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THE DISTANCE TO L^∞ IN SOME FUNCTION SPACES AND APPLICATIONS*

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Abstract. L^∞ is not dense in some function spaces like: the space EXP of exponentially integrable functions; the Marcinkiewicz space $L^{q,\infty}$=weak-L^q; the Orlicz space L^A when the convex continuously increasing function A, does not satisfy the so-called Δ_2-condition. We find formulas for the distance to L^∞ in these spaces. Using the simple observation that if a bounded linear operator $T{:}L^q\to W$ satisfies $T(L^\infty)\subset L^\infty$, then $\mathrm{dist}_W(Tf,L^\infty)=0,\quad \forall f\in L^q$, we give some applications of previous results (see Section 5) to integrability properties of Riesz potential and of solutions to linear elliptic equations.

1. Introduction. The Marcinkiewicz space $L^{q,\infty}(\Omega)$ = weak-$L^q(\Omega)$ $(q>1)$ and the Orlicz space $EXP_\alpha(\Omega)$, $(\alpha>0)$, where $\Omega\subset\mathbb{R}^n$ is an open set, are defined according to the norms

$$||f||_{L^{q,\infty}}=\sup_{E\subseteq\Omega}|E|^{\frac1q}\fint_E|f|\,dx,\qquad ||f||_{EXP_\alpha}=\inf\{\lambda>0:\fint_\Omega e^{|\frac{f}{\lambda}|^\alpha}\,dx\le2\},$$

where $\fint_E h\,dx=\frac{1}{|E|}\int_E h\,dx$. These spaces arise in a natural way when one considers the behaviour of the Riesz potential

$$Ig(x)=\int_\Omega g(y)|x-y|^{1-n}dy,\qquad g\in L^p(\Omega)\tag{1.1}$$

in the limiting cases $p=1$ and $p=n$ respectively.

Namely, the operator (1.1) is bounded from:

$$L^p(\Omega)\quad\text{to}\quad L^{\frac{np}{n-p}}(\Omega)\quad\text{for}\quad 1<p<n\tag{1.2}$$
$$L^1(\Omega)\quad\text{to}\quad L^{\frac{n}{n-1},\infty}(\Omega)\tag{1.3}$$
$$L^n(\Omega)\quad\text{to}\quad EXP_{\frac{n}{n-1}}(\Omega)\tag{1.4}$$

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(see [8], [15]). It is well known (see [12], Sect. 0, and [13], Chap. 3) that L^∞ is dense neither in $L^{q,\infty}$ nor in EXP_α. A characterization of the closure of L^∞ in $L^{q,\infty}$ denoted by $L_0^{q,\infty}$, is given for example in [1]: if Ω is a bounded open set in $\mathbb{R}^n$, then $f \in L_0^{q,\infty}(\Omega)$ if and only if

$$\lim_{|E|\to 0} |E|^{\frac{1}{q}} \fint_E |f|\,dx = 0. \tag{1.5}$$

One of the results of the present paper (Theorem 3.1) gives a formula for the distance to L^∞ in $L^{q,\infty}$. This generalizes (1.5):

$$\operatorname{dist}_{L^{q,\infty}}(f, L^\infty) = \limsup_{|E|\to 0} |E|^{\frac{1}{q}} \fint_E |f|\,dx. \tag{1.6}$$

Another result (Theorem 2.1 combined with Proposition 1 in [6]) gives a similar formula for the space EXP_α

$$\begin{aligned} \operatorname{dist}_{EXP_\alpha}(f, L^\infty) &= \inf\{\lambda > 0 : \fint_\Omega e^{|\frac{f}{\lambda}|^\alpha}\,dx < \infty\} \\ &= e \limsup_{q\to\infty} \frac{1}{q} (\fint_\Omega |f|^{\alpha q}\,dx)^{\frac{1}{q}}. \end{aligned} \tag{1.7}$$

It is interesting to compare (1.6), (1.7) with the respective definitions of the norms.

There are some immediate applications of the above formulas. Here is one.

Example. Since $L^\infty(\Omega)$ is dense in $L^p(\Omega)$, actually (1.3), (1.4) imply that the Riesz potential is bounded into $L_0^{\frac{n}{n-1},\infty}(\Omega)$ and $exp_{\frac{n}{n-1}}(\Omega)$ respectively (see Lemma 5.1), where $exp_\alpha(\Omega)$ denotes the closure of $L^\infty(\Omega)$ in $EXP_\alpha(\Omega)$. In particular, formula (1.5) implies that, for any $g \in L^1(\Omega)$,

$$\lim_{|E|\to 0} |E|^{1-\frac{1}{n}} \fint_E |Ig(x)|\,dx = 0$$

and formula (1.7) implies that, for any $g \in L^n$,

$$\lim_{q\to\infty} \frac{1}{q^{1-\frac{1}{n}}} (\fint_\Omega |Ig(x)|^q\,dx)^{\frac{1}{q}} = 0$$

or, equivalently,

$$\fint_\Omega e^{\frac{|Ig|^{\frac{n}{n-1}}}{\lambda}}\,dx < \infty, \qquad \forall\lambda > 0.$$

For the other applications, see Section 5.

2. The distance in Orlicz spaces to L^∞. The usual norm that is introduced in the Orlicz space L^A, where $A : [0,\infty) \to [0,\infty)$ is a convex continuously increasing function

$$||f||_A = \inf\{\lambda > 0 : \fint_\Omega A(\frac{|f|}{\lambda})(x)\,dx \le 1\} \tag{2.1}$$

is the well-known Luxemburg norm.

We say that $A(s)$ satisfies the Δ_2 condition if there exists a positive constant k, such that $A(2s) \le kA(s), \quad \forall s \ge 0$. If $A(s)$ does not have Δ_2-property, then $L^\infty(\Omega)$ is not dense in $L^A(\Omega)$ (see [13], Chapter 3). Now, let us define

$$E_A(f) = \inf\{\lambda > 0 : \fint_\Omega A(\frac{|f|}{\lambda})\,dx < \infty\}.$$

We observe that for every f, $g \in L^A(\Omega)$ we have

$$E_A(f) \le ||f||_A \quad E_A(\lambda f) = |\lambda| E_A(f), \quad E_A(f+g) \le E_A(f) + E_A(g)$$

and easily, we obtain

$$E_A(f-g) = E_A(f) \quad \text{and} \quad E_A(g) = 0 \quad \forall g \in L^\infty.$$

Theorem 2.1. *Let Ω be a bounded open set in $\mathbb{R}^n$. Let A be a Young function. For every f belonging to $L^A(\Omega)$, we have*

$$\operatorname{dist}_A(f, L^\infty) = E_A(f),$$

where the distance is evaluated with respect to the norm (2.1).

Corollary 2.1. *$E_A(f) = 0$ if and only if f belongs to the closure of $L^\infty(\Omega)$ in $L^A(\Omega)$.*

Remark 2.1. In the case of the space $\mathrm{EXP}(\Omega) = \mathrm{EXP}_1(\Omega)$ this result was obtained in ([6], Proposition 2).

Proof of Theorem 2.1. Let f belong to L^A. For every $g \in L^\infty$, we have

$$E_A(f) = E_A(f-g) \le ||f-g||_A$$

and hence

$$E_A(f) \le \operatorname{dist}_A(f, L^\infty).$$

Now it suffices to prove that if $\lambda > 0$ satisfies

$$\fint_\Omega A(\frac{|f|}{\lambda}) < \infty$$

then

$$\mathrm{dist}_A(f, L^\infty) \leq \lambda.$$

To this end observe that, from the absolute continuity of the integral, there exists $h_\lambda > 0$ such that

$$\frac{1}{|\Omega|}\int_{\{|f|>h_\lambda\}} A(\frac{|f|}{\lambda}) \leq 1.$$

If we set $f_\lambda(x) = f(x)$ when $|f(x)| \leq h_\lambda$ and $|f_\lambda(x)| = 0$ when $|f(x)| > h_\lambda$, then

$$\begin{aligned}\fint_\Omega A(\frac{|f-f_\lambda|}{\lambda}) &= \frac{1}{|\Omega|}(\int_{\{|f|>h_\lambda\}} A(\frac{|f-f_\lambda|}{\lambda}) + \int_{\{|f|\leq h_\lambda\}} A(\frac{|f-f_\lambda|}{\lambda}))\\ &= \frac{1}{|\Omega|}\int_{\{|f|>h_\lambda\}} A(\frac{|f-f_\lambda|}{\lambda}) \leq 1\end{aligned}$$

and finally

$$\mathrm{dist}_A(f, L^\infty) \leq ||f - f_\lambda||_A \leq \lambda.$$

Let us compare the preceding result with the following one (see [13], Chapter 3, Proposition 3).

Theorem 2.2. *Let Ω a bounded open set in $\mathbb{R}^n$. Then for every $f \in L^A(\Omega)$,*

$$dist_A(f, L^\infty) = \lim_{k\to\infty} ||f - f_k||_A,$$

where $f_k(x) = f(x)$ when $|f(x)| \leq k$, and $f_k(x) = 0$ when $|f(x)| > k$.

3. The distance in weak-L^q to L^∞. Let us consider the Banach space $L^{q,\infty}(\Omega)$ with the norm

$$||f||_{q,\infty} = \sup_{E\subseteq\Omega} |E|^{\frac{1}{q}-1}\int_E |f|\,dx.$$

It is well known that $L^\infty(\Omega)$ is not dense in $L^{q,\infty}(\Omega)$.

We shall indicate by $L_0^{q,\infty}(\Omega)$ the closure of $L^\infty(\Omega)$ in $L^{q,\infty}(\Omega)$. In [5] (see also [1]) the following seminorm in $L^{p,\infty}(\Omega)$ was introduced

$$N_q(f) = \limsup_{|E|\to 0} |E|^{\frac{1}{q}} \fint_E |f|\,dx \tag{3.1}$$

and it was shown that if Ω is a bounded open set in $\mathbb{R}^n$, a function f belongs to $L_0^{q,\infty}(\Omega)$ if and only if $N_q(f) = 0$. In this section we will improve this result by showing an explicit formula for the distance in $L^{q,\infty}$ to L^∞. Namely we prove the following result:

Theorem 3.1. *Let Ω be a bounded open set in $\mathbb{R}^n$. Then for $f \in L^{q,\infty}(\Omega)$,*

$$\mathrm{dist}_{L^{q,\infty}}(f, L^\infty) = N_q(f). \tag{3.2}$$

Proof. First of all, it is simple to check that $N_q(f)$ has the following properties:

$$N_q(f+g) = N_q(f) \quad \forall g \in L^\infty \tag{3.3}$$

$$N_q(f) = \lim_{\lambda\to\infty} ||f - f_\lambda||_{L^{q,\infty}}, \tag{3.4}$$

where $f_\lambda(x) = \lambda$ when $|f(x)| \geq \lambda$, $f_\lambda(x) = 0$ when $|f(x)| < \lambda$. Now, to prove (3.2), let us note that if $f \in L^{q,\infty}$ and $g \in L^\infty$, we have

$$N_q(f) = N_q(f-g) \leq ||f-g||_{q,\infty}$$

and then $N_q(f) \leq \mathrm{dist}_{L^{q,\infty}}(f, L^\infty)$. Now it suffices to prove that $\lambda > N_q(f) \Longrightarrow \mathrm{dist}_{L^{q,\infty}}(f, L^\infty) \leq \lambda$. Let $\delta > 0$ be such that

$$|E| < \delta \Longrightarrow |E|^{\frac{1}{q}-1} \int_E |f|\, dx < \lambda.$$

By absolute continuity of the integral, there exists $h_\lambda > 0$ with

$$\int_{\{|f|>h_\lambda\}} |f|\, dx < \lambda\delta^{1-\frac{1}{q}}.$$

If we define $f_{h_\lambda}(x)$ as above, then for E with $|E| < \delta$, we deduce

$$|E|^{\frac{1}{q}-1} \int_E |f_{h_\lambda} - f|\, dx = |E|^{\frac{1}{q}-1} \int_{E\cap\{|f|>h_\lambda\}} |f|\, dx \leq |E|^{\frac{1}{q}-1} \int_E |f|\, dx < \lambda;$$

if $|E| \geq \delta$, we have

$$|E|^{\frac{1}{q}-1} \int_E |f_{h_\lambda} - f|\, dx \leq \delta^{\frac{1}{q}-1} \int_{\{|f|>h_\lambda\}} |f|\, dx < \delta^{\frac{1}{q}-1}\lambda\delta^{1-\frac{1}{q}} = \lambda$$

as desired.

4. The distance to L^∞ from a space "close" to L^q. Let Ω be an open set in $\mathbb{R}^n$. The function space, "close" to $L^q(\Omega)$,

$$L^{q)}(\Omega) = \Big\{ f \in L^1(\Omega) : ||f||_{L^{q)}} = \sup_{0<\varepsilon\leq q-1} \Big(\varepsilon \fint_\Omega |f|^{q-\varepsilon}\Big)^{\frac{1}{q-\varepsilon}} < \infty \Big\},$$

was introduced by T. Iwaniec and C. Sbordone in order to get new properties of the Jacobian determinant (see [2], [7], [9]).

Also for this space, we get a formula for the distance to L^∞.

Theorem 4.1. *For every function $f \in L^{q)}(\Omega)$, Ω an open bounded set of $\mathbb{R}^n$, we have*

$$\mathrm{dist}_{L^{q)}}(f, L^\infty) = \limsup_{\varepsilon \to 0^+} \big(\varepsilon \fint_\Omega |f|^{q-\varepsilon}\big)^{\frac{1}{q-\varepsilon}}. \tag{4.1}$$

Proof. Set

$$(f)_q = \limsup_{\varepsilon \to 0^+} \big(\varepsilon \fint_\Omega |f|^{q-\varepsilon}\big)^{\frac{1}{q-\varepsilon}}.$$

The thesis becomes $(f)_q = \mathrm{dist}_{L^{q)}}(f, L^\infty)$. For $g \in L^\infty$ and $f \in L^{q)}$, we have $(f)_q = (f-g)_q \leq ||f-g||_{L^{q)}}$ and then $(f)_q \leq \mathrm{dist}_{L^{q)}}(f, L^\infty)$. On the other hand $\lambda > (f)_q \Longrightarrow \mathrm{dist}_{L^{q)}}(f, L^\infty) \leq \lambda$. In order to prove that, fix $\lambda > (f)_q$; then there exists $\varepsilon_q > 0$ such that

$$\big(\frac{\varepsilon}{|\Omega|} \int_{\{|f|>h\}} |f|^{q-\varepsilon}\big)^{\frac{1}{q-\varepsilon}} < \lambda \tag{4.2}$$

for $0 < \varepsilon < \varepsilon_q$ and for any $h > 0$. After that, if $\varepsilon_q \leq \varepsilon \leq q-1$, we get

$$\big(\frac{\varepsilon}{|\Omega|} \int_{\{|f|>h\}} |f|^{q-\varepsilon}\big)^{\frac{1}{q-\varepsilon}} \leq \big(q \frac{1+|\Omega|}{|\Omega|} \int_{\{|f|>h\}} |f|^{q-\varepsilon_q}\big)^{\frac{1}{q-\varepsilon_q}}.$$

Now, let h_λ be such that

$$\big(q \frac{1+|\Omega|}{|\Omega|} \int_{\{|f|>h_\lambda\}} |f|^{q-\varepsilon_q}\big)^{\frac{1}{q-\varepsilon_q}} < \lambda \tag{4.3}$$

and set $f_\lambda(x) = f(x)$ if $|f(x)| \leq h_\lambda$, $f_\lambda(x) = 0$ when $|f(x)| > h_\lambda$. Finally, we obtain

$$\big(\varepsilon \fint_\Omega |f - f_\lambda|^{q-\varepsilon}\big)^{\frac{1}{q-\varepsilon}} = \big(\frac{\varepsilon}{|\Omega|} \int_{\{|f|>h_\lambda\}} |f|^{q-\varepsilon}\big)^{\frac{1}{q-\varepsilon}} \leq \lambda$$

in both cases $0 < \varepsilon < \varepsilon_q$ and $\varepsilon \geq \varepsilon_q$, by (4.2) and (4.3) respectively.

Remark 4.1. It is known that $L^{q,\infty}(\Omega) \subset L^{q)}(\Omega)$ (see [9]). Trivially we deduce

$$f \in L_0^{q,\infty}(\Omega) \Longrightarrow f \in \Sigma^q(\Omega),$$

where $\Sigma^q(\Omega) = clos_{L^{q)}} L^\infty(\Omega)$. Then by the characterizations of $L_0^{q,\infty}(\Omega)$ and $\Sigma^q(\Omega)$ previously given, we get that the condition

$$\lim_{|E| \to 0} |E|^{\frac{1}{q}} \fint_E |f|\, dy = 0 \quad \text{implies} \quad \lim_{\varepsilon \to 0^+} \varepsilon \fint_\Omega |f|^{q-\varepsilon}\, dy = 0.$$

In particular, coming back to the example given in the Introduction, we deduce that for any $g \in L^1(\Omega)$,

$$\lim_{\varepsilon \to 0^+} \varepsilon \fint_\Omega |Ig|^{\frac{n}{n-1}-\varepsilon}\, dx = 0.$$

5. Applications. The following lemma is a direct consequence of density of $L^\infty(\Omega)$ in $L^q(\Omega)$.

Lemma 5.1. *Let $T : L^q \to W$ be a bounded linear operator with values in the Banach space W. Assume that $T(L^\infty) \subset L^\infty$. Then $\forall f \in L^q$*

$$\text{dist}_W(Tf, L^\infty) = 0.$$

Example 1. (*Exponential integrability of solutions to Dirichlet problems*) Let Ω be a bounded open set of $\mathbb{R}^n$. Consider the class $\text{EXP}(\Omega) = \text{EXP}_1(\Omega)$. It follows from Theorem 2.1 that

$$g \in exp(\Omega) \quad \Leftrightarrow \quad \forall \lambda > 0 \quad e^{\frac{g}{\lambda}} \in L^1(\Omega), \tag{5.1}$$

where $exp(\Omega) = exp_1(\Omega)$ (see also [6], paragraph 2).

Recently Chanillo and Li (see [4]) have considered the following Dirichlet problem:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where $\Omega \subset \mathbb{R}^2$ is an arbitrary bounded open set,

$$Lu = -D_i(a_{ij}(x)D_j u)$$
$$a_{ij} = a_{ji} \in L^\infty, \quad \lambda^{-1}|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2$$

and $f \in L^1(\Omega)$. They obtained that, if u is a very weak solution, in the sense of Stampacchia (see [14]), of (5.2), then u belongs to $EXP(\Omega)$ and

$$||u||_{EXP} \le c||f||_{L^1}. \tag{5.3}$$

The same result in the case $L = -\Delta$ was obtained earlier by Brezis and Merle (see [3], Theorem 1). Moreover, they proved the following theorem that we deduce here in a different way:

Theorem 5.1. *If u is the weak solution of the problem* (5.2) *above, then*

$$\int_\Omega e^{\beta|u|}\, dx < \infty, \qquad \forall \beta > 0 \quad . \tag{5.4}$$

Proof. Let T be the operator which associates to any $f \in L^1(\Omega)$ the solution of the Dirichlet problem (5.2). Since T maps $L^\infty(\Omega)$ in $L^\infty(\Omega)$ (see [22], Theorem 2) the theorem follows from Lemma 5.1 and from equivalence (5.1). ∎

It was recently proven by V.A. Liskevich (see [11], Theorem 2) that the solution to the problem

$$\begin{cases} -D_i(a_{ij}(x)D_j u) = D_i f_i & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.5}$$

where Ω is a bounded set in $\mathbb{R}^n$, $n \ge 3$, and

$$|f| = (\sum_{i=1}^n f_i^2)^{\frac{1}{2}} \in L^{n,\infty}(\Omega),$$

satisfies $u \in EXP(\Omega)$.

We get the following reformulation of Liskevich's result (where ω_n = volume of the unit ball in $\mathbb{R}^n$).

Theorem 5.2. *If u is the weak solution of the problem* (5.4)*, then*

$$\mathrm{dist}_{EXP}(u, L^\infty) \leq \frac{||f||_{n,\infty}}{n\omega_n^{\frac{1}{n}}}. \tag{5.6}$$

Inequality (5.6) is sharp in the sense that the equality may occur. Following [11], note that the solution of the homogeneous Dirichlet problem in the unit ball of $\mathbb{R}^n$ for the equation $-\Delta u = f = x|x|^{-2}$ is $u(x) = -\log|x|$. In this case

$$\mathrm{dist}_{\mathrm{EXP}}(u, \mathrm{L}^\infty) = \frac{1}{n}, \quad ||f||_{n,\infty} = \omega_n^{\frac{1}{n}}.$$

Example 2. (*Imbeddings for functions u such that $|Du| \in L_0^{n,\infty}(\Omega)$*) The space $BMO(\Omega)$ consists of functions $g \in L^1(\Omega)$ such that

$$\sup_{B\subseteq\Omega} \fint_B |g - g_B|\, dy < \infty \quad .$$

Moreover, $VMO(\Omega)$ is the class of functions $g \in BMO(\Omega)$ such that

$$\lim_{r\to 0} \fint_{B_r(x)} |g - g_{B_r}|\, dy = 0$$

uniformly with respect to x, where $B_r = B_r(x)$ is the ball with radius r centered at x. It follows easily from the Poincarè inequality

$$\fint_{B_r} |u - u_{B_r}|\, dy \leq cr \fint_{B_r} |Du|\, dy$$

that $W^{1,n}(\Omega) \subset VMO(\Omega)$. If we put a weaker condition on u, namely if we assume that $u \in W^{1,1}(\Omega)$ and $|Du| \in L^{n,\infty}(\Omega)$, then $u \in BMO(\Omega)$. In the second case, in general, it does not follow that $u \in VMO(\Omega)$. However, as a direct consequence of the Poincarè inequality and Theorem 3.1 we obtain

Proposition 5.1. *Let Ω be a bounded open set of $\mathbb{R}^n$. If $|Du| \in L_0^{n,\infty}(\Omega)$, then $u \in VMO(\Omega)$.*

Example 3. (*Imbeddings for functions u such that $|Du| \in L^{n)}(\Omega)$*) In [6] it was proved by Fusco-Lions-Sbordone that the Riesz potential I satisfies

$$||If||_{\mathrm{EXP}(\Omega)} \leq c||f||_{L^{n)}(\Omega)}. \tag{5.7}$$

It was also directly proved that if $u \in W_0^{1,1}(\Omega)$ satisfies the condition

$$\lim_{\varepsilon\to 0^+} \varepsilon \fint_\Omega |Du|^{n-\varepsilon}\, dx = 0 \tag{5.8}$$

(in particular, if $|Du| \in L^n(logL)^{-1}$) then

$$\fint_\Omega e^{\frac{|u|}{\lambda}}\, dx < \infty \qquad \forall \lambda > 0. \tag{5.9}$$

We wish to deduce the implication (5.8) $\Longrightarrow$ (5.9) directly from (5.7), with the same idea as in Lemma 5.1.

According to the notation of the previous section:

$$\Sigma^n(\Omega) = clos_{L^{n)}} L^\infty(\Omega), \qquad \exp(\Omega) = clos_{\mathrm{EXP}} L^\infty(\Omega).$$

Then (5.7) implies that I is bounded from $\Sigma^n(\Omega)$ to $exp(\Omega)$. In particular if $|Du|$ satisfies (5.8) then, by Theorem 4.1 $|Du| \in \Sigma^n(\Omega)$ and so $u \in exp(\Omega)$ which, in turn, implies (5.9) by Corollary 2.1.

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