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A DEGREE-THEORETIC APPROACH FOR THE STUDY OF EIGENVALUE PROBLEMS IN VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

D. Goeleven
Dept. of Mathematics, Facultés Universitaires N.D. de la Paix, Namur, Belgium

D. Motreanu
Dept. of Mathematics, University "Al.I. Cuza," Iasi, Romania

(Submitted by: Viorel Barbu)

Abstract. The aim of this paper is to study a class of eigenvalue problems of variational-hemivariational type.

1. Introduction. In this paper we extend the results of Szulkin ([19]–[21]) regarding the noncoercive elliptic variational inequalities to the case of variational-hemivariational inequalities. Our approach is based on the use of Leray-Schauder degree in a way that is suitable in the setting of nonsmooth analysis. The use of Leray-Schauder degree for the study of variational inequalities has been initiated by Szulkin ([20]) and Quittner ([18]). This topological approach has later been revisited by Goeleven, Nguyen and Théra ([5], [6]) for the study of eigenvalue problems of von Kármán type. For the study of hemivariational inequalities by means of a Leray-Schauder degree method the first attempt in this direction was made by Goeleven, Motreanu and Panagiotopoulos ([9]) where a problem at resonance was treated. Concerning the eigenvalue problems for hemivariational inequalities different results can be found in Goeleven, Motreanu and Panagiotopoulos [7]–[9], Motreanu and Naniewicz [12], Motreanu and Panagiotopoulos [13], [14], Naniewicz and Panagiotopoulos [15], and Panagiotopoulos [16], [17]. Here our study is carried out over the hemivariational inequalities subject to variational constraints. Our results extend and sharpen some relevant contributions in the field (see, for comparison, Boccardo [1], Friedman [4], Kinderlehrer and Stampacchia [11], and Szulkin [19]–[21]).

The rest of the paper is organized as follows. Section 2 deals with an abstract eigenvalue problem of variational-hemivariational type. Section 3 is devoted to the nonlinear elliptic problems with Dirichlet boundary conditions.

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2. An abstract eigenvalue problem. Let V be a real Hilbert space, with the norm $\| \|_V$, which is densely and compactly embedded in $L^2(\Omega)$ for a bounded domain Ω in $\mathbb{R}^N, N \geq 1$, of Lebesgue measure $\mid \Omega \mid$. Let $a : V \times V \to \mathbb{R}$ be a continuous bilinear form which is coercive:

$$a(u,u) \geq \alpha \parallel u \parallel_V^2, \quad \forall\, u \in V, \tag{1}$$

with a constant $\alpha > 0$. We denote by $k^{1/2}$ the best constant of continuity corresponding to the embedding $V \hookrightarrow L^2(\Omega)$, i.e., $\parallel u \parallel_{L^2} \leq k^{1/2} \parallel u \parallel_V, \forall u \in V$.

Let $j : \Omega \times \mathbb{R} \to \mathbb{R}$ denote a (Carathéodory) function satisfying

(H_0)

(a) $j(.,y)\colon \Omega \to \mathbb{R}$ is measurable, $\forall y \in \mathbb{R}$ and $j(.,0) \in L^1(\Omega)$;

(b) $j(x,.)\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, $\forall x \in \Omega$;

(c) there exists a continuous mapping $W\colon L^2(\Omega) \to L^2(\Omega)$ such that

$$W(v)(x) \in \partial_y j(x, v(x)), \quad \forall v \in V, \text{ a.e. } x \in \Omega;$$

and

(H_1) $|z| \leq c_1|y| + c_2$, $\forall y \in \mathbb{R}$, almost every $x \in \Omega, \forall z \in \partial_y j(x,y)$ for constants $c_1 \geq 0$ and $c_2 \geq 0$.

In (H_0), (H_1) the notation $\partial_y j(x,y)$ stands for the generalized gradient of $j(x,.)$ at $y \in \mathbb{R}$ (see Clarke [3]). We recall also the definition of the generalized directional derivative in Clarke [3] (in fact a characterization of it)

$$j^0(x,y;\xi) = \max\{z\xi\colon z \in \partial_y j(x,y)\}, \quad \forall x,y,\xi \in \mathbb{R}. \tag{2}$$

Let $C \subset V$ be a convex and closed subset of V that satisfies

$$0 \in C. \tag{3}$$

Given $g \in L^2(\Omega)$ we consider the variational-hemivariational inequality: find $u \in C$ and $\lambda \in \mathbb{R}$ such that

$$\text{(P)} \quad a(u,v-u) + \int_\Omega j^0(x,u;v-u)\,dx \geq \lambda \int_\Omega u(v-u)\,dx + \int_\Omega g(v-u)\,dx, \forall v \in C.$$

We associate to problem (P) the linear eigenvalue problem

$$a(u,v) = \lambda(u,v)_{L^2}, \quad \forall v \in V. \tag{4}$$

The compactness of the embedding $V \subset L^2(\Omega)$ and relation (1) ensure that if in addition a is symmetric then (4) has the spectrum consisting of a sequence (λ_n) of eigenvalues with $\lambda_n \to \infty$ and

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots. \tag{5}$$

Our first result is formulated below.

Theorem 1. *Under assumptions* (H_0), (H_1), *for every* $0 < \lambda < \alpha k^{-1} - c_1$ *problem* (P) *possesses at least a solution* $u \in C$.

Proof. We claim that for every $t \in [0,1]$ the variational inequality

$$\begin{cases} a(u, v-u) \geq \lambda t \int_\Omega u(v-u)\,dx \\ \qquad\qquad -t\int_\Omega w(v-u)\,dx + \int_\Omega g(v-u)\,dx, \quad \forall v \in C, \\ w \in L^2(\Omega), w(x) \in \partial_y j(x, u(x)) \quad \text{for a.e. } x \in \Omega, \end{cases} \tag{6}$$

has no solution $u \in C$ such that $\| u \|_V = r$ if $r > 0$ is sufficiently large, where $0 < \lambda < \alpha k^{-1} - c_1$.

Arguing by contradiction we assume that sequences $(u_n, t_n) \in C \times [0,1]$ and $w_n \in L^2(\Omega)$ can be found provided $\| u_n \|_V \to \infty$ as $n \to \infty$,

$$\begin{aligned} a(u_n, v-u_n) \geq \lambda t_n \int_\Omega u_n(v-u_n)\,dx - t_n \int_\Omega w_n(v-u_n)\,dx \\ + \int_\Omega g(v-u_n)\,dx, \quad \forall v \in C, \end{aligned} \tag{7}$$

and

$$w_n(x) \in \partial_y j(x, u_n(x)) \quad \text{for a.e. } x \in \Omega. \tag{8}$$

In view of (3) it is permitted to put $v = 0$ in (7). This yields

$$\alpha \| u_n \|_V^2 \leq a(u_n, u_n) \leq \lambda t_n \| u_n \|_{L^2}^2 - t_n \int_\Omega w_n u_n\,dx + \int_\Omega g u_n\,dx, \forall n \geq 1,$$

where (1) was used. From (H_1) and (8) we then derive

$$\begin{aligned} \alpha \| u_n \|_V^2 &\leq \lambda t_n \| u_n \|_{L^2}^2 + t_n(c_1 \| u_n \|_{L^2}^2 + c_2 \| u_n \|_{L^1}) + \| g \|_{L^2} \| u_n \|_{L^2} \\ &\leq k\lambda t_n \| u_n \|_V^2 + t_n(c_1 k \| u_n \|_V^2 + c_2 \mid \Omega \mid k^{1/2} \| u_n \|_V) \\ &\quad + k^{1/2} \| g \|_{L^2} \| u_n \|_V, \quad \forall n \geq 1. \end{aligned}$$

If, in the foregoing inequality, we divide by $\| u_n \|_V^2$, it is seen that

$$\alpha \leq (\lambda + c_1)k t_n + (c_2 \mid \Omega \mid + \| g \|_{L^2}) k^{1/2} \| u_n \|_V^{-1}, \quad \forall n \geq 1.$$

Since $t_n \in [0,1]$ and $\| u_n \|_V \to \infty$ we obtain $\alpha \leq (\lambda + c_1)k$ which contradicts the assumptions of the theorem. Thus the claim follows.

Define the map $\Pi_C \colon L^2(\Omega) \to V$ by

$$\Pi_C(v) = u \in C, \quad \forall\, v \in L^2(\Omega), \tag{9}$$

where u is the unique solution of the variational inequality

$$u \in C\colon a(u, z-u) \geq \int_\Omega v(z-u)\, dx, \quad \forall\, z \in C. \tag{10}$$

For the existence and the uniqueness of the solution to (10), see e.g. Brezis [2]. The map Π_C is continuous. Indeed, let $v_n \to v$ in $L^2(\Omega)$. By (9), (10) we know

$$a(\Pi_C(v), z - \Pi_C(v)) \geq -\int_\Omega v(\Pi_C(v) - z)\, dx, \quad \forall\, z \in C$$

and

$$a(-\Pi_C(v_n), -z + \Pi_C(v_n)) \geq \int_\Omega v_n(z - \Pi_C(v_n))\, dx, \quad \forall\, z \in C.$$

The sum of the inequalities above with z equal to $\Pi_C(v_n)$ and $\Pi_C(v)$ respectively shows

$$\begin{aligned} &-a(\Pi_C(v_n) - \Pi_C(v), \Pi_C(v_n) - \Pi_C(v)) \\ &\qquad \geq \int_\Omega (\Pi_C(v) - \Pi_C(v_n))(v_n - v)\, dx, \quad \forall n \geq 1. \end{aligned}$$

Combining with (1) we obtain

$$\alpha \parallel \Pi_C(v_n) - \Pi_C(v) \parallel_V^2 \leq \int_\Omega (\Pi_C(v_n) - \Pi_C(v))(v_n - v)\, dx, \quad \forall n \geq 1,$$

so that

$$\parallel \Pi_C(v_n) - \Pi_C(v) \parallel_V \leq \frac{k^{1/2}}{\alpha} \parallel v_n - v \parallel_{L^2}, \quad \forall n \geq 1.$$

This implies that $\Pi_C(v_n) \to \Pi_C(v)$ in V as $n \to \infty$. The continuity of $\Pi_C\colon L^2(\Omega) \to V$ follows.

We define the homotopy $H : V \times [0,1] \to V$ by

$$H(v,t) = \Pi_C(\lambda t v - tW(v) + g), \quad \forall (v,t) \in V \times [0,1], \tag{11}$$

where Π_C is the mapping in (9) and W is the mapping supplied by $(\mathrm{H}_0)(c)$. The map H is compact because the same property is valid for the embedding $V \subset L^2(\Omega)$. Moreover, because (6) has no solution on $\partial B(0,r)$, where

$$B(0,r) = \{x \in V\colon \parallel x \parallel_V < r\}, \tag{12}$$

if $r > 0$ is large enough, the Leray-Schauder degree

$$\deg(\mathrm{Id}_V - H(.,t), B(0,r), 0) \tag{13}$$

is well defined for all $t \in [0,1]$. The homotopy invariance of Leray-Schauder degree (see, e.g., Kavian [10]) insures that

$$\deg(\mathrm{Id}_V - H(.,1), B(0,r), 0) = \deg(\mathrm{Id}_V - H(.,0), B(0,r), 0). \tag{14}$$

On the other hand, as seen from (11), one has

$$H(.,0) = \Pi_C(g). \tag{15}$$

Let us choose the radius $r > 0$ of the ball $B(0,r)$ in (12) so large that

$$\Pi_C(g) \in B(0,r). \tag{16}$$

Then relations (15) and (16) imply

$$\deg(\mathrm{Id}_V - H(.,0), B(0,r), 0) = 1. \tag{17}$$

It follows from (14) and (17) that the fixed-point problem

$$u = H(u,1) \tag{18}$$

has at least a solution. Taking into account (11), (18) this means that there exists $u \in C$ such that

$$\begin{aligned} a(u, v-u) &+ \int_\Omega W(u)(v-u)\,dx \\ &\geq \lambda \int_\Omega u(v-u)\,dx + \int_\Omega g(v-u)\,dx, \quad \forall v \in C. \end{aligned} \tag{19}$$

Comparing (H_0) (c) and (2) we infer that

$$\int_\Omega j^0(x,u;v-u)\,dx = \int_\Omega \max_{z\in\partial_y j(x,u(x))} z(v-u)\,dx \geq \int_\Omega W(u)(v-u)\,dx.$$

From (19) it turns out that $u \in C$ solves (P). This completes the proof.

Remark 1. Theorem 1 extends well-known results in the theory of variational inequalities (see, e.g., Theorem 1.2.7 in Friedman [4], Theorem II.2.1. in Kinderlehrer and Stampacchia [11], Theorem 1 (i) in Szulkin [21]). As an illustration of this aspect, taking $V = H_0^1(\Omega), a = (.,.)_{H_0^1}, j = 0, C = \{v \in H_0^1(\Omega) : v \geq 0$ almost everywhere in $\Omega\}$, our assumption $0 < \lambda < \frac{\alpha}{k} - c_1$ becomes $0 < \lambda < \lambda_1$ and Theorem 1 reduces to Theorem 1 (i) in Szulkin [21].

Notice that Theorem 1 expresses the solvability of problem (P) for each $g \in L^2(\Omega)$ if $0 < \lambda < \frac{\alpha}{k} - c_1$. This is no longer true for $\lambda \geq \frac{\alpha}{k} - c_1$. For instance, in the situation described in Remark 1 this has been pointed out by Boccardo ([1]) and Szulkin ([19], [21]) for $\lambda = \lambda_1$ and by Szulkin ([19], [21]) if $\lambda \in (\lambda_1, \lambda_2)$ (see (5)).

The next result deals with the solvability of problem (P) for every $\lambda > 0$.

Theorem 2. *Assume* (H_0), (H_1),

(H_2) $zy \geq -c_0 \mid y \mid^2, \forall y \in \mathbb{R}, \forall z \in \partial_y j(x,y)$, *almost every* $x \in \Omega$, *with a constant* $c_0 \geq 0$;

(H_3) $\int_\Omega g(x)v(x)\,dx < 0$ *for all* $v \in L^2(\Omega) \setminus \{0\}$ *in the* $L^2(\Omega)$*-closure of* $\{\frac{z}{\|z\|_V} : z \in C \setminus \{0\}\}$.

Then for each $\lambda > 0$ *problem* (P) *admits at least a solution.*

Proof. In contrast to the proof of Theorem 1 we argue with the ball $B(0,r)$ in (12) of a small radius $r > 0$.

Fix any $\lambda > 0$. We claim that the problem (6) for all $t \in [0,1]$ has no solution $u \in C$ if $r = \| u \|_V > 0$ is sufficiently small. If we suppose the contrary we find sequences $(u_n, t_n) \in C \times [0,1]$ and $w_n \in L^2(\Omega)$ satisfying $u_n \neq 0, u_n \to 0$ in V and (7), (8). Let us put

$$z_n := \frac{u_n}{\| u_n \|_V}, \quad n \geq 1. \tag{20}$$

Taking $v = 0$ in (7), which is possible by (3), we obtain

$$a(z_n, z_n) \leq \lambda t_n \int_\Omega z_n^2\,dx - \frac{t_n}{\| u_n \|_V^2} \int_\Omega w_n u_n\,dx + \frac{1}{\| u_n \|_V} \int_\Omega g z_n\,dx.$$

Since by assumption (H_2) we know that

$$w_n(x)u_n(x) \geq -c_0 \mid u_n(x) \mid^2 \quad \text{for a.e. } x \in \Omega,$$

then we get

$$\alpha \leq \lambda t_n \| z_n \|_{L^2}^2 + c_0 t_n \| z_n \|_{L^2}^2 + \frac{1}{\| u_n \|_V} \int_\Omega g z_n\,dx, \quad \forall n \geq 1. \tag{21}$$

In view of (20) and the compact embedding $V \subset L^2(\Omega)$, by passing to a subsequence we may assume

$$z_n \to \overline{z} \quad \text{in } L^2(\Omega) \tag{22}$$

for some $\overline{z} \in L^2(\Omega)$. Assumption ($H_3$) entails

$$\int_\Omega g z_n\,dx = \frac{1}{\| u_n \|_V} \int_\Omega g u_n\,dx < 0.$$

Thus from (21) we derive

$$\alpha < (\lambda + c_0) t_n \| z_n \|_{L^2}^2 . \tag{23}$$

If we suppose that $\overline{z} = 0$, then by (22) the right-hand side of (23) tends to zero and we arrive at a contradiction. Therefore $\overline{z} \neq 0$. Then, taking the limit as $n \to \infty$

in (21), we obtain $\alpha \leq -\infty$ because $u_n \to 0$ in V and (H$_3$) applies for $v = \overline{z}$. The contradiction proves the claim.

Consider now the map $\Pi_C\colon L^2(\Omega) \to V$ introduced by (9), (10). As shown in the proof of Theorem 1 the map Π_C is continuous. Then we can construct the homotopy $H : V \times [0,1] \to V$ given in (11). The compactness of the embedding $V \subset L^2(\Omega)$ insures that H is compact. The validity of the claim in the first part of the proof enables us to conclude that $H(u,t) \neq u$ for all $(u,t) \in \partial B(0,r) \times [0,1]$ with $r > 0$ sufficiently small ($B(0,r)$ represents the ball in (12)). Thus the Leray-Schauder degree in (13) is well defined for any small-enough $r > 0$. As in the proof of Theorem 1 we deduce that equalities (14), (15) hold true. Furthermore, one has

$$\Pi_C(g) = 0 \tag{24}$$

due to the fact that the variational inequality

$$u \in C\colon a(u, z-u) \geq \int_\Omega g(z-u)\,dx, \quad \forall z \in C,$$

possesses a unique solution $u = 0$ (see, e.g., Brezis [2]). The assertion is true on the basis of (H$_3$). Hence by (24) property (16) is verified. Consequently, (15), (16) lead to (17). From now we argue exactly as in the proof of Theorem 1. The proof is thus complete.

The next consequence of Theorem 2 provides a relevant special case of application.

Corollary 1. *Assume* (H$_0$), (H$_1$), (H$_2$) *and*

(H$_3'$)

(i) $\lambda C \subset C, \forall \lambda > 0$*;*

(ii) $\int_\Omega g(x)v(x)\,dx < 0$*, for all* $v \in C \cap \partial B(0,1)$*.*

Then the conclusion of Theorem 2 *holds.*

Proof. It suffices to check that (H$_3'$) implies (H$_3$). From (H$_3'$) (i) it follows that

$$\Big\{\frac{z}{\| z \|_V}\colon z \in C \setminus \{0\}\Big\} \subset C \cap \partial B(0,1).$$

Then the inclusion above and (H$_3'$) (ii) ensures (H$_3$).

Remark 2. For the situation $V = H_0^1(\Omega), a = (.,.)_{H_0^1}, j = 0$ and $C = \{v \in H_0^1(\Omega)\colon v \geq 0$ almost everywhere in $\Omega\}$ hypothesis (H$_3$) implies $g \leq 0$ and $g \not\equiv 0$. Denoting by $e \in H_0^1(\Omega)$ the first eigenfunction of $-\Delta$ on $H_0^1(\Omega)$ with $e > 0$ on Ω, it follows that $\int_\Omega ge\,dx < 0$. It is known from Boccardo [1] and Szulkin [19], [21], that for $\lambda \geq \lambda_1$ this condition is essential for the existence of solutions. Thus, generally, hypothesis (H$_3$) cannot be dropped. Assumption (H$_1$) assures that the integral of j^0 in the statement of problem (P) is well defined. Assumption (H$_2$), which is compatible with the growth condition (H_1), can be regarded as a generalized sign condition. Indeed, for $c_0 = 0$ in (H_2), it appears the monotonicity of the generalized gradient $\partial_y j(.,y)$. Another interesting particular case of (H_2) is the following:

(H_2') $\mid z \mid \leq c \mid y \mid, \forall y \in \mathbb{R}, \forall z \in \partial_y j(x,y)$, a.e. $x \in \Omega$, with a constant $c > 0$.

3. Noncoercive elliptic variational-hemivariational inequalities. In this section we are concerned with the case where $V = H_0^1(\Omega)$,

$$C = \{u \in H_0^1(\Omega) \colon u \geq 0 \text{ a.e. on } \Omega\} \tag{25}$$

and $a \colon H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is the scalar product on $H_0^1(\Omega)$ defined by

$$a(u,v) = \int_\Omega \nabla u . \nabla v \, dx, \forall u, v \in H_0^1(\Omega). \tag{26}$$

We consider the variational-hemivariational inequality

(P') $u \in C \colon \int_\Omega \nabla u \cdot \nabla (v-u) \, dx + \int_\Omega j^0(x,u;v-u) \, dx \geq \lambda \int_\Omega u(v-u) \, dx + \int_\Omega g(v-u) \, dx$, $\forall v \in C$,

with $g \in L^2(\Omega), C$ given in (25) and $j \colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying (H_0). Notice that the inequality written in (P') is obtained from (P) by taking a in (26) and C in (25).

Denoting by λ_1 the first eigenvalue of the linearized problem (4) with a given by (26), it is well known that λ_1 is simple and the corresponding eigenfunction e can be chosen so that $e(x) > 0$ for all $x \in \Omega$ (see, e.g., Brezis [2]).

Due to the Sobolev Embedding Theorem we can weaken the growth condition (H_1) for the following one:

(H_1'') $\mid z \mid \leq c(1 + \mid y \mid^{p-1}), \forall y \in \mathbb{R}, z \in \partial_y j(x,y)$, almost every $x \in \Omega$, for constants $c \geq 0$ and $1 \leq p \leq 2N/(N-2)$ if $N \geq 3$ and any $1 \leq p < \infty$ if $N = 1, N = 2$.

We note that, in view of (2) and the inclusion $H_0^1(\Omega) \subset L^2(\Omega)$, assumption ($\mathrm{H}_1''$) assures the existence of the integral

$$\int_\Omega j^0(x,u;v-u) \, dx.$$

We need also a version of (H_2) for $y \geq 0$ and an adaptation of (H_3). Namely, we suppose

(H_2'') $-K_0 y \leq z \leq K_1(y^\sigma + 1)$, $\forall y \geq 0, z \in \partial_y j(x,y)$, almost every $x \in \Omega$, with constants $K_0 \geq 0, K_1 \geq 0, 0 \leq \sigma < 1$; and

(H_3'') $g(x) < 0$ for almost every $x \in \Omega$.

Our result regarding the existence of solutions to problem (P') is formulated below.

Theorem 3. *Assume that conditions* (H_0), $(\mathrm{H}_1^{''})$, $(\mathrm{H}_2^{''})$ *and* $(\mathrm{H}_3^{''})$ *are satisfied. Then for each* $\lambda > \lambda_1$ *problem* (P') *has at least two solutions.*

Proof. From (25), $(\mathrm{H}_2^{''})$ with $y = 0$ and $(\mathrm{H}_3^{''})$ we see that

$$\int_\Omega j^0(x,0;v)\,dx = \int_\Omega v(x)j^0(x,0;1)\,dx \geq 0 \geq \int_\Omega gv\,dx, \forall v \in C,$$

so $u = 0$ is a solution to problem (P') for all λ.

Fix now a number $\lambda > \lambda_1$. We claim that the problem: find $u \in C, t \in [0,1], w \in L^{p/(p-1)}(\Omega)$ such that

$$\begin{aligned} a(u,v-u) &\geq \lambda \int_\Omega u(v-u)\,dx - (1-t)\int_\Omega w(v-u)\,dx \\ &\quad + (1-2t)\int_\Omega g(v-u)\,dx, \quad \forall v \in C, \\ w(x) &\in \partial_y j(x,u(x)) \quad \text{for a.e. } x \in \Omega, \end{aligned} \tag{27}$$

has no solution $u \in V :\, = H_0^1(\Omega)$ for $r = \| u \|_{H_0^1}$ large enough.

If this were not true we would find sequences $(u_n, t_n) \in C \times [0,1]$ and $w_n \in L^2(\Omega)$ so that $\| u_n \|_{H_0^1} \to \infty$,

$$\begin{aligned} a(u_n, v-u_n) \geq \lambda \int_\Omega u_n(v-u_n)\,dx - (1-t_n)\int_\Omega w_n(v-u_n)\,dx \\ + (1-2t_n)\int_\Omega g(v-u_n)\,dx, \forall v \in C, \end{aligned} \tag{28}$$

$$w_n(x) \in \partial j(x,u_n(x)) \quad \text{for a.e. } x \in \Omega. \tag{29}$$

Putting $v = 0$ in (28) it turns out that

$$a(u_n,u_n) \leq \lambda \| u_n \|_{L^2}^2 - (1-t_n)\int_\Omega w_n u_n\,dx + (1-2t_n)\int_\Omega g u_n\,dx.$$

With the notation in (20), where $V = H_0^1(\Omega)$, the preceding inequality can be written as follows:

$$\begin{aligned} \alpha \leq a(z_n,z_n) \leq \lambda \| z_n \|_{L^2}^2 - (1-t_n)\frac{1}{\| u_n \|_{H_0^1}^2}\int_\Omega w_n u_n\,dx \\ + (1-2t_n)\frac{1}{\| u_n \|_{H_0^1}}\int_\Omega g z_n\,dx, \quad \forall n \geq 1. \end{aligned}$$

By $(\mathrm{H}_2^{''})$ this implies

$$\alpha \leq \lambda \| z_n \|_{L^2}^2 + K_0(1-t_n) \| z_n \|_{L^2}^2 + (1-2t_n)\frac{1}{\| u_n \|_{H_0^1}}\int_\Omega g z_n\,dx, \ \forall n \geq 1. \tag{30}$$

The compactness of the embedding $V \subset L^2(\Omega)$ allows us to assume the existence of some $\overline{z} \in L^2(\Omega)$ having (22) along a subsequence. We check that $\overline{z} \neq 0$. This is clear from (30) and (22) because $\| u_n \|_{H_0^1} \to \infty$ and $\alpha > 0$.

Using the first eigenfunction $e > 0$ of $-\Delta$ on $H_0^1(\Omega)$ we set $v = e + u_n$ in (28). Then we obtain

$$a(u_n, e) \geq \lambda \int_\Omega u_n e \, dx - (1 - t_n) \int_\Omega w_n e \, dx + (1 - 2t_n) \int_\Omega g e \, dx, \quad \forall n \geq 1.$$

Dividing by $\| u_n \|_{H_0^1}$ we arrive at

$$\begin{aligned} a(z_n, e) &+ \frac{1}{\| u_n \|_{H_0^1}} (1 - t_n) \int_\Omega w_n e \, dx \\ &\geq \lambda \int_\Omega z_n e \, dx + (1 - 2t_n) \frac{1}{\| u_n \|_{H_0^1}} \int_\Omega g e \, dx, \quad \forall n \geq 1. \end{aligned} \tag{31}$$

By means of (H_2'') and the Sobolev Embedding Theorem we derive

$$\int_\Omega w_n e \, dx \leq K_1 \int_\Omega (u_n^\sigma + 1) e \, dx \leq K(\| u_n \|_{H_0^1}^\sigma + 1),$$

with a constant $K \geq 0$. Then (31) yields the inequality

$$\begin{aligned} a(z_n, e) &+ K(1 - t_n) \frac{1}{\| u_n \|_{H_0^1}} (\| u_n \|_{H_0^1}^\sigma + 1) \\ &\geq \lambda \int_\Omega z_n e \, dx + (1 - 2t_n) \frac{1}{\| u_n \|_{H_0^1}} \int_\Omega g e \, dx, \quad \forall n \geq 1. \end{aligned}$$

Letting $n \to \infty$ and taking into account (22), $\| u_n \|_{H_0^1} \to \infty$ and $\sigma < 1$ we get

$$\lambda_1 \int_\Omega \overline{z} e \, dx = a(\overline{z}, e) \geq \lambda \int_\Omega \overline{z} e \, dx.$$

Since $\lambda_1 < \lambda$ and $\overline{z} \geq 0$ almost everywhere in $\Omega, \overline{z} \not\equiv 0$, this is a contradiction. Consequently, the claim concerning the unsolvability of (27) for a large $r = \| u \|_{H_0^1}$ is proved.

Consider now for the set C in (25) the map $\Pi_C \colon L^2(\Omega) \to H_0^1(\Omega)$ determined by (9), (10) with a in (26) and $V = H_0^1(\Omega)$. We introduce the homotopy $H \colon H_0^1(\Omega) \times [0, 1] \to H_0^1(\Omega)$ by

$$H(v, t) = \Pi_C(\lambda v - (1 - t) W(v) + (1 - 2t) g), \ \forall (v, t) \in H_0^1(\Omega) \times [0, 1], \tag{32}$$

where $W \colon L^2(\Omega) \to L^2(\Omega)$ stands for the continuous mapping provided by $(\mathrm{H}_0)(c)$. The compactness of the embedding $H_0^1(\Omega) \subset L^2(\Omega)$ insures that H is a compact map. Recall the notation $B(0,r)$ for the open ball in (12) with $V = H_0^1(\Omega)$. The discussion above concludes that the Leray-Schauder degree, in (13), in $B(0,r)$ is well defined with respect to 0 if $r > 0$ is sufficiently large. Then, by (32), the homotopy invariance of the Leray-Schauder degree implies

$$\begin{aligned} &\deg(\mathrm{Id}_{H_0^1} - H(.,0), B(0,r), 0) = \deg(\mathrm{Id}_{H_0^1} - H(.,1), B(0,r), 0) \\ = &\deg(\mathrm{Id}_{H_0^1} - \Pi_C(\lambda. - g), B(0,r), 0). \end{aligned} \tag{33}$$

We observe that the variational inequality

$$u \in C\colon a(u, v-u) - \lambda \int_\Omega u(v-u)\,dx \geq -\int_\Omega g(v-u)\,dx,\ \forall v \in C, \tag{34}$$

has no solution. Otherwise, there would exist $\overline{u} \in C$ satisfying (34). By setting $v = \overline{u} + e$ in (34) we would find

$$(\lambda_1 - \lambda) \int_\Omega \overline{u} e\,dx + \int_\Omega ge\,dx \geq 0,$$

which contradicts ($\mathrm{H}_3^{''}$). The definition of Π_C in (9), (10) and relations (33), (34) show that

$$\deg(\mathrm{Id}_{H_0^1} - H(.,0), B(0,r), 0) = 0. \tag{35}$$

On the other hand, from the proof of Theorem 2 it is known that for small $r > 0$ we have

$$\deg(\mathrm{Id}_{H_0^1} - H(.,0), B(0,r), 0) = 1. \tag{36}$$

Formulas (35) for a large $r > 0$ and (36) for a small $r > 0$ insure the existence of $u^* \in C \setminus \{0\}$ such that $H(u^*, 0) = u^*$. We deduce as in the proof of Theorem 2 that $u^* \in C$ is a nontrivial solution to (P′). This completes the proof.

Remark 3. In the proof of Theorem 3 we referred to the reasoning in the proof of Theorem 2. To this end, by the definition of the set C in (25), it suffices to fulfil (H_2) only for $y \geq 0$, which is assured by ($\mathrm{H}_2^{''}$). A useful special case of ($\mathrm{H}_2^{''}$) is given by ($\mathrm{H}_2^{'}$) for $y \geq 0$ and the following decreasing property of function j :

$$z \in \partial_y j(x,y) \Rightarrow z \leq 0 \quad \text{a.e. } x \in \Omega, \forall y > 0$$

and

$$z \in \partial_y j(x,0) \Rightarrow z = 0.$$

Theorem 3 extends Theorem 7.1 in Szulkin [20] (or Theorem 3 in Szulkin [21]). Our approach simplifies also the argument in Szulkin [20], [21] based on the concept of solution index.

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