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SMOOTH SOLUTIONS OF THE VECTOR BURGERS EQUATION IN NONSMOOTH DOMAINS

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Abstract. We prove the existence and uniqueness of smooth solutions of the vector Burgers equation in arbitrary two- and three-dimensional domains. The only assumption about the spatial domain is that it should be an open set. The underlying estimates for these results are proved using new "elliptic-Sobolev" inequalities of Xie ([13], [15]) for the Laplacian. Our purpose in giving these results is to develop methods that we think can be eventually transferred to the Navier-Stokes equations. Indeed, the only missing point is the proof of analogous "elliptic-Sobolev" inequalities for the Stokes operator, which we conjecture to be valid.

1. Introduction and main result. The vector Burgers equation is important for its similarity with the Navier-Stokes equations. It was studied, for instance, in the landmark paper of Kiselev and Ladyzhenskaya ([7]), to illuminate the question of the possible global (in time) existence and uniqueness of solutions of the Navier-Stokes equations. This is also discussed in a recent review paper of Heywood ([4]). Our present reason for studying the vector Burgers equation concerns another less-than-satisfactory aspect of the current Navier-Stokes theory. That is the question of the regularity of nonstationary solutions in domains with nonsmooth boundaries. Current proofs of the classical regularity of a nonstationary solution utilize spatially global estimates of the nonlinear term, which are made with the aid of elliptic inequalities for the Stokes operator proven ([1], [11], [12]). These elliptic inequalities provide the regularity of solutions of the steady Stokes equations up to smooth portions of the boundary. They provide spatially global estimates only if the entire boundary is smooth (or, where not smooth, at least convex—there are counter examples to these inequalities in the case of reentrant corners). In striking contrast, the unique generalized solution of Kiselev and Ladyzhenskaya is obtained in an arbitrary open set; see Ladyzhenskaya's book ([8]). But again, the proof of

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the classical regularity of their solution uses elliptic estimates that require a globally regular domain ([8]). Also, curiously, their existence theorem seems to depend in an essential way on an extraordinary assumption concerning the smoothness of the initial data, namely, that it should belong to the Sobolev space $W_2^2(\Omega)$. Thus, the current existence and uniqueness theory for a local generalized solution requires either a smooth boundary or initial data in $W_2^2(\Omega)$, but not both. This contrast between the results of Kiselev and Ladyzhenskaya, and those of other authors (we have in mind, particularly, Prodi ([10]), and Kato and Fujita ([6]), and their successors) seems to show that the regularity of the boundary should not play such a crucial role as it currently does in the theory of the nonstationary Navier-Stokes equations.

In an effort to clarify these matters, and free up the theory of the nonstationary Navier-Stokes equations, Xie ([13], [15]) has introduced a new type of elliptic inequality (see Lemmas 1 and 5 below) which is domain independent. But, to date, this new type of inequality is proven only for the Laplacian, and not the Stokes operator. We conjecture, however, that the desired analogues for the Stokes operator are valid, and refer to Xie ([14], [16]) for some suggestive partial results.

At this time it seems appropriate to draw attention to these efforts, by showing the improvements which will be possible in the Navier-Stokes theory once the conjectured inequalities for the Stokes operator are proven. We can do this by developing an existence, uniqueness, and regularity theory for the vector Burgers equation, analogous to what we expect for the Navier-Stokes equations, using the already available new, domain-independent, elliptic inequalities for the Laplacian. All the methods of this paper, except the continuation argument in Section 5 (which is included to complete our theory of the Burgers equation) will carry over with only very minor changes once the conjectured results for the Stokes operator are proven. Indeed, even now they carry over to the Navier-Stokes equations in some domains, using domain-dependent variants of Lemmas 1 and 5 for the Stokes operator, which are well known for bounded domains with smooth boundaries, and have been recently proven for exterior three-dimensional domains with smooth boundaries, by Maremonti ([9]).

Thus, in this paper we investigate the initial boundary value problem

$$\frac{\partial u}{\partial t} + u\cdot\nabla u = \Delta u, \quad x \in \Omega,\ t > 0,$$
$$u|_{x\in\partial\Omega} = 0, \quad u|_{t=0} = u_0,$$

in an arbitrary two- or three-dimensional domain Ω, for a solution that is, correspondingly, two- or three-dimensional vector valued. Since the domain is only required to be an open set in $\mathbb{R}^2$ or $\mathbb{R}^3$, we impose the boundary condition in a generalized sense. In two dimensions we require that $u(t)$ belong to the Sobolev space $H_0^1(\Omega)$. In three dimensions we require that $u(t)$ belong to $\hat{H}_0^1(\Omega)$, which is defined as the completion of $C_0^\infty(\Omega)$ in the Dirichlet norm. We note that the space

$\hat{H}_0^1(\Omega)$ is less restrictive than $H_0^1(\Omega)$ in that its members need not belong to $L^2(\Omega)$ if the domain is one for which Poincaré's inequality fails to hold.

The following theorem is our main result for three-dimensional domains. Here, and throughout the paper, we denote the $L^2(\Omega)$ norm by $|\cdot|$ and the $L^\infty(\Omega)$ norm by $|\cdot|_\infty$.

Theorem 1. *Suppose that Ω is an open set in $\mathbb{R}^3$ and that $u_0 \in \hat{H}_0^1(\Omega)$. Let*

$$T = \frac{256\pi^2}{27|\nabla u_0|^4}. \tag{1}$$

Then there exists a unique smooth function $u \in C^\infty(\Omega \times (0,T))$ satisfying

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\cdot\nabla u = \Delta u, \qquad x \in \Omega, \quad t \in (0,T),\\
&u(t) \in \hat{H}_0^1(\Omega), \qquad t \in (0,T),\\
&\lim_{t\to 0^+} |\nabla u(t) - \nabla u_0| = 0,
\end{aligned}$$

and the bounds

$$\begin{aligned}
|\nabla u(t)|^2 &\le \frac{|\nabla u_0|^2}{\sqrt{1-t/T}},\\
\int_0^t |\Delta u(s)|^2\, ds &\le \frac{|\nabla u_0|^2}{2\left(1-\sqrt[6]{t/T}\right)\sqrt{1-\sqrt{t/T}}},
\end{aligned}$$

as well as all the estimates given in Lemmas 2, 3 and 4 below. Furthermore, if a portion of the spatial boundary is $C^{m+1,1}$ for some integer $m \ge 0$, then u is H^{m+2} and $C^{m,1/2}$ up to that portion of the boundary.

Our proof is an adaptation and refinement of known proofs of existence theorems for the Navier-Stokes equations, based on a differential inequality for the Dirichlet norm, and its analogue for Galerkin approximations. The method originated with Prodi ([10]), who used it to prove the existence of generalized solutions in bounded domains. Heywood ([3]) introduced a further infinite sequence of differential inequalities for the Galerkin approximations, to obtain classically smooth solutions. He also extended the method to unbounded domains. Heywood and Rannacher ([5]) developed the method further through the use of weight functions dependent on t, to give more precise estimates as $t \to 0^+$. All of these developments are incorporated into the existence theorem given here. We also introduce the use of a space-time Laplacian in proving the solution's regularity.

As mentioned, the principle innovation here is that the nonlinear term is estimated in a new way, using the following inequality of Xie ([13], [15]), to give results that are not only sharper but also valid in arbitrary domains.

Lemma 1. *Suppose that* Ω *is an open set in* $\mathbb{R}^3$*. Then every* $u \in \hat{H}^1_0(\Omega)$ *with* $\Delta u \in L^2(\Omega)$ *satisfies*

$$|u|_\infty \leq \frac{1}{\sqrt{2\pi}} |\nabla u|^{1/2} |\Delta u|^{1/2}. \tag{2}$$

An analogous lemma is given for two-dimensional domains in Section 4, where we treat the two-dimensional Burgers equation.

Finally, we mention that the methods of this paper seem to offer the first unified treatment of both the Navier-Stokes equations and the Burgers equation. The Galerkin method used by Kiselev and Ladyzhenskaya ([7]) to treat the Navier-Stokes equations is not applicable to the multi dimensional Burgers equation, because the equation lacks an appropriate energy estimate. Instead, they constructed their solution of Burgers equation by a time stepping scheme, using the maximum principle yielded, of course, a global solution. But, as is well known, a similar maximum principle does not hold for the Navier-Stokes equations. In Section 5, we use the maximum principle for Burgers equation, not in the construction of the solution, but merely to provide a continuation argument, to continue the already constructed local solution globally in time.

2. Estimates. In this section, we assume Ω to be bounded. In a bounded domain, we can use eigenfunctions of the Laplacian in constructing Galerkin approximations.

Let $\{\phi_1, \phi_2, \dots\}$ be an orthonormal basis for $L^2(\Omega)$ consisting of eigenfunctions of the Laplacian,

$$\Delta \phi_n = -\lambda_n \phi_n, \qquad \phi_n \in H^1_0(\Omega). \tag{3}$$

Let $(\cdot, \cdot)$ denote the L^2 inner product. The N-th Galerkin approximation u_N is a function of the form

$$u_N(x,t) = \sum_{n=1}^{N} c_{N,n}(t) \phi_n(x)$$

that satisfies

$$\Big(\frac{\partial u_N}{\partial t} - \Delta u_N + u_N \cdot \nabla u_N, \phi_n\Big) = 0, \quad (u_N(0) - u_0, \phi_n) = 0,$$

for all $1 \leq n \leq N$. This amounts to a system of first-order ordinary differential equations for the coefficients $c_{N,1}(t), \dots, c_{N,N}(t)$. Clearly, any such system has a solution on at least a short time interval. We will obtain uniform bounds on the Galerkin approximations, independent of N and Ω, so that we can use weak compactness theorems to prove the existence of a true solution. Henceforth, for convenience, we will drop the subscript N.

Taking linear combinations, we have

$$\Big(\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u, v\Big) = 0, \tag{4}$$

$$(u(0) - u_0, v) = 0, \tag{5}$$

for all $v \in$ span $\{\phi_1, \dots, \phi_N\}$. Because of (3), we can take v to be $-\Delta u$ or u_t in deriving estimates.

The bounds in (6) and (7) below are continuous functions of $(t, |\nabla u_0|) \in [0, T) \times [0, \infty)$ that are independent of N and Ω. We will generically denote such bounds by $b(t)$. In each occurrence, $b(t)$ can be explicitly calculated if one wishes to, just as in (6) and (7).

Lemma 2. *Let T be as given in* (1). *For $0 < t < T$, we have*

$$|\nabla u|^2 \leq \frac{|\nabla u_0|^2}{\sqrt{1 - t/T}}, \tag{6}$$

$$\int_0^t |\Delta u|^2 \, ds \leq \frac{|\nabla u_0|^2}{2(1 - \sqrt[6]{t/T})\sqrt{1 - \sqrt{t/T}}}, \tag{7}$$

$$\int_0^t |u|_\infty^2 \, ds \leq b(t), \tag{8}$$

$$\int_0^t |u_t|^2 \, ds \leq b(t), \tag{9}$$

$$|u(t) - u(0)| \leq b(t)\sqrt{t}. \tag{10}$$

Proof. Let $v = -\Delta u$ in (4) and integrate by parts. We obtain

$$\frac{1}{2}\frac{d}{dt}|\nabla u|^2 + |\Delta u|^2 = (u \cdot \nabla u, \Delta u).$$

By Lemma 1 and Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{11}$$

with $p = 4$ and $q = 4/3$, we have

$$\begin{aligned}(u \cdot \nabla u, \Delta u) &\leq |u|_\infty |\nabla u| |\Delta u| \leq \frac{1}{\sqrt{2\pi}} |\nabla u|^{3/2} |\Delta u|^{3/2} \\ &\leq \frac{1}{\sqrt{2\pi}} \left(\frac{1}{4} |\alpha^{-1} \nabla u|^6 + \frac{3}{4} |\alpha \Delta u|^2 \right) = C_\epsilon |\nabla u|^6 + \epsilon |\Delta u|^2,\end{aligned}$$

where $\alpha > 0$ is arbitrary, and

$$\epsilon = \frac{3\alpha^2}{4\sqrt{2\pi}}, \qquad C_\epsilon = \frac{\alpha^{-6}}{4\sqrt{2\pi}} = \frac{27}{1024\pi^2\epsilon^3}.$$

Therefore we have

$$\frac{1}{2}\frac{d}{dt}|\nabla u|^2 + (1-\epsilon)|\Delta u|^2 \le C_\epsilon |\nabla u|^6.$$

Let $0 < \epsilon \le 1$, and let

$$\phi(t) = |\nabla u(t)|^2 + 2(1-\epsilon)\int_0^t |\Delta u(s)|^2\, ds.$$

Then $\phi(t)$ is nonnegative and satisfies

$$\frac{d\phi}{dt} \le 2C_\epsilon \phi^3. \tag{12}$$

In (5), let $v = -\Delta u(0)$ to obtain

$$(\nabla u(0), \nabla u(0)) = (\nabla u_0, \nabla u(0)).$$

Therefore $|\nabla u(0)| \le |\nabla u_0|$, and hence

$$\phi(0) \le |\nabla u_0|^2. \tag{13}$$

From (12) and (13) we obtain

$$\phi(t) \le \frac{|\nabla u_0|^2}{\sqrt{1 - 4C_\epsilon |\nabla u_0|^4 t}},$$

or in other words,

$$|\nabla u(t)|^2 + 2(1-\epsilon)\int_0^t |\Delta u(s)|^2\, ds \le \frac{|\nabla u_0|^2}{\sqrt{1 - t/(\epsilon^3 T)}}.$$

Now, letting $\epsilon = 1$ we obtain (6), and letting $\epsilon = \sqrt[6]{t/T}$ we obtain (7).

Inequality (8) follows immediately from (2), (6) and (7).

Letting $v = u_t$ in (4), we get

$$|u_t|^2 = (\Delta u - u\cdot\nabla u, u_t) \le |\Delta u - u\cdot\nabla u||u_t|.$$

Hence

$$|u_t| \le |\Delta u| + |u|_\infty |\nabla u|,$$

and therefore we have (9).

Inequality (10) follows from (9) since

$$|u(t) - u(0)| = \left|\int_0^t u_t\, ds\right| \le \sqrt{\int_0^t ds \int_0^t |u_t|^2\, ds}.$$

Hereafter, for convenience, we will denote $\partial^k u/\partial t^k$ by u^k, and use c to generically denote a constant that may depend on k but on nothing else. In each occurrence the value of c can be calculated if one wishes to.

From (4) we obtain

$$(u^{k+1} + (u\cdot\nabla u)^k - \Delta u^k, v) = 0. \tag{14}$$

Letting $v = u^k$, $-\Delta u^k$ and u^{k+1} respectively, we have

$$\frac{1}{2}\frac{d}{dt}|u^k|^2 + |\nabla u^k|^2 = -\left((u\cdot\nabla u)^k, u^k\right), \tag{15}$$

$$\frac{1}{2}\frac{d}{dt}|\nabla u^k| + |\Delta u^k|^2 = \left((u\cdot\nabla u)^k, \Delta u^k\right), \tag{16}$$

$$|u^{k+1}|^2 = \left(\Delta u^k - (u\cdot\nabla u)^k, u^{k+1}\right). \tag{17}$$

By estimating the right sides of these equations, we derive the following inequalities.

$$\frac{d}{dt}|u^k|^2 + |\nabla u^k|^2 \le c(1 + |\nabla u|^4)|u^k|^2 + c\sum_{i=1}^{k-1}|\nabla u^i|^2|\nabla u^{k-i}|^2, \tag{18}$$

$$\frac{d}{dt}|\nabla u^k|^2 + |\Delta u^k|^2 \le c(|\nabla u|^4 + |u|_\infty^2)|\nabla u^k|^2 + c\sum_{i=1}^{k-1}|u^i|_\infty^2|\nabla u^{k-i}|^2, \tag{19}$$

$$|u^{k+1}|^2 \le c|\Delta u^k|^2 + c\sum_{i=0}^{k}|u^i|_\infty^2|\nabla u^{k-i}|^2. \tag{20}$$

Proof of (18). Let $|\cdot|_p$ denote the $L^p(\Omega)$ norm. By Hölder's inequality we have

$$-(u^k, u^i\cdot\nabla u^{k-i}) \le |u^k|_3|u^i|_6|\nabla u^{k-i}|.$$

By Sobolev inequalities (see e.g. [8]) we have:

$$|u^k|_3 \le c|u^k|^{1/2}|\nabla u^k|^{1/2} \quad \text{and} \qquad |u^i|_6 \le c|\nabla u^i|.$$

Therefore,

$$\begin{aligned}
-\left(u^k, (u\cdot\nabla u)^k\right) &\le c|u^k|^{1/2}|\nabla u^k|^{1/2}\sum_{i=0}^{k}|\nabla u^i||\nabla u^{k-i}| \\
&= c|\nabla u^k|^{3/2}|\nabla u||u^k|^{1/2} + c|u^k|^{1/2}|\nabla u^k|^{1/2}\sum_{i=1}^{k-1}|\nabla u^i||\nabla u^{k-i}| \\
&\le \frac{1}{4}|\nabla u^k|^2 + c|\nabla u|^4|u^k|^2 + |\nabla u^k||u^k| + c\left(\sum_{i=1}^{k-1}|\nabla u^i||\nabla u^{k-i}|\right)^2 \\
&\le \frac{1}{2}|\nabla u^k|^2 + c(1 + |\nabla u|^4)|u^k|^2 + c\sum_{i=1}^{k-1}|\nabla u^i|^2|\nabla u^{k-i}|^2,
\end{aligned}$$

where we have used the Leibniz formula and Young's inequality, (11). Inequality (18) is obtained from this and (15).

Proof of (19). By the Leibniz formula, we have

$$|(u\cdot\nabla u)^k|^2 \le c\sum_{i=0}^{k}|u^i|_\infty^2|\nabla u^{k-i}|^2.$$

From Lemma 1, we have

$$|u^k|_\infty^2 \le c|\nabla u^k||\Delta u^k|, \tag{21}$$

and therefore

$$|u^k|_\infty^2|\nabla u|^2 \le \frac{1}{4}|\Delta u^k|^2 + c|\nabla u|^4|\nabla u^k|^2.$$

Hence we have

$$|(u\cdot\nabla u)^k|^2 \le \frac{1}{4}|\Delta u^k|^2 + c|\nabla u|^4|\nabla u^k|^2 + c\sum_{i=0}^{k-1}|u^i|_\infty^2|\nabla u^{k-i}|^2. \tag{22}$$

Now, the right-hand side of (16) is estimated as follows.

$$\begin{aligned}((u\cdot\nabla u)^k, \Delta u^k) &\le |(u\cdot\nabla u)^k||\Delta u^k| \le \frac{1}{4}|\Delta u^k|^2 + |(u\cdot\nabla u)^k|^2\\ &\le \frac{1}{2}|\Delta u^k|^2 + c|\nabla u|^4|\nabla u^k|^2 + c\sum_{i=0}^{k-1}|u^i|_\infty^2|\nabla u^{k-i}|^2.\end{aligned}$$

Thus Inequality (19) is obtained.

Inequality (20) follows immediately from (17) by using the Schwarz inequality.

Lemma 3. *We have*

$$t^{2k-1}|u^k|^2 + \int_0^t s^{2k-1}|\nabla u^k|^2\,ds \le b(t) \qquad (k\ge 1), \tag{23}$$

$$t^{2k}|\nabla u^k|^2 + \int_0^t s^{2k}|\Delta u^k|^2\,ds \le b(t) \qquad (k\ge 0), \tag{24}$$

$$\int_0^t s^{2k}|u^k|_\infty^2\,ds \le b(t) \qquad (k\ge 0), \tag{25}$$

$$\int_0^t s^{2k}|u^{k+1}|^2\,ds \le b(t) \qquad (k\ge 0). \tag{26}$$

Proof. We use mathematical induction. The case $k = 0$ has been proven in Lemma 2. Suppose now that $k \ge 1$ and that (23)–(26) have been proven for all lower order cases.

We proceed to prove (23). Let

$$\phi(t) = t^{2k-1}|u^k|^2 + \int_0^t s^{2k-1}|\nabla u^k|^2\,ds.$$

Then, by (18), we have

$$\begin{aligned}\frac{d\phi}{dt} &= t^{2k-1}\left(\frac{d}{dt}|u^k|^2 + |\nabla u^k|^2\right) + (2k-1)t^{2k-2}|u^k|^2 \\ &\le c(1+|\nabla u|^4)\phi + f,\end{aligned}$$

where

$$f = c\sum_{i=1}^{k-1}(t^{2i}|\nabla u^i|^2)(t^{2k-2i-1}|\nabla u^{k-i}|^2) + (2k-1)t^{2k-2}|u^k|^2.$$

Hence we have

$$\phi(t) \le \int_0^t f(s)\,ds \exp\int_0^t (1+|\nabla u(s)|^4)\,ds \le b(t).$$

Therefore (23) is obtained.

Inequality (24) is obtained similarly, from (19), using the bound on the integral in (23). Inequality (25) then follows from Lemma 1. Finally, Inequality (26) is obtained by using (20).

Lemma 4. *We have*

$$\begin{aligned} t^{2k+1}|\Delta u^k|^2 &\le b(t) \qquad (k \ge 0), \\ t^{2k+1/2}|u^k|_\infty^2 &\le b(t) \qquad (k \ge 0). \end{aligned}$$

Proof. Letting $v = \Delta u^k$ in (14) and using the Schwarz inequality, we get

$$|\Delta u^k| \le |u^{k+1} + (u\cdot\nabla u)^k|.$$

Hence

$$|\Delta u^k|^2 \le 2|u^{k+1}|^2 + 2|(u\cdot\nabla u)^k|^2.$$

Using (22), we obtain

$$|\Delta u^k|^2 \le 4|u^{k+1}|^2 + c\sum_{i=0}^{k-1}|u^i|_\infty^2|\nabla u^{k-i}|^2.$$

Using this and Inequality (21) and the estimates in Lemma 3, we obtain Lemma 4 by mathematical induction.

3. Proof of Theorem 1. Using the estimates obtained above, following somewhat standard procedures (cf. [3]), it is not difficult to prove the following. There exists a subsequence of the Galerkin approximations which tend to a weak limit. The weak limit, denoted again by u, belongs to $C^\infty((0,T),\hat{H}_0^1(\Omega))$ and satisfies Lemmas 2, 3 and 4. We have

$$u^{k+1} + (u\cdot\nabla u)^k = \Delta u^k,$$

weakly, for all $k \geq 0$.

Next, we show that u satisfies the initial condition. From (10) we have $u(t) - u_0 \to 0$ weakly as $t \to 0^+$. Hence $\nabla u(t) \to \nabla u_0$ weakly. But from (6) we have $\lim_{t\to 0^+} |\nabla u(t)| \leq |\nabla u_0|$. Therefore $\nabla u(t) \to \nabla u_0$ strongly.

Now, suppose Ω is an arbitrary open set in $\mathbb{R}^3$, and $u_0 \in \hat{H}_0^1(\Omega)$ is given. Choose $u_{m,0} \in C_0^\infty(\Omega)$ to be a sequence of functions such that $|\nabla u_{m,0} - \nabla u_0| \to 0$ and $|\nabla u_{m,0}| \leq |\nabla u_0|$. Choose a sequence of bounded open subsets $\{\Omega_m\}$ expanding to Ω such that supp $u_{m,0} \subset \Omega_m$. In each subdomain Ω_m, with initial value $u_{m,0}$, we find a solution u_m of the Burgers equation as above. Then, again, there exists a subsequence of u_m tending to a weak limit, which is the solution u of Theorem 1.

Next, we prove the smoothness of the solution. Let Q denote the space-time domain $\Omega \times (0,T)$. Let Δ_Q denote the space-time Laplacian $\Delta + \partial^2/\partial t^2$. Let

$$f = u_{tt} + u_t + u\cdot\nabla u.$$

Then we have

$$\Delta_Q u^k = f^k, \qquad \text{for all } k = 0, 1, 2, \dots .$$

From the estimates in the lemmas, we know every f^k is in $L^2_{loc}(Q)$. By the interior regularity theorem for the Poisson equation, we conclude that every u^k is in $H^2_{loc}(Q)$. Then, it follows that every f^k is in $H^1_{loc}(Q)$, which in turn implies that every u^k is in $H^3_{loc}(Q)$. Continuing this bootstrapping argument, we conclude that u is in $H^m_{loc}(Q)$ for all $m > 0$. Therefore, by a well-known Sobolev imbedding theorem, we have $u \in C^\infty(Q)$.

If a portion of the spatial boundary is $C^{m+1,1}$ for some integer $m \geq 0$, then, by well-known theorems of elliptic boundary regularity and Sobolev imbeddings (see, for example, Grisvard, [2]), one concludes that $u(\cdot,t)$ is H^{m+2}, and hence $C^{m,1/2}$, up to that portion of the spatial boundary. The smoothness in the time variable is not restricted.

Finally, we prove uniqueness. Suppose v is another solution. Let $w = v - u$. Then we have

$$\frac{\partial w}{\partial t} + v\cdot\nabla w + w\cdot\nabla u = \Delta w, \tag{27}$$

$$\lim_{t\to 0+} |\nabla w(t)| = 0. \tag{28}$$

Multiplying (27) with Δw and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}|\nabla w|^2 + |\Delta w|^2 = (v\cdot\nabla w + w\cdot\nabla u, \Delta w).$$

Using the Schwarz inequality, Lemma 1 and Young's inequality, we obtain

$$\frac{d}{dt}|\nabla w|^2 \leq c\left(|v|_\infty^2 + |\nabla u|^4\right)|\nabla w|^2.$$

Hence, for any $0 < t_1 < t < T$, we have

$$|\nabla w(t)|^2 \leq |\nabla w(t_1)|^2 \exp\int_{t_1}^{t}\left(|v|_\infty^2 + |\nabla u|^4\right)\,ds \leq b(t)|\nabla w(t_1)|^2.$$

Letting $t_1 \to 0^+$ and using (28), we have $|\nabla w(t)| = 0$. Therefore $v \equiv u$.

Remark. Since the solution is unique, it follows that in fact the whole sequence of u_m tends to the solution u. And, in the case of a bounded domain, the whole sequence of Galerkin approximations u_N tends to the solution.

4. Two-dimensional domains. In two-dimensional domains, Lemma 1 is replaced by the following inequality from [15].

Lemma 5. *Suppose that Ω is an open set in $\mathbb{R}^2$. Then every $u \in H_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ satisfies*

$$|u|_\infty \leq \frac{1}{\sqrt{\pi}}|u|^{1/2}|\Delta u|^{1/2}. \tag{29}$$

A result for the two-dimensional case, analogous to Theorem 1, can be established using essentially the same methods of proof. However, we must assume that $u_0 \in H_0^1(\Omega)$. In other words, we must make an additional assumption, that u_0 is square-summable. The only significant new difficulty in the proof concerns the first, basic, *a priori* estimate.

Letting $v = u$ in (4) and using the well-known Hölder's inequality and Sobolev inequality, we have

$$\frac{1}{2}\frac{d}{dt}|u|^2 + |\nabla u|^2 = -(u\cdot\nabla u, u) \leq \|u\|_{L^4}^2|\nabla u| \leq c|u|\,|\nabla u|^2.$$

Writing the last expressions as $(c|u|\,|\nabla u|^{1/2})||\nabla u|^{3/2}$ and using Young's inequality, we obtain

$$\frac{d}{dt}|u|^2 \leq c|u|^4|\nabla u|^2. \tag{30}$$

Letting $v = -\Delta u$ in (4) and using Inequality (29), we have

$$\frac{1}{2}\frac{d}{dt}|\nabla u|^2 + |\Delta u|^2 = (u\cdot\nabla u, \Delta u)$$
$$\leq |u|_\infty |\nabla u||\Delta u| \leq c|u|^{1/2}|\nabla u||\Delta u|^{1/2} \leq \frac{1}{4}|\Delta u|^2 + c|u|^2|\nabla u|^4.$$

From this and (30), we obtain

$$\frac{d}{dt}\left(|u|^2 + |\nabla u|^2\right) + |\Delta u|^2 \leq c\left(|u|^2 + |\nabla u|^2\right)^3.$$

Assuming that the initial data u_0 is given in $H_0^1(\Omega)$, we can solve this differential inequality and obtain the first basic *a priori* estimates. The remainder of the two-dimensional theory parallels the three-dimensional case.

5. Global existence. Local solutions of the Burgers equation can be continued in time indefinitely, thanks to a maximum principle given in [7]. For the sake of completeness, we present this continuation argument here. Unfortunately, it does not apply to the Navier-Stokes equations.

Assume first that Ω is bounded and smooth. The local solution that we obtained in the foregoing sections is smooth and satisfies the Burgers equation

$$\frac{\partial u}{\partial t} + u\cdot\nabla u = \Delta u. \tag{31}$$

Multiplying (30) by u, one obtains

$$\frac{\partial}{\partial t}\frac{u^2}{2} + u\cdot\nabla\frac{u^2}{2} = \Delta\frac{u^2}{2} - (\nabla u)^2.$$

Let $\alpha > 0$ be arbitrary and let $\phi = e^{-\alpha t}u^2/2$. Then we have

$$\frac{\partial}{\partial t}\phi + \alpha\phi + u\cdot\nabla\phi = \Delta\phi - e^{-\alpha t}(\nabla u)^2. \tag{32}$$

Let $t_1 = T/2$, and suppose that the solution exists on at least $[t_1, t_2]$. Over the compact set $\overline{\Omega}\times[t_1, t_2]$, the smooth function $\phi(x,t)$ must attain a maximum value. If a maximum point is in $\Omega\times(t_1, t_2]$, then it is clear from (31) that the value of ϕ there must be zero. Therefore, the maximum value must be attained at $t = t_1$. Since we can let $\alpha \to 0^+$, we conclude that

$$|u(t)|_\infty \leq |u(t_1)|_\infty \qquad (t_1 \leq t \leq t_2). \tag{33}$$

Multiplying (30) by $-\Delta u$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}|\nabla u|^2 + |\Delta u|^2 = (u\cdot\nabla u, \Delta u) \leq |u|_\infty|\nabla u||\Delta u| \leq \frac{1}{4}|u|_\infty^2|\nabla u|^2 + |\Delta u|^2.$$

Therefore, using (32), we get

$$\frac{d}{dt}|\nabla u(t)| \leq \frac{1}{4}|u(t_1)|_\infty^2|\nabla u(t)| \qquad (t_1 \leq t \leq t_2).$$

Hence

$$|\nabla u(t)| \leq |\nabla u(t_1)| \exp \frac{|u(t_1)|_\infty^2 (t-t_1)}{4}, \tag{34}$$

for $t_1 \leq t \leq t_2$. In the three-dimensional case, we can now apply the local existence theorem to extend the solution beyond t_2, for a short time depending only on $|\nabla u(t_2)|$. In the mean time, the estimate (33) continues to hold. It is clear that the solution can be extended globally in time.

In the two-dimensional case, we need an additional estimate for $|u(t_2)|$. That can be obtained by multiplying (30) by u, and following a parallel argument. We obtain

$$|u(t)| \leq |u(t_1)| \exp \frac{|u(t_1)|_\infty^2 (t-t_1)}{4}.$$

The case of an arbitrary domain (nonsmooth or unbounded) is treated by applying the foregoing argument to each of the solutions u_m (which were defined in an expanding sequence of smoothly bounded domains Ω_m) used in the construction of the solution u in Section 3.

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