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ON A FREE-BOUNDARY PROBLEM FOR BURGERS EQUATION: THE LARGE-TIME BEHAVIOUR

M. Bertsch
Dipartimento di Matematica, Università di Roma Tor Vergata
Viale della Ricerca Scientifica, 00133 Rome, Italy

D. Hilhorst
Laboratoire d'Analyse Numérique, Bâtiment 425
CNRS et Université Paris-Sud, 91405 Orsay Cédex, France

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1. Introduction. In this paper we describe the large-time behavior of the solution $(u(x,t),\zeta(t))$ of the free-boundary problem

$$\text{(FB)}\qquad \begin{cases} u_t = u_{xx} + uu_x & x \in \mathbf{R}\setminus\{\zeta(t)\},\ t>0 \\ u(\zeta(t)^-,t) = u(\zeta(t)^+,t) = q & t>0 \\ u_x(\zeta(t)^-,t) - u_x(\zeta(t)^+,t) = 1 & t>0 \\ u(x,0) = u_0(x) & x \in \mathbf{R} \\ \zeta(0) = \zeta_0, \end{cases}$$

where q is a positive constant, ζ_0 a given real number and u_0 is a given initial function satisfying the hypothesis

H. $u_0 \in C(\mathbf{R}) \cap \{C^3((-\infty,\zeta_0]) \cup C^3([\zeta_0,\infty))\}$, $0 \le u_0 < q$ in $(-\infty,\zeta_0)$, $q < u_0 \le A$ in (ζ_0,∞) for some $A > q$, $u_0'(\zeta_0^-) - u_0'(\zeta_0^+) = 1$, $u_0'(x) \to 0$ as $x \to \pm\infty$, $u_0'(\zeta_0^+) > 0$, and $u_0 - AH \in L^1(\mathbf{R})$, where H denotes the Heaviside function.

Problem (FB) arises in combustion theory. For a brief account of the physical background of the problem we refer to [3].

The well-posedness of Problem (FB) has been proved by Bertsch, Hilhorst and Schmidt-Lainé ([1]). Their main observation was that, if u_0 satisfies H, Problem (FB) is formally equivalent to the problem

$$\text{(P)}\qquad \begin{cases} u_t = u_{xx} + uu_x + (H(u-q))_x & x \in \mathbf{R}, t>0 \\ u(x,0) = u_0(x) & x \in \mathbf{R}, \end{cases}$$

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where H is the Heaviside function:

$$H(s) = \begin{cases} 1 & \text{if } s \geq 0 \\ 0 & \text{if } s < 0. \end{cases}$$

In Section 2 we recall the main results of [1].

In the present paper we study the behavior of the solution of Problem (FB) as $t \to \infty$. It turns out that this behavior depends critically on the values of the parameters A and q. If $q > 2/A$, Problem (FB) has a travelling-wave solution, i.e., a solution of the type $u(x,t) = U(x - \mu - ct)$, where μ is a translation parameter and $(U(z), c)$ is the unique solution of the problem

$$\text{(TW)} \quad \begin{cases} U_{zz} + (U+c)U_z + (H(U-q))_z = 0 & z \in \mathbf{R} \\ U(-\infty) = 0, \quad U(0) = q, \quad U(+\infty) = A. \end{cases}$$

Our first main result is the convergence of u to a travelling wave as $t \to \infty$:

Theorem A. *Let $q > 2/A$, let u_0 satisfy hypothesis* H, *let $(U(z), c)$ be the unique solution of Problem* (TW), *and let $\mu_0 \in \mathbf{R}$ be determined by*

$$\int_{-\infty}^{\infty} (u_0(x) - U(x - \mu_0))\, dx = 0.$$

Then the solution (u, ζ) of Problem (FB) *satisfies*

$$u(\cdot, t) - U(\cdot - \mu_0 - ct) \to 0 \quad \text{in } L^1(\mathbf{R}) \cap L^\infty(\mathbf{R}) \quad \text{as } t \to \infty, \tag{1.1}$$

and

$$\zeta(t) - ct \to \mu_0 \text{ as } t \to \infty. \tag{1.2}$$

If $q < 2/A$, the situation changes drastically. The travelling-wave solution of Problem (TW) does not exist, and the asymptotic behavior is determined by two travelling waves with different velocities. Roughly speaking, for $x \leq \zeta(t)$ the solution $u(x,t)$ converges to a travelling wave with speed $c_- < 0$, given by $u(x,t) = U_-(x - \mu_- - c_- t)$, where μ_- is a translation parameter and $(U_-(z), c_-)$ is the unique solution of

$$\text{(TW}_-\text{)} \quad \begin{cases} U_{zz} + (U+c)U_z = 0 & z \in \mathbf{R}^- \\ U(-\infty) = 0, \quad U(z) = q \ \text{ for } z \geq 0, \quad U'(0^-) = 1, \end{cases}$$

while for $x > \zeta(t)$ the solution converges to a travelling wave with speed $c_+ \in (c_-, 0)$, given by $u(x,t) = U_+(x - c_+ t)$, where $(U_+(z), c_+)$ is a solution of the problem

$$\text{(TW}_+\text{)} \quad \begin{cases} U_{zz} + (U+c)U_z = 0 & z \in \mathbf{R} \\ U(-\infty) = q, \quad U(+\infty) = A. \end{cases}$$

More precisely, we have the following result:

Theorem B. *Let $q < 2/A$, let u_0 satisfy hypothesis* H, *and let $(U_-(z), c_-)$ be the unique solution of Problem TW_-. Then there exist a $\mu_1 \in \mathbf{R}$ and a solution $(U_+(z), c_+)$ of Problem TW_+ such that the solution (u, ζ) of Problem* (FB) *satisfies*

$$\min\{u(\cdot,t), q\} - U_-(\cdot - \mu_1 - c_- t) \to 0 \quad in\ L^1(\mathbf{R}) \cap L^\infty(\mathbf{R}) \quad as\ t \to \infty, \tag{1.3}$$

$$\max\{u(\cdot,t), q\} - U_+(\cdot - c_+ t) \to 0 \quad in\ L^1(\mathbf{R}) \cap L^\infty(\mathbf{R}) \quad as\ t \to \infty, \tag{1.4}$$

and

$$\zeta(t) - c_- t \to \mu_1 \ \ as\ t \to \infty. \tag{1.5}$$

The proof of Theorem A, which we give in Section 5, is based on the L^1-contraction of solutions of Problem (P), which enables us to construct a Lyapunov functional; this approach was originally introduced by Osher and Ralston ([8]). The proof of Theorem B, given in Sections 6 and 7, is based on a generalization of the contraction property. The contraction property and its generalizations are studied in Section 3; in Section 4 we collect the results about the travelling waves.

We are not able to describe the precise asymptotic behavior in the critical case $q = 2/A$, but in the last section we briefly discuss why this behavior is not as simple as one might think.

The solution of Problem TW_- is a travelling-wave solution of the "one-phase" problem

$$(\mathrm{FB}_1) \quad \begin{cases} u_t = u_{xx} + u u_x & x < \zeta(t), t > 0 \\ u(\zeta(t)^-, t) = q & t > 0 \\ u_x(\zeta(t)^-, t) = 1 & t > 0 \\ u(x,0) = u_0(x) & x < \zeta_0 \\ \zeta(0) = \zeta_0, \end{cases}$$

which can be thought of as a solution of our "two-phase" Problem (FB) if we extend it by $u(x,t) = q$ for $x \geq \zeta(t)$. It was proved by Hilhorst and Hulshof ([6]) that, under suitable conditions on u_0, the solution of Problem (FB_1) converges to a travelling wave. Other results in this spirit were proved by Ricci and Xie Weiqing ([9]).

Brauner, Lunardi and Schmidt-Lainé ([2, 3]) have studied the local stability of travelling waves, without imposing the condition that $u - q$ is positive for $x > \zeta(t)$ and negative for $x < \zeta(t)$.

Finally, we observe that our results can be easily extended to the case in which the convection term uu_x is replaced by $g(u)_x$, where g is a smooth convex function (see the last remark of Section 4).

2. Preliminaries. Throughout the paper we shall use the notation

$$\begin{aligned} &Q = \mathbf{R} \times \mathbf{R}^+, &&Q^+ = \{(x,t) \in Q, x > \zeta(t)\}, \\ &Q_T = \mathbf{R} \times (0,T], &&Q^- = \{(x,t) \in Q, x < \zeta(t)\}. \end{aligned}$$

We first recall some results about Problem (FB).

Definition 2.1. The pair (u, ζ) is a solution of Problem (FB) if

(i) $\zeta \in C([0,\infty))$, $u \in C(\overline{Q}) \cap L^\infty(Q) \cap C^{2,1}(Q^+ \cup Q^-)$;
(ii) the one-sided limits $u_x(\zeta(t)^-, t)$ and $u_x(\zeta(t)^+, t)$ exist for all $t > 0$ and are continuous functions in $\mathbf{R}^+$;
(iii) $u_t \in L^1_{loc}(\overline{Q})$, $u_x \in L^2_{loc}(\overline{Q})$;
(iv) the equations in Problem (FB) are satisfied pointwise.

Proposition 2.2 ([1])**.** *Let u_0 satisfy hypothesis* H*. Then Problem* (FB) *has a unique solution (u, ζ). In addition, u satisfies*

$$0 \le u \le A \text{ in } Q,\ u < q \text{ in } Q^- \text{ and } u > q \text{ in } Q^+, \tag{2.1}$$

u is a weak solution of Problem (P) *in the sense that for any $\phi \in C^1(\overline{Q})$ which vanishes for large values of $|x|$ and for $T > 0$*

$$\iint_{Q_T} \left(u_t\phi + (u_x + \tfrac{1}{2}u^2 + H(u-q))\phi_x\right) = 0,$$

and u satisfies a comparison principle: if $\underline{u}_0$ satisfies hypothesis H and $(\underline{u}, \underline{\zeta})$ is the corresponding solution of Problem (FB)*, then*

$$\underline{u}_0 \le u_0 \quad \text{in } \mathbf{R} \Rightarrow \underline{u} \le u \quad \text{in } Q.$$

In the following theorem we collect some results about the properties of the solution $u(x,t)$.

Proposition 2.3. *Let hypothesis H be satisfied and let (u, ζ) be the solution of Problem* (FB)*. Then, for any $T > 0$,*

(i) $\|u(x,\cdot)\|_{L^\infty(0,T)} \to 0$ *as* $x \to -\infty$*;*
(ii) $\|u(x,\cdot) - A\|_{L^\infty(0,T)} \to 0$ *as* $x \to +\infty$*;*
(iii) $\|u_x(x,\cdot)\|_{L^\infty(0,T)} \to 0$ *as* $x \to \pm\infty$*;*
(iv) *the map $t \mapsto u(\cdot,t) - AH(\cdot)$ is continuous as a function from $[0,T]$ into $L^1(\mathbf{R})$, and*

$$\|u_1(\cdot,t) - u_2(\cdot,t)\|_{L^1(\mathbf{R})} \le \|u_{01} - u_{02}\|_{L^1(\mathbf{R})};$$

(v) $u_x \in L^\infty(Q)$.

Proof. (i) Let $\epsilon \in (0,q)$ be arbitrary. We choose a constant M such that $|\zeta(t)| < M$ for any $t \in [0,T]$. Then, by (2.1), $u < q$ in $D = (-\infty, -M] \times [0,T]$, and u is a smooth solution of the Burgers equation

$$u_t = u_{xx} + uu_x \tag{2.2}$$

in D. It is well-known that the Burgers equation has travelling-wave solutions $U(x-\mu-ct)$, where U is a function satisfying $U(-\infty)=\epsilon$, $U(+\infty)=q$, $c<0$ is the uniquely determined wave speed, and μ is a translation parameter. Hence, choosing μ such that

$$\begin{aligned} &u_0(x) \le U(x-\mu) \quad \text{for } x \le -M,\\ &u(-M,t) \le U(-M-\mu-ct) \quad \text{for } 0 \le t \le T, \end{aligned}$$

it follows from the classical comparison principle for parabolic equations applied to (2.2) in D that $u(x,t) < U(x-\mu-ct)$ for $(x,t) \in D$. Hence $\limsup_{x\to-\infty} u(x,t) \le \epsilon$, and, since ϵ is arbitrary, the proof of (i) is complete.

(ii) The proof is similar to the one of (i).

(iii) The proof follows from the results in (i) and (ii) and the assumption that $u_0'(x) \to 0$ as $x \to \pm\infty$, applying a straightforward Bernstein argument to the Burgers equation (see for example [5]).

(iv) Let M be defined as in the proof of (i). Then, for any $R > M$,

$$\int_{-R}^{-M} u(x,T)\,dx = \int_{-R}^{-M} u_0(x)\,dx + \int_0^T \left[u_x + \tfrac{1}{2}u^2\right]_{x=-R}^{x=-M}\,dt.$$

From (iii) it follows that the right-hand side is uniformly bounded with respect to R and hence $u(\cdot,T) \in L^1(\mathbf{R}^-)$.

In a similar way we obtain that $u(\cdot,T) - A \in L^1(\mathbf{R}^+)$. Combined with the continuity of u this yields the continuity of $u(\cdot,t) - AH(\cdot)$ in $L^1(\mathbf{R})$. The contraction in $L^1(\mathbf{R})$ follows as in the proof of Theorem 3.3 in [1] in which we set $\chi_k = 1$.

(v) Let $\tau > 0$ be arbitrary, and let $R > 0$ be chosen. Following the proof of Lemma 5.3(iv) in [1] it follows that $z(x,t) = u_x + \frac{1}{2}u^2 + H(u-q)$ attains its maximum and minimum in the rectangle $[-R,R] \times [0,\tau]$ on the parabolic boundary $\Gamma_R = ([-R,R] \times \{0\}) \cup (\{-R,R\} \times (0,\tau])$. In view of (iii) there exist a constant C which does not depend on τ and a constant R_τ such that $|z| < C$ on Γ_R for any $R > R_\tau$. This completes the proof of Proposition 2.3.

We shall need some results which concern the one-phase Problem (FB_1):

Proposition 2.4 ([6]). *Let $\zeta_0 \in \mathbf{R}$ and let u_0 be a locally Lipschitz-continuous function in $(-\infty,\zeta_0)$ such that $u_0 \in L^1(-\infty,\zeta_0)$, $u_0(x) \to 0$ as $x \to -\infty$, $0 \le u_0 < q$ in $(-\infty,\zeta_0)$ and $u_0(\zeta_0) = q$. Then Problem* (FB_1) *has a unique classical solution $u(x,t)$.*

Proposition 2.5. *Let u_0 satisfy hypothesis* H *and let $\underline{u}_0$ satisfy the assumptions of Proposition* 2.4 *with ζ_0 replaced by $\underline{\zeta}_0$. Let (u,ζ) be the solution of Problem* (FB) *with data u_0 and ζ_0 and let $(\underline{u},\underline{\zeta})$ be the solution of Problem* (FB_1) *with data $\underline{u}_0$ and $\underline{\zeta}_0$. If*

$$\underline{u}_0 \le u_0 \quad \text{in } (-\infty,\zeta_0) \quad \text{and} \quad \zeta_0 \le \underline{\zeta}_0,$$

then for any $t > 0$

$$\zeta(t) < \underline{\zeta}(t) \quad \textit{and} \quad \underline{u}(x,t) < u(x,t) \ \textit{if}\ x < \zeta(t).$$

Proof. If we prove that $\zeta(t) < \underline{\zeta}(t)$ for $t > 0$ it follows at once from the classical maximum principle applied to the Burgers equation in the set $\{(x,t) : x < \zeta(t), t > 0\}$ that $\underline{u}(x,t) < u(x,t)$ if $x < \zeta(t)$ and $t > 0$.

First we assume that $\zeta_0 < \underline{\zeta}_0$. Arguing by contradiction we define

$$t_0 = \sup\{t > 0 : \zeta(t) < \underline{\zeta}(t)\} \in \mathbf{R}^+.$$

Hence it follows from the maximum principle that $\underline{u}(x,t_0) < u(x,t_0)$ for $x < \zeta(t_0)$. Since $\underline{u}(\zeta(t_0),t_0) = u(\zeta(t_0),t_0) = q$, this implies that $\underline{u}_x(\zeta(t_0)^-,t_0) \geq u_x(\zeta(t_0)^-,t_0)$, which leads to a contradiction, since $\underline{u}_x(\zeta(t_0)^-,t_0) = 1$ while, by [1, Theorem 3.1], $u_x(\zeta(t_0)^-,t_0) > 1$.

Finally, we observe that if $\zeta_0 = \underline{\zeta}_0$, we may change $\underline{u}_0$ and $\underline{\zeta}_0$ in such a way that, for a fixed $\epsilon > 0$, $u_0 - \epsilon < \underline{u}_0 < u_0$ and $\zeta_0 < \underline{\zeta}_0 < \zeta_0 + \epsilon$, and hence we may apply the result which we have just proved. Since we may choose ϵ arbitrarily small, we obtain that $\zeta(t) \leq \underline{\zeta}(t)$ for $t > 0$. Since $u_x(\zeta(t),t) > 1 = \underline{u}_x(\underline{\zeta}(t),t)$ for any $t > 0$, we have necessarily that $\zeta(t) < \underline{\zeta}(t)$.

In the rest of this paper we shall need solutions of Problem (FB) under weaker assumptions on u_0:

H_1. u_0 is uniformly Lipschitz continuous in $\mathbf{R}$, $u_0 - AH \in L^1(\mathbf{R})$, $u_0(x) \to 0$ as $x \to -\infty$, $u_0(x) \to A$ as $x \to +\infty$, and there exists a sequence $\{u_{0i}, i = 1, 2, \dots\}$ such that u_i satisfies hypothesis H for all i, u'_{0i} is uniformly bounded in $\mathbf{R}$, $u_{0,i+1} \leq u_{0i}$ in $\mathbf{R}$ for all i, and

$$u_{0i} - u_0 \to 0 \ \text{ in } L^1(\mathbf{R}) \cap L^\infty(\mathbf{R}) \text{ as } i \to \infty.$$

If u_0 satisfies hypothesis H_1, it is more convenient to consider Problem (P). First we define what we mean by a solution of Problem (P).

Definition 2.6. The function $u \in C(\overline{Q})$ is a solution of Problem (P) if for any $T > 0$ and for any $\phi \in C^2(\overline{Q}_T)$ which vanishes for large values of $|x|$

$$\iint_{Q_T} \left(u\phi_t + u\phi_{xx} - (\tfrac{1}{2}u^2 + H(u-q))\phi_x\right) dx\, dt$$
$$= \int_{\mathbf{R}} (u(x,T)\phi(x,T) - u_0(x)\phi(x,0))\, dx, \tag{2.3}$$

where H is the Heaviside function with $H(0) = 1$:

$$H(s) = \begin{cases} 1 & \text{if } s \geq 0 \\ 0 & \text{if } s < 0. \end{cases} \tag{2.4}$$

We need the following, rather specific existence result:

Theorem 2.7. *Let u_0 satisfy hypothesis* H_1 *and let u_{0i} be as in hypothesis* H_1. *Let $u_i(x,t)$ be the solution of Problem* (FB) *with initial data u_{0i}. Then there exists a function $u \in C(\overline{Q})$ such that for any $T > 0$*

$$u_i - u \to 0 \quad \text{in } C(0,T;L^1(\mathbf{R})) \text{ as } i \to \infty,$$

and u is a solution of Problem (P). In addition,

$$u_i \to u \quad \text{in } C^{2,1}_{loc}(\{(x,t) \in Q : u(x,t) \neq q\}), \tag{2.5}$$

and u does not depend on the choice of the approximating sequence $\{u_{0i}\}$.

Proof. By the comparison principle in Proposition 2.2, $0 \le u_{i+1} \le u_i$ in $\overline{Q}$ and thus the pointwise limit of $u_i(x,t)$ exists:

$$u(x,t) = \lim_{i\to\infty} u_i(x,t) \quad \text{for } (x,t) \in \overline{Q}.$$

By Proposition 2.3(v), u_{ix} is uniformly bounded in $\overline{Q}$ and thus also

$$|u_x| \le C \quad \text{in } \overline{Q} \tag{2.6}$$

for some $C \in \mathbf{R}$.

The functions u_i satisfy the integral equality (2.3) with u_0 replaced by u_{0i}. Since u_i is nonincreasing with respect to i it follows from the definition of $H(0)$ that $H(u_i(x,t) - q) \to H(u(x,t) - q)$ as $i \to \infty$. Hence, by the Dominated Convergence Theorem, also u satisfies (2.3). Integrating by parts, we obtain that u satisfies

$$\begin{aligned} &\iint_{Q_T} \left(u\phi_t - (u_x + \tfrac{1}{2}u^2 + H(u-q))\phi_x\right)\, dx\, dt \\ &\qquad = \int_{\mathbf{R}} (u(x,T)\phi(x,T) - u_0(x)\phi(x,0))\, dx. \end{aligned} \tag{2.7}$$

We deduce from (2.7) and (2.6) that $u \in C(\overline{Q})$: substituting into (2.7) a smooth but arbitrary function $\phi(x)$ with compact support, we obtain that the integral $\int_{\mathbf{R}} u(t)\phi\, dt$ is uniformly continuous with respect to t, which, combined with the uniform continuity of u with respect to x, implies the uniform continuity of $u\phi$. Since the equation is smooth at points where $u \neq q$, the proof of (2.5) is standard.

If $i < j$, it follows from Lemma 2.8 below that for any $t > 0$

$$\begin{aligned} \int_{\mathbf{R}} |u_i - u_j|(x,t)\, dx &= \int_{\mathbf{R}} (u_i - u_j)(x,t)\, dx \\ &= \int_{\mathbf{R}} (u_{0i} - u_{0j})\, dx \to 0 \quad \text{as } i, j \to \infty, \end{aligned}$$

and hence $u_i - u \to 0$ in $C(0,T;L^1(\mathbf{R}))$.

The independence on u_{0i} follows at once from the contraction property which we shall prove in the next section (see Lemma 3.1).

Lemma 2.8. *Let* (u_1, ζ_1) *and* (u_2, ζ_2) *be two solutions of Problem* (FB) *with initial data* u_{01} *and, respectively,* u_{02} *which satisfy hypothesis* H. *Then, for all* $t > 0$,

$$\int_{\mathbf{R}} (u_1 - u_2)(x,t)\,dx = \int_{\mathbf{R}} (u_{01} - u_{02})(x)\,dx.$$

Proof. Let $|\zeta_1| + |\zeta_2| < M$ in $[0,t]$. Then, for any $R > M$,

$$\int_{-R}^{R} (u_1 - u_2)(t) - \int_{-R}^{R} (u_{01} - u_{02}) = \int_0^t \Big[(u_{1x} - u_{2x})(x,s) + \tfrac{1}{2}(u_1^2 - u_2^2)(x,s) + (H(u_1 - q) - H(u_2 - q))(x,s)\Big]_{x=-R}^{x=R} ds,$$

which, in view of Proposition 2.3, vanishes as $R \to \infty$, and we have found the desired result.

For the proof of Lemma 3.1 below we shall use the following comparison principle:

Lemma 2.9. *Let* u *be a solution of Problem* (P) (*constructed as in Theorem* 2.7) *with initial function* u_0 *satisfying hypothesis* H_1, *and let* v *be the unique solution of Problem* (FB) *with initial function* v_0 *satisfying hypothesis* H. *Then*

$$u_0 \geq v_0 \quad in\ \mathbf{R} \Rightarrow u \geq v \quad in\ Q, \tag{2.8}$$

$$u_0 \leq v_0 \quad in\ \mathbf{R} \Rightarrow u \leq v \quad in\ Q. \tag{2.9}$$

Proof. Let u_{0i} be the approximating sequence in hypothesis H_1 with solution u_i of Problem (FB). If $u_0 \geq v_0$, then $u_{0i} \geq v_0$ and (2.8) follows from the comparison principle in Proposition 2.2.

If $u_0 \leq v_0$, we approximate v_0 by v_{0i}, satisfying hypothesis H and such that $u_{0i} \leq v_{0i}$ in $\mathbf{R}$. Let v_i be the corresponding solution of Problem (FB). By Proposition 2.2, $u_i \leq v_i$ in Q, and passing to the limit and using the uniqueness of v, we obtain (2.9).

Arguing as in the proof of Lemma 2.9, the comparison principle for "classical" solutions (i.e., Propositions 2.2 and 2.5), combined with the uniqueness of solutions, implies a comparison principle for the following solutions of Problem (P):

1) the solution of Problem (P) defined by Proposition 2.2, if u_0 satisfies H or H_1;
2) the unique solution of Problem $(\mathrm{FB})_1$ extended by q to the points at which $x > \zeta(t)$, if u_0 satisfies the assumptions of Proposition 2.4;
3) the unique solution of the Burgers equation, if $q \leq u_0 \leq A$ on $\mathbf{R}$ and u_0 is locally Lipschitz continuous.

In the rest of the paper we shall need precisely these three classes of solutions, which we shall commonly refer to as the solution of Problem (P), and to which we shall often apply the Comparison Principle.

Remark. This "classification" of solutions of Problem (P) might appear rather arbitrary, but we emphasize that it is not possible to prove a comparison principle for "general" solutions of Problem (P), since there is a lack of uniqueness. Choosing for example an initial function u_0 as in Proposition 2.4, we obtain both the solution of Problem (FB_1) as a solution of Problem (P) (which is equal to q for $x > \zeta(t)$), as the solution of the Burgers equation with the same initial function (which, by the strong maximum principle, is strictly smaller than q for $t > 0$, and thus is also a solution of Problem (P)). Of course the nonuniqueness is caused by the discontinuity of the function $H(u - q)$.

3. Contraction properties. As we have indicated in the introduction, an important tool to determine the large-time behavior of solutions is the contraction of solutions in L^1.

Lemma 3.1. *Let u_1 and u_2 be two solutions of Problem (P) (given by Theorem 2.7) with initial data u_{01} and, respectively, u_{02} which satisfy hypothesis H_1. Then, for all $t > 0$,*

$$\int_{\mathbf{R}} (u_1 - u_2)(x,t)\,dx = \int_{\mathbf{R}} (u_{01} - u_{02})(x)\,dx \tag{3.1}$$

and

$$\int_{\mathbf{R}} |u_1 - u_2|(x,t)\,dx \le \int_{\mathbf{R}} |u_{01} - u_{02}|(x)\,dx. \tag{3.2}$$

If $u_{01} - u_{02}$ changes sign in an interval in which $u_{01}, u_{02} \ne q$, then the inequality in (3.2) is strict.

Proof. If u_{01} and u_{02} satisfy hypothesis H, then (3.1) follows from Lemma 2.8.

If u_{01} and u_{02} satisfy merely hypothesis H_1, (3.1) follows from the construction of the solutions u_1 and u_2 of Problem (P) in Theorem 2.7 and the convergence in $C(0,T;L^1(\mathbf{R}))$:

$$\begin{aligned}\int_{\mathbf{R}} (u_1 - u_2)(t)\,dx &= \lim_{i\to\infty} \int_{\mathbf{R}} (u_{1i} - u_{2i})(t)\,dx \\ &= \lim_{i\to\infty} \int_{\mathbf{R}} (u_{01i} - u_{02i})\,dx = \int_{\mathbf{R}} (u_{01} - u_{02})\,dx.\end{aligned}$$

In order to prove (3.2), we define

$$u_0^-(x) = \min\{u_{01}(x), u_{02}(x)\} \quad\text{and}\quad u_0^+(x) = \max\{u_{01}(x), u_{02}(x)\}.$$

Then u_0^- and u_0^+ satisfy hypothesis H_1. Let u^- and u^+ be two corresponding solutions of Problem (P), constructed as in Theorem 2.7. Then, by Lemma 2.9,

$$u^- \le u_1,\, u_2 \le u^+ \quad\text{in } \overline{Q},$$

from which we obtain (3.2): in view of (3.1)

$$\int_{\mathbf{R}} |u_1(t) - u_2(t)| \le \int_{\mathbf{R}} (u^+(t) - u^-(t)) = \int_{\mathbf{R}} (u_0^+ - u_0^-) = \int_{\mathbf{R}} |u_{01} - u_{02}|. \tag{3.3}$$

It remains to prove the strict inequality if $u_{01} - u_{02}$ changes sign in an interval I in which $u_{01}, u_{02} \neq q$. We may suppose that, for example, there exists an $\epsilon > 0$ such that

$$u_{01}(x), u_{02}(x) \ge q + \epsilon \quad \text{for } x \in I.$$

Then also $u_0^+ \ge u_0^- \ge q + \epsilon$ in I and there exists a $\tau > 0$ such that

$$u^+ \ge u^- \ge q + \tfrac{1}{2}\epsilon \quad \text{in } I \times [0, \tau];$$

hence u^- and u^+ are classical solutions of the Burgers equation. Since $u_{01} - u_{02}$ changes sign in I, the functions u_0^+ and u_0^- do not coincide with u_{01} or u_{02} in I. Thus, by the strong maximum principle,

$$u^- < u_1, u_2 < u^+ \quad \text{in } I \times (0, \tau],$$

and therefore the inequality in (3.3) is strict. This completes the proof of Lemma 3.1.

For the proof of Theorem B, we shall need the following extension of Lemma 3.1.

Lemma 3.2. *Let u_0 satisfy hypothesis H and let (u, ζ) be the solution of Problem* (FB). *Let u_1 be the solution of Problem* (FB$_1$) *with initial data $u_{01} \le q$ satisfying the conditions of Proposition* 2.4, *and let u_2 be the solution of the Burgers equation with Lipschitz-continuous initial function $q \le u_{02} \le A$ such that $u_{02} - q - (A - q)H \in L^1(\mathbf{R})$. Setting*

$$\hat{u}(x,t) = \min\{u(x,t), q\} \qquad \overline{u}(x,t) = \max\{u(x,t), q\}, \tag{3.4}$$

we have for all $t > 0$

$$\int_{\mathbf{R}} (\hat{u} - u_1)(x,t)\,dx = \int_{\mathbf{R}} (\hat{u}_0 - u_{01})\,dx + \int_0^t u_x(\zeta(s)^+, s)\,ds, \tag{3.5}$$

$$\int_{\mathbf{R}} (\overline{u} - u_2)(x,t)\,dx = \int_{\mathbf{R}} (\overline{u}_0 - u_{02})\,dx - \int_0^t u_x(\zeta(s)^+, s)\,ds, \tag{3.6}$$

and

$$\int_{\mathbf{R}} |\hat{u} - u_1|(x,t)\,dx \le \int_{\mathbf{R}} |\hat{u}_0 - u_{01}|\,dx + \int_0^t u_x(\zeta(s)^+, s)\,ds, \tag{3.7}$$

$$\int_{\mathbf{R}} |\overline{u} - u_2|(x,t)\,dx \le \int_{\mathbf{R}} |\overline{u}_0 - u_{02}|\,dx + \int_0^t u_x(\zeta(s)^+, s)\,ds. \tag{3.8}$$

Proof. The proofs of (3.5) and (3.6) are straightforward if, for large values of R, we use expressions of the type

$$\frac{d}{dt}\int_{-R}^{+R} \hat{u}(x,t)\,dx = \int_{-R}^{+R} \hat{u}_t(x,t)\,dx = \int_{-R}^{\zeta(t)} \hat{u}_t(x,t)\,dx = \left[u_x + \tfrac{1}{2}u^2\right]_{-R}^{\zeta(t)}.$$

It remains to prove (3.7) (the proof of (3.8) is similar). We set

$$f(t) = u_x(\zeta(t)^+, t), \tag{3.9}$$

and, for $x \in \mathbf{R}$,

$$u_0^-(x) = \min\{\hat{u}_0(x), u_{01}(x)\} \quad \text{and} \quad u_0^+(x) = \max\{\hat{u}_0(x), u_{01}(x)\}.$$

Let $u^-(x,t)$ be the unique solution of Problem (FB_1) (extended by q) with initial function u_0^-, and let $u^+(x,t)$ be the solution of Problem (FB_1) (extended by q) with the free boundary condition $u_x(\zeta(t)^-, t) = 1$ replaced by

$$u_x(\zeta(t)^-, t) = 1 + f(t) \quad \text{for } t > 0,$$

and with initial function u_0^+ (this free boundary problem has been studied by Hulshof and Peletier ([7]), and its solution is well-defined, unique, and satisfies the comparison principle). Then, since $f \geq 0$, we obtain from the comparison principle ([7, Theorem 3]) that

$$u^- \leq \hat{u}, u_1 \leq u^+ \quad \text{in } Q,$$

and thus, using an equality which is quite similar to (3.5),

$$\begin{aligned}\int_{\mathbf{R}} |\hat{u} - u_1|(x,t)\,dx &\leq \int_{\mathbf{R}} (u^+ - u^-)(x,t)\,dx = \int_{\mathbf{R}} (u_0^+ - u_0^-)\,dx + \int_0^t f(s)\,ds \\ &= \int_{\mathbf{R}} |\hat{u}_0 - u_{01}| + \int_0^t u_x(\zeta(s)^+, s)\,ds.\end{aligned}$$

Finally, we mention some strict contractions for the solutions of Problem (FB_1) and of the Burgers equation:

Lemma 3.3. *Let u_1 and u_2 be two solutions of Problem* $(\mathrm{FB})_1$ (*extended to Q by q*) *with initial data u_{01} and, respectively, u_{02} which satisfy the assumptions of Proposition* 2.4. *If $u_{01} - u_{02}$ changes sign in an interval in which $u_{01}, u_{02} < q$, then, for all $t > 0$,*

$$\int_{\mathbf{R}} |u_1 - u_2|(x,t)\,dx < \int_{\mathbf{R}} |u_{01} - u_{02}|(x)\,dx.$$

Lemma 3.4. *Let u_1 and u_2 be two solutions of the Burgers equation with uniformly bounded and Lipschitz-continuous initial data u_{01} and, respectively, u_{02} such that $u_{01} - u_{02} \in L^1(\mathbf{R})$. If $u_{01} - u_{02}$ changes sign in $\mathbf{R}$, then, for all $t > 0$,*

$$\int_{\mathbf{R}} |u_1 - u_2|(x,t)\,dx < \int_{\mathbf{R}} |u_{01} - u_{02}|(x)\,dx.$$

The proof of Lemma 3.3 is given in [6], and the proof of Lemma 3.4 is standard.

4. Travelling waves. First we look for solutions $u(x,t) = U(x - ct)$ of Problem (P), where $U(z)$ is a solution of Problem (TW).

Integrating the ordinary differential equation for $U(z)$ it is easy to solve $U(z)$ explicitly and we find the following result.

Theorem 4.1. *If $q > 2/A$, the solutions of Problem (TW) are $(U(z-\mu), c)$, where $\mu \in \mathbf{R}$, $c = -(\frac{1}{2}A + 1/A)$ and*

$$U(z) = \begin{cases} \dfrac{-2cqe^{-cz}}{-2c - q + qe^{-cz}} & \text{if } z \le 0 \\ \dfrac{2(A-q) + (qA-2)Ae^{(A+c)z}}{A(A-q) + (qA-2)e^{(A+c)z}} & \text{if } z > 0. \end{cases}$$

If $q \le 2/A$, then Problem (TW) does not have a solution, and we need the solutions of the Problems ($\mathrm{TW}_\pm$). Again the results follow from integrating twice the equation for U:

Theorem 4.2. (i) *Problem (TW_-) has a unique solution $(U_-(z), c_-)$, given by*

$$U_-(z) = \begin{cases} \dfrac{-2c_-qe^{-c_-z}}{-2c_-q + qe^{-c_-z}} & \text{for } z < 0 \\ q & \text{for } z \ge 0, \end{cases} \qquad c_- = -\frac{2+q^2}{2q}\,.$$

(ii) *The solutions of Problem (TW_+) are $(U(z-\mu_+), c_+)$ where $\mu_+ \in \mathbf{R}$ and*

$$U_+(z) = \frac{q + Ae^{(A+c_+)z}}{1 + e^{(A+c_+)z}} \quad \text{for } z \in \mathbf{R}, \quad c_+ = -\frac{q+A}{2}\,.$$

Remarks. (i) In Theorem 4.2 we did not impose the condition $q \le 2/A$, but later it shall be used as the condition which implies that $c_- \le c_+$. (ii) The fact that we can determine the travelling waves explicitly in the Theorems 4.1–2 has no significance at all for the rest of the paper. As a matter of fact all results in this paper can be easily extended to the case in which the term $\frac{1}{2}u^2$ in Problem (P) is replaced by a convex function:

$$u_t = u_{xx} + (g(u) + H(u-q))_x \quad \text{in } Q,$$

where g is a smooth function satisfying $g(0) = 0$ and $g'' > 0$ in $\mathbf{R}$. In particular, the necessary and sufficient condition $q > 2/A$ for the existence of a travelling wave which connects 0 and A is replaced by $Ag(q) + A < qg(A) + q$, which can be deduced from the requirement that in the interval $(0, A)$ the graph of the function $g(u) + H(u - q)$ is strictly below the straight line which connects the points $(0, 0)$ and $(A, g(A) + H(A - q)) = (A, g(A) + 1)$, which is the necessary and sufficient condition to solve the travelling-wave problem.

5. The case $q > 2/A$. We introduce the travelling-wave variable $z = x - ct$ and we define $\tilde{u}(z, t) = u(z + ct, t)$. Writing $u(z, t)$ instead of $\tilde{u}(z, t)$, we obtain the equation

$$u_t = u_{zz} + \left(\tfrac{1}{2}u^2 + cu + H(u - q)\right)_z \quad \text{in } Q. \tag{5.1}$$

We define the space

$$\mathcal{L} = \{v \in L^\infty(\mathbf{R}) : 0 \le v \le A \text{ a.e } \in \mathbf{R}, v - AH \in L^1(\mathbf{R})\} \tag{5.2}$$

with the metric

$$d(v, w) = \int_{-\infty}^{\infty} |v(z) - w(z)|\, dz,$$

and, for a constant μ to be determined, the continuous functional $V_\mu : \mathcal{L} \to \mathbf{R}$, given by

$$V_\mu(v) = \int_{-\infty}^{\infty} |v(z) - U(z - \mu)|\, dz. \tag{5.3}$$

Denoting the solution $u(z, t)$ with initial function u_0 by $u(t; u_0)$, it follows from Lemma 3.1 that if u_0 satisfies hypothesis H_1,

$$V_\mu(u(t; u_0)) \quad \text{is nonincreasing in } \mathbf{R}^+, \tag{5.4}$$

and, if $u_0 - U(z - \mu)$ changes sign in an interval in which $u_0, U(\cdot - \mu) \ne q$,

$$V_\mu(u(t; u_0)) < V_\mu(u_0). \tag{5.5}$$

We define the omega-limit set of u_0 in the usual way:

$$\omega(u_0) = \{v \in \mathcal{L} : d(u(t_n; u_0), v) \to 0 \text{ as } t_n \to \infty \text{ for some sequence } \{t_n\}\}. \tag{5.6}$$

Now we are ready to prove Theorem A.

Proof of Theorem A. First we assume that u_0 satisfies the additional condition

$$U_1(z) \le u_0(z) \le U_2(z) \quad \text{for } z \in \mathbf{R}, \tag{5.7}$$

where $U_1(z) = U(z-\alpha_1)$ and $U_2(z) = U(z-\alpha_2)$ for some $\alpha_1, \alpha_2 \in \mathbf{R}$. By the comparison principle

$$U_1(z) \le u(z,t) \le U_2(z) \quad \text{for } z \in \mathbf{R}, t > 0. \tag{5.8}$$

By Proposition 2.3(v) and (5.8) the set $\{u(t;u_0); t \ge 0\}$ is precompact in $C(\mathbf{R})$ (and hence in $\mathcal{L}$), which implies that

$$\omega(u_0) \ne \emptyset. \tag{5.9}$$

In addition, by standard theory (see for example [4])

$$V_\mu \;\text{ is constant on } \omega(u_0). \tag{5.10}$$

Let $v \in \omega(u_0)$. One easily checks that v satisfies hypothesis H_1. Let $u(x,t;v)$ be the solution of Problem (P) given by Theorem 2.7, and let $\overline{u}(z,t;v) = u(x+ct,t;v)$, which we denote by $u(t;v)$. Then also $u(t;v)$ satisfies (5.4) and it follows from standard Lyapunov theory ([4]) that

$$u(t;v) \in \omega(u_0) \quad \text{for all } t > 0. \tag{5.11}$$

By (5.10) and (5.11)

$$V_\mu(u(t;v)) = V_\mu(v) \quad \text{for } t > 0. \tag{5.12}$$

We claim that

$$v(z) = U(z-\mu_0) \quad \text{for } z \in \mathbf{R}. \tag{5.13}$$

Arguing by contradiction we suppose that $v \ne U(\cdot - \mu_0)$. By the conservation law (3.1)

$$\int_{\mathbf{R}} (v-u_0)(z)\, dz = 0,$$

and thus $v \ne U(\cdot - \mu)$ for $\mu \ne \mu_0$. Using that v satisfies hypothesis H_1, we conclude that there are two possibilities left:

(a) for some μ, $v - U(\cdot - \mu)$ changes sign in an interval I in which $v, U(\cdot - \mu) \ne q$;

(b) there exist $\lambda_1 < \lambda_2$ such that

$$v(z) = \begin{cases} U(z-\lambda_1) & \text{for } z < \lambda_1 \\ q & \text{for } \lambda_1 \le z \le \lambda_2 \\ U(z-\lambda_2) & \text{for } z > \lambda_2. \end{cases} \tag{5.14}$$

We consider first case (a). Then, by (5.5),

$$V_\mu(u(t;v)) < V_\mu(v) \;\text{ for } t > 0,$$

and we have found a contradiction with (5.12).

There remains case (b). Observe that if, for some $t_0 > 0$, $u(t_0; v)$ intersects a function $U(z-\mu)$ for some μ, we are in case a again, and a contradiction follows. Thus we may assume without loss of generality that

$$u(t;v) = \begin{cases} U(z-\lambda_1(t)) & \text{for } z < \lambda_1(t) \\ q & \text{for } \lambda_1(t) \le z \le \lambda_2(t) \\ U(z-\lambda_2(t)) & \text{for } z > \lambda_2(t) \end{cases}$$

for some functions $\lambda_1(t) \le \lambda_2(t)$ with $\lambda_1(0) = \lambda_1 < \lambda_2 < \lambda_2(0)$. But such a function $u(t;v)$ is not a solution of Problem (P) and we obtain a contradiction. Hence we have proved (5.13).

We observe that (1.1) is an immediate consequence of (5.13). Since $U(0) = q$ and U is strictly increasing in $\mathbf{R}$, (5.1) implies that $\zeta(t) - ct - \mu_0 \to 0$ as $t \to \infty$, and we have found (1.2).

To complete the proof of Theorem A we have to remove condition (5.7). If u_0 satisfies hypothesis H, there exists a sequence $\{u_{0k}, k = 1, 2, \dots\}$ such that, for all k, u_{0k} satisfies hypothesis H, condition (5.7) (with α_1 and α_2 replaced by α_{1k} and α_{2k}), and

$$\int_{\mathbf{R}} (u_{0k}(x) - u_0(x))\, dx = 0,$$

and such that

$$\int_{\mathbf{R}} |u_{0k}(x) - u_0(x)|\, dx \to 0 \quad \text{as } k \to \infty.$$

Hence the solutions $u(\cdot, t; u_{0k}) - U(\cdot - \mu_0 + ct) \to 0$ in $L^1(\mathbf{R})$ and (1.1) follows from the contraction lemma: for all k

$$\begin{aligned} &\int_{\mathbf{R}} |u(x,t;u_0) - U(x-\mu_0-ct)|\, dx \\ &\le \int_{\mathbf{R}} |u(x,t;u_0) - u(x,t;u_{0k})|\, dx + \int_{\mathbf{R}} |u(x,t;u_{0k}) - U(x-\mu_0-ct)|\, dx \\ &\le \int_{\mathbf{R}} |u_0(x) - u_{0k}(x)|\, dx + \int_{\mathbf{R}} |u(x,t;u_{0k}) - U(x-\mu_0-ct)|\, dx \\ &\to \int_{\mathbf{R}} |u_0(x) - u_{0k}(x)|\, dx \quad \text{as } t \to \infty. \end{aligned}$$

6. The case $q < 2/A$: $u \le q$. In this section we begin our study of the large-time behavior of u in the case that there does not exist a travelling-wave solution of Problem (TW). In this section we shall prove the convergence of $\min\{u, q\}$ to a travelling-wave solution of Problem (TW_-); i.e., we shall prove (1.3) and (1.5) of Theorem B.

As in the previous section, the proof is based on a Lyapunov argument, but the details are more delicate, because the function $\min\{u, q\}$ does not satisfy a contraction property; the basic idea of the proof is that it satisfies an approximate contraction property for large values of t.

Let (u, ζ) be the solution of Problem (FB) . Let $c_\pm$ be the speed of the travelling waves of Problem ($\mathrm{TW}_\pm$). We remind the reader that, since $q < 2/A$, $c_- < c_+ < 0$. We set $z = x - c_-t$ and $\chi(t) = \zeta(t) - c_-t$, and we denote the solution $\underline{u}(z,t) = u(z + ct, t)$ by $u(z,t)$ or by $u(t; u_0)$. Hence u is a solution of the problem

$$(\mathrm{P}_1) \quad \begin{cases} u_t = u_{zz} + (u + c_-)u_z + (H(u-q))_z & \text{in } Q \\ u(z,0) = u_0(z) & z \in \mathbf{R}. \end{cases}$$

Before we start the proof of (1.3) and (1.5), we state several preliminary lemmas, in which we shall always assume that u_0 satisfies the additional assumption that there exist constants α_1 and α_2 such that, for all $z \in \mathbf{R}$,

$$U_-(z - \alpha_1) \le u_0(z) \le U_+(z), \tag{6.1}$$

$$\min\{u_0(z), q\} \le U_-(z - \alpha_2), \tag{6.2}$$

where U_+ is a solution of Problem (TW_+). Hence, by the comparison principle,

$$U_-(z - \alpha_1) \le u(z,t) \le U_+(z - (c_+ - c_-)t) \ \text{ in } Q. \tag{6.3}$$

Lemma 6.1. *There exist positive constants $a > \alpha_1$, C_1 and γ such that*

$$\chi(t) \le a \quad \textit{for } t > 0, \tag{6.4}$$

$$\limsup_{t\to\infty} u(z,t) \le q \quad \textit{for } z \in \mathbf{R}, \tag{6.5}$$

and

$$q \le u(z,t) \le q + C_1 e^{-\gamma t} \quad \textit{for } a \le z \le 4a. \tag{6.6}$$

About the proof we only observe that (6.4) and the first inequality in (6.6) follow at once from the first inequality in (6.3), while (6.5) and the second inequality in (6.6) are a consequence of the second inequality in (6.3).

From (6.6) and a standard Bernstein argument we obtain the following result:

Lemma 6.2. *There exists a constant C_2 such that*

$$u_z(z,t) \le C_2 e^{-\gamma t} \quad \textit{in } [2a, 3a] \times [0, \infty). \tag{6.7}$$

An important step in the proof of Theorem B is the boundedness of $\chi(t)$ from below. For this purpose we need the following result.

Lemma 6.3. *Let* $b \in (2a, 3a)$ *and*

$$J(t) = \int_{-\infty}^{b} (u(z,t) - U_-(z-b))\, dz \quad \text{for } t \geq 0.$$

Then $J' \in L^1(\mathbf{R}^+)$.

Proof. We set, for large values of R,

$$J_R(t) = \int_{-R}^{b} (u(z,t) - U_-(z-b))\, dz \quad \text{for } t \geq 0,$$

and we determine its derivative

$$J_R'(t) = \left[u_z(z,t) + \tfrac{1}{2}u^2(z,t) + c_- u(z,t) + H(u(z,t) - q)\right]_{z=-R}^{z=b}.$$

Passing to the limit $R \to \infty$ we obtain

$$J'(t) = u_z(b,t) + \tfrac{1}{2}u^2(b,t) + c_- u(b,t) + 1,$$

and, since $\frac{1}{2}q^2 + c_- q + 1 = 0$, it follows from (6.6) and (6.7) that $|J'(t)| < C_3 e^{-\gamma t}$ for some C_3.

Lemma 6.4. *There exists a constant* $d \leq \alpha_2$ *such that*

$$\chi(t) \geq d \quad \text{for } t \geq 0 \tag{6.8}$$

and

$$\min\{u(z,t), q\} \leq U_-(z-d) \quad \text{in } Q. \tag{6.9}$$

Proof. We set

$$h(y) = \int_{y}^{b} (q - U_-(z-b))\, dz \quad \text{for } y < b.$$

Then

$$h(\chi(t)) \leq J(t) \quad \text{for } t > 0,$$

and (6.8) is a consequence of the uniform boundedness of $J(t)$, given by Lemma 6.3, and the fact that $h(y) \to \infty$ as $y \to -\infty$. Finally, (6.9) is a consequence of (6.2), (6.8) and the comparison principle applied to the Burgers equation in the set $(-\infty, d) \times \mathbf{R}^+$.

The proof of Theorem B is based on Lemma 3.2 and the following result:

Lemma 6.5. *Let u_0 satisfy hypothesis* H *and let*

$$f(t) = u_z(\chi(t)+, t). \tag{6.10}$$

Then $f \in L^1(\mathbf{R}^+)$.

Proof. Setting

$$\hat{u}(z, t; u_0) = \min\{q, u(z, t; u_0)\}, \tag{6.11}$$

we obtain from (3.5) and (6.9) that

$$\begin{aligned}\int_0^t f(s)\, ds + \int_{\mathbf{R}} (\hat{u}_0 - U_-(z-b))\, dz &= \int_{\mathbf{R}} (\hat{u}(z,t) - U_-(z-b))\, dz \\ &\leq \int_{\mathbf{R}} (U_-(z-d) - U_-(z-b))\, dz.\end{aligned}$$

Proof of (1.3) **and** (1.5)**.** First we consider the case that u_0 satisfies (6.1) and (6.2). Let

$$\mathcal{L}_1 = \{v \in L^\infty : 0 \leq v \leq q,\ v - qH \in L^1(\mathbf{R})\}$$

with the metric $d(v, w) = \int_{\mathbf{R}} |v - w|\, dx$, and let the ω-limit set $\omega(u_0)$ be the set

$$\{v \in \mathcal{L}_1 : \hat{u}(t_n; u_0) \to v \text{ in } \mathcal{L}_1 \text{ as } t_n \to \infty \text{ for some sequence } \{t_n\}\},$$

where $\hat{u}$ is defined by (6.11). Since $u_z(z, t; u_0)$ is uniformly bounded in Q, the ω-limit set is nonempty. We claim that

$$v \in \omega(u_0) \Rightarrow v \text{ is a solution of Problem (TW}_-). \tag{6.12}$$

Let $v \in \omega(u_0)$. In view of (6.3), the fact that $b > a > \alpha_1$, and (6.9), we have that

$$U_-(z-b) \leq \hat{u}(z, t; u_0),\ v(z) \leq U_-(z-d) \quad \text{for } z \in \mathbf{R}, t > 0.$$

In addition, it follows from classical regularity theory applied to the Burgers equation that v is smooth in the set where $v < q$.

We argue by contradiction and suppose that v is not a solution of Problem (TW$_-$). Then there exists a $\mu \in (d, b)$ such that $v - U_-(z-\mu)$ changes sign. We define the functional $V : \mathcal{L}_1 \to \mathbf{R}$ by

$$V(w) = \int_{\mathbf{R}} |w(z) - U_-(z-\mu)|\, dz.$$

Let $u(t; v)$ denote the solution (as a function of z and t) of the one-phase Problem (FB$_1$) introduced in the introduction, extended by q to points (z, t) at the right-hand side of the free boundary. Then, by Lemma 3.3,

$$V(u(t; v)) < V(v) \quad \text{for } t > 0.$$

This leads to a contradiction if we are able to prove that

$$u(t;v) \in \omega(u_0) \quad \text{for } t > 0 \tag{6.13}$$

and

$$V \text{ is constant on } \omega(u_0). \tag{6.14}$$

We begin with the proof of (6.13). There exists a sequence $\{t_n\}$ such that

$$\hat{u}(t_n;u_0) \to v \quad \text{as } t_n \to \infty. \tag{6.15}$$

Using the notation (6.10), we obtain from Lemma 3.2 that for $t > 0$

$$\int_{\mathbf{R}} |\hat{u}(t_n+t;u_0) - u(t;v)|\,dx \le \int_{\mathbf{R}} |\hat{u}(t_n;u_0) - v|\,dx + \int_{t_n}^{t_n+t} f(s)\,ds,$$

which, by (6.15) and Lemma 6.5, vanishes as $t_n \to \infty$. Hence $\hat{u}(t_n+t;u_0) \to u(t;v)$ and we have proved (6.13).

To prove (6.14), we suppose that $V(\hat{u}(t_n;u_0)) \to V_1$ as $t_n \to \infty$ and that $V(\hat{u}(t_m;u_0)) \to V_2$ as $t_m \to \infty$. Applying (3.7) first between t_m and t_n with $t_m \le t_n$, and then between $\tilde{t}_n$ and $\tilde{t}_m$ with $\tilde{t}_n \le \tilde{t}_m$, we deduce from Lemma 6.5 that $V_1 = V_2$.

In order to show that $\omega(u_0)$ consists of only one element we suppose that $\hat{u}(t_n) \to \hat{u}_1$ as $t_n \to \infty$ and $\hat{u}(t_m) \to \hat{u}_2$ as $t_m \to \infty$. Letting $t_m, t_n \to \infty$ in the inequality

$$\|\hat{u}(t_n) - \hat{u}_2\|_{L^1(\mathbf{R})} \le \|\hat{u}(t_m) - \hat{u}_2\|_{L^1(\mathbf{R})} + \int_{t_m}^{t_n} u_x(\zeta(t)^+, t)\,dt$$

we obtain the desired result.

So we have completed the proof of (6.12). From (6.12) and Lemma 3.2 and 6.5 we obtain that $\omega(u_0)$ consists of only one element, and thus we have proved (1.3).

Since $U_-(z-\mu_1) < q$ if $z < \mu_1$, (1.3) implies that

$$\liminf_{t\to\infty}(\zeta(t) - c_- t) \ge \mu_1.$$

Arguing as in [6, Theorem 5.11], it is not difficult to prove that also

$$\limsup_{t\to\infty}(\zeta(t) - c_- t) \le \mu_1,$$

from which we obtain (1.5).

It remains to prove (1.3) if u_0 does not satisfy conditions (6.1) and (6.2). We approximate u_0 in $\mathcal{L}$ by initial functions u_{0i} which satisfy hypothesis H and conditions (6.1) and (6.2). Then there exists for any i a constant μ_i such that the corresponding solutions $u_i(z,t)$ satisfy

$$\hat{u}_i(z,t) - U_-(z-\mu_i) \to 0 \quad \text{in } L^1(\mathbf{R}) \text{ as } t \to \infty.$$

Since, by Lemma 3.1,

$$\|\hat{u}_i(\cdot,t) - \hat{u}_j(\cdot,t)\|_{L^1(\mathbf{R})} \le \|u_i(\cdot,t) - u_j(\cdot,t)\|_{L^1(\mathbf{R})} \le \|u_{0i} - u_{0j}\|_{L^1(\mathbf{R})},$$

it follows that $\{U_-(\cdot - \mu_i) - qH(z)\}$ is a Cauchy sequence in $\mathcal{L}_1$, from which (1.3) follows.

7. The case $q < 2/A$: $u > q$. In this section we complete the proof of Theorem B by showing (1.4). We set $y = x - c_+ t$ and denote the solution of Problem (FB) as a function of y and t by $u(y,t;u_0)$. In addition, we use the notation $\overline{u}(y,t) = \max\{u(y,t),q\}$.

Proof of (1.4)**.** First we assume that u_0 satisfies the additional conditions (6.1), (6.2) and

$$u_0(y) \ge U_+(y - y_0) \quad \text{for } y \ge y_1 \tag{7.1}$$

for certain constants y_0 and y_1, where U_+ is the function defined by (6.1). Hence, by (6.3),

$$u(y,t) \le U_+(y) \quad \text{in } Q. \tag{7.2}$$

We claim that there exist constants $y_2 \ge y_0$ and $y_3 \ge y_1$ such that

$$u(y,t) \ge U_+(y - y_2) \quad \text{if } y \ge y_3, t > 0. \tag{7.3}$$

To prove (7.3) we observe that, by Lemmas 3.2 and 6.5, for all $t > 0$

$$\int_{\mathbf{R}} (U_+(y) - \overline{u}(y,t))\,dy = \int_{\mathbf{R}} (U_+(y) - \overline{u}_0(y))\,dy + \int_0^t f(s)\,ds \le C \tag{7.4}$$

for some $C > 0$. By [1, proof of Lemma 5.2], there exists an $\epsilon > 0$ such that for any $t > 0$ there exists a unique value $y_\epsilon(t)$ such that

$$u(y,t)\begin{cases} < q + \epsilon & \text{if } y < y_\epsilon(t) \\ > q + \epsilon & \text{if } y > y_\epsilon(t). \end{cases}$$

Hence, defining $y_4 < y_\epsilon(t)$ by $U_+(y_4) = q + \epsilon$, we have

$$\int_{\mathbf{R}} (U_+(y) - \overline{u}(y,t))\,dy \ge \int_{y_4}^{y_\epsilon(t)} (U_+(y) - (q+\epsilon))\,dy. \tag{7.5}$$

It follows from (7.4) that the integral on the left-hand side of (7.5) is uniformly bounded with respect to t, and thus $y_\epsilon(t)$ is uniformly bounded. Choosing $y_3 \ge y_\epsilon(t)$ for all $t > 0$ and $y_2 \ge y_3 - y_4$, we find that

$$U_+(y_3 - y_2) \le U_+(y_4) = q + \epsilon \le u(y_3,t) \quad \text{for } t > 0.$$

Hence, if $y_2 \geq y_0$ and $y_3 \geq y_1$, then $u \geq U(\cdot - y_2)$ on the parabolic boundary of the set $\{(y,t) : y > y_3, t > 0\}$, and (7.3) follows from the comparison principle.

We define the space

$$\mathcal{L}_2 = \{v \in L^\infty(\mathbf{R}) : q \leq v \leq A \text{ a.e. in } \mathbf{R}, v - q - (A-q)H \in L^1(\mathbf{R})\}$$

with the metric $d(v,w) = \int_{\mathbf{R}} |v - w|\, dy$. Let $\omega(u_0)$ denote the set

$$\{v \in \mathcal{L}_2 : d(\overline{u}(t_n; u_0), v) \to 0 \text{ as } t_n \to \infty \text{ for some sequence } \{t_n\}\}.$$

By (7.2) and (7.3), $\omega(u_0) \neq \emptyset$.

Following the lines of the proof in the preceding section, we find that

$$v \in \omega(u_0) \Rightarrow v \text{ is a solution of Problem TW}_+. \tag{7.6}$$

Indeed, arguing by contradiction, v intersects a solution $U_+(y-\mu)$ of Problem (TW$_+$), and defining the functional $V(v) = \int_{\mathbf{R}} |v(y) - U_+(y-\mu)|\, dy$, it follows easily from Lemmas 3.2 and 6.5 that

$$V(u(t;v)) = V(v) = \lim_{t\to\infty} V(\overline{u}(t;u_0)),$$

where $\overline{u}(t;v)$ denotes the solution of the Burgers equation with initial function v. But this leads to a contradiction since, by Lemma 3.4, $V(u(t;v)) < V(v)$. It follows from (6.7) and Lemma 3.2 that $\omega(u_0)$ consists of only one point.

It remains to prove (1.4) if u_0 does not satisfy conditions (6.1), (6.2) and (7.1). Approximating u_0 in $\mathcal{L}$ by functions u_{0i} which do satisfy these conditions, the functions $\overline{u}(t;u_{0i})$ converge to a travelling wave $U_+(\cdot - \mu_i)$ in $\mathcal{L}_2$ as $t \to \infty$, and it follows from

$$\int_{\mathbf{R}} |\overline{u}_i - \overline{u}_j|(t)\, dy \leq \int_{\mathbf{R}} |u_i - u_j|(t)\, dy \leq \int_{\mathbf{R}} |u_{0i} - u_{0j}|\, dy \to 0 \quad \text{as } i,j \to \infty,$$

that $\{U_+(\cdot - \mu_i)\}$ is a Cauchy sequence in $\mathcal{L}_2$, from which (1.4) follows.

8. The case $q = 2/A$. In this section we briefly indicate the proof of the following result:

Theorem 8.1. *Let $q = 2/A$ and let $c = c_- = c_+$ be the velocity of the travelling-wave solutions of Problems (TW$_-$) and TW$_+$. Let u_0 be an initial function which satisfies conditions* (6.1) *and* (6.2), *and let (u,ζ) be the solution of Problem* (FB). *Then*

$$\zeta(t) - ct \to -\infty \quad \text{as } t \to \infty \tag{8.1}$$

and

$$\zeta(t) - ct = o(t) \text{ as } t \to \infty. \tag{8.2}$$

Proof. We set $z = x - ct$, $\chi(t) = \zeta(t) - ct$, and we denote the solution $u(x,t)$ as a function of z and t by $u(z,t)$. Arguing as in section 6 it follows easily that

$$\chi(t) \to -\infty \;\text{ as } t \to \infty \quad \text{if and only if} \quad u_z(\chi(\cdot)^+, \cdot) \notin L^1(\mathbf{R}^+). \tag{8.3}$$

To prove (8.1) we argue by contradiction and suppose that

$$\chi(t) > z_0 \text{ for } t > 0 \text{ and } u_z(\chi(\cdot)^+, \cdot) \in L^1(\mathbf{R}^+).$$

We observe that, by (6.3), $\overline{u}(z,t) \le U_+(z)$ in Q, and, by Lemma 3.2, there exists a constant C such that for all $t > 0$

$$\int_{\mathbf{R}} (U_+(z) - \overline{u}(z,t))\, dz \le C. \tag{8.4}$$

Let $u_1(z,t)$ be the classical solution of the problem

$$\begin{cases} u_t = u_{zz} + (u+c)u_z & \text{for } z > z_0, t > 0 \\ u(z_0, t) = q & \text{for } t > 0 \\ u(z,0) = U_+(z) & \text{for } z > z_0. \end{cases}$$

Since $\chi(t) \ge z_0$, we have that $U_+(\chi(t)) \ge q = u(\chi(t), t)$ for $t > 0$; since also $u_0 \le U_+$ in (ζ_0, ∞), it follows from the classical maximum principle that

$$\overline{u} \le u_1 \in (z_0, \infty) \times \mathbf{R}^+. \tag{8.5}$$

Since the initial function $U_+(z)$ is a supersolution of the problem for u_1, the function $u_1(z,t)$ is decreasing with respect to t, and since $u_1(z,t) \ge q$ we find that

$$w(z) = \lim_{t \to \infty} u_1(z,t) \ge q \quad \text{for } z \ge z_0.$$

On the other hand $w \in C^2([z_0, \infty))$, and w satisfies

$$w'' + (w+c)w' = 0 \quad \text{for } z > z_0$$
$$w(z_0) = q \quad \text{and} \quad q \le \lim_{z \to \infty} w(z) \le A.$$

Integrating the differential equation for w and using the value of c we obtain that, for some $C \ge 0$, $w' = C - g(w)$ with $g(s) = \frac{1}{2}s^2 + cs + 1$, $g(q) = g(A) = 0$ and $g(s) < 0$ for $q < s < A$. If $C > 0$, we obtain a contradiction with the boundedness of w; therefore $C = 0$; i.e., $w(z) = q$ for all $z \ge z_0$. Finally, it follows from (8.5) that

$$\int_{\mathbf{R}} (U_+(z) - \overline{u}(z,t)) dz \ge \int_{z_0}^{\infty} (U_+(z) - u_1(z,t))\, dz \to \infty \;\text{ as } t \to \infty,$$

and we have found a contradiction with (8.4).

It remains to prove (8.2). We choose an arbitrary constant $\epsilon \in (0, q)$. Then there exists a constant $A_\epsilon > A$ such that

$$A_\epsilon \to A \quad \text{as} \quad \epsilon \to 0, \tag{8.6}$$

and such that the travelling-wave problem

$$\begin{cases} U_{zz} + (U + c)U_z + (H(U - q))_z = 0 & z \in \mathbf{R} \\ U(-\infty) = \epsilon, \quad U(0) = q, \quad U(+\infty) = A_\epsilon \end{cases}$$

has a unique solution (U_ϵ, c_ϵ), with $c_\epsilon < c$. Choosing z_1 such that $U_\epsilon(z - z_1) \geq u_0(z)$, we obtain from the maximum principle that $U_\epsilon(z - z_1 - (c_\epsilon - c)t) \geq u_1(z, t)$ in Q, which implies that

$$\chi(t) \geq z_1 + (c_\epsilon - c)t \quad \text{for } t > 0. \tag{8.7}$$

Since, by (8.6), $c_\epsilon \to c$ as $\epsilon \to 0$, it follows from (8.7) that $\chi(t) = o(t)$ as $t \to \infty$, and we have proved (8.2).

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