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SPACELIKE GRAPHS WITH PRESCRIBED MEAN CURVATURE

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Abstract. We describe all the rotationally symmetric spacelike graphs $G_u = \big(x, u(|x|)\big)$ in Minkowsky's space whose mean curvature is a prescribed function f of u. In particular, we prove the existence of regular and singular solutions by means of a fixed-point theorem, and we study the global behaviour of solutions in the case $f(u) \cdot u$ does not change sign.

I. Introduction. Let $L^{N+1} = \big(\mathbb{R}^{N+1}, dx_1^2 + dx_2^2 + \cdots + dx_N^2 - dt^2\big)$ be the $(N+1)$-dimensional Minkowsky space and $N > 1$. For any C^1 function $u : \mathbb{R}^N/\{0\} \to \mathbb{R}$, its graph

$$G_u = \big\{(x,t) = \big(x_1, \dots, x_n, t\big) \in L^{N+1}, \qquad t = u(x)\big\} \tag{1}$$

is said to be spacelike whenever $|Du(x)| < 1$ everywhere in $\mathbb{R}^N/\{0\}$.

In this paper we study the rotationally symmetric spacelike graphs in L^{N+1} whose mean curvature is a prescribed function of u. We are led to looking for radial solutions in $\mathbb{R}^N/\{0\}$ of the equation

$$\operatorname{div} \frac{Du}{\sqrt{1 - |Du|^2}} = f(u), \tag{2}$$

where the left-hand side represents (up to the factor N) the mean curvature of G_u and f is any given C^1 function on $\mathbb{R}$. The special case that $f(u) = \kappa u$ ($\kappa \in \mathbb{R}^*$) provides a version of the capillarity equation in Minkowsky's space.

The Dirichlet problem in an open set Ω of $\mathbb{R}^N$ associated to equation (2) has been thoroughly studied: Bartnik and Simon ([2]) proved that when f is bounded (possibly depending on $x \in \Omega$), equation (2) has a spacelike solution u in Ω with prescribed boundary value ϕ, if and only if ϕ admits a spacelike extension to Ω.

By way of contrast, the problem of determining all the entire solutions of (2) is interestingly open. When $f(u) = 0$, Cheng and Yau ([6]) proved a remarkable

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Bernstein property: all the entire spacelike solutions are linear. When $f(u)$ is a positive constant NH, Stumbles ([14]) and Treibergs ([15]) constructed many solutions different from the standard hyperboloids

$$u_k(x) = \Big(\sum_{j=1}^{k} x_j^2 + k^2/N^2H^2\Big)^{1/2}, \quad k = 1, 2, \dots, N. \tag{3}$$

More recently Bartnik ([1]) gave the behaviour of any spacelike graph G_u near an isolated singularity x_0: either x_0 is removable, or G_u is tangent to the light cone at x_0.

In this article we describe all the radial solutions of (2) and, in particular, we prove their existence, a fact which complements the results of [1]. Moreover, we recover the classification of the singularities in the radial case with an elementary proof. In the last section we study the global behaviour of solutions under the assumption that $f(u) \cdot u$ does not change sign.

Denoting $u(r) = u\big(|x|\big)$, the equation takes the form

$$\frac{u''}{\big(1-u'^2\big)^{3/2}}(r) + \frac{N-1}{r}\frac{u'}{\sqrt{1-u'^2}}(r) = f\big(u(r)\big), \quad \text{on } (0,+\infty). \tag{4}$$

Let u be any solution of (4) on an interval $(0, r_0)$. We shall say that u is *regular* with initial data $u_0 \in \mathbb{R}$ if

$$\lim_{r\to 0} u(r) = u_0, \quad \lim_{r\to 0} u'(r) = 0. \tag{5}$$

We shall say that u is *singular* with initial data u_0 if

$$\lim_{r\to 0} u(r) = u_0, \quad \lim_{r\to 0} u'(r) = \pm 1. \tag{6}$$

Our existence result is the following:

Theorem 1.

(i) *Any local solution of equation* (4) *has a unique extension to* $(0,+\infty)$ *and is either regular or singular.*

(ii) *For any $u_0 \in \mathbb{R}$, there exist a unique local regular solution, and an infinity of singular ones, with initial data u_0. All of them have the following form:*

$$u(r) = u_0 + \int_0^r \frac{z(s)}{\sqrt{1+z^2(s)}}\, ds, \tag{7}$$

where

$$z(r) = ar^{1-N} + \frac{f(u_0)}{N} r + O(r^2) \quad \textit{near the origin} \tag{8}$$

and $a \in \mathbb{R}$ is arbitrary. The regular solution is characterized by $a = 0$.

This theorem can be applied to the solutions of equation (2) which depend only on the first k coordinates ($k = 1, \dots, N$) and are radial with respect to $(x_1, \dots, x_k)$ (simply replace r by $r_k = \big(\sum_{j=1}^k x_j^2\big)^{1/2}$). In this way we find, for each k, one regular solution and an infinite family of solutions which admit $\sum = \mathbb{R}^{N-k}$ as a singular set. The expansion of these solutions near $\sum$ is again (7), (8) with N replaced by k and r by r_k.

It is interesting to compare equation (2) with the classical equation of hypersurfaces of the Euclidean space with prescribed mean curvature:

$$\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = f(u). \tag{9}$$

In particular, when $f(u) = \kappa u$ and κ is positive, equation (9) describes a liquid drop resting on a horizontal hyperplane; all the solutions are bounded near the origin and any singularity is removable. When κ is negative, it describes a pendent drop under a horizontal hyperplane, and admits (at least) two opposite solutions, unbounded near 0 (see [4], [9], [10]). Problem (2) is different because any solution of (2) has bounded gradient, hence is bounded near the origin. In particular, the singularity appears at the level of the gradient, and equation (2) admits both regular and singular solutions. As a special case of our analysis, when $f(u) \cdot u > 0$, we shall find among the singular ones that some tend to zero at infinity (see Theorem 2 below), so providing examples of *singular ground states*. This result should be compared with the main theorem of [12]. By contrast, the behaviour near infinity of bounded solutions of (2) and (9) appears to be quite similar, at least when $f(u)$ behaves like a power for large u. In particular, when f is a power and u tends to 0 at infinity, we find that the solutions of (2) behave as those of the linearized equation $\Delta u = f(u)$. The details concerning our results on the global behaviour of solutions can be found in Section 3, Theorems 2, 3 and 4.

2. Proof of Theorem 1 and related remarks. Let us introduce the auxiliary function

$$z(r) = u'(r)/\sqrt{1-u'^2(r)}; \tag{10}$$

then equation (4) can be rewritten as

$$z'(r) + \frac{N-1}{r} z(r) = r^{1-N}\big(r^{N-1}z\big)'(r) = f\big(u(r)\big), \tag{11}$$

and (4) is equivalent to (11) together with

$$u'(r) = z(r)/\sqrt{1+z^2(r)}. \tag{12}$$

Proof of Theorem 1. (i) For any $r_1 > 0$, $u_1 \in \mathbb{R}$, $v_1 \in (-1, +1)$, we have local existence and uniqueness of a solution u of (4) such that $u(r_1) = u_1$, $u'(r_1) = v_1$, since $f \in C^1(\mathbb{R})$. Let (α,β) be its maximal existence interval. Since u' is bounded, u has a finite limit at α; from (11) the same is true for $(r^{N-1}z)'$, hence for $r^{N-1}z$. Suppose $\alpha > 0$; then z has a limit at α; from (12), u' has a limit $\ell \in (-1,1)$, which contradicts the maximality. In the same way, β cannot be finite and so $(\alpha,\beta) = (0,+\infty)$. Next, let $u_0 = \lim_{r\to 0} u(r)$ and $a = \lim_{r\to 0} r^{N-1}z(r)$. Integrating (11) we obtain

$$r^{N-1}z(r) - a = \int_0^r s^{N-1} f\big(u(s)\big)\, ds; \tag{13}$$

hence near the origin

$$z(r) = ar^{1-N} + \frac{f\big(u_0\big)}{N} r + o(r). \tag{14}$$

If $a \neq 0$ then $\lim_{r\to 0} z(r) = \pm\infty$, which implies $\lim_{r\to 0} u'(r) = \pm 1$ and so u is singular. If $a = 0$, then $\lim_{r\to 0} z(r) = 0$, $\lim_{r\to 0} u'(r) = 0$ and u is regular.

(ii) Suppose that u is a singular (respectively regular) solution on an interval $[0, r_0]$ with initial data u_0. Then u satisfies (11), (12) and (13) for some $a \in \mathbb{R}/\{0\}$ (respectively $a = 0$), from which (7) follows immediately. Moreover

$$z(r) = ar^{1-N} + \frac{f\big(u_0\big)}{N} r + r^{1-N} \int_0^r s^{N-1} \Big(f\Big(u_0 + \int_0^s \frac{z(\theta)}{\sqrt{1+z^2(\theta)}}\, d\theta\Big) - f\big(u_0\big)\Big)\, ds. \tag{15}$$

We set

$$z(r) = \psi_a(r) + w(r), \text{with } \psi_a(r) = a\ r^{1-N} + \frac{f\big(u_0\big)}{N} r. \tag{16}$$

Then

$$w(r) = \mathbf{T}(w)(r), \quad on\ (0, r_0], \tag{17}$$

where

$$\mathbf{T}(w)(r) = r^{1-N} \int_0^r s^{N-1} \Big(f\Big(u_0 + \int_0^s \frac{z(\theta)}{\sqrt{1+z^2(\theta)}}\, d\theta\Big) - f\big(u_0\big)\Big)\, ds, \tag{18}$$

and z is defined by (16). This also implies the estimate

$$|w(r)| \leq M_0 r^2, \quad \text{on } (0, r_0], \tag{19}$$

where

$$M_0 = \frac{1}{N+1} \max_{\xi \in [-r_0, r_0]} |f'\big(u_0 + \xi\big)|. \tag{20}$$

Conversely, let $a \in \mathbb{R}$ and $r_0 > 0$ be arbitrary, and M_0 be given by (20). For any $R < r_0$, we define

$$\mathbf{B_R} = \Big\{ w \in C^\circ([0,R]) : \|w\| = \sup_{r\in(0,R]} \frac{|w(r)|}{r^2} \leq M_0 \Big\}, \tag{21}$$

and consider the function $\mathbf{T}$ defined on $\mathbf{B_R}$ by (16) and (18). It is clear that $\mathbf{T}$ maps $\mathbf{B_R}$ into itself. Moreover, for any w_1, $w_2 \in \mathbf{B_R}$, denoting $z_i = \psi_a + w_i$, $i = 1, 2$, we have

$$\mathbf{T}(w_1)(r) - \mathbf{T}(w_2)(r) = \\ r^{1-N} \int_0^r s^{N-1} \big(f\big(u_0 + \int_0^s \frac{z_1(\theta)}{\sqrt{1+z_1^2(\theta)}}\, d\theta\big) - f\big(u_0 + \int_0^s \frac{z_2(\theta)}{\sqrt{1+z_2^2(\theta)}}\, d\theta\big)\big)\, ds.$$

Since the function $x \to x/\sqrt{1+x^2}$ is a contraction on $\mathbb{R}$, it follows that

$$|\mathbf{T}(w_1)(r) - \mathbf{T}(w_2)(r)| \leq (N+1) M_0 r^{1-N} \int_0^r s^{N-1} \big(\int_0^s |w_1(\theta) - w_2(\theta)|\, d\theta \big)\, ds,$$

and

$$\|\mathbf{T}(w_1) - \mathbf{T}(w_2)\| \leq \frac{(N+1)}{3(N+3)} M_0^2 R^2 \|w_1 - w_2\|. \tag{22}$$

Then for R small enough $\mathbf{T}$ is a strict contraction, and it has a unique fixed point w. Defining z by (16) and u by (7) it is easy to verify that u is a solution of (4) with initial data u_0; moreover, u is singular if $a \neq 0$, regular if $a = 0$.

Remark 1. Using the differentiability of f at u_0, we can give a more precise expansion for z near the origin. Indeed,

$$z(r) = a r^{1-N} + \frac{f\big(u_0\big)}{N} r + \frac{f'\big(u_0\big)}{N+1} r^2 + o(r^2), \tag{23}$$

which implies the following for the corresponding solution u of (4): when u is regular,

$$u(r) = u_0 + \frac{f\big(u_0\big)}{2N} r^2 + \frac{f'\big(u_0\big)}{3(N+1)} r^3 + o(r^3); \tag{24}$$

when u is singular and $N > 3$,

$$u(r) = u_0 + (\operatorname{sign} a)\big(r - \frac{r^{2N-1}}{2(2N-1)a^2} + \frac{f\big(u_0\big)}{N(3N-1)a^3} r^{3N-1} \\ + \frac{f'\big(u_0\big)}{3N(N+1)a^3} r^{3N} + o\big(r^{3N}\big)\big), \tag{25}$$

when $N = 3$,

$$u(r) = u_0 + (\operatorname{sign} a)\big(r - \frac{r^5}{10a^2} + \frac{f\big(u_0\big)}{24a^3} r^8 + \frac{2af'\big(u_0\big) + 3}{72a^4} r^9 + o(r^9)\big), \tag{26}$$

and when $N = 2$,

$$u(r) = u_0 + (\operatorname{sign} a)\big(r - \frac{r^3}{6a^2} + \frac{4af(u_0) + 3}{40a^4} r^5 + \frac{f'(u_0)}{18a^3} r^6 + o(r^6)\big). \tag{27}$$

Notice that to prove local existence of such solutions with a fixed $u_0 \in \mathbb{R}$ we need only that f be a C^1 function near u_0.

Remark 2. All the radial spacelike graphs with constant mean curvature H and initial data u_0 are given by

$$v_a(r) = u_0 + \int_0^r \frac{\zeta_a(s)}{\sqrt{1 + \zeta_a^2(s)}}\, ds, \tag{28}$$

where

$$\zeta_a(r) = a\ r^{1-N} + Hr\ , \tag{29}$$

and a is arbitrary (in particular, when H is positive, the regular graph v_0 is u_N defined by (3), up to some obvious translation in the vertical direction). Each singular (respectively regular) solution of (4) given by (7), (8) is tangent to the corresponding curve v_a. Moreover the four (respectively the two) first terms of the expansion are the same, as follows from (24)–(27).

Next, we prove the continuous dependence of the solutions with respect to the parameter a. This fact will be applied in Section 3 below to prove the existence of singular ground states, but we found it preferable to insert it here because its proof uses heavily the notation and the estimates of this section. For a given $u_0 \in \mathbb{R}$, we denote by $u(a, .)$ the solution of equation (4) given by (7), (8). We have

Lemma 1. *For any $R > 0$ the function $a \to u(a, .)$ is continuous from R^* to $C^1([0, R])$.*

Proof. We only consider the case that $a > 0$ ($a < 0$ is analogous). Let r_0 be arbitrary and M_0 be defined by (20). For any $r \le r_0$, let

$$z(a, r) = u'(a, r)/\sqrt{1 + u'^2(a, r)} = a\ r^{1-N} + \frac{f(u_0)}{N} r + w(a, r). \tag{30}$$

Then from (17), (18), for any $b > 0$,

$$\begin{aligned} |w(b, r) - w(a, r)| = &\Big| r^{1-N} \int_0^r s^{N-1}\big(f\big(u_0 + \int_0^s \frac{z(b,\theta)}{\sqrt{1 + z^2(b,\theta)}}\, d\theta\big) \\ &- f\big(u_0 + \int_0^s \frac{z(a,\theta)}{\sqrt{1 + z^2(a,\theta)}}\, d\theta\big)\big)\, ds \Big| \\ \le &M_0(N+1) r^{1-N} \int_0^r s^{N-1}\big(\int_0^s \Big|\frac{z(b,\theta)}{\sqrt{1 + z^2(b,\theta)}} - \frac{z(a,\theta)}{\sqrt{1 + z^2(a,\theta)}}\Big|\, d\theta\big)\, ds. \end{aligned} \tag{31}$$

But $z(a,r) \geq r^{1-N}\big(a - |f(u_0)| r^N/N - M_0 r^{N+1}\big)$, from (19), hence $z(b,r) > r^{1-N}a/2$ for any $r \leq R_a$ small enough, and for small $|b-a|$. Then, for any $\theta \in [0, R_a]$,

$$\begin{aligned} \Big|\frac{z(b,\theta)}{\sqrt{1+z^2(b,\theta)}} - \frac{z(a,\theta)}{\sqrt{1+z^2(a,\theta)}}\Big| &\leq 8a^{-3}\theta^{3(N-1)}\big|z(b,\theta)-z(a,\theta)\big| \\ &\leq 8a^{-3}\big(\theta^{2(N-1)}|b-a| + \theta^{3N-1}\|w(b,.)-w(a,.)\|_{C^o([0,R_a])}\big). \end{aligned} \tag{32}$$

Using (32) in (31) and performing the integrations we obtain

$$\begin{aligned} \|w(b,\cdot)-w(a,\cdot)\| \leq{}& 8a^{-3}(N+1)M_0 \\ &\times \big(R_a^{2N-2}|b-a| + R_a^{3N-1}\|w(b,\cdot)-w(a,\cdot)\|_{C^o([0,R_a])}\big). \end{aligned} \tag{33}$$

Hence it is easy to deduce from (32), (33) and (7) that, for R_a small enough,

$$\|w(b,\cdot)-w(a,\cdot)\|_{C^o([0,R_a])} \leq |b-a|, \tag{34}$$

and

$$|u'(b,r)-u'(a,r)| + |u(b,r)-u(a,r)| \leq |b-a| \quad \textit{on } [0,R_a]. \tag{35}$$

Now we observe that equation (4) satisfies the usual Lipschitz conditions away from $r=0$ and $|u'|=1$. Thus it is not difficult to deduce that there exists a positive constant $C = C(a,R)$ such that

$$|u'(b,r)-u'(a,r)| + |u(b,r)-u(a,r)| \leq C|b-a| \quad \text{on } [0,R], \tag{36}$$

so ending the lemma.

3. Qualitative study of solutions. In this section we study the global behaviour of the solutions under simple assumptions on f. We have a special interest in the solutions which tends to 0 at infinity. Our main results are stated in Theorems 2, 3 and 4 below.

First we suppose that f is a constant NH. Then the solutions are given by (28), (29). When $H=0$, we get

$$u(a,r) = u_0 + |a|^{(2-N)/(N-1)} a \int_0^{|a|^{-1/(N-1)}r} \frac{dt}{\sqrt{1+t^{2(N-1)}}}; \tag{37}$$

hence u goes monotonically to a finite limit when $N>2$, to infinity when $N=2$ and $a \neq 0$. When $N>2$, for any real u_0 there is a unique a such that $\lim_{r\to+\infty} u(a,r) = 0$. When $H>0$, then $u(a,\cdot)$ is increasing from u_0 to $+\infty$ when a is nonnegative, and $u(a,\cdot)$ is decreasing to a minimum point and then increasing to $+\infty$ when a is negative. In each case we have $\lim_{r\to+\infty} u'(r) = 1$.

Now we consider the case where $f(u)$ has the sign of u. In particular, in the following theorem we prove the existence of an infinity of positive singular ground states:

Theorem 2. *Suppose that $f(u) \cdot u > 0$ for any real u. Then*

(i) *If $u_0 \geq 0$ and $a \geq 0$, then either $u(a,\cdot)$ is increasing from u_0 to $+\infty$, or $u_0 = a = 0$ and $u \equiv 0$.*

(ii) *If $u_0 \geq 0$ and $a < 0$, then either $u(a,\cdot)$ is decreasing to $-\infty$, or $u(a,\cdot)$ is decreasing to a positive minimum point and then increasing to $+\infty$, or $u(a,\cdot)$ is positive and decreasing to 0 (under the condition that $u_0 > 0$).*

(iii) *All those types of solutions exist for any fixed $u_0 > 0$.*

(iv) *Suppose moreover that f is nondecreasing. Then, if $u_0 > 0$, there is a unique $a < 0$ such that $u(a,\cdot)$ is positive and decreasing to 0 at infinity.*

Proof. (i) First we observe that no solution can have a finite limit $\ell \neq 0$ at infinity. Indeed this implies $\lim_{r\to+\infty} r^{1-N}\left(r^{N-1}z\right)'(r) = f(\ell)$ from (11), hence $\lim_{r\to+\infty} Nr^{-1}z(r) = f(\ell) \neq 0$, and $\lim_{r\to+\infty} |u'(r)| = 1$, which is impossible. Moreover any extremal point is a minimum point if $u > 0$ (and a maximum point if $u < 0$), from (4), hence there is at most one of them. If $u_0 \geq 0$ and $a \geq 0$, with $u_0 + a \neq 0$, then $u(a,\cdot)$ is necessarily increasing from u_0 to $+\infty$, since $\lim_{r\to 0} u'(r) = +1$ when $a > 0$, and $\lim_{r\to 0} u'(r) = 0$ and $\lim_{r\to 0} u''(r) = f(u_0)/N > 0$ from (4) when $a = 0$.

(ii) Suppose $u_0 \geq 0$ and $a < 0$. Let

$$I = \{a < 0 : \exists \rho > 0, u'(a,\rho) = 0 \text{ and } u(a,\rho) > 0\}, \tag{38}$$

$$J = \{a < 0 : u(a,r) > 0 \text{ and } u'(a,r) < 0\ \forall r > 0\}, \tag{39}$$

$$K = \{a < 0 : \exists \rho > 0, u(a,\rho) = 0\}. \tag{40}$$

Clearly if $a \in I$, then ρ is unique, $u(a,.)$ decreases on $(0,\rho)$ and increases on $(\rho,+\infty)$, with $\lim_{r\to+\infty} u(a,r) = +\infty$. If not, then $u(a,.)$ is decreasing to 0 or $-\infty$, hence $I \cup J \cup K = (-\infty, 0)$.

(iii) Let us prove that I *is nonempty*. Consider any $a \in J \cup K$. Then $u(a,\cdot)$ is decreasing from u_0 to 0 or $-\infty$, hence there is an r_a such that $u(a,r_a) = u_0/2$. From (13) we have

$$|a| \geq \int_0^{r_a} s^{N-1} f(u(a,s))\, ds \geq \frac{r_a^N}{N} \min_{[u_0/2,u_0]} f. \tag{41}$$

From (7) we have $u_0 - u(a,r_a) = u_0/2 \leq r_a$, hence

$$|a| \geq \frac{(u_0/2)^N}{N} \min_{[u_0/2,u_0]} f,$$

and I is a neighborhood of 0 in $(-\infty,0)$. Let us prove now that K *is nonempty*. Consider any $a \in I \cup J$. If $a \in I$, let ρ_a the minimum point of $u(a,\cdot)$. Then from (13) we have

$$|a| = \int_0^{\rho_a} s^{N-1} f(u(a,s))\, ds \leq \frac{\rho_a^N}{N} M, \tag{42}$$

where $M = \max_{[0,u_0]} f$. For any $a \in I \cup J$, u decreases on $[0, s_a]$, where $s_a = (N|a|/2M)^{1/N}$. On this interval we have

$$r^{N-1} z(a,r) \leq -|a| + M s_a^N / N = -|a|/2, \tag{43}$$

by (13), and

$$u_0 - u(a, s_a) = \int_0^{s_a} \frac{ds}{\sqrt{1 + z^{-2}(a,s)}}, \tag{44}$$

hence

$$\begin{aligned} u_0 &\geq \int_0^{s_a} \frac{ds}{\sqrt{1 + 4a^{-2} s^{2(N-1)}}} \geq \int_0^{s_a} \left(1 - 2a^{-2} s^{2(N-1)}\right) ds \\ &\geq \left(\frac{N}{2M}\right)^{1/N} |a|^{1/N} - \frac{2}{2N-1} \left(\frac{N}{2M}\right)^{2-1/N} |a|^{-1/N}, \end{aligned} \tag{45}$$

and K is a neighborhood of $-\infty$. We claim that I and K are *open* in $(-\infty, 0)$. Let $b \in I$; since $\lim_{r \to +\infty} u(b, r) = +\infty$, there is a $t_b > 0$ such that $u(b, t_b) \geq 3u_0$. From Lemma 1, the function $a \to u(a, .) \in C^1\big([0, t_b]\big)$ is continuous. Then for small $|a - b|$, we have $u(a, t_b) \geq 2u_0$, so that $a \in I$. In the same way let $c \in K$; then there is a $t_c > 0$ such that $u(c, t_c) \leq -1$, and for small $|a - c|$ we have $u(a, t_c) \leq -1/2$, hence $a \in K$. Then J is nonempty.

(iv) When f is nondecreasing, the solutions are ordered with respect to a: if $a_1 > a_2$, then, for small r, we get $z(a_1, r) > z(a_2, r)$ from (13), hence $u(a_1, r) > u(a_2, r)$. Suppose that there is a first point $\rho > 0$ where $u(a_1, \rho) = u(a_2, \rho)$. From (13) this implies $u'(a_1, r) > u'(a_2, r)$ on $(0, \rho]$, which is impossible. Hence we get $u(a_1, r) > u(a_2, r)$ and $u'(a_1, r) > u'(a_2, r)$ for any $r > 0$. Suppose now that a_1, $a_2 \in J$ with $a_1 > a_2$. Then, from (7),

$$u_0 = \int_0^{+\infty} \frac{ds}{\sqrt{1 + z^{-2}(a_1, s)}} = \int_0^{+\infty} \frac{ds}{\sqrt{1 + z^{-2}(a_2, s)}}, \tag{46}$$

which is impossible, since the integrals are ordered.

Remark 3. i) Regular solutions fall into i) of Theorem 2 ($a = 0$).

ii) Suppose that $f(u) \cdot u > 0$ on $\mathbb{R}$, and $\liminf_{|u| \to \infty} |f(u)| > 0$. Then we have $\lim_{r \to +\infty} |u'(a, r)| = 1$ for any $a \in I \cup K$, by integration of (11). In particular, that is the case when f is nondecreasing.

Let us consider in more detail the case where u tends to zero near infinity when f is a power.

Theorem 3. *Consider any solution $u \not\equiv 0$ of (4) such that $\lim_{r \to +\infty} u(r) = 0$. Then*

(i) *When $f(u) = \kappa u$ ($\kappa > 0$), u has an exponential decay:*

$$\lim_{r \to +\infty} r^{(N-1)/2} e^{\sqrt{\kappa}\, r} u(r) = c > 0. \tag{47}$$

(ii) *When* $f(u) = \kappa|u|^{q-1}u$ $(\kappa > 0,\ q > 1)$, *then*

$$\lim_{r\to+\infty} r^{2/(q-1)}u(r) = \pm\big(2\big(N-(N-2)q\big)/\kappa(q-1)^2\big)^{1/(q-1)} \quad \textit{when } q < N/(N-2), \tag{48}$$

$$\lim_{r\to+\infty} r^{N-2}u(r) = c \neq 0 \quad \textit{when } q > N/(N-2). \tag{49}$$

Proof. We can suppose $u > 0$, hence $u' < 0$, and $u(\cdot) = u(a,\cdot)$ for some $a < 0$. From (13) we have $|z(r)| \le ar^{1-n}$ on $(0,+\infty)$, hence $u'(r) = O\big(r^{1-N}\big)$ at infinity.

(i) When f is linear we can derive from (11) and (12)

$$z''(r) + \frac{N-1}{r}z'(r) - \frac{N-1}{r^2}z(r) = \frac{\kappa z(r)}{\sqrt{1+z^2(r)}}. \tag{50}$$

Setting $z(r) = r^{(1-N)/2}y(r)$, we get an equation of the form

$$y''(r) - (\kappa + \phi(r))y(r) = 0, \tag{51}$$

where

$$\phi(r) = \big(N^2-1\big)/4r^2 + \kappa\big(\big(1+z^2(r)\big)^{-1/2} - 1\big). \tag{52}$$

As $\lim_{r\to+\infty}\phi(r) = 0$ and $y(r) = O\big(r^{(1-N)/2}\big)$ we deduce, from [3, p. 126], that y decreases exponentially. Then ϕ is integrable at infinity, and we get more precisely $\lim_{r\to+\infty} e^{\sqrt{\kappa}\,r}y(r) = \bar{c} < 0$, hence (47).

(ii) Suppose $f(u) = \kappa|u|^{q-1}u$. We can write (4) in the form

$$-u''(r) + \frac{N-1}{r}|u'(r)|\big(1-u'^2(r)\big) + \kappa\big(1-u'^2(r)\big)^{3/2}u^q(r) = 0. \tag{53}$$

It follows that, for large r,

$$-\Big(u''(r) + (N-2)\frac{u'(r)}{r}\Big) + \frac{\kappa}{2}u^q(r) \le 0. \tag{54}$$

Then we obtain the classical Osserman-type estimate $u(r) = O\big(r^{-2/(q-1)}\big)$ at infinity. This implies the estimate $u'(r) = O\big(r^{-(q+1)/(q-1)}\big)$. Indeed that is true when $q \ge N/(N-2)$, since $u'(r) = O\big(r^{1-N}\big)$. When $q < N/(N-2)$ we use the fact that $\big(r^{N-1}z\big)'(r) = \big(r^{N-1-2q/(q-1)}\big)$ from (11). Then there is a constant $\alpha > 0$ such that the function $r \to r^{N-1}z(r) + \alpha r^{N-2q/(q-1)}$ is decreasing to a finite limit ℓ, and $\lim_{r\to+\infty} r^{N-1}z(r) = \ell \le 0$. If $\ell < 0$, then $\lim_{r\to+\infty} r^{N-2}u(r) = \ell/(2-N)$, which is impossible from Osserman's estimate. Then $\ell = 0$ and $|z(r)| \le \alpha r^{-(q+1)/(q-1)}$, hence the estimate of u'. Now let us make the classical change of variables (see [4]):

$$u(r) = r^{-\delta}v(t), \quad t = \operatorname{Log} r, \quad \delta = 2/(q-1). \tag{55}$$

Then the equation becomes

$$v_{tt} + (N-2-2\delta)v_t + \delta(\delta+2-N)v - \kappa v^q = K(t), \tag{56}$$

where

$$\begin{aligned} K(t) &= r^{\delta+2}\big((N-1)r^{-1}u'^3(r) - \kappa u^q(r)\big(1-\big(1-u'^2(r)\big)^{3/2}\big)\big) \\ &= \big(v(t)+|v_t(t)|\big)O\big(e^{-2(\delta+1)t}\big). \end{aligned} \tag{57}$$

Suppose first that $q > N/(N-2)$. From the estimate $u'(r) = O(r^{1-N})$, we get $v(t) + |v_t(t)| = O\big(e^{(\delta+2-N)t}\big)$. Let $v(t) = e^{(\delta-N+2)t}y(t)$. Then

$$\big(e^{-(N-2)t}y_t\big)_t(t) = e^{-\delta t}\big(K(t)+\kappa v^q(t)\big) = O\big(e^{-(N+2\delta)t}\big) + O\big(e^{(2-(N-2)q)t}\big), \tag{58}$$

and $y_t(t) = O\big(e^{-(2\delta+2)t}\big) + O\big(e^{(N-(N-2)q)t}\big)$. Then y has a finite limit c at infinity, and $\lim_{r\to+\infty} r^{N-2}u(r) = c$. Suppose that $c = 0$. Then there is a $k_0 > (N-2)$ such that $v(t) = O\big(e^{(\delta-k_0)t}\big)$, hence $u(r) \le c_0 r^{-k_0}$ for large r, and we can suppose $c_0 < 1$. But (11) implies $|u'(r)| \le |z(r)| \le \big(c_0^q/(k_0q-N)\big)r^{1-k_0q}$, and $u(r) \le \big(c_0^q/(k_0q-N)(k_0q-2)\big)r^{2-k_0q}$. Consequently we construct some sequences k_n, c_n such that $u(r) = c_nr^{-k_n}$ and $k_n = k_{n-1}q-2$ and $c_{n+1} = c_n^q/(k_nq-N)(k_nq-2)$. We have $\lim_{n\to+\infty} k_n = +\infty$ and $\lim_{n\to+\infty} c_n = 0$; then $u \equiv 0$, which is impossible. Then c is nonzero, and (49) holds. Now suppose that $q < N/(N-2)$. Multiplying (56) by v_t and integrating between t and $T > t$, we get the energy relation

$$E(T) - E(t) = (2\delta+2-N)\int_t^T v_t^2(s)\,ds + \int_t^T K(s)v_t(s)\,ds, \tag{59}$$

where

$$E(t) = \frac{v_t^2(t)}{2} + \frac{\delta(\delta+2-N)}{2}v^2(t) - \frac{\kappa}{q+1}v^{q+1}(t). \tag{60}$$

Now E is bounded near infinity, and from (57) we have

$$E(T)-E(t) \ge \delta\int_t^T v_t^2(s)\,ds - \int_t^T |K(t)|\,|v_t(s)|\,ds \ge \frac{\delta}{2}\int_t^T v_t^2(s)\,ds + O(1); \tag{61}$$

hence, $v_t \in L^2((t,+\infty))$. Now v_{tt} is bounded near infinity, then classically we get $\lim_{t\to+\infty} v_t(t) = 0$, and v has a limit ℓ from (58), (59), and $\delta(\delta+2-N)\ell - \kappa\ell^q = 0$ from (56). Then either $\lim_{t\to+\infty} v(t) = (\delta(\delta+2-N)/\kappa)^{1/(q-1)}$, or $\lim_{t\to+\infty} v(t) = 0$. Consider the latter case. The function v is decreasing to zero for large t, from (56), (57); and from (61), there is a constant C such that

$$\begin{aligned} E(T)-E(t) &\ge \delta\int_t^T v_t^2(s)\,ds + C\int_t^T e^{-2(\delta+1)s}v(s)v_t(s)\,ds \\ &- C\int_t^T e^{-2(\delta+1)s}v_t^2(s)\,ds \ge \frac{\delta}{2}\int_t^T v_t^2(s)\,ds - \frac{C}{2}e^{-2(\delta+1)t}v^2(t) \end{aligned} \tag{62}$$

for large t. Passing to the limit when T goes to infinity, we get $E(t) - \frac{C}{2}e^{-2(\delta+1)t}$ $v^2(t) < 0$; hence, $v \equiv 0$ for large t, which is impossible. Then (48) holds.

To end this section we consider the case where $f(u)$ has the opposite sign of u. More details concerning the techniques of the following theorem can be found in references [6], [11] and [13].

Theorem 4.

i) *Suppose that* $f(u) \cdot u < 0$ *for any real* u, *and* $\lim_{|u|\to+\infty} |\int_0^u f(v)dv| = +\infty$. *Then any solution* u *of* (4) *satisfies* $\lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} u'(r) = 0$.

ii) *When* $f(u) = \kappa u (k < 0)$, u *is oscillatory. The distance between two extremal points* $r_n < r_{n+1}$ *satisfies* $\lim_{n\to+\infty}\big(r_{n+1} - r_n\big) = \pi/\sqrt{|\kappa|}$. *Moreover* $\lim_{n\to+\infty} r_n^{(N-1)/2}|u(r_n)| = M \neq 0$.

iii) *When* $f(u) = \kappa|u|^{q-1}u(\kappa < 0, q > 1)$, *then, denoting* $\lambda = (2((N-2)q - N)/|\kappa|(q-1)^2)^{1/(q-1)}$,

$$\text{either } \lim_{r\to+\infty} r^{2/(q-1)}u(r) = \pm\lambda, \text{ or } \lim_{r\to+\infty} r^{N-2}u(r) = c \neq 0, \quad \text{when } q > \frac{N+2}{N-2}; \tag{63}$$

$$\text{either } u \text{ is oscillatory, or } \lim_{r\to+\infty} r^{2/(q-1)}u(r) = \pm\lambda, \quad \text{or } \lim_{r\to+\infty} r^{N-2}u(r) = c \neq 0, \text{ when } \frac{N}{N-2} < q < \frac{N+2}{N-2}; \tag{64}$$

$$u \text{ is oscillatory when } q \leq N/(N-2). \tag{65}$$

Proof. i) is proved in [13]. ii) As in the proof of Theorem 3, we derive the equation and set $z(r) = r^{(1-N)/2}y(r)$. Then we get equation (51) with $\kappa < 0$, and ϕ still satisfies $\lim_{r\to+\infty}\phi(r) = 0$. Adapting the proof of [13] (Theorems 6, 7), we get the estimates $u(r) + |u'(r)| = 0\big(r^{\frac{1-N}{2}+\epsilon}\big)$ for any $\epsilon > 0$. Now $\phi'(r) = -(N^2-1)/2r^3 + |\kappa|u'(r)u''(r)/(1-u'^2(r))^{3/2}$; hence, $\phi'(r) = 0(r^{-N+\epsilon}) + 0(r^{-3})$ and ϕ' is integrable. And $\phi(r) = 0(r^{1-N+\epsilon})$, hence ϕ^2 is integrable (and also ϕ whenever $N > 2$). From [3, page 135], we deduce that $y(r) = a\cos\big(r + \frac{1}{2}\int_1^r \phi(s)\,ds + b\big) + 0(r)(a \in \mathbb{R}^*, b \in \mathbb{R})$; hence, the estimate of $(r_{n+1} - r_n)$. Moreover, from (51), $y'^2(r) + |\kappa|y^2(r)$ has a limit L, necessarily positive; then $\lim_{n\to+\infty} r_n^{(N-1)/2}|u(r_n)| = \sqrt{L}/|\kappa| \neq 0$.

iii) From [13] we know that u is oscillatory when $q \leq N/(N-2)$, and that any constant sign solution satisfies the estimates

$$u(r) = O(r^{-2/(q-1)}), \quad u'(r) = O\big(r^{-2(q+1)/(q-1)}\big). \tag{66}$$

Let us consider the case where $q > (N+2)/(N-2)$. The Pohožaev-type function

$$V(r) = r^N\big(\sqrt{1+z^2(r)} - 1 + |\kappa|\frac{|u(r)|^{q+1}}{q+1} + \frac{N}{q+1}r^{-1}u(r)z(r)\big) \tag{67}$$

satisfies

$$V'(r) = r^{N-1}\big(N\big(\sqrt{1+z^2(r)}-1\big) + \big(\frac{N}{q+1} - N + 1\big)\frac{z^2(r)}{\sqrt{1+z^2(r)}}\big); \tag{68}$$

hence, V is decreasing near infinity, since $\lim r^{1-N}z^{-2}(r)V'(r) = \frac{N}{q+1} - \frac{N-2}{2}$. If u is oscillatory, then V is positive for large r, because it is positive at each zero of u. Then V has a limit, and

$$\sqrt{1+z^2}(r) - 1 + |\kappa|\frac{|u(r)|^{q+1}}{q+1} + \frac{N}{q+1}r^{-1}u(r)z(r) = 0\big(r^{-N}\big) = 0\big(r^{-2(q+1)/(q-1)}\big).$$

But for any $\epsilon > 0$ there is a $C(\epsilon)$ such that $|r^{-1}u(r)z(r)| \le \epsilon|u(r)|^{q+1} + \epsilon z(r)^2 + r^{-2(q+1)/(q-1)}$. Hence we get estimates (65), no matter if u is oscillatory or not. Now consider any solution u if $q > (N+2)/(N-2)$, or any positive solution u if $q < (N+2)/(N-2)$. Let us make the change of variables (55): we get

$$v_{tt} + (N-2-2\delta)v_t - \delta(N-2-\delta)v + |\kappa||v|^{q-1}v = \tilde{K}(t), \tag{69}$$

where

$$\tilde{K}(t) = r^{\delta+2}\big((N-1)r^{-1}u'^3(r) + |\kappa||u(r)|^{q-1}u(r)\big(1-(1-u'^2(r))^{3/2}\big)\big). \tag{70}$$

Then, from (66), $\tilde{K}(t) = \big(|v(t)|+|v_t(t)|\big)O\big(e^{-2(\delta+1)t}\big)$. The energy relation becomes, for any $t < T$,

$$\tilde{E}(T) - E(t) = (2\delta+2-N)\int_t^T v_t^2(s)\,ds + \int_t^T \tilde{K}(t)v_t(s)\,ds, \tag{71}$$

where

$$\tilde{E}(t) = \frac{v_t^2}{2}(t) - \frac{\delta(N-2-\delta)}{2}v^2(t) + \frac{|\kappa|}{q+1}|v^{q+1}(t)|. \tag{72}$$

Since $q \ne (N+2)/(N-2)$ we have $2\delta+2-N \ne 0$; hence,

$$|\tilde{E}(T) - E(t)| \ge \frac{|2\delta+2-N|}{2}\int_t^T v_t^2(s)\,ds + O(1), \tag{73}$$

and we get again $v_t \in L^2([t,+\infty))$. As in Theorem 3 we deduce that $\lim_{t\to+\infty} v_t(t) = 0$, and $\lim_{t\to+\infty} v(t) = 0$, or $\lim_{t\to+\infty} v(t) = \lambda = (\delta(N-2-\delta)/|\kappa|)^{1/(q-1)}$. Let us consider the case where v tends to zero. Then from (69) v is monotonous for large t; hence, v keeps a constant sign. We can suppose that v decreases to zero. Then for any $\epsilon > 0$ and $t \ge t(\epsilon)$ we deduce the inequality

$$(v_{tt} + (N-2-2\delta-\epsilon)v_t - (\delta(N-2-\delta)-\epsilon)v)(t) \ge 0. \tag{74}$$

Consequently, we get as in [6] an estimate of the form $v(t) = O\big(e^{(\delta-N+2+\alpha)t}\big)$ for any $\alpha > 0$; hence, $u(r) = O\big(r^{2-N+\alpha}\big)$. From (11), u and $r^{N-1}z$ are decreasing at infinity, and for any $r < s$ large enough, we have $s^{1-N}r^{N-1}z(r) \geq z(s) \geq 2u'(s)$. Integrating between r and $+\infty$ we find $rz(r) \geq -2(N-2)u(r)$; hence, $|u'(r)| \leq 2(N-2)u(r)/r$ for large r. Then v_t satisfies the same estimate as v near infinity. Let $v(t) = e^{(\delta-N+2)t}y(t)$. Then we have

$$\big(e^{-(N-2)t}y_t\big)_t(t) = e^{-\delta t}\big(\tilde{K}(t) - |\kappa|v^q(t)\big) = O\big(e^{-N-2\delta+\alpha)t}\big) + O\big(e^{2+(\alpha-N+2)q)t}\big). \tag{75}$$

Choosing $\alpha < \min\big(N-2-Nq^{-1}, 2\delta+2\big)$, we get, as in [6], $y_t(t) = O\big(e^{(-2\delta-2+\alpha)t}\big) + O\big(e^{(N-(N-2)q+\alpha q)t}\big)$. Then y has a finite limit c at infinity, and $\lim_{r\to+\infty} r^{2-N} u(r) = c$, with a c different from 0 from [11].

Remark 4. Consider any oscillatory solution and denote by (r_n) the sequence of its extremal points and by (ρ_n) the sequence of its zeros. Adapting the proofs of Theorems 6 and 7 of [3] one can get the estimates $u(r_n) = O\big(r_n^{-2(N-1)/(q+3)+\epsilon}\big)$, $u'(\rho_n) = O\big(\rho_n^{-(N-1)(q+1)/(q+3)+\epsilon}\big)$ for any $\epsilon > 0$.

Remark 5. When $N/(N-2) < q < (N+2)/(N-2)$, then any solution u such that $u_0\, a \geq 0$ is oscillating. Indeed, it is easy to see that the function V defined in (67) is increasing, with $\lim_{r\to 0} V(r) = Nu_0 a/(q+1)$, and $\lim_{r\to+\infty} V(r) = 0$ from (66) if u keeps a constant sign. In particular, *there does not exist any regular ground state*. On the other hand, *there do exist singular ground states*. Indeed, using the Fixed-Point Theorem, one can construct positive solutions such that $\lim_{r\to+\infty} r^{N-2}u(r) = c \neq 0$, because 0 is a saddle point of the linearized equation of (67).

Remark 6. When $q > (N+2)/(N-2)$, then *there do exist regular ground states*. Indeed, the function V is decreasing, whenever $z^2(r) < N(q+1)((N-2)q-(N+2))/(N+q+1)^2$. But from (4) we get easily the following energy inequality for regular solutions:

$$\sqrt{1+z^2(r)} + |\kappa||u|^{q+1}/(q+1) \leq |\kappa||u_0|^{q+1}/(q+1) + 1. \tag{76}$$

Then if $|u_0|$ is not too large, V is decreasing on $(0,+\infty)$, with $V(0) = 0$. It follows that u keeps a constant sign, because V takes a positive value at each zero of u. Moreover, we have $\lim_{r\to+\infty} r^{2/(q-1)}u(r) = \pm\lambda$. Indeed, if $\lim_{r\to+\infty} u(r) = c \neq 0$, then $V(r) = O(r^{2-N})$ at infinity, which is impossible.

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