)$ we deduce that

$$I(u_0) < \gamma < I(u_0) + S^\infty. \tag{4.2}$$

Ekeland's variational principle insures the existence of a Palais-Smale sequence $\{u_n\}$ for I at level γ. Using Lemma 4.2 and Proposition 3.1 we deduce that

$$\gamma = \lim_{n\to+\infty} I(v_n) = I(v_0) + \sum_{i=1}^{m} \tilde{I}(v^i),$$

for some v_0, v^i satisfying $I^{'}(v_0) = 0$ and $\tilde{I}^{'}(v^i) = 0$, for $1 \leq i \leq m$. To get a second critical point we just need to show that $v_0 \neq u_0$. Assume for the sake of contradiction that $v_0 = u_0$ then either $\gamma = I(u_0)$ or $\gamma \geq I(u_0) + S^\infty$ must hold. This contradicts (4.2).

Lemma 4.4. *When we assume that* $f(x,t) = 0$ *for almost every* $x \in \mathbb{R}^N$, $\forall t \leq 0$ *any critical point* v *of* I *is a positive weak solution of* (1.1). *In particular the critical point* v_0 *obtained in Lemma* 4.3 *is a positive weak solution of* (1.1).

Proof. Setting $v^- = \min\{0, v\}$ since v solves (1.1) and $\int_{\mathbb{R}^N} f(x,v)v^- = 0$ we obtain

$$\int_{\mathbb{R}^N} |\nabla v^-|^2 + |v^-|^2 = \int_{\mathbb{R}^N} hv^- \leq 0.$$

Thus $v \geq 0$ and as in Lemma 3.4 it follows that $v > 0$.

5. Proofs of the main results. In this final section we give the proofs of Theorems 1.1, 1.2 and 1.3. Clearly Theorem 1.1 follows directly from Lemmas 3.3, 3.4, 4.3 and 4.4. To establish the other two theorems we need some preliminaries:

Lemma 5.1. *For all* $u \in \Lambda$,

$$\{\frac{p-2}{p-1}\}||u||_b \geq \tilde{c}\ S_b^{\frac{p}{2(p-2)}}.$$

Proof. Let $u \in \Lambda$ be arbitrary. Using the fact that $u \in \Lambda$ we get

$$||u||_p = \frac{||u||_b^{\frac{2}{p}}}{[a(p-1)]^{\frac{1}{p}}}$$

and since, $S_b^{\frac{1}{2}}||u||_p \leq ||u||_b$ (by definition of S_b),

$$||u||_b \geq S_b^{\frac{1}{2}} \; ||u||_p \geq S_b^{\frac{1}{2}} \frac{||u||_b^{\frac{2}{p}}}{[a(p-1)]^{\frac{1}{p}}},$$

or equivalently that

$$||u||_b \geq S_b^{\frac{p}{2(p-2)}} \big[\frac{1}{a(p-1)}\big]^{\frac{1}{p-2}}.$$

Thus by definition of $\tilde{c}$ we indeed get that

$$\{\frac{p-2}{p-1}\}||u||_b \geq \tilde{c} \; S_b^{\frac{p}{2(p-2)}}, \text{ for all } u \in \Lambda.$$

Lemma 5.2. *Conditions* (1.3) *and* (1.5) *hold if* $||h||_b < \tilde{c} \; S_b^{\frac{p}{2(p-2)}}$ *and* $||h||_b \leq \tilde{c} \; S_b^{\frac{p}{2(p-2)}}$ *respectively.*

Proof. First notice that, since Λ is bounded away from 0,

$$\begin{aligned}(1.3) &\iff \tilde{c} \; \frac{||u||_b^{\frac{2(p-1)}{p-2}}}{||u||_p^{\frac{p}{p-2}}} - \int_{\mathbb{R}^N} hu > 0, \text{ for } ||u||_p = 1 \\ &\iff \tilde{c} \; \frac{||u||_b^{\frac{2(p-1)}{p-2}}}{||u||_p^{\frac{p}{p-2}}} - \int_{\mathbb{R}^N} hu > 0, \text{ for } \; u \in \Lambda \\ &\iff \{\frac{p-2}{p-1}\}||u||_b^2 - \int_{\mathbb{R}^N} hu > 0, \text{ for } \; u \in \Lambda,\end{aligned}$$

where the last equivalence is obtained using the fact that $u \in \Lambda$. Thus if we insure that

$$\inf_{u \in \Lambda} \{\frac{p-2}{p-1}\}||u||_b^2 > \sup_{u \in \Lambda} \int_{\mathbb{R}^N} hu, \tag{5.1}$$

it is sufficient for (1.3) to hold. Equivalently if

$$\inf_{u \in \Lambda} \{\frac{p-2}{p-1}\}||u||_b^2 \geq \sup_{u \in \Lambda} \int_{\mathbb{R}^N} hu, \tag{5.2}$$

then (1.5) holds. But since,

$$\int_{\mathbb{R}^N} hu \leq ||h||_b \; ||u||_b, \; \forall u \in H, \tag{5.3}$$

in view of Lemma 5.1, this is indeed the case.

Proof of Theorem 1.3. If we add (H3) and (H4) to its assumptions Theorem 1.3 is just a combination of Lemmas 3.3, 3.4 and 5.2. These two assumptions, however, are not needed. Indeed, under (H0)–(H2) and (1.3), Lemma 3.1 gives the existence of a bounded Palais-Smale sequence $\{u_n\} \subset U_1$ for I at level c_0. Passing to a subsequence there is $u \in H$ such that $u_n \rightharpoonup u$ weakly in H and it is standard to show that u is a critical point for I. Moreover, using (H2), (3.1) and the fact that $\{u_n\} \subset U_1$ we get

$$\begin{aligned} 0 > c_0 &= \liminf_{n\to+\infty} I(u_n) \geq \liminf_{n\to+\infty} J(u_n) \\ &\geq \{\frac{1}{2} - \frac{1}{p(p-1)}\}||u_n||_b^2 - ||h||_b||u_n||_b. \end{aligned}$$

Thus

$$||u||_b \leq \{\frac{1}{2} - \frac{1}{p(p-1)}\}^{-1}||h||_b.$$

Now if (1.5) holds but not (1.3), following the approximation argument developed in the proof of Lemma 3.3 we obtain a Palais-Smale sequence $\{u_n\} \subset H$ satisfying

$$||u_n|| \leq \{\frac{1}{2} - \frac{1}{p(p-1)}\}^{-1}(1-\frac{1}{n})\ ||h||_b.$$

Consequently $\{u_n\}$ is bounded and passing to a subsequence we again have $u_n \rightharpoonup u$ with u a critical point of I. Finally if $h \geq 0$, as in Lemma 4.4, we get that $u > 0$.

Proof of Theorem 1.2. When $||h||_b < \tilde{c}\ S_b^{\frac{p}{2(p-2)}}$, using Theorem 1.1 and Lemma 5.2, we immediately see that the claim of Theorem 1.2 holds. To complete the proof let us show that (1.3) also holds when h satisfies

$$||h||_b = \tilde{c}\ S_b^{\frac{p}{2(p-2)}} \text{ and } \int_{\mathbb{R}^N} hu < ||h||_b\ ||u||_b \text{ for } u \in F_b. \tag{5.4}$$

Since $||h||_b = \tilde{c}\ S_b^{\frac{p}{2(p-2)}}$ we see by (5.3) and Lemma 5.1 that the inequality (5.2) holds. In turn this clearly implies that

$$\inf_{||u||_p=1} \tilde{c}\ ||u_n||_b^{\frac{2(p-1)}{p-2}} \geq \sup_{||u||_p=1} \int_{\mathbb{R}^N} hu. \tag{5.5}$$

Let us show there is $\delta > 0$ such that, $\forall u \in H,\ ||u||_p = 1$ if

$$\tilde{c}\ ||u||_b^{\frac{2(p-1)}{p-2}} \leq \inf_{||u||_p=1} \tilde{c}\ ||u||_b^{\frac{2(p-1)}{p-2}} + \delta,$$

then

$$\int_{\mathbb{R}^N} hu \leq \sup_{||u||_p=1} \int_{\mathbb{R}^N} hu - \delta.$$

If this is true, because of (5.5) condition (1.3) holds. Let us assume for the sake of contradiction that there is no such $\delta > 0$. Then there exists $\{u_n\} \subset H$, $||u_n||_p = 1$ such that

$$\tilde{c}\ ||u_n||_b^{\frac{2(p-1)}{p-2}} \to \inf_{||u||_p=1} \tilde{c}\ ||u||_b^{\frac{2(p-1)}{p-2}} \tag{5.6}$$

and

$$\int_{\mathbb{R}^N} hu_n \to \sup_{||u||_p=1} \int_{\mathbb{R}^N} hu. \tag{5.7}$$

Clearly if $\{u_n\}$ satisfies (5.6) it is a minimizing sequence for S_b. Using the concentration compactness principle of P.L. Lions ([6]), it is standard to show that neither the vanishing nor the dichotomy of $\{u_n\}$ can occurs. Moreover (5.7) prevents a possible loss of compactness due to the translational invariance of (5.6). Thus the sequence $\{u_n\}$ converges strongly to a $\bar{u} \in H$, $||\bar{u}||_p = 1$ satisfying

$$||\bar{u}||_b = S_b^{\frac{1}{2}} \quad \text{and} \quad \int_{\mathbb{R}^N} h\bar{u} = ||h||_b ||\bar{u}||_b.$$

But this contradicts (5.4) and the proof of Theorem 1.2 is completed.

Acknowledgment. The author thanks Professors M.J. Esteban and H.S. Zhou for stimulating discussions on the subject. He also expresses his gratitude to Professor C.A. Stuart whose comments and help improved an earlier version of the work.

REFERENCES

[1] V. Benci and G. Cerami, *Positive solutions of some nonlinear elliptic problems in exterior domains,* Arch. Rat. Mech. Anal., 99 (1987), 283–300.

[2] H. Berestycki and P.L. Lions, *Nonlinear scalar field equations I,* Arch. Rat. Mech. Anal., 82 (1983), 313–346.

[3] D.M. Cao and H.S. Zhou, *Multiple positive solutions of nonhomogeneous semilinear elliptic equations in* $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh, 126 A (1996), 443–463.

[4] D.M. Cao and H.S. Zhou, *On the existence of multiple solutions on nonhomogeneous elliptic equations involving critical Sobolev exponents,* ZAMP, 47 (1996), 89–97.

[5] D. Gilbarg and N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order," second edition, Springer-Verlag, 1983.

[6] P.L. Lions, *The concentration compactness principle in the calculus of variation. The locally compact case,* Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 1, 109–145.

[7] J. Mahwin and M. Willem, "Critical Point Theory and Hamiltonian Systems," Appli. Math. Sci., 74, Springer, 1989.

[8] G. Tarantello, *On nonhomogeneous elliptic equations involving critical Sobolev exponent,* Ann. Inst. H. Poincaré Anal. Non Linéaire, 9 (1992), 281–304.

[9] X.P. Zhu, *A perturbation result on positive entire solutions of a semilinear elliptic equation,* J.D.E., 92 (1991), 163–178.

[10] X.P. Zhu and H.S. Zhou , *Existence of multiple positive solutions of inhomogeneous semilinear elliptic problems in unbounded domains,* Proc. Roy. Soc. Edinburgh, 115, A (1990), 301–318.