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FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS AND HENSTOCK-KURZWEIL INTEGRALS

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Abstract. The existence and uniqueness of solutions to the Cauchy problem for a first-order quasi-linear partial differential equation is studied in this paper using the Henstock-Kurzweil integral. The classical theory requires certain differentiability and continuity conditions on the coefficients of the derivatives in the equation. It is shown here that in the Henstock-Kurzweil integral setting these conditions can be relaxed and that the resulting solution is differentiable though the derivatives need not be continuous. This sharpens the classical result and provides a bridge between classical and weak solutions in the linear case.

1. Introduction. The Henstock-Kurzweil integral is a generalization of the Newton, Riemann, and Lebesgue integrals (cf. [5, 12, 16, 18, 20, 22, 31]). A particular feature of this integral is that it can be used to define the integrals of highly oscillating functions such as $F'(t)$, where $F(t) = t^2 \sin t^{-2}$, $t \neq 0, F(0) = 0$. The Henstock-Kurzweil integral thus provides a natural framework for the study of differential equations which involve highly oscillating functions. The Henstock-Kurzweil integral provides a bridge between the classical theory which generally focuses on solutions with continuous derivatives and the more modern theory, which concentrates on weak solutions. In the Henstock-Kurzweil setting solutions are differentiable but their derivatives are not necessarily continuous. There has been much research on the applications of this integral to ordinary differential equations ([2, 4, 6, 7, 15, 21, 27]) and some initial investigations on applications to functional differential equations ([8]); however, applications to partial differential equations are largely unexplored. In this paper we study first order partial differential equations involving highly oscillating functions in the Henstock-Kurzweil setting.

In this section we introduce notation which will be used throughout the paper, describe the general problem and further motivate the use of the Henstock-Kurzweil

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integral. The definition of the Henstock-Kurzweil integral will be given in the next section. For succinctness we shall refer to a function which is Henstock-Kurzweil integrable as simply HK integrable. All intervals of the real line are assumed compact unless otherwise noted.

Let $\mathbb{R}$ denote the set of all real numbers, I_1, I_2, I_3 be intervals in $\mathbb{R}$, and $I = I_1 \times I_2 \times I_3$. The space of functions $h : I_k \to \mathbb{R}$ which are HK integrable on I_k will be denoted by $HK(I_k)$. Let $h, g \in HK(I_1)$. A function $f : I \to \mathbb{R}$ belongs to the class $Car(h, g)$ if: (a) for each $(y, z) \in I_1 \times I_2$, $f(\cdot, y, z)$ is measurable; (b) $f(x, \cdot, \cdot)$ is continuous for all $x \in I_1$; and (c) the inequality $g(x) \le f(x, y, z) \le h(x)$ is satisfied for all $(x, y, z) \in I$. Let $\ell : I_1 \to \mathbb{R}$ be a function which is Lebesgue integrable on I_1. A function $f : I_1 \to \mathbb{R}$ belongs to the class $Lip(\ell)$ if:

$$|f(x, y_1, z_1) - f(x, y_2, z_2)| \le \ell(x)(|y_1 - y_2| + |z_1 - z_2|),$$

for any two $(x, y_1, z_1), (x, y_2, z_2) \in I$. Recall that a function $f : I_2 \times I_3 \to \mathbb{R}$ is differentiable (i.e., has a total differential) at $(y, z) \in I_2 \times I_3$ if there exists a $(q_1, q_2) \in \mathbb{R}^2$ such that $f(y + \Delta y, z + \Delta z) - f(y, z) = q_1 \Delta y + q_2 \Delta z + \epsilon(|\Delta y| + |\Delta z|)$, where $\epsilon = \epsilon(y, z, \Delta y, \Delta z)$, and $\epsilon \to 0$ as $\Delta y, \Delta z \to 0$. It is well known that if f is differentiable at (y, z) then the partial derivatives exist and $q_1 = f_y, q_2 = f_z$. A function $f : I \to \mathbb{R}$ is said to belong to the class $D(I_2 \times I_3)$ if for each $x \in I_1$, the function $f(x, \cdot, \cdot)$ is differentiable on $I_2 \times I_3$; i.e.,

$$f(x, y + \Delta y, z + \Delta z) - f(x, y, z) = f_y(x, y, z)\Delta y + f_z(x, y, z)\Delta z + \epsilon(|\Delta y| + |\Delta z|),$$

where $\epsilon = \epsilon(x, y, z, \Delta y, \Delta z)$, and $\epsilon \to 0$ as $\Delta y, \Delta z \to 0$.

Let $m : I_1 \to \mathbb{R}^+$ be Lebesgue integrable on I_1. A function $f : I \to \mathbb{R}$ is said to belong to the class $DB(I_2 \times I_3, m)$ if $f \in D(I_2 \times I_3)$ and there exists a $\rho > 0$ such that the function $\epsilon(x, y, z, \Delta y, \Delta z)$ in the above inequality satisfies

$$|\epsilon(x, y, z, \Delta y, \Delta z)| \le m(x)$$

for all $(x, y, z) \in I$ and $\Delta y, \Delta z \in [-\rho, \rho]$.

Throughout this paper the point $(x_0, y_0, z_0) \in \mathbb{R}^3$ is regarded as fixed. The intervals I_1, I_2 and I_3 are defined as $I_1 = \{x : |x - x_0| \le \alpha\}, I_2 = \{y : |y - y_0| \le \beta\}, I_3 = \{z : |z - z_0| \le \gamma\}$. The symbols a and b represent throughout this paper real-valued functions defined on I satisfying the following conditions:

(a) $a \in Car(h_1, g_1), b \in Car(h_2, g_2)$;
(b) $a, b \in Lip(\ell)$.

For certain applications we will also require

(c) $a, b \in D(I_2 \times I_3)$;
(d) $a_y, a_z, b_y, b_z \in Car(-h_3, h_3)$.

Here, the functions h_k, g_k are in $HK(I_1)$. We remark that if both $h_3 \in HK(I_1)$ and $-h_3 \in HK(I_1)$, then h_3 is Lebesgue integrable on this interval (see Section 2).

We shall prove that if $a : I \to \mathbb{R}$ satisfies conditions (b), (c), and (d), then $a \in DB(I_2 \times I_3, m)$ (see Lemma 11). Throughout this paper the function $w : I_2 \to \mathbb{R}$ is assumed to have a continuous derivative on I_2 with $w(y_0) = z_0$.

In this paper we consider the Cauchy problem for a first-order quasi-linear partial differential equation

$$\frac{\partial z}{\partial x} + a(x, y, z)\frac{\partial z}{\partial y} = b(x, y, z), \tag{1}$$

$$z(x_0, y) = w(y). \tag{2}$$

The Cauchy problem is said to have a solution z if there exist intervals $E_1 \subset I_1, E_2 \subset I_2$ with $x_0 \in E_1, y_0 \in E_2$ such that $z : E_1 \times E_2 \to I_3$ is differentiable almost everywhere in $E_1 \times E_2$, and satisfies equation (1) almost everywhere in $E_1 \times E_2$, and equation (2) everywhere in E_2.

We prove that the Cauchy problem has a unique solution if a and b satisfy conditions (a)–(d). Our results are a generalization of classical results (cf. [3, 11, 14, 29, 32]). There is a paucity of results regarding the analogous problem in the Lebesgue framework; however, there are many references for second-order partial differential equations, e.g., [1, 10, 24, 25, 26, 33]. We now state the main results of the paper; the proofs will be given in Section 4.

Theorem 1. *If the functions a and b satisfy conditions* (a)–(d)*, then the Cauchy problem* (1)*,* (2) *has a unique solution.*

Theorem 2. *Let $\bar{a}, \bar{b}$ be real-valued functions on I with continuous partial derivatives, and let P, Q be real-valued functions on I_1 such that $a = \bar{a} + P', b = \bar{b} + Q'$. Then the Cauchy problem* (1)*,* (2) *has a unique solution z. Furthermore, the solution z is differentiable on $I_1 \times I_2$ and satisfies* (1) *everywhere in $I_1 \times I_2$.*

Corollary 3. *Let a, b be real-valued functions on I with continuous partial derivatives. Then the Cauchy problem* (1)*,* (2) *has a unique solution which satisfies equation* (5) *everywhere in $I_1 \times I_2$.*

Remark 1. The above corollary corresponds to a classical result for the Cauchy problem (cf. [3, 11, 14, 29, 32]). Theorem 2 is an extension of this result because the functions a and b need be in only $HK(I_1)$ and therefore can be of a highly oscillatory nature. Since the Henstock-Kurzweil integral encompasses the Lebesgue integral, this theorem is also an extension on the analogous problem under the more restrictive assumption that a and b are Lebesgue integrable on I_1. Note also that the solution z guaranteed by Theorem 2 is differentiable everywhere on $I_1 \times I_2$ but that the derivatives need not be continuous there.

The following theorem refines this result for the linear partial differential equation:

Theorem 4. *Let $a : I_1 \times I_2 \to \mathbb{R}$ be a function such that $a \in Car(h_1, g_1), a \in Lip(\ell), a \in D(I_2)$, and $a_y \in Car(h_2, -h_2)$, where the $h_1, h_2, g_1 \in HK(I_1)$. Let c and d be real-valued functions on $I_1 \times I_2$ such that $c \in Car(h_2, -h_2), d \in Car(h_3, g_3)$, c*

,$d \in Lip(\ell)$, c, $d \in D(I_2)$ and $c_y, d_y \in Car(h_2, -h_2)$, where $h_3, g_3 \in HK(I_1)$. Then the Cauchy problem

$$\frac{\partial z}{\partial x} + a(x,y)\frac{\partial z}{\partial y} = c(x,y)z + d(x,y), \tag{3}$$

$$z(x_0, y) = w(y), \tag{4}$$

has a unique solution z given by $z(x,y) = \bar{z}(x, \theta(x,y))$, where

$$\bar{z}(x,\eta) = e^F\{w(\eta) + \int_{x_0}^{x} d(t, y(t,\eta))e^{-F}\,dt\},$$

$$F = F(x,\eta) = \int_{x_0}^{x} c(t, y(t,\eta))\,dt,$$

$y(x,\eta)$ is the solution to the ordinary differential equation $y' = a(x,y)$ with $y(x_0) = \eta$, and $\theta(x,y)$ is an inverse of $y(x,\eta)$.

Remark 2. In the above theorem it is assumed that $c \in Car(h_2, -h_2)$ and hence that c is Lebesgue integrable on I_1. This means that for any fixed η the function $F(\cdot, \eta)$ is of bounded variation on I_1 and this ensures that $d(t, y(t,\eta))e^{-F(t,\eta)}$ is in $HK(I_1)$. If this restriction is relaxed to say $c \in Car(h_3, g_3)$ then the result does not follow because $d(t, y(t,\eta))e^{-F(t,\eta)}$ may not even be HK integrable on I_1. What is needed here is that $c(x,y)z + d(x,y) \in Car(h_4, g_4)$ for some functions $h_4, g_4 \in HK(I_1)$; if we assume that $c \in Car(h_3, g_3)$, then, in I_1,

$$g_2(x) \le c(x,y) \le h_2(x),$$

but this does not imply that

$$g_2(x)z \le c(x,y)z \le h_2(x)z,$$

since z may be negative. If, however, $g_2(x) = -h_2(x)$, then $|c(x,y)| \le h_2(x)$ on I_1 and since I_3 is compact there is an $M \ge 0$ such that $|c(x,y)z| \le Mh_3(x)$ for $z \in I_3$.

Remark 3. The solution in Theorem 4 is a weak solution ([11, 13, 30]); i.e., for any infinitely differentiable function ϕ with compact support

$$\int_{I_2}\int_{I_1} z(x,y)\{-\frac{\partial \phi(x,y)}{\partial x} - \frac{\partial \phi(x,y)a(x,y)}{\partial y}\}\,dx\,dy$$
$$= \int_{I_2}\int_{I_1} z(x,y)c(x,y)\phi(x,y)\,dx\,dy + \int_{I_2}\int_{I_1} d(x,y)\phi(x,y)\,dx\,dy.$$

We shall not prove this here. The proof involves two-dimensional Henstock-Kurzweil integrals, integration by parts, and a Tonelli-type theorem ([5, 12, 27]) as in the case of Lebesgue integrals. The solutions to the Cauchy problems guaranteed by the above theorems, however, are in fact differentiable everywhere in the region of existence for the weak solutions. This is an advantage of using the Henstock-Kurzweil integral. It may be, however, that the derivatives are not continuous. The solutions thus lie somewhere between the classical and weak solutions. This case is not contained in the Lebesgue theory.

Example. It is not difficult to construct specific examples of partial differential equations the solution of which require the Henstock-Kurzweil integral setting. Consider the partial differential equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = F'(x), \tag{5}$$

for $(x, y, z) \in I = \{[-1, 1] \times [-1, 1] \times [-4, 4]\}$ subject to the initial condition

$$z(0, y) = -y. \tag{6}$$

Here,

$$F(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Evidently, the solution is $z = x - y + F(y)$. The solution is differentiable on $[-1, 1] \times [-1, 1]$; however, the partial derivative $z_x(x, y) = 1 + F'(x)$ is not continuous at $x = 0$ and is highly oscillating in a neighbourhood of zero. This simple example is not covered by the classical theory (Corollary 3) or any result using the Lebesgue integral since $F'(x)$ is not even Lebesgue integrable on $[-1, -1]$.

Remark 4. Examples such as the above involving highly oscillating functions emphasize the natural rôle the HK integral plays in the theory. The need for the HK integral is more acute for the quasi-linear case. For the linear case, partial differential equations involving highly oscillating functions such as F in the above example can be tackled using the theory of distributions. Nonetheless, even in these cases the HK integral setting gleans a sharper result, viz., the differentiability of the solution.

2. Henstock-Kurzweil integrals and ordinary differential equations. The solution of a quasi-linear first-order partial differential equation can be reduced to the solution of a system of ordinary differential equations using the familiar method of characteristics ([3, 11, 14, 29, 32]). As the method of characteristics will play a central rôle in the proofs of the existence results, we summarize in this section some of the known results of use in later sections concerning ordinary differential equations and Henstock-Kurzweil integrals. We also establish some results concerning the differentiability of solutions, which will be used in the study of the Cauchy problem.

Definition 1. A real-valued function f is said to be HK integrable to A on $[u, v]$ if, for every $\epsilon > 0$, there is a positive function $\delta(\xi)$ such that for any division D given by $u = x_0 < x_1 < \ldots < x_n = v$ and $\{\xi_1, \xi_2, \ldots, \xi_n\}$ satisfying

$$\xi_i - \delta(\xi_i) < x_{i-1} \leq \xi_i \leq x_i < \xi_i + \delta(\xi_i)$$

for $i = 1, 2, \ldots, n$, the inequality

$$|\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A| < \epsilon$$

is satisfied.

The relationship between the Henstock-Kurzweil and the Lebesgue integrals is detailed in the following theorems:

Theorem 5. *If f is Lebesgue integrable on the interval $[u, v]$, then $f \in HK([u, v])$.*

Theorem 6. *If $f \in HK([u, v])$ and nonnegative on $[u, v]$, then it is Lebesgue integrable on $[u, v]$.*

Theorem 7. *If $f \in HK([u, v])$ and g is a function of bounded variation on $[u, v]$, then $fg \in HK([u, v])$. For a fixed $f \in HK([u, v])$, the linear functional $\langle f, g\rangle = \int_u^v fg$ is continuous on the space of functions of bounded variation on $[u, v]$.*

Proofs for the above results can be found in [5, 12, 16, 18, 20, 31].

The next theorem concerns the existence, uniqueness, and continuous dependence on initial values of solutions to ordinary differential equations using the Henstock-Kurzweil integral:

Theorem 8. *Let e,f be real-valued functions on I and suppose that $e \in Car(h_1, g_1)$, $f \in Car(h_2, g_2)$, and that $e, f \in Lip(\ell)$. Let ϕ be a continuous function on I_2 with $\phi(y_0) = z_0$. Then there exist subintervals $J_k \subset I_k$ with $x_0 \in J_1, y_0 \in J_2$ such that on J_1, for each $\eta \in J_2$ there exists a unique solution $(y(x, \eta), z(x, \eta))$ to the system of ordinary differential equations*

$$\frac{dy}{dx} = e(x, y, z), \frac{dz}{dx} = f(x, y, z) \tag{7}$$

such that $y(x_0, \eta) = \eta$ and $z(x_0, \eta) = \phi(\eta)$.

The proof of this result can be found in [7, 27]. We note here that from the proof of the above existence theorem the subintervals J_1, J_2 can be chosen such that on J_1 a solution $(y(x, \eta), z(x, \eta))$ exists for each $\eta \in J_2$. We may choose $J_1 = [x_0 - \alpha_1, x_0 + \alpha_1]$ and $J_2 = [y_0 - \beta_1, y_0 + \beta_1]$ in the following way. The function ϕ is continuous at y_0 and we can therefore choose a β_1 so that $0 < \beta_1 < \beta/2$ and $|\phi(\eta) - \phi(y_0)| = |\phi(\eta) - z_0| < \gamma/2$ for all $\eta \in [y_0 - \beta_1, y_0 + \beta_1]$. We can also choose an α_1 such that $0 < \alpha_1 \leq \alpha$ and for all $(x, \eta) \in J_1 \times J_2$ we have

$$(x, \eta + \int_{x_0}^{x} h_1(t)\, dt),\ (x, \eta + \int_{x_0}^{x} g_1(t)\, dt) \in I_1 \times I_2,$$

and

$$(x, \phi(\eta) + \int_{x_0}^{x} h_2(t)\, dt),\ (x, \phi(\eta) + \int_{x_0}^{x} g_2(t)\, dt) \in I_1 \times I_3.$$

Theorem 9. *Let $(y(x,\eta), z(x,\eta))$ be the solution given in Theorem 8. Then, for each $x \in J_1$, the solution is continuous in η on J_2.*

The proof of this result can be found in [27]. It is similar to that used for the Lebesgue case.

The next theorem concerns the continuous dependence of solutions on a parameter. We state it only for linear systems and in the Lebesgue integral framework as this is sufficient for our purposes.

Theorem 10. *Let E_1, E_2 and E_3 be intervals in $\mathbb{R}$ with $x_0 \in E_1$ and f_n, S be real-valued functions on E_1, for $n = 1,2,3,4$, and Lebesgue integrable on E_1. Let $\epsilon_k : E_1 \times [-\rho,\rho] \to \mathbb{R}$, $k = 1,2$ be measurable functions such that, for $k = 1,2$, $|\epsilon_k(x,\theta)| \le S(x)$ for all $(x,\theta) \in E_1 \times [-\rho,\rho]$ and for each $x \in E_1$, $\epsilon_k(x,\theta) \to 0$ on E_1 as $\theta \to 0$. Suppose that, for each $\theta \in [-\rho,\rho]$, the functions $u(x,\theta), v(x,\theta)$ are a solution of the system*

$$\begin{aligned} \frac{du}{dx} &= f_1(x)u + f_2(x)v + \epsilon_1(x,\theta)(u+v), \\ \frac{dv}{dx} &= f_3(x)u + f_4(x)v + \epsilon_2(x,\theta)(u+v) \end{aligned} \tag{8}$$

for $(x,u,v) \in E_1 \times E_2 \times E_3$ with $u(x_0,\theta) = p(\theta), v(x_0,\theta) = q(\theta)$, where $p(\theta) \to p_0, q(\theta) \to q_0$ as $\theta \to 0$. Then $\lim_{\theta\to 0}(u(x,\theta), v(x,\theta)) = (u(x), v(x))$ exists and is the solution of

$$\begin{aligned} \frac{du}{dx} &= f_1(x)u + f_2(x)v, \\ \frac{dv}{dx} &= f_3(x)u + f_4(x)v \end{aligned} \tag{9}$$

for $(x,u,v) \in E_1 \times E_2 \times E_3$ with $u(x_0) = p_0$ and $v(x_0) = q_0$.

Lemma 11. *Let a be a real-valued function on I such that $a \in Lip(\ell)$, $a \in D(I_2 \times I_3)$, and $a_y, a_z \in Car(h_3, -h_3)$. Then $a \in DB(I_2 \times I_3, M)$, where $M(x) = \ell(x) + h_3(x)$.*

Proof. Since $a \in Lip(\ell)$,

$$|a(x, y+\Delta y, z+\Delta z) - a(x,y,z)| \le \ell(x)\{|\Delta y| + |\Delta z|\},$$

and, since $a \in D(I_2 \times I_3)$,

$$a(x, y+\Delta y, z+\Delta z) - a(x,y,z) = a_y(x,y,z)\Delta y + a_z(x,y,z)\Delta z + \epsilon\{|\Delta y| + |\Delta z|\},$$

where $\epsilon = \epsilon(x,y,z,\Delta y,\Delta z)$, and $\epsilon \to 0$ as $\Delta y, \Delta z \to 0$. Hence,

$$\begin{aligned} |\epsilon|\{|\Delta y| + |\Delta z|\} &\le |a(x, y+\Delta y, z+\Delta z) - a(x,y,z)| \\ &\quad + |a_y(x,y,z)\Delta y| + |a_z(x,y,z)\Delta z| \\ &\le \ell(x)(|\Delta y| + |\Delta z|) + h_3(x)(|\Delta y| + |\Delta z|) \\ &\le (\ell(x) + h_3(x))(|\Delta y| + |\Delta z|), \end{aligned}$$

and therefore there exists a $\rho > 0$ such that for all $\Delta y, \Delta x \in [-\rho, \rho]$ and all $(x, y, z) \in I$

$$|\epsilon(x, y, z, \Delta y, \Delta z)| \le M(x),$$

so that $a \in DB(I_2 \times I_3, M)$.

Theorem 12. *Let a and b be real-valued functions on I satisfying conditions* (a) *and* (b). *Let $w : I_2 \to \mathbb{R}$ be differentiable with a continuous derivative and $w(y_0) = z_0$. Then there exist subintervals $J_1 \subset I_1, J_2 \subset I_2$ with $x_0 \in J_1, y_0 \in J_2$ such that on J_1, for each $\eta \in J_2$, there exists a unique solution $(y(x, \eta), z(x, \eta))$ of the system*

$$\frac{dy}{dx} = a(x, y, z),\ \frac{dz}{dx} = b(x, y, z), \tag{10}$$

satisfying the conditions $y(x_0, \eta) = \eta, z(x_0, \eta) = w(\eta)$. The solution is continuous on $J_1 \times J_2$. If, in addition, a and b also satisfy conditions (c) *and* (d), *then there exist subintervals $K_1 \subset J_1, K_2 \subset J_2$ with $x_0 \in K_1, y_0 \in K_2$ such that the partial derivatives y_η, z_η exist and are continuous on $K_1 \times K_2$.*

Proof. The existence, uniqueness, and continuity of the solution follow from Theorems 8 and 9. Theorem 10 can be used to prove that the solution is differentiable (with respect to η). Let $\eta \in J_2$ be fixed; for any $x \in J_1$ and any nonzero real number τ we have

$$\Delta y = \tau + \int_{x_0}^{x} [a(t, y(t, \eta + \tau), z(t, \eta + \tau)) - a(t, y(t, \eta), z(t, \eta))]\, dt, \tag{11}$$

$$\Delta z = \Delta w + \int_{x_0}^{x} [b(t, y(t, \eta + \tau), z(t, \eta + \tau)) - b(t, y(t, \eta), z(t, \eta))]\, dt, \tag{12}$$

where $\Delta y = y(t, \eta+\tau) - y(t, \eta)$, $\Delta z = z(t, \eta+\tau) - z(t, \eta)$, and $\Delta w = w(\eta+\tau) - w(\eta)$.

Let $f_1(t, \eta)$, $f_2(t, \eta)$, $f_3(t, \eta)$, $f_4(t, \eta)$ denote the values of a_y, a_z, b_y, b_z at $(t, y(t, \eta)$, $z(t, \eta))$ respectively, and let $u = u(t, \eta, \tau) = \Delta y/\tau$ and $v = v(t, \eta, \tau) = \Delta z/\tau$. Condition (c) and equations (11) and (12) imply

$$u = 1 + \int_{x_0}^{x} [f_1(t, \eta)u + f_2(t, \eta)v + \epsilon_1(u + v)]\, dt \tag{13}$$

$$v = \frac{\Delta w}{\tau} + \int_{x_0}^{x} [f_3(t, \eta)u + f_4(t, \eta)v + \epsilon_2(u + v)]\, dt, \tag{14}$$

where $\epsilon_1 = \epsilon_1(t, \eta, \tau), \epsilon_2 = \epsilon_2(t, \eta, \tau)$ and $\epsilon_1, \epsilon_2 \to 0$ as $\Delta y, \Delta z \to 0$. Note that $y(t, \eta), z(t, \eta)$ are continuous on $J_1 \times J_2$; consequently, for each η, $\epsilon_1(t, \eta, \tau) \to 0$ and $\epsilon_2(t, \eta, \tau) \to 0$ as $\tau \to 0$. Furthermore, by Lemma 11, $|\epsilon_k(t, \eta, \tau)| \le \ell(t) + h_3(t)$ for $k = 1, 2$ and all (t, η, τ).

On the other hand, by condition (d) we have $a_y, a_z, b_y, b_z \in Car(h_3, -h_3)$, for each $\eta \in J_2$, and there exists a unique solution $(u(x,\eta), v(x,\eta))$ on $J_1 \times J_2$ of

$$\begin{aligned} \frac{du}{dx} &= f_1(x)u + f_2(x)v, \\ \frac{dv}{dx} &= f_3(x)u + f_4(x)v, \end{aligned} \tag{15}$$

with $u(x_0,\eta) = 1$ and $v(x_0,\eta) = w'(\eta)$. In order to apply Theorem 10, however, we must restrict the values of $u(x,\eta)$ and $v(x,\eta)$ to bounded sets, say $u(x,\eta) \in A = [0,2]$, and $v(x,\eta) \in B = [w'(y_0) - 1, w'(y_0) + 1]$. With this restriction Theorem 8 indicates that there exist subintervals $K_1 \subset J_1$, $K_2 \subset J_2$ with $x_0 \in K_1$, $y_0 \in K_2$ such that on K_1, for each $\eta \in K_2$, there exists a unique solution $(u(x,\eta), v(x,\eta)$ of the system (15) with $u(x,\eta) \in A$, $v(x,\eta) \in B$ for $(x,\eta) \in K_1 \times K_2$. Now, for each $\eta \in K_2$, $x \in K_1$, $u \in A$, and $v \in B$ we can apply Theorem 10 to equations (13), (14), and (15) and this implies that $\lim_{\tau\to 0}(u(x,\eta,\tau), v(x,\eta,\tau))$ exists on $K_1 \times K_2$. Hence the derivatives y_η and z_η exist on $K_1 \times K_2$, and (u,v) is the solution of (15). By Theorem 9, for each x, the functions $y_\eta(x,\eta)$ and $z_\eta(x,\eta)$ are continuous in η on K_2.

Corollary 13. *Let $y(x,\eta), z(x,\eta) : K_1 \times K_2 \to \mathbb{R}$ be given as in Theorem* 12. *Then $(y_\eta(x,\eta), z_\eta(x,\eta))$ is the solution of*

$$\frac{du}{dx} = a_y(x, y(x,\eta), z(x,\eta))u + a_z(x, y(x,\eta), z(x,\eta))v \tag{16}$$

$$\frac{dv}{dx} = b_y(x, y(x,\eta), z(x,\eta))u + b_z(x, y(x,\eta), z(x,\eta))v, \tag{17}$$

with $u(x_0,\eta) = 1$ and $v(x_0,\eta) = w'(\eta)$.

The proof of this corollary follows from the last part of the proof of Theorem 12.

Theorem 14. *Let $y(x,\eta)$ and $z(x,\eta)$ be real-valued functions on $K_1 \times K_2$ which satisfy the differential equations* (10) *in Theorem* 12. *Then $y(x,\eta)$ and $z(x,\eta)$ are differentiable almost everywhere on $K_1 \times K_2$. More precisely, $y(x,\eta)$ is differentiable on $K_1 \times K_2 \setminus (S_1 \cup S_2)$ and $z(x,\eta)$ is differentiable on $K_1 \times K_2 \setminus (S_1 \cup S_3)$, where*

$$\begin{aligned} S_1 &= \{(x,\eta) : \frac{d}{dx}\int_{x_0}^{x} \ell(t)\,dt \quad \textit{does not exist at } x\} \\ S_2 &= \{(x,\eta) : y_x \quad \textit{does not exist at } (x,\eta)\} \\ S_3 &= \{(x,\eta) : z_x \quad \textit{does not exist at } (x,\eta)\}. \end{aligned}$$

Proof. Let $(x,\eta) \in K_1 \times K_2 \setminus (S_1 \cup S_2)$ and p, q be any two nonzero numbers small in modulus. Then

$$\begin{aligned} &y(x+p,\eta+q) - y(x,\eta) - py_x(x,\eta) - qy_\eta(x,\eta) \\ =&y(x+p,\eta) - y(x,\eta) - py_x(x,\eta) + y(x,\eta+q) - y(x,\eta) - qy_\eta(x,\eta) \\ &+ \int_x^{x+p} \{a(t, y(t,\eta+q), z(t,\eta+q)) - a(t, y(t,\eta), z(t,\eta))\}\,dt. \end{aligned}$$

Now $a \in Lip(\ell)$, and $y_\eta(x,\eta)$ and $z_\eta(x,\eta)$ exist and are continuous on $K_1 \times K_2$. Therefore,

$$\begin{aligned}\int_x^{x+p} &\{a(t,y(t,\eta+q),z(t,\eta+q)) - a(t,y(t,\eta),z(t,\eta))\}\,dt \\ &\leq \int_x^{x+p} \ell(t)\{|y(t,\eta+q)-y(t,\eta)| + |z(t,\eta+q)-z(t,\eta)|\}\,dt \\ &\leq \int_x^{x+p} \ell(t)|q|\{|y_\eta(t,\xi_1)| + |z_\eta(t,\xi_2)|\}\,dt \leq M|q|\int_x^{x+p}\ell(t)\,dt,\end{aligned}$$

for some $M > 0$, $\eta \leq \xi_k \leq \eta + q$, $k = 1, 2$. Consequently,

$$y(x+p,\eta+q) - y(x,\eta) - py_x(x,\eta) - qy_\eta(x,\eta) = \epsilon(x,\eta,p,q)(|p|+|q|),$$

where $\epsilon(x,\eta,p,q) \to 0$ as $p, q \to 0$. We conclude thus that $y(x,\eta)$ is differentiable on $K_1 \times K_2 \setminus (S_1 \cup S_2)$. Similarly, $z(x,\eta)$ is differentiable on $K_1 \times K_2 \setminus (S_1 \cup S_3)$. It is clear that for each η, the sets $\{\bar{x} : y_x$ does not exist at $(\bar{x},\eta)\}$, and $\{\bar{x} : \frac{d}{dx}\int_{x_0}^x \ell(t)\,dt$ does not exist at $(\bar{x},\eta)\}$ are sets of measure zero in $\mathbb{R}^2$; consequently, S_1 and S_2 are sets of measure zero in $\mathbb{R}^2$. Similarly, S_3 is a set of measure zero in $\mathbb{R}^2$.

3. Ancillary results. In this section, we prove that the function $y(x,\eta)$ given in Theorem 12 has an inverse $\theta(x,y)$, i.e., $y = y(x,\theta(x,y))$, and that $u(x,y) = z(x,\eta) = z(x,\theta(x,y))$ is differentiable almost everywhere. First, we state a theorem on inverse functions (cf. [28], page 6).

Theorem 15. *Let N be a neighbourhood of (p_0,q_0,s_0) in $\mathbb{R}^3$. Suppose that $f(p,q,s)$ is a continuous function in N and that f_s exists in N and is continuous at (p_0,q_0,s_0). Then, if $f_s(p_0,q_0,s_0) \neq 0$, and $f(p_0,q_0,s_0) = 0$ there exists a unique function θ which is continuous on some neighbourhood N_1 of (p_0,q_0) such that $f(p,q,\theta(p,q)) = 0$ for all $p, q \in N_1$.*

Lemma 16. *Let $y : K_1 \times K_2 \to \mathbb{R}$ be given as in Theorem 12. Then:*

(i) *there exist subintervals $E_1 \subset K_1$, $E_2 \subset K_2$ with $x_0 \in E_1$, $y_0 \in E_2$ such that, on $E_1 \times E_2$, there exists a unique continuous function $\theta(x,y)$ satisfying $y = y(x,\theta(x,y))$ on $E_1 \times E_2$;*

(ii) *if $y(x,\eta)$ is differentiable at $(\bar{x},\bar{\eta}) \in E_1 \times E_2$, then $\theta(x,y)$ is differentiable at $(\bar{x},\bar{y})$, where $\bar{y} = y(\bar{x},\bar{\eta})$, and*

$$\theta_y(\bar{x},\bar{y}) = \frac{1}{y_\eta(\bar{x},\bar{\eta})}, \quad \theta_x(\bar{x},\bar{y}) = -\frac{y_x(\bar{x},\bar{\eta})}{y_\eta(\bar{x},\bar{\eta})}.$$

Proof. Let $f(x,y,\eta) = y(x,\eta) - y$. then f is continuous on $K_1 \times I_2 \times K_2$, and $(x_0,y_0,\eta_0) \in K_1 \times I_2 \times K_2$, where $\eta_0 = y_0$. Furthermore, by Theorem 12 and Corollary 13, $f_\eta(x_0,y_0,\eta_0) = y_\eta(x_0,\eta_0) = 1$. Thus, by Theorem 15 we obtain the result (i). Since $y_\eta(x_0,\eta_0) = 1$ and $y_\eta(x,\eta)$ is continuous at (x_0,η_0), it can be assumed that $y_\eta(x,\eta) \neq 0$ on $E_1 \times E_2$. The proof of (ii) is standard and can be found in any book on real analysis (e.g. [23], page 343).

Lemma 17. *Let $\theta : E_1 \times E_2 \to \mathbb{R}$ be as given in Lemma* 16*. Then $\theta(x,y)$ is differentiable almost everywhere on $E_1 \times E_2$. More precisely, $\theta(x,y)$ is differentiable on $E_1 \times E_2 \setminus (S_2 \cup S_4)$, where $S_4 = \{(x,y) : (x,\eta) \in S_2, y = y(x,\eta)\}$ and S_1, S_2 are given as in Theorem* 14*.*

Proof. The result follows from Lemma 16 and Theorem 14.

Theorem 18. *Let $\theta : E_1 \times E_2 \to \mathbb{R}$ be given as in Lemma* 16 *and $z(x,\eta) : K_1 \times K_2 \to \mathbb{R}$ be as given in Theorem* 12*. Let $u(x,y) = z(x,\theta(x,y))$ for $(x,y) \in E_1 \times E_2$. Then $u(x,y)$ is differentiable on $E_1 \times E_2 \setminus (S_1 \cup S_3 \cup S_4)$.*

Proof. The result follows from Lemma 17 and Theorem 14.

4. Proofs of the main results.

Proof of Theorem 1. The equations (10) are the characteristic equations for the partial differential equation (1) and their solutions satisfy the Cauchy data (2); therefore, the function $u(x,y) = z(x,\theta(x,y))$ given in Theorem 18 is the unique solution to the Cauchy problem (1), (2) (cf. [3], [11], [14], [29], [32]). We can verify this directly. First note that $y(x_0,\theta(x_0,y)) = \theta(x_0,y)$ and consequently from the definition of θ we have that $y = \theta(x_0,y)$. Thus, $u(x_0,y) = z(x_0,\theta(x_0,y)) = z(x_0,y) = w(y)$ and therefore u satisfies the Cauchy data (1).

Now we will prove that u satisfies the differential equation (2) almost everywhere in $E_1 \times E_2$. Let u be differentiable at $(\bar{x},\bar{y})$; then, at $(\bar{x},\bar{y})$,

$$\begin{aligned} &u_x(\bar{x},\bar{y}) + a(\bar{x},\bar{y},u(\bar{x},\bar{y}))u_y(\bar{x},\bar{y}) \\ =&z_x(\bar{x},\theta(\bar{x},\bar{y})) + z_\eta(\bar{x},\theta(\bar{x},\bar{y}))\theta_x(\bar{x},\bar{y}) \\ &+ a(\bar{x},\bar{y},u(\bar{x},\bar{y}))z_\eta(\bar{x},\theta(\bar{x},\bar{y}))\theta_y(\bar{x},\bar{y}) \\ =&z_\eta(\bar{x},\theta(\bar{x},\bar{y}))[\theta_x(\bar{x},\bar{y}) + a(\bar{x},\bar{y},u(\bar{x},\bar{y}))\theta_y(\bar{x},\bar{y})] + z_x(\bar{x},\theta(\bar{x},\bar{y})). \end{aligned} \tag{18}$$

By Lemma 16 we have

$$\theta_y(\bar{x},\bar{\eta}) = \frac{1}{y_\eta(\bar{x},\bar{\eta})}, \quad \theta_x(\bar{x},\bar{\eta}) = -\frac{y_x(\bar{x},\bar{\eta})}{y_\eta(\bar{x},\bar{\eta})},$$

where $\bar{\eta} = \theta(\bar{x},\bar{y})$. By Theorem 12,

$$y_x(\bar{x},\bar{\eta}) = a(\bar{x},y(\bar{x},\bar{\eta}),z(\bar{x},\bar{\eta})) = a(\bar{x},\bar{y},z(\bar{x},\theta(\bar{x},\bar{y})) = a(\bar{x},\bar{y},u(\bar{x},\bar{y}));$$

therefore, $\theta_x(\bar{x},\bar{y}) + a(\bar{x},\bar{y},u(\bar{x},\bar{y}))\theta_y(\bar{x},\bar{y}) = 0$. Equation (18) implies that

$$\begin{aligned} u_x(\bar{x},\bar{y}) + a(\bar{x},\bar{y},u(\bar{x},\bar{y}))u_y(\bar{x},\bar{y}) &= z_x(\bar{x},\theta(\bar{x},\bar{y})) \\ &= b(\bar{x},y(\bar{x},\theta(\bar{x},\bar{y})),z(\bar{x},\theta(\bar{x},\bar{y})) = b(\bar{x},\bar{y},u(\bar{x},\bar{y})), \end{aligned}$$

and hence u satisfies differential equation (1) at $(\bar{x},\bar{y})$. The function u thus satisfies the differential equation (1) almost everywhere on $E_1 \times E_2$.

Proof of Theorem 2. It is clear that $\bar{a}, \bar{b}$ satisfy conditions (a), (c), and (d). The mean value theorem (or Taylor's theorem) indicates that these functions also satisfy condition (b). Note that P' and Q' are in $HK(I_1)$ and therefore $\bar{a}+P'$ and $\bar{b}+Q'$ also satisfy conditions (a), (b), (c), and (d). Hence the Cauchy problem (1), (2) has a unique solution.

Finally, we need to prove that the solution is differentiable everywhere on $K_1 \times K_2$. This follows from Theorem 18 since the sets S_1, S_2 and S_3 are empty.

Corollary 3 follows immediately from Theorem 2

Proof of Theorem 4. The functions a and b satisfy conditions (a), (b), (c), and (d), and the result follows from Theorem 1 and the theory of nonhomogeneous linear ordinary differential equations (cf. [9]).

5. Conclusions and further remarks. In this paper we studied the Cauchy problem

$$\frac{\partial z}{\partial x} + a(x,y,z)\frac{\partial z}{\partial y} = b(x,y,z), \tag{19}$$

$$z(x_0, y) = w(y), \tag{20}$$

in the framework of the Henstock-Kurzweil integral. The functions a and b in the above differential equation were not assumed to be differentiable with respect to x as in the classical theory; rather, this condition was replaced by certain requirements which allowed a and b to be HK integrable but not necessarily Lebesgue integrable. The theory encompasses the classical results. The linear case can be tackled using distributions, but using the HK integral the weak solution can be sharpened to a differentiable solution.

It is clear that the results follow if the Cauchy data are of the form $z(p(y), y) = w(y)$ provided p and w have continuous derivatives. It may be, however, that the conditions on p and w can be relaxed so that "Henstock"-type initial data is admissible.

The partial differential equation (19) includes the general quasi-linear equation

$$a(x,y,z)\frac{\partial z}{\partial x} + b(x,y,z)\frac{\partial z}{\partial y} = c(x,y,z), \tag{21}$$

provided either a or b do not vanish. In the more general case where a and b may vanish we must resort to the autonomous system of characteristic equations

$$x'(t) = a(x,y,z), \quad y'(t) = b(x,y,z), \quad z'(t) = c(x,y,z),$$

with Cauchy data $x(0,s) = x_0(s)$, $y(0,s) = y_0(s)$, $z(0,s) = z_0(s)$ for the initial curve parameter s in some interval. Generally, in such cases the continuity of a, b, and c are required. Without these conditions we can nonetheless proceed "locally" in regions where a or b do not vanish and establish the result in the restricted regions by converting the partial differential equation to the form (19). It is thus only at points on the initial curve for which a and b both vanish that cause problems.

The general first-order nonlinear partial differential equation

$$F(x, y, z, p, q) = 0,$$

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$ poses a different problem. Here, we can use the Charpit equations for the characteristics

$$x'(t) = F_p, \quad y'(t) = F_q, \quad z'(t) = pF_p + qF_q,$$
$$p'(t) = -F_x - pF_z, \quad q'(t) = -F_y - qF_z,$$

and apply the Henstock-Kurzweil theory, but the analysis is more involved since these equations contain second-order derivatives. The nonlinear case will be treated in a separate paper.

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