Differential and Integral Equations Volume 10, Number 5, September 1997, pp. 853–892.

ABSTRACT CONVOLUTION SYSTEMS ON THE LINE

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(Submitted by: Herbert Amann)

Abstract. A quite general abstract system of an integral and an integrodifferential equation is studied. Necessary conditions and sufficient conditions for its well posedness are given in many functional spaces. Examples and applications are considered.

1. Introduction. The starting motivation of this paper was the study of an elliptic integrodifferential boundary value problem with dynamic boundary conditions of the form

$$\begin{cases} A(x,\partial_x)u(t,x) + \int_{\mathbb{R}} K(t-s,x,\partial_x)u(s,x)\,ds - r^2 e^{i\theta_0} u(t,x) = f(t,x), \\ x \in \Omega, t \in \mathbb{R}, \\ \partial_t u(t,x') - B(x',\partial_x)u(t,x') - \int_{\mathbb{R}} H(t-s,x',\partial_x)u(s,x')\,ds = g(t,x'), \\ x' \in \partial\Omega, t \in \mathbb{R}. \end{cases} \tag{1.1}$$

Here Ω is an open bounded subset of $\mathbb{R}^n$ with suitably regular boundary $\partial\Omega$, $A(x,\partial_x)$ is a second-order properly elliptic differential operator and, for $x' \in \partial\Omega$, $B(x',\partial_x)$ is a first-order differential operator; for $(t,x) \in \mathbb{R}\times\overline{\Omega}$, $K(t,x,\partial_x)$ is a linear differential operator of order not exceeding two whose coefficients are, for each $x \in \overline{\Omega}$, integrable in $\mathbb{R}$; for $(t,x') \in \mathbb{R}\times\partial\Omega$, $H(t,x',\partial_x)$ is a linear differential operator of order not exceeding one whose coefficients are, for each $x' \in \partial\Omega$, integrable in $\mathbb{R}$. Starting from suitable data f and g defined respectively in $\mathbb{R}\times\Omega$ and in $\mathbb{R}\times\partial\Omega$ one can look for conditions guaranteeing the existence and the uniqueness of solutions of system (1.1) defined in $\mathbb{R}\times\Omega$ and (roughly speaking) with a certain behaviour at infinity, in order to be able to apply the Fourier transform. Certain particular cases of (1.1) were for example considered in [1] in connection with certain quasi-static problems with viscous boundary conditions in linear viscoelasticity. More specifically, the authors of [1] treated problems in variational form, took $f \in L^2(\mathbb{R};L^2(\Omega))$, $g \equiv 0$ and looked for suitably defined weak solutions in the space $L^2(\mathbb{R};H^1(\Omega))$ (here and in the sequel we shall follow the usual convention of identifying complex-valued functions of domain $\mathbb{R}\times\Omega$ or $\mathbb{R}\times\partial\Omega$ with functions of domain $\mathbb{R}$ and values in functional spaces in Ω or $\partial\Omega$). Strengthening properly the assumptions on the coefficients, we can study (for example) under what conditions, for a certain $p \in (1,+\infty)$,

Accepted for publication August 1996.
AMS Subject Classifications: 45N05, 47G20.

given $f \in L^p(\mathbb{R}; L^p(\Omega))$, $g \in L^p(\mathbb{R}; W^{1-\frac{1}{p},p}(\partial\Omega))$, (1.1) has a unique solution $u \in L^p(\mathbb{R}; W^{2,p}(\Omega))$ with trace $\gamma u \in W^{1,p}(\mathbb{R}; W^{1-\frac{1}{p},p}(\partial\Omega))$. Put now $X = W^{2,p}(\Omega)$, $Y = L^p(\Omega)$, $Z = W^{1-\frac{1}{p},p}(\partial\Omega)$, $\mathcal{A} = A(x, \partial_x) - r^2 e^{i\theta_0}$, $K(t) = K(t, x, \partial_x)$, $\mathcal{B} = B(x', \partial_x)$, $H(t) = H(t, x', \partial_x)$ and indicate with γ the trace operator on $\partial\Omega$; then we can rewrite (1.1) in the abstract form

$$\begin{cases} \mathcal{A}u(t) + (K * u)(t) = f(t), \quad t \in \mathbb{R} \\ (\gamma u)'(t) - \mathcal{B}u(t) - (H * u)(t) = g(t), \quad t \in \mathbb{R}, \end{cases} \tag{1.2}$$

with $f(t)(x) := f(t, x), g(t)(x') := g(t, x')$.

So, we shall study abstract problem (1.2) under the following general conditions: we have three Banach spaces X,Y and Z. If E and F are Banach spaces, we shall indicate with $\mathcal{L}(E, F)$ the Banach space of linear bounded operators from E to F and we shall write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$. We assume $\mathcal{A} \in \mathcal{L}(X, Y)$, $\mathcal{B} \in \mathcal{L}(X, Z)$, $\gamma \in \mathcal{L}(X, Z)$, $K \in L^1(\mathbb{R}; \mathcal{L}(X, Y))$, $H \in L^1(\mathbb{R}; \mathcal{L}(X, Z))$; $*$ is the convolution in $\mathbb{R}$; the derivative is intended in the sense of vector-valued distributions . Remark that the structure of (1.2) is very general; if, for example, one takes $Y = \{0\}$ in such a way that the first equation can be neglected, $X \subseteq Z$ and γ is the embedding of X into Z, we obtain the problem

$$u'(t) - \mathcal{B}u(t) - (H * u)(t) = g(t). \tag{1.3}$$

When $\mathcal{B}$ generates an analytic semigroup in Z and $\text{supp}\,(H) \subseteq [0, +\infty)$ ($\text{supp}\,(H)$ is the support of H), (1.3) is a particular case of a class of abstract problems treated in the book [13] (in particular Chapter III is dedicated to problems on the line). For further results concerning equation (1.3) we mention also [5] where some resonance problems which we do not consider are studied.

There is a point which deserves to be explained: if the variable t is interpreted as time, as “the present does not depend on future,” it is reasonable to assume that in (1.2) $K(t) = 0$ and $H(t) = 0$ if $t < 0$; other restrictions, essentially concerning the solvability of the stationary problem depending on a parameter obtained from (1.2) applying formally the Fourier transform with respect to t, must be imposed if we want the solution u at time t depending only on $f(s)$ and $g(s)$ for $s \leq t$. All these conditions imply (more or less) that, at least concerning problem (1.3), the further assumptions that $\mathcal{B}$ generates an analytic semigroup in Z and $\text{supp}\,(H) \subseteq [0, +\infty)$ are satisfied. However, there exist problems where the variable t has not the meaning of time which do not seem to be solvable through the theory developed in [13]; an example is the problem treated in 5.2, which is applied in 5.3 to solve a boundary value problem for the Laplace operator in an unbounded stripe in spaces of integrable functions (we shall come back to this problem later in this introduction). Such a problem allows us to get information for an analogous problem in an angle, following an idea by P. Grisvard.

Finally, from the purely mathematical point of view, it seems natural to study (1.2) on the real line under assumptions which do not attribute any special role to the positive half-line.

We pass now to illustrating the content of the paper. We are looking for necessary conditions and sufficient conditions assuring that (1.2) is well posed in various functional frameworks (see 1.1 for precise definitions). In the second section we start with necessary conditions; there appear the natural conditions (C) in the non-periodic case (see 2.5) and (C_T) in the T-periodic case (see 2.6), concerning the well-posedness of the stationary problem depending on a parameter obtained applying the Fourier transform with respect to t. In the second section it is shown that these conditions are (in their own context) necessary for the well-posedness of the problem. The result seems to be new even for equation (1.3). An analogous result is given in [13] (Proposition 11.5), but only in the simplest case $p = +\infty$.

In the third section we consider the problem whether the necessary conditions found in the second section are also sufficient; the main tools are certain results on Fourier multipliers and singular integrals for vector-valued functions, stated and proved in 3.9 and 3.10. These are simple generalizations of results that in the scalar case are essentially known, but we have preferred to treat them in detail, because we have found it difficult to indicate a precise reference where they are stated in a way suitable to the applications we aim to. Most of them are obtained following the (to our knowledge) unpublished lecture notes by Metivier ([12]). These results allow us to treat directly our problem in the case of sufficiently regular coefficients; the general case is obtained applying the well-known Paley-Wiener's theorem (see [13], Theorem 0.6), following the idea already here and there used in [13]. The main conclusions of the third section are that the necessary conditions introduced in the second are also sufficient in the case of fractional order spaces. A similar result is given in [13] in a very general setting, covering problem (1.3) if $\mathcal{B}$ generates an analytic semigroup and $H(t) = 0$ for $t < 0$ ([4] contains the oldest results in this direction). What about the case of spaces of integer order? In such a case the given necessary conditions are (generally speaking) not sufficient (see 3.16). However, it is shown that, if the problem is well posed in some L^p space, with $1 \leq p < +\infty$, it is also well posed in any L^q space with $1 < q < +\infty$ (see 3.14). Using this fact, it is very easy to see that in case X, Y and Z are Hilbert spaces, the necessary conditions are also sufficient for $1 < p < +\infty$.

In the fourth section a further sufficient condition assuring the well-posedness in the integer-order case is given; this is done under the assumption (D) that the operator $x \to (\mathcal{A}x, \gamma x)$ is a bijection between X and $Y \times Z_1$, with $Z_1 = \gamma(X)$. The main result is stated in 4.7 and 4.8 and, roughly speaking, consists in the fact that the necessary conditions are also sufficient if we replace Z with some real interpolation space between Z and Z_1. The key step is Theorem 4.5, which is an extension of a result by P. Grisvard (see [9]), where (using the notations of 4.5) the case $\mathcal{D} = \frac{d}{dt}$ and $F = \mathbb{R}$ is considered. To prove 4.5, we use the technique of Dunford's integrals, extensively developed in [4]. The abstract result obtained seems particularly well suited to treating problems with dynamic boundary conditions, as spaces of traces are usually real interpolation spaces.

Other theorems of well-posedness for $\theta \in \mathbb{Z}$ can be certainly obtained applying the result of [6], but we have not considered them, to keep the paper in a reasonable length.

The fifth and final section contains some examples and applications of the abstract theory; the first example (see 5.1), in itself very simple, shows that assumption (D) is not a necessary consequence of (C), and so, also certain (in some sense) singular problems can be treated employing the foregoing results.

In the second example (paragraphs 5.2 and 5.3), starting from the Dirichlet problem for the Laplace operator in an angle with an integrable datum, we obtain by a change of variable the Dirichlet problem for the Laplace operator in a stripe, again with an integrable datum; the problem is reduced to an equation of type (1.3) with $H \equiv 0$ (following an idea of P. Grisvard, who studied the equation treated in 5.2 in another functional framework; see [9]). In this case we have, for a certain positive ω, $\sigma(\mathcal{B}) = \{\frac{k\pi}{\omega} : k \in \mathbb{Z}\} - \{0\}$; so it seems tempting, as alternative strategy, to try to split the space Z in the direct sum of two closed subspaces $Z_\pm$ which are invariant with respect to $\mathcal{B}$ and such that, if $\mathcal{B}_\pm$ is the part of $\mathcal{B}$ in $Z_\pm$, $\sigma(\mathcal{B}_\pm) = \sigma(\mathcal{B}) \cap \mathbb{R}^\pm$; in this case one could reduce by projection the problem in Z into two problems in Z_+ and Z_- which could be solved using the results of [13]; unfortunately two subspaces with all these properties do not exist. This fact is proved in detail in [7] and justifies the theory which we have developed.

The third and final example concerns problem (1.1), which inspired the general structure (1.2). Remark that the final result (Theorem 5.9), in contrast with [11], does not require (in the spirit of the paper) any strong ellipticity of the operator $A(x, \partial_x)$, so that the Cauchy problem studied in [11] is not necessarily well posed for the elliptic operator and the boundary conditions we consider.

We introduce now some notations we shall use in the following: first of all, $\mathbb{N}_0$ is the set of nonnegative integers; next, let E be a Banach space with norm $||.||$; we indicate with $W^{k,p}(E)$ the space $W^{k,p}(\mathbb{R}; E)$ if $k \geq 0, k \in \mathbb{Z}$, $1 \leq p < +\infty$, while $W^{k,\infty}(E)$ will be the space of functions from $\mathbb{R}$ to E which are of class C^k with all the derivatives of order less than or equal to k bounded and uniformly continuous. The definition of $W^{k,p}(E)$ is given in such a way that for every $p \in [1, +\infty]$ $C^\infty(\mathbb{R}; E) \cap W^{k,p}(E)$ is dense in $W^{k,p}(E)$. If $u \in W^{k,p}(E)$, we set $||u||_{k,p,E} := \sum_{j=0}^k ||u^{(j)}||_{L^p(\mathbb{R};E)}$. If $0 < \theta < 1$, $W^{\theta,p}(E) := \{u \in W^{0,p}(E) : (t,s) \to |t-s|^{-\theta-1/p}||u(t) - u(s)|| \in L^p(\mathbb{R}^2)\}$. If $\theta \geq 1$, $W^{\theta,p}(E) := \{u \in W^{\theta-1,p}(E) : u' \in W^{\theta-1,p}(E)\}$. For $\theta \geq 0, \theta \notin \mathbb{Z}$, we set

$$||u||_{\theta,p,E} := ||u||_{[\theta],p,E} + ||(t,s) \to |t-s|^{-(\theta-[\theta])-1/p}||u^{[\theta]}(t) - u^{[\theta]}(s)||||_{L^p(\mathbb{R}^2)},$$

where we have indicated with $[\theta]$ the largest integer less than θ. If $E = \mathbb{C}$, we shall omit E. Finally, we set $W^{\infty,p}(E) := \bigcap_{\theta>0} W^{\theta,p}(E)$. We shall consider also problem (1.2) in the periodic case; in such a case we could also take

$K \in L^1_{loc}(\mathbb{R}; \mathcal{L}(X, Y)), H \in L^1_{loc}(\mathbb{R}; \mathcal{L}(X, Z))$ and $T-$periodic $(T > 0)$. Remark that we can reduce this case to (1.2) replacing K and H with elements K_1 and H_1, respectively in $L^1(\mathbb{R}; \mathcal{L}(X, Y))$ and in $L^1(\mathbb{R}; \mathcal{L}(X, Z))$ such that $K(t) = \sum_{j=1}^{\infty} K_1(t + jT)$, $H(t) = \sum_{j=1}^{\infty} H_1(t + jT)$. It is well known that for $j \in \mathbb{Z}$ the Fourier transforms of K_1 and H_1 in $\dfrac{2\pi j}{T}$ are proportional to the Fourier coefficients of K and H. Therefore, even in the periodic case we shall assume K and H integrable. For $\theta \geq 0$ and $1 \leq p \leq +\infty$ we set $W^{\theta,p}_{loc}(E) := \{u : \mathbb{R} \to E | \forall \phi \in \mathcal{D}(\mathbb{R}) \quad \phi u \in W^{\theta,p}(E)\}$, where $\mathcal{D}(\mathbb{R})$ is the space of scalar C^∞ functions with compact support. We put, for $\theta \geq 0$, $1 \leq p \leq +\infty$, $T > 0$, $W^{\theta,p}_T(E) := \{u \in W^{\theta,p}_{loc}(E) : \ u \text{ is } T\text{-periodic}\}$, with norm $||u||_{\theta,p,T,E} := ||\phi_0 u||_{\theta,p,E}$, where ϕ_0 is a fixed element in $\mathcal{D}(\mathbb{R})$ such that $\phi_0(t) = 1$ if $0 \leq t \leq T$. In 3.10 we consider also the Besov spaces $B^\theta_{p,q}(F)$ and $B^\theta_{p,q,T}(F)$ with F a Banach space, $\theta > 0, 1 \leq p, q \leq +\infty, T > 0$. If $\theta = (1 - \rho)k_0 + \rho k_1$ with $\rho \in (0, 1), k_0 \in \mathbb{N}_0, k_1 \in \mathbb{N}_0, k_0 < k_1$, we set

$$B^\theta_{p,q}(F) = (W^{k_0,p}(F), W^{k_1,p}(F))_{\rho,q},$$

where we have indicated with $(.,.)_{\rho,q}$ the real interpolation functor. It can be shown that the definition is, for given θ, independent of k_0 and k_1; using the fact that the space $W^{1,p}(F)$ is the domain of the infinitesimal generator of the group of translations in $W^{0,p}(F)$, one can see that, if $\theta \notin \mathbb{Z}$, $B^\theta_{p,p}(F) = W^{\theta,p}(F)$ and, if $0 < \theta < 1$, $B^\theta_{p,\infty}(F) = \{u \in W^{0,p}(F) : \sup_{h>0} |h|^{-\theta} ||u(. + h) - u||_{0,p,E} < +\infty\}$. Analogous definitions and characterizations are true in the T-periodic case (for these facts see [2]).

Other notations that we shall use are the following: if u is a distribution with values in a Banach space E $(u \in \mathcal{D}'(\mathbb{R}; E))$ we shall indicate with $\text{supp}\,(u)$ its support; if u is tempered, we shall indicate with $\hat{u}$ or $\mathcal{F}u$ its Fourier transform; if E is a Banach space, $\mathcal{S}(\mathbb{R}; E)$ is the set of C^∞ functions from $\mathbb{R}$ to E which are rapidly decreasing at infinity with all their derivatives. If Ω is an open subset of $\mathbb{R}^n$ lying on one side of its boundary $\partial\Omega$, which is a submanifold of class C^1 of $\mathbb{R}^n$ and $x' \in \partial\Omega$, we shall indicate with $T_{x'}(\partial\Omega)$ the set of vectors in $\mathbb{R}^n$ which are tangent to $\partial\Omega$ in x', with $\nu(x')$ the unit vector normal to $\partial\Omega$ in x' pointing outside Ω. If A is a linear operator in a Banach space E, we shall indicate with $\rho(A)$ and $\sigma(A)$, respectively, its resolvent set and its spectrum. The composition of functions γ and u will be indicated (if defined) with $\gamma \circ u$ or, simply, with γu. Finally, we shall indicate with C a positive constant which may be different from time to time. To stress the fact that C depends on $\alpha, \beta \ldots$ we shall use $C(\alpha, \beta, \ldots)$.

We are now going to make precise the various notions of well-posedness of (1.2) we shall employ:

Definition 1.1. We shall say that (1.2) is (θ, p)-well posed if for any $f \in W^{\theta,p}(Y)$, for any $g \in W^{\theta,p}(Z)$ there exists a unique solution u in $W^{\theta,p}(X)$, such that $\gamma u \in W^{1+\theta,p}(Z)$.

Let $T > 0$. We shall say that (1.2) is (θ, p, T)-well posed if for any $f \in W_T^{\theta,p}(Y)$, for any $g \in W_T^{\theta,p}(Z)$ there exists a unique solution u in $W_T^{\theta,p}(X)$, such that $\gamma u \in W_T^{1+\theta,p}(Z)$.

2. Necessary conditions. The aim of this section is to determine necessary conditions in order that (1.2) be well posed. Applying formally the Fourier transform to (1.2) one has

$$\begin{cases} \mathcal{A}\hat{u}(\omega) + \hat{K}(\omega)\hat{u}(\omega) = \hat{f}(\omega), \\ i\omega\gamma\hat{u}(\omega) - \mathcal{B}\hat{u}(\omega) - \hat{H}(\omega)\hat{u}(\omega) = \hat{g}(\omega), \quad \omega \in \mathbb{R}. \end{cases}$$

So, we set, for $\omega \in \mathbb{R}$, $x \in X$,

$$\begin{aligned} &S(\omega)x := (\mathcal{A}x + \hat{K}(\omega)x, i\omega\gamma x - \mathcal{B}x - \hat{H}(\omega)x) \\ &S_0(\omega)x := (\mathcal{A}x, i\omega\gamma x - \mathcal{B}x). \end{aligned}$$

We set also, given a function u from $\mathbb{R}$ to X, $Su = (\mathcal{A}u + K * u, (\gamma u)' - \mathcal{B}u - H * u)$ whenever this expression has a meaning.

The main results of this section are 2.5 and 2.6, giving necessary conditions for the well-posedness of (1.2) in the two cases we consider. We start with some simple technical lemmata:

Lemma 2.1. *Let E and F be Banach spaces, $0 \le \theta < +\infty$, $1 \le p \le +\infty$, $T > 0$. Let $K \in L^1(\mathbb{R}; \mathcal{L}(E, F)), u \in W^{0,p}(E)(W_T^{0,p}(E))$, $h \in L^1(\mathbb{R})$. Then*

$$h * (K * u) = K * (h * u).$$

Proof. We consider the nonperiodic case; from Hausdorff-Young theorem, we have that $|h(\cdot)| * (||K(\cdot)||_{\mathcal{L}(E,F)} * ||u(\cdot)||_E) \in L^p$. So, for almost every $t \in \mathbb{R}$ $(s, \sigma) \to |h(t-s)|||K(s-\sigma)||_{\mathcal{L}(E,F))}||u(\sigma)||_E \in L^1(\mathbb{R}^2; \mathbb{C})$.

It follows from Fubini's theorem that for almost every t

$$\begin{aligned} [h * (K * u)](t) &= \int_{\mathbb{R}} h(t-s)(\int_{\mathbb{R}} K(s-\sigma)u(\sigma)\, d\sigma)\, ds \\ &= \int_{\mathbb{R}} K(t-\sigma)(\int_{\mathbb{R}} h(\sigma - s)u(s)\, ds)\, d\sigma = [K * (h * u)](t). \end{aligned}$$

The periodic case can be treated analogously, remarking that $|h(\cdot)| * (||K(\cdot)||_{\mathcal{L}(E,F)} * ||u(\cdot)||_E) \in L^p([0,T]); \mathbb{C})$, so that, as the convolution is T-periodic, again for almost every t $(s, \sigma) \to |h(t-s)|||K(s-\sigma)||_{\mathcal{L}(E,F))}||u(\sigma)||_E \in L^1(\mathbb{R}^2; \mathbb{C})$.

Lemma 2.2. *Assume that, for some $(\theta, p) \in [0, +\infty) \times [1, +\infty]$, (1.2) is (θ, p)-well posed. Then for every $k \in \mathbb{N}$ (1.2) is $(\theta + k, p)$-well posed. An analogous result is true in the periodic case. Moreover, for $\omega \in \mathbb{R}$, put $K_\omega(t) := e^{-i\omega t} K(t)$, $H_\omega(t) := e^{-i\omega t} H(t)$. Then, the problem*

$$\begin{cases} \mathcal{A}u(t) + (K_\omega * u)(t) = f(t), \quad t \in \mathbb{R} \\ (\gamma u)'(t) + i\omega\gamma u(t) - \mathcal{B}u(t) - (H_\omega * u)(t) = g(t), \quad t \in \mathbb{R} \end{cases} \tag{2.1}$$

is (θ, p)-well posed. In the periodic case an analogous result is true for $\omega = \frac{2k\pi}{T}$, for every $k \in \mathbb{Z}$.

Proof. Concerning the first statement, it clearly suffices to prove the result for $k = 1$.

Let $f \in W^{\theta+1,p}(Y)$, $g \in W^{\theta+1,p}(Z)$. The uniqueness of the solution $u \in W^{\theta+1,p}(X)$ such that $\gamma u \in W^{\theta+2,p}(Z)$ follows immediately from the fact that (1.2) is (θ, p)-well posed. To prove the existence, let v be the solution in $W^{\theta,p}(X)$ such that $\gamma v \in W^{\theta+1,p}(Z)$ with data $\phi := f + f'$, $\psi := g + g'$. Set $\rho(t) := e^{-t}\theta(t)$, where θ is the Heaviside function. Remark that $\hat{\rho}(\omega) = (1 + i\omega)^{-1}$. Put $u := \rho * v$. We have from 2.1

$$f = \rho * \phi = \rho * (\mathcal{A}v + K * v) = \mathcal{A}u + K * u.$$

Analogously, $g = (\gamma u)' - \mathcal{B}u - H * u$, and from the expression of $\hat{\rho}$ one can see that u has the desired regularity.

The same proof works also in the periodic case.

For what concerns the well-posedness of (2.1), it suffices to introduce the new unknown function $v(t) := e^{i\omega t}u(t)$ to come back to the original problem.

In the periodic case, one has to observe that $e^{i\omega t}$ is a multiplier for $W_T^{\theta,p}(E)$ if and only if $\omega = \frac{2k\pi}{T}$, with $k \in \mathbb{Z}$.

Remark 2.3. From the previous result and proof one can see that, if $f \in W^{\theta+1,p}(Y)$, $g \in W^{\theta+1,p}(Z)$ and u is the solution with data f and g, the solution with data f' and g' is u'. An analogous remark is true in the periodic case.

In the next lemma, we collect a certain number of estimates which will be crucial to prove the main results of this section:

Lemma 2.4. *Let $h \in \mathcal{D}(\mathbb{R})$, $h \not\equiv 0$, E a Banach space, $k_j (j \in \mathbb{N}), k \in W^{0,1}(E)$, $k_j \to k (j \to +\infty)$ in $W^{0,1}(E)$. Then*

(I) $\forall \theta \in [0, +\infty)$, $p \in [1, +\infty]$,

$$0 < \liminf_{|\xi| \to \infty} |\xi|^{-\theta} ||e^{i\xi \cdot} h(\cdot)||_{\theta,p} \le \limsup_{|\xi| \to \infty} |\xi|^{-\theta} ||e^{i\xi \cdot} h(\cdot)||_{\theta,p} < +\infty;$$

(II) $||e^{i\xi \cdot} h(\cdot) * k||_{\theta,p,E} = o(|\xi|^\theta)(|\xi| \to +\infty)$;

(III) *set*

$$\zeta_j(t) := \int_{\mathbb{R}} [h(\frac{s}{j}) - h(\frac{t}{j})] k_j(t - s)\, ds;$$

then, $\forall p \in [1, +\infty], \theta \geq 0$

$$||\zeta_j||_{\theta,p,E} = o(j^{1/p})(j \to +\infty);$$

(IV) *let* $\theta \geq 0$; *there exist* C_1 *and* C_2 *strictly positive, such that for any* $\omega \in \mathbb{R}$

$$C_1(1 + |\omega|)^\theta \leq ||e^{i\omega t}||_{\theta,\infty} \leq C_2(1 + |\omega|)^\theta.$$

Proof. (I) The case $0 < \theta < 1$ was proved in [10], Lemma 2.11. For $\theta \geq 1$, assume $\theta = r + \alpha$, with $r \in \mathbb{N}, \alpha \in [0, 1)$. We have, for some $C > 0$,

$$\begin{aligned} ||e^{i\xi\cdot} h(\cdot)||_{\theta,p} &\leq \sum_{j=0}^{r} || \sum_{k=0}^{j} \binom{j}{k} (i\xi)^k e^{i\xi\cdot} h^{(j-k)}(\cdot)||_{\alpha,p} \\ &\leq C \sum_{j=0}^{r} \sum_{k=0}^{j} \binom{j}{k} |\xi|^k ||e^{i\xi\cdot} h^{(j-k)}||_{\alpha,p} \end{aligned}$$

so that $\limsup_{|\xi|\to\infty} |\xi|^{-\theta}||e^{i\xi\cdot}h(\cdot)||_{\theta,p} < +\infty$. On the other hand,

$$\begin{aligned} ||e^{i\xi\cdot} h(\cdot)||_{\theta,p} &\geq || \sum_{k=0}^{r} \binom{r}{k} (i\xi)^k e^{i\xi\cdot} h^{(r-k)}(\cdot)||_{\alpha,p} \\ &\geq |\xi|^r ||e^{i\xi\cdot} h||_{\alpha,p} - \sum_{k=0}^{r-1} \binom{r}{k} |\xi|^k ||e^{i\xi\cdot} h^{(r-k)}||_{\alpha,p} \end{aligned}$$

implying, using the first part of the proof,

$$\liminf_{|\xi|\to\infty} |\xi|^{-\theta}||e^{i\xi\cdot}h(\cdot)||_{\theta,p} \geq \liminf_{|\xi|\to\infty} |\xi|^{-\alpha}||e^{i\xi\cdot}h(\cdot)||_{\alpha,p}.$$

Consider now (II); we have

$$||e^{i\xi\cdot}h(\cdot) * k||_{\theta,p,E} \leq ||e^{i\xi\cdot}h(\cdot)||_{\theta,p}||k||_{0,1,E}$$

so that, from (I),

$$\limsup_{|\xi|\to\infty} |\xi|^{-\theta}||e^{i\xi\cdot}h(\cdot) * k||_{\theta,p,E} \leq C||k||_{0,1,E} \quad (C > 0).$$

Take now $k \in W^{1,1}(E)$. Integrating by parts,

$$[e^{i\xi\cdot}h(\cdot)] * k = (1 + i\xi)^{-1}[(e^{i\xi\cdot}h(\cdot)) * (k + k') - (e^{i\xi\cdot}h') * k]$$

implying $|\xi|^{-\theta}||[e^{i\xi\cdot}h(\cdot)]*k||_{\theta,p} = O(|\xi|^{-1})(|\xi|\to+\infty)$. The result follows from the density of $W^{1,1}(E)$ in $W^{0,1}(E)$.

We show (III); we have for any $l \in \mathbb{N}$

$$\zeta_j^{(l)}(t) = j^{-l}\int_{\mathbb{R}}[h^{(l)}(\frac{s}{j}) - h^{(l)}(\frac{t}{j})]k_j(t-s)\,ds;$$

it follows that, choosing $m \in \mathbb{N}$, $m \geq \theta$, one has, for some $C > 0$,

$$||\zeta_j||_{\theta,p,E} \leq C||\zeta_j||_{m,p,E} = C\sum_{l=0}^{m} j^{-l}||\int_{\mathbb{R}}[h^{(l)}(\frac{s}{j}) - h^{(l)}(\frac{\cdot}{j})]k_j(\cdot - s)\,ds||_{0,p,E}$$

implying that it suffices to prove the result in case $\theta = 0$. Now, we have

$$\zeta_j(t) = [h(\frac{\cdot}{j})*(k_j - k)](t) - h(\frac{t}{j})\int_{\mathbb{R}}[k_j(s) - k(s)]ds + \int_{\mathbb{R}}[h(\frac{t-s}{j}) - h(\frac{t}{j})]k(s)ds.$$

$$\begin{aligned}&||h(\frac{\cdot}{j})*(k_j-k)||_{0,p,E} \leq ||h(\frac{\cdot}{j})||_{0,p}||k_j-k||_{0,1,E}\\&= j^{1/p}||h||_{0,p}||k_j-k||_{0,1,E} = o(j^{1/p}) \quad (j\to+\infty).\end{aligned}$$

Next,

$$\begin{aligned}&||h(\frac{\cdot}{j})\int_{\mathbb{R}}[k_j(s)-k(s)]\,ds||_{0,p,E}\\&= j^{1/p}||h||_{0,p}||\int_{\mathbb{R}}[k_j(s)-k(s)]\,ds||_E = o(j^{1/p}) \quad (j\to+\infty).\end{aligned}$$

Finally, consider first the case $p = 1$; fix $\eta > 0$; there exists $\delta(\eta) > 0$ such that, if $|\sigma-\tau| \leq \delta(\eta)|$, $|h(\sigma)-h(\tau)| \leq \eta$. Assume also $h(t) = 0$ if $|t| \geq M$. We have

$$||\int_{\mathbb{R}}[h(\frac{\cdot - s}{j}) - h(\frac{\cdot}{j})]k(s)\,ds||_{0,1,E} \leq j^2\int_{\mathbb{R}^2}|h(s)-h(t)|||k(j(t-s))||_E\,ds\,dt.$$

We have

$$\begin{aligned}&j^2\int_{\mathbb{R}}(\int_{|t-s|\leq\delta(\eta)}|h(s)-h(t)|||k(j(t-s))||_E\,ds)\,dt\\&\leq j^2\eta\int_{|t|\wedge|s|\leq M}||k(j(t-s))||_E\,ds\,dt\\&= \eta j^2(\int_{\mathbb{R}}(\int_{|s|\leq M}||k(j(t-s))||_E\,ds)\,dt + \int_{-M}^{M}(\int_{|s|\geq M}||k(j(t-s))||_E\,ds)\,dt)\\&\leq 4M\eta j||k||_{0,1,E}.\end{aligned}$$

Next,

$$j^2 \int_{\mathbb{R}} (\int_{|t-s|>\delta(\eta)} |h(s) - h(t)| ||k(j(t-s))||_E \, ds) dt$$
$$= j^2 \int_{\mathbb{R}} (\int_{|t|>\delta(\eta)} |h(s) - h(s+t)| ||k(jt)||_E \, dt) \, ds$$

$$\leq j^2 \int_{\mathbb{R}} (\int_{|t|>\delta(\eta)} |h(s)| ||k(jt)||_E dt) \, ds + j^2 \int_{\mathbb{R}} (\int_{|t|>\delta(\eta)} |h(s+t)| ||k(jt)||_E dt) ds$$
$$\leq 2j ||h||_{0,1} \int_{|t|>j\delta(\eta)} ||k(t)||_E dt.$$

So the result is proved in case $p = 1$. We consider now the case $p = +\infty$. For $t \in \mathbb{R}$

$$|| \int_{\mathbb{R}} [h(\frac{t-s}{j}) - h(\frac{t}{j})] k(s) \, ds ||_E = j || \int_{\mathbb{R}} [h(\frac{t}{j} - s) - h(\frac{t}{j})] k(js) \, ds ||_E.$$

We have $\forall \eta > 0$

$$j || \int_{\mathbb{R}} [h(t-s) - h(t)] k(js) \, ds ||_E$$
$$\leq j\eta \int_{|s|\leq\delta(\eta)} ||k(js)||_E \, ds + 2j ||h||_{0,\infty} \int_{|s|>\delta(\eta)} ||k(js)||_E \, ds$$
$$\leq \eta ||k||_{0,1,E} + 2||h||_{0,\infty} \int_{|s|>j\delta(\eta)} ||k(s)||_E \, ds.$$

So the result is proved also in case $p = +\infty$. The case $1 < p < +\infty$ follows from

$$||\zeta_j||_{0,p,E} \leq ||\zeta_j||_{0,1,E}^{1/p} ||\zeta_j||_{0,\infty,E}^{1-1/p}.$$

(IV) is straightforward.

We are now going to state and prove the main result of this section:

Theorem 2.5. *Assume that, for some $\theta \geq 0$, $p \in [1, +\infty]$, (1.2) is (θ, p)-well posed. Then the following conditions* (C) *and* (C′) *are satisfied:*

$$\forall \omega \in \mathbb{R} \;\; S(\omega) \textit{ is a bijection between } X \textit{ and } Y \times Z, \textit{ and}$$
$$\sup_{\omega \in \mathbb{R}} \{ ||S(\omega)^{-1}||_{\mathcal{L}(Y\times Z;X)} + |\omega| ||\gamma S(\omega)^{-1}||_{\mathcal{L}(Y\times Z;Z)} \} < +\infty. \tag{C}$$

and

$$\exists \omega_0 \in \mathbb{R}^+, \textit{ such that, if } |\omega| \geq \omega_0, S_0(\omega) \textit{ is a bijection}$$
$$\textit{between } X \textit{ and } Y \times Z \textit{ and}$$
$$\sup_{|\omega|\geq\omega_0} \{ ||S_0(\omega)^{-1}||_{\mathcal{L}(Y\times Z;X)} + |\omega| ||\gamma S_0(\omega)^{-1}||_{\mathcal{L}(Y\times Z;Z)} \} < +\infty. \tag{C′}$$

Proof. We prove the result in three steps:

First step: $\exists \omega_0 > 0, M > 0$ such that if $|\omega| \geq \omega_0$, $\forall x \in X$

$$||x||_X + |\omega|||\gamma x||_Z \leq M||S(\omega)x||_{Y\times Z},\ ||x||_X + |\omega|||\gamma x||_Z \leq M||S_0(\omega)x||_{Y\times Z}.$$

To prove this fact, for $x \in X, \omega \in \mathbb{R}$, we set

$$u_\omega(t) := e^{i\omega t}h(t)x,$$

with $h \in \mathcal{D}(\mathbb{R})$, $h(t) = 1$ if $|t| \leq 1$. If (1.2) is (θ, p)-well posed there exists $M_1 > 0$ such that for any $u \in W^{\theta,p}(X)$ such that $\gamma u \in W^{1+\theta,p}(Z)$

$$||u||_{\theta,p,X} + ||(\gamma u)'||_{\theta,p,Z} \leq M_1||Su||_{\theta,p,Y\times Z}. \tag{2.2}$$

We have, by 2.4(I), that, if $|\omega|$ is large enough, for certain $M_2 > 0, M_3 > 0$, $||u_\omega||_{\theta,p,X} \geq M_2|\omega|^\theta||x||_X$, $||(\gamma u_\omega)'||_{\theta,p,Z} = ||i\omega e^{i\omega\cdot}h(\cdot)\gamma x + e^{i\omega\cdot}h'(\cdot)\gamma x||_{\theta,p,Z} \geq (M_2|\omega|^{\theta+1} - M_3|\omega|^\theta)||\gamma x||_Z$; moreover,

$$Su_\omega = e^{i\omega\cdot}h(\cdot)S_0(\omega)x + ((e^{i\omega\cdot}h(\cdot)) * K(\cdot)x, e^{i\omega\cdot}h'(\cdot)\gamma x - (e^{i\omega\cdot}h(\cdot)) * H(\cdot)x).$$

It follows from 2.4 (I)–(II) that for some $M_4 > 0$,

$$||Su_\omega||_{\theta,p,Y\times Z} \leq M_4|\omega|^\theta(||S_0(\omega)x||_{Y\times Z} + ||\gamma x||_Z) + \mu(\omega)||x||_X,$$

with $\mu(\omega) = o(|\omega|^\theta)(|\omega| \to +\infty)$. Therefore the estimate is proved in the case of $S_0(\omega)$. The estimate with $S(\omega)$ can be obtained from the estimate of $S_0(\omega)$ just remarking that $||\hat{K}(\omega)||_{\mathcal{L}(X,Y)} + ||\hat{H}(\omega)||_{\mathcal{L}(X,Z)} \to 0$ ($|\omega| \to +\infty$) through a simple perturbation argument.

Second step: for $|\omega| \geq \omega_0$, $S(\omega)$ and $S_0(\omega)$ are onto $Y \times Z$.

Fix $\omega \in \mathbb{R}$, with $|\omega| \geq \omega_0$. Then, by 2.2, problem (2.1) is (θ, p)-well posed. Fix $(y, z) \in Y\times Z$ and put, for $j \in \mathbb{N}$, $f_j(t) := h(\frac{t}{j})y$, $g_j(t) := h(\frac{t}{j})z$, with h as in the first step. We indicate with u_j the solution of (2.1) with data f_j and g_j. As for any $k \in \mathbb{N}$ $f_j^{(k)} \in W^{\theta,p}(Y)$ and $g_j^{(k)} \in W^{\theta,p}(Z)$, by 2.3 $\forall k$ $u_j^{(k)} \in W^{\theta,p}(X)$ and is the solution with data $f_j^{(k)}, g_j^{(k)}$. Remark now that, for $k \geq 2$, $||f_j^{(k)}||_{\theta,p,Y} + ||g_j^{(k)}||_{\theta,p,Z} \to 0$ ($j \to +\infty$), which implies that $||u_j^{(k)}||_{\theta,p,X} \to 0$ ($j \to +\infty$) for $k \geq 2$ (for $p > 1$ this happens also for $k = 1$). By the Sobolev embedding theorem, for $k \geq 2$ $||u_j^{(k)}||_{0,\infty,X} \to 0$ ($j \to +\infty$). As $f_j'(0) = 0$ and $g_j'(0) = 0$, we have

$$S(\omega)u_j'(0) = (-\int_{\mathbb{R}} e^{i\omega s}K(-s)[u_j'(s) - u_j'(0)]\,ds,$$
$$\int_{\mathbb{R}} e^{i\omega s}H(-s)[u_j'(s) - u_j'(0)]\,ds - \gamma u_j^{(2)}(0)).$$

Assume for a moment that K and H have compact support, say $K(t) = 0$ and $H(t) = 0$ if $|t| > r$. As $||u_j^{(2)}||_{0,\infty,X} \to 0$ as $j \to +\infty$, it is easily seen that

$$||\int_{\mathbb{R}} e^{i\omega s} K(-s)[u_j'(s) - u_j'(0)]\, ds||_Y + ||\int_{\mathbb{R}} e^{i\omega s} H(-s)[u_j'(s) - u_j'(0)]\, ds)||_Z \to 0$$

as $j \to +\infty$; that is, $||S(\omega)u_j'(0)||_{Y\times Z} \to 0$ as $j \to \infty$. As $|\omega| \geq \omega_0$ (see the first step), we have that $||u_j'(0)||_X \to 0$ as $j \to \infty$. From this we have that $||u_j'(t)||_X \to 0$ as $j \to \infty$ uniformly on bounded subsets of $\mathbb{R}$; from

$$\begin{aligned} S(\omega)u_j(0) = (y - \int_{\mathbb{R}} e^{i\omega s} K(-s)[u_j(s) - u_j(0)]\, ds, z \\ + \int_{\mathbb{R}} e^{i\omega s} H(-s)[u_j(s) - u_j(0)]\, ds - \gamma u_j'(0)) \end{aligned}$$

we have that $||S(\omega)u_j(0) - (y, z)||_{Y\times Z} \to 0$ as $j \to \infty$. So the range of $S(\omega)$ is dense in $Y\times Z$. From the estimate proved in the first step, we have that $S(\omega)$ is onto $Y\times Z$, at least when K and H have a compact support. In general, consider two sequences $(K_r)_{r\in\mathbb{N}}$ and $(H_r)_{r\in\mathbb{N}}$ respectively in $L^1(\mathbb{R}; \mathcal{L}(X, Y))$ and in $L^1(\mathbb{R}; \mathcal{L}(X, Z))$, with K_r and H_r compactly supported for every r and converging respectively to K and H. With a simple perturbation argument, it is easily seen that, if (1.2) is well posed, the problem obtained replacing K with K_r and H with H_r for r sufficiently large is well posed also. It is also easily seen that, for r large enough there exists $S_r(\omega)^{-1}$ ($S_r(\omega)$ is the correspondent of $S(\omega)$ in the perturbed problem) and the operators $(S_r(\omega)^{-1})_{r\geq r_0}$ are equibounded in $\mathcal{L}(Y \times Z; X)$. Using these facts it is not difficult to show that, if $S_r(\omega)x_r = (y, z)$, the sequence $(x_r)_{r\in\mathbb{N}}$ converges in X to an element x such that $S(\omega)x = (y, z)$.

The surjectivity of $S_0(\omega)$ follows by perturbation from that of $S(\omega)$ if $|\omega|$ is "large," and from the continuity method (using the estimate in the first step) afterwards.

Third step: for any $\omega \in \mathbb{R}$, $S(\omega)$ is an isomorphism between X and $Y \times Z$.

We put $\Omega := \{\omega \in \mathbb{R} : S(\omega)$ is an isomorphism between X and $Y \times Z\}$. We already know that $\mathbb{R} - \Omega$ is bounded and it is not difficult to see that it is even closed. Therefore, if it is not empty, it has a maximum, say ν. It is not difficult to see that necessarily $\lim_{\omega\to\nu^+} ||S(\omega)^{-1}||_{\mathcal{L}(Y\times Z;X)} = +\infty$. So, by the Banach-Steinhaus theorem, there exist $(y, z) \in Y \times Z$ and a sequence $(\omega_j)_{j\in\mathbb{N}}$ with $\omega_j > \nu$ for any j and $\lim_{j\to+\infty} \omega_j = \nu$ such that $||S(\omega_j)^{-1}(y, z)||_X \to +\infty$ as $j \to +\infty$. We set $x_j := S(\omega_j)^{-1}(y, z)$ and $u_j(t) := e^{i\omega_j t} h(\frac{t}{j})x_j$, where h is the function that we have already used in the two first steps. We employ again inequality (2.2) with $u = u_j$. We have $||u_j||_{\theta,p,X} \geq ||u_j||_{0,p,X} = j^{1/p}||h||_{0,p}||x_j||_X$. Moreover,

$$\mathcal{A}u_j(t) + (K * u_j)(t) = h(\frac{t}{j})e^{i\omega_j t} y + e^{i\omega_j t}(\int_{\mathbb{R}} [h(\frac{s}{j}) - h(\frac{t}{j})]e^{i\omega_j (s-t)} K(t - s)\, ds)x_j,$$

$$(\gamma u_j)'(t) - \mathcal{B}u_j(t) - (H * u_j)(t) = h(\frac{t}{j})e^{i\omega_j t}z + j^{-1}e^{i\omega_j t}h'(\frac{t}{j})\gamma x_j$$
$$- e^{i\omega_j t}(\int_{\mathbb{R}}[h(\frac{s}{j}) - h(\frac{t}{j})]e^{i\omega_j(s-t)}H(t-s)\,ds)x_j,$$

so that, by 2.4 (III), for some $C > 0$,

$$||Su_j||_{\theta,p,Y\times Z} \le Cj^{1/p}(||y||_Y + ||z||_Z) + o(j^{1/p})||x_j||_X, \quad (j \to +\infty).$$

So, for j sufficiently large, $||x_j||_X \le 2C(||y||_Y + ||z||_Z)$, in contradiction with $\lim_{j\to+\infty} ||x_j||_X = +\infty$.

Theorem 2.6. *Assume that, for some $\theta \ge 0$, $p \in [1, +\infty]$, $T > 0$ (1.2) is (θ, p, T)-well posed. Then,*

$$\begin{aligned}&\forall j \in \mathbb{Z}\ \ S(\frac{2\pi j}{T})\ \text{is a bijection between}\ \ X\ \text{and}\ Y \times Z\ \text{and}\\ &\sup_{j\in\mathbb{Z}}\{||S(\frac{2\pi j}{T})^{-1}||_{\mathcal{L}(Y\times Z;X)} + |j|||\gamma S(\frac{2\pi j}{T})^{-1}||_{\mathcal{L}(Y\times Z;Z)}\} < +\infty;\end{aligned} \tag{C$_T$}$$

moreover, condition (C′) *is satisfied.*

Proof. We start by showing that, if (1.2) is (θ, p, T)-well posed, $S(0)$ is surjective: let $(y, z) \in Y \times Z$, $f(t) = y$, $g(t) = z$ for any $t \in \mathbb{R}$, and let u be the solution of (1.2) in $W_T^{\theta,p}(X)$ such that $\gamma u \in W_T^{\theta+1,p}(Z)$; by Remark 2.3, u' is the solution with zero data and so $u'(t) = 0$ for any $t \in \mathbb{R}$. This means that $u(t) = x$ for some $x \in X$ (constant function) and one can easily verify that $S(0)x = (y, z)$. On the other hand, if $x \in X$ and $S(0)x = (0, 0)$, the constant function $u(t) \equiv x$ solves (1.2) with $f(t) \equiv 0$ and $g(t) \equiv 0$; this implies that $S(0)$ must be injective. The same argument can be used to show that for every $j \in \mathbb{Z}$ $S(\frac{2\pi j}{T})$ is onto $Y \times Z$, using the second statement of Lemma 2.2, with $\omega = \frac{2\pi j}{T}$. Next, for $\omega \in \mathbb{Z}$, $x \in X$, set $u_\omega(t) = \exp(\frac{2\pi\omega i t}{T})x$; arguing as in the first step of 2.5 one gets (C$_T$).

The fact that condition (C′) is necessary even in the periodic case can be seen with the following perturbation argument: let $\omega \in \mathbb{R}$. Let $j(\omega) \in \mathbb{Z}$ be such that $\frac{2j(\omega)\pi}{T} \le \omega < \frac{2(j(\omega)+1)\pi}{T}$. We have for any $x \in X$, for some $C > 0$, using (C$_T$),

$$\begin{aligned}||x||_X + |\omega|||\gamma x||_Z &\le C||S(\frac{2j(\omega)\pi}{T})x||_{Y\times Z} + (\omega - \frac{2j(\omega)\pi}{T})||\gamma x||_Z\\ &\le C(||S(\omega)x||_{Y\times Z} + ||[S(\frac{2j(\omega)\pi}{T}) - S(\omega)]x||_{Y\times Z} + (\omega - \frac{2j(\omega)\pi}{T})||\gamma x||_Z)\\ &\le C(||S(\omega)x||_{Y\times Z} + \mu(\omega)||x||_X + (\omega - \frac{2j(\omega)\pi}{T})||\gamma x||_Z)\end{aligned}$$

with $\mu(\omega) = o(1)$ for $|\omega| \to +\infty$. From this estimate we have $||x||_X + |\omega|||\gamma x||_Z \le 2C||S(\omega)x||_{Y\times Z}$, for $|\omega|$ large enough. The surjectivity of $S(\omega)$ can be proved using the continuation method. From this, again by perturbation, one can obtain the analogous result for $S_0(\omega)$.

3. Sufficient conditions. In this section we look for sufficient conditions assuring that (1.2) is well posed. We show that, if condition (C) is satisfied, we have that the solution is unique and in the T-periodic case this is true just assuming (C_T) (3.3 and 3.6). If we perturb a well-posed problem in such a way that (C) is still satisfied ((C_T) in the T-periodic case), the new problem is well posed (3.4 and 3.7). Next, if $\theta \notin \mathbb{Z}$ condition (C) is also sufficient for the well-posedness ((C_T) in the T-periodic case) (see 3.12 and 3.17). If $\theta \in \mathbb{Z}$, we show that, if (1.2) is $(0,p)$-well posed for some $p \in [1,+\infty)$, it is $(0,q)$-well posed for any $q \in (1,+\infty)$ (see 3.14). An easy consequence of this is that, in case X, Y and Z are Hilbert spaces, condition (C) is also sufficient for any $\theta \geq 0$, $p \in (1,+\infty)$ (see 3.15).

We start with

Lemma 3.1. *Assume that condition* (C) *is satisfied. Let* $\hat{\chi} \in \mathcal{D}(\mathbb{R})$. *Then,*

$$\mathcal{F}^{-1}(\hat{\chi}(\cdot)S(\cdot)^{-1}) \in W^{\infty,1}(\mathcal{L}(Y \times Z; X)).$$

Proof. The result is obvious if $K \in \mathcal{S}(\mathbb{R}; \mathcal{L}(X,Y))$ and $H \in \mathcal{S}(\mathbb{R}; \mathcal{L}(X,Z))$, because in such a case $S(\cdot)^{-1} \in C^{\infty}(\mathbb{R}; \mathcal{L}(Y \times Z; X))$. In general pick two sequences $(K_j)_{j\in\mathbb{N}}$ and $(H_j)_{j\in\mathbb{N}}$ in $\mathcal{S}(\mathbb{R}; \mathcal{L}(X,Y))$ and in $\mathcal{S}(\mathbb{R}; \mathcal{L}(X,Z))$ converging respectively to K in $L^1(\mathbb{R}; \mathcal{L}(X,Y))$ and to H in $L^1(\mathbb{R}; \mathcal{L}(X,Z))$. It is easily seen that there exists $j_o \in \mathbb{N}$ such that for $j \geq j_o$ (C) is satisfied if we replace K with K_j and H with H_j. Moreover, $||S_j(\omega)^{-1}||_{\mathcal{L}(Y\times Z;X)} \leq C$ with C independent of j. So there exists $j_0 \in \mathbb{N}$ such that $||[S(\omega) - S_j(\omega)]S_j(\omega)^{-1}||_{\mathcal{L}(Y\times Z)} < 1$ for any $j \geq j_0$. We have, if we fix $j \geq j_0$, for any $\omega \in \mathbb{R}$,

$$S(\omega)^{-1} = S_j(\omega)^{-1}\{I_{Y\times Z} + [S(\omega) - S_j(\omega)]S_j(\omega)^{-1}\}^{-1}$$

so that

$$\hat{\chi}(\omega)S(\omega)^{-1} = \hat{\chi}(\omega)S_j(\omega)^{-1}\{I_{Y\times Z} + [S(\omega) - S_j(\omega)]\psi(\omega)S_j(\omega)^{-1}\}^{-1}$$

if $\psi \in \mathcal{D}(\mathbb{R})$ is such that $\psi(\omega) = 1$ for $\omega \in \operatorname{supp}(\hat{\chi})$ and $|\psi(\omega)| \leq 1$ for any $\omega \in \mathbb{R}$. Set, for $t \in \mathbb{R}, x \in X$, $R_j(t) = (K(t)x - K_j(t)x, H_j(t)x - H(t)x)$. Then,

$$[S(\omega) - S_j(\omega)]\psi(\omega)S_j(\omega)^{-1} = \hat{T}_j(\omega)$$

with $T_j := R_j * \mathcal{F}^{-1}(\psi S_j^{-1}) \in L^1(\mathbb{R}; \mathcal{L}(Y \times Z))$. For any $\omega \in \mathbb{R}$ $I_{Y\times Z} + \hat{T}_j(\omega)$ is an isomorphism of $Y \times Z$ onto itself. It follows from Paley-Wiener's theorem that

$$[I_{Y\times Z} + \hat{T}_j(\omega)]^{-1} = I_{Y\times Z} + \hat{U}_j(\omega)$$

with $U_j \in L^1(\mathbb{R}; \mathcal{L}(Y \times Z))$. So,

$$\begin{aligned}\mathcal{F}^{-1}(\hat{\chi}(\cdot)S(\cdot)^{-1}) &= \mathcal{F}^{-1}(\hat{\chi}(\cdot)S_j(\cdot)^{-1}) + \mathcal{F}^{-1}(\hat{\chi}(\cdot)S_j(\cdot)^{-1}) * U_j \\ &\in W^{\infty,1}(\mathcal{L}(Y \times Z; X)).\end{aligned}$$

An immediate consequence of 3.1 is the following:

Corollary 3.2. *Assume that* (C) *is satisfied,* $\chi \in \mathcal{S}(\mathbb{R})$ *such that* $\hat{\chi} \in \mathcal{D}(\mathbb{R})$*. Then, there exist* $R_1 \in W^{\infty,1}(\mathcal{L}(Y,X))$*,* $R_2 \in W^{\infty,1}(\mathcal{L}(Z,X))$ *such that for any* $u \in W^{0,p}(X)$ *such that* $\gamma u \in W^{1,p}(Z)$ *for some* $p \in [1,+\infty]$*, if* $Su = (f,g)$*,*

$$\chi * u = R_1 * f + R_2 * g.$$

Proposition 3.3. *Assume that condition* (C) *is satisfied. Then, for any* $p \in [1,+\infty]$ *problem* (1.2) *has at most one solution in* $W^{0,p}(E)$ *such that* $\gamma u \in W^{1,p}(Z)$ *for any* $f \in W^{0,p}(Y)$*,* $g \in W^{0,p}(Z)$*.*

Proof. It follows immediately from 3.2.

Proposition 3.4. *Assume that for certain* $\theta \geq 0$*,* $1 \leq p \leq +\infty$*,* (1.2) *is* (θ,p)*-well posed; then substituting for* K K_1 $(K_1 \in L^1(\mathbb{R};\mathcal{L}(X,Y)))$ *and for* H H_1 $(H_1 \in L^1(\mathbb{R};\mathcal{L}(X,Z)))$ *such that condition* (C) *is still satisfied, the new problem is* (θ,p)*-well posed.*

Proof. Let $\chi \in \mathcal{S}(\mathbb{R})$ be such that $\hat{\chi} \in \mathcal{D}(\mathbb{R})$, $\hat{\chi}(\omega) = 1$ if $|\omega| \leq 1$, $\hat{\chi}(\omega) = 0$ if $|\omega| \geq 2$ and put $\chi_k(t) := \mathcal{F}^{-1}(\hat{\chi}(\frac{\cdot}{k}))(t)$. Set

$$K_o := \chi_k * (K - K_1) + K_1, \quad H_o := \chi_k * (H - H_1) + H_1 \tag{3.1}$$

where k is chosen so large that problem (1.2) with K_o instead of K and H_o instead of H is well posed (using the fact that $\chi_k * (K - K_1) + K_1 \to K$ and $\chi_k * (H - H_1) + H_1 \to H$ respectively in $L^1(\mathbb{R};\mathcal{L}(X,Y))$ and in $L^1(\mathbb{R};\mathcal{L}(X,Z))$ as $k \to +\infty$); remark that, for $|\omega| \geq 2k$, $\hat{K}_o(\omega) = \hat{K}_1(\omega)$, $\hat{H}_o(\omega) = \hat{H}_1(\omega)$. Given $\phi \in W^{\theta,p}(Y)$, $\psi \in W^{\theta,p}(Z)$ indicate with $S(\phi,\psi)$ the solution u in $W^{\theta,p}(X)$ such that $\gamma u \in W^{1+\theta,p}(Z)$ of

$$\begin{cases} \mathcal{A}u(t) + (K_o * u)(t) = \phi(t), \quad t \in \mathbb{R} \\ (\gamma u)'(t) - \mathcal{B}u(t) - (H_o * u)(t) = \psi(t), \quad t \in \mathbb{R}. \end{cases} \tag{3.2}$$

We look for a solution of the perturbed problem (that is, with K_1 instead of K and H_1 instead of H) in the form $u = S(\phi,\psi)$. It is easily seen that ϕ and ψ must solve the following system:

$$\begin{cases} \phi + (K_1 - K_o) * S(\phi,\psi) = f, \\ \psi - (H_1 - H_o) * S(\phi,\psi) = g. \end{cases} \tag{3.3}$$

Put now $\rho := \chi_{2k}$; then, observe that, by 2.1, 3.1 and 3.2,

$$\begin{aligned} (K_1 - K_o) * S(\phi,\psi) &= (K_1 - K_o) * [\rho * S(\phi,\psi)] \\ &= (K_1 - K_o) * [R_1 * \phi + R_2 * \psi], \\ (H_1 - H_o) * S(\phi,\psi) &= (H_1 - H_o) * [\rho * S(\phi,\psi)] \\ &= (H_1 - H_o) * [R_1 * \phi + R_2 * \psi] \end{aligned}$$

for certain $R_1 \in W^{\infty,1}(\mathcal{L}(Y,X))$, $R_2 \in W^{\infty,1}(\mathcal{L}(Z,X))$. So (3.3) is a convolution system in $W^{\theta,p}(Y \times Z)$ with an integrable kernel. Applying the assumptions, one can verify that for any $\omega \in \mathbb{R}$ the operator

$$(y,z) \to (y + (\hat{K}_1(\omega) - \hat{K}_o(\omega)))[\hat{R}_1(\omega)y + \hat{R}_2(\omega)z],$$

$$z - (\hat{H}_1(\omega) - \hat{H}_o(\omega))[\hat{R}_1(\omega)y + \hat{R}_2(\omega)z])$$

is an isomorphism of $Y \times Z$ onto itself; so, by Paley-Wiener's theorem, (3.3) has a unique solution in $W^{\theta,p}(Y \times Z)$, which is of the form $(\phi,\psi) = (f,g) + U * (f,g)$ with $U \in L^1(\mathbb{R}; \mathcal{L}(Y \times Z))$. This gives the result in the (θ,p)-case.

We are now going to extend the foregoing results to the periodic case; indicate with Ω the set of real numbers ω such that $S(\omega)$ is an isomorphism of X onto $Y \times Z$.

Lemma 3.5. *Assume that condition* (C_T) *is satisfied for some* $T > 0$; *let* $\chi \in \mathcal{S}(\mathbb{R}^n)$ *be such that* $\hat{\chi} \in \mathcal{D}(\mathbb{R})$ *and* $\operatorname{supp}(\hat{\chi}) \subseteq \Omega$. *Then,* $\mathcal{F}^{-1}(\hat{\chi}(\cdot)S(\cdot)^{-1}) \in W^{\infty,1}(\mathcal{L}(Y \times Z; X))$, *and there exist* $R_1 \in W^{\infty,1}(\mathcal{L}(X,Y))$, $R_2 \in W^{\infty,1}(\mathcal{L}(X,Z))$ *such that for any* $u \in W^{0,p}_T(X)$ *such that* $\gamma u \in W^{1,p}_T(Z)$ *for some* $p \in [1,+\infty]$, *if* $Su = (f,g)$,

$$\chi * u = R_1 * f + R_2 * g.$$

Proof. Analogous to the proof of 3.1 and 3.2.

Proposition 3.6. *Assume that condition* (C_T) *is satisfied, for some* $T > 0$. *Then, for any* $p \in [1,+\infty]$ *problem* (1.2) *has at most one solution in* $W^{0,p}_T(E)$ *such that* $\gamma u \in W^{1,p}_T(Z)$.

Proof. If u is a solution in the declared class of (1.2) with $f \equiv 0$ and $g \equiv 0$, by 3.5 $\chi * u = 0$ for every $u \in \mathcal{S}(\mathbb{R})$ such that $\hat{\chi} \in \mathcal{D}(\mathbb{R})$, $\operatorname{supp}(\hat{\chi}) \subseteq \Omega$. This implies that $u = 0$, because $\operatorname{supp}(\hat{u}) \subseteq \Omega$.

Proposition 3.7. *Assume that for certain* $\theta \geq 0$, $1 \leq p \leq +\infty$, $T > 0$ (1.2) *is* (θ,p,T)*-well posed; then substituting for* K K_1 $(K_1 \in L^1(\mathbb{R}; \mathcal{L}(X,Y))$ *and for* H H_1 $(H_1 \in L^1(\mathbb{R}; \mathcal{L}(X,Z)))$ *such that condition* (C_T) *is satisfied, the new problem is* (θ,p,T)*-well posed.*

Proof. The proof follows the same lines of 3.4 until the formulation of system (3.3). At this point, fix $\zeta \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp}(\hat{\zeta}) \subseteq \Omega_o, |\hat{\zeta}(\omega)| \leq 1$ for any $\omega \in \mathbb{R}$ and $\hat{\zeta}(\omega) \equiv 1$ in an open neighbourhood of $\{ \frac{2j\pi}{T} : j \in \mathbb{Z}$ and $|\frac{2j\pi}{T}| \leq 2k\}$, where Ω_o is the analog of Ω in problem (3.2). Then, (3.3) can be rewritten in the form

$$\begin{cases} \phi + (K_1 - K_o) * [\zeta * S(\phi,\psi)] = f, \\ \psi - (H_1 - H_o) * [\zeta * S(\phi,\psi)] = g \end{cases} \tag{3.4}$$

and the conclusion is the same as in 3.4.

Lemma 3.8. *Assume that* (C) *is satisfied and* $K \in \mathcal{S}(\mathbb{R};\mathcal{L}(X,Y)), H \in \mathcal{S}(\mathbb{R};\mathcal{L}(X,Z))$; *then, for any* $k \in \mathbb{N}_0$ *there exists* $C(k) \geq 0$ *such that, for any* $\omega \in \mathbb{R}$,

$$||(S(\cdot)^{-1})^{(k)}(\omega)||_{\mathcal{L}(Y\times Z;X)} + |\omega|||\gamma(S(\cdot)^{-1})^{(k)}(\omega)||_{\mathcal{L}(Y\times Z;Z)} \leq C(k)(1+|\omega|)^{-k}.$$

Proof. The result is true if $k=0$. For $k \geq 1$, observe that

$$(S^{-1})^{(k)}(\omega) = -\sum_{j=1}^{k} \binom{k}{j} S(\omega)^{-1} S^{(j)}(\omega)(S^{-1})^{(k-j)}(\omega)$$

so that, if the estimate is true for $l < k$, we have, for $j=1$,

$$\begin{aligned}||S(\omega)^{-1}S'(\omega)(S^{-1})^{(k-1)}(\omega)||_{\mathcal{L}(Y\times Z;X)} &\leq C(||\hat{K}'(\omega)(S^{-1})^{(k-1)}(\omega)||_{\mathcal{L}(Y\times Z;Y)}\\ &+ ||(i\gamma - \hat{H}'(\omega))(S^{-1})^{(k-1)}(\omega)||_{\mathcal{L}(Y\times Z;Z)}) \leq C(1+|\omega|)^{-k}.\end{aligned}$$

Analogously, for $1 < j \leq k$,

$$\begin{aligned}||S(\omega)^{-1}S^{(j)}(\omega)(S^{-1})^{(k-j)}(\omega)||_{\mathcal{L}(Y\times Z;X)} &\leq C(||\hat{K}^{(j)}(\omega)(S^{-1})^{(k-j)}(\omega)||_{\mathcal{L}(Y\times Z;Y)}\\ &+ ||\hat{H}^{(j)}(\omega)(S^{-1})^{(k-j)}(\omega)||_{\mathcal{L}(Y\times Z;Z)})\end{aligned}$$

which is rapidly decreasing at infinity. So, we have

$$||(S(\cdot)^{-1})^{(k)}(\omega)||_{\mathcal{L}(Y\times Z;X)} \leq C(k)(1+|\omega|)^{-k}$$

for any k. The corresponding estimate for $||\gamma(S(\cdot)^{-1})^{(k)}(\omega)||_{\mathcal{L}(Y\times Z;Z)}$ can be obtained analogously using the estimate

$$||\gamma S(\omega)^{-1}||_{\mathcal{L}(Y\times Z;Z)} \leq C(1+|\omega|)^{-1}.$$

Proposition 3.9. *Let* E *be a Banach space,* $U \in C^\infty(\mathbb{R};E)$, *such that for any* $k \in \mathbb{N}\cup\{0\}$ *there exists* $C(k) > 0$ *such that* $||U^{(k)}(\omega)||_E \leq C(k)(1+|\omega|)^{-k}$. *Set* $T := \mathcal{F}^{-1}U$, $F := T|_{\mathbb{R}-\{0\}}$; *then,*

(I) $F \in C^\infty(\mathbb{R}-\{0\};E)$;

(II) *for any* $k \in \mathbb{N}\cup\{0\}$ *there exists* $C(k) > 0$ *such that for any* $t \in \mathbb{R}-\{0\}$ $||F^{(k)}(t)||_E \leq C(k)|t|^{-k-1}$;

(III) *for any* $k \in \mathbb{N}\cup\{0\}$ $||F^{(k)}(t)||_E$ *is rapidly decreasing at infinity;*

(IV) *for any* $\phi \in \mathcal{D}(\mathbb{R})$ $\sup_{t>0}||T(\phi(t.))||_E < +\infty$ *(here* $\phi(t.)$ *indicates the mapping* $s \to \phi(ts)$*);*

(V)

$$\sup_{0<r_1<r_2} ||\int_{r_1<|t|<r_2} F(t)\,dt||_E < +\infty;$$

(VI) *fix* $\psi \in \mathcal{D}(\mathbb{R})$ *such that* $\psi(t) = 1$ *if* $|t| \leq \delta$ *for some* $\delta > 0$; *then there exists* $a \in E$ *such that for any* $\phi \in \mathcal{D}(\mathbb{R})$

$$T(\phi) = \int_{\mathbb{R}} [\phi(t) - \phi(0)\psi(t)]F(t)\, dt + \phi(0)a;$$

(VII) *there exists* $C > 0$ *such that for any* $s \in \mathbb{R} - \{0\}$

$$\int_{|t| \geq 2|s|} ||F(t-s) - F(t)||_E \, dt \leq C.$$

Proof. For any k and j in $\mathbb{N}_0$ we have $(-it)^k T^{(j)} = \mathcal{F}^{-1}(\partial^{(k)}((i\omega)^j U(\cdot)))$, and so $(-it)^k T^{(j)}$ is bounded and uniformly continuous if $k \geq j + 2$. This implies (I) and (III); fix now $\chi \in \mathcal{S}(\mathbb{R})$, such that $\hat{\chi} \in \mathcal{D}(\mathbb{R})$, $\hat{\chi}(\omega) = 1$ if $|\omega| \leq 1$, $\hat{\chi}(\omega) = 0$ if $|\omega| \geq 2$; set, for $N \in \mathbb{N}$, $U_N := \hat{\chi}(\frac{\cdot}{N})U$, $T_N := \mathcal{F}^{-1}U_N$. Remark that $T_N \in \mathcal{S}(\mathbb{R}; E)$, $T_N \to T$ as $N \to +\infty$ in $\mathcal{S}'(\mathbb{R}; E)$; a simple estimate gives $||U_N^{(j)}(\omega)||_E \leq C(j)(1 + |\omega|)^{-j}$ for any $\omega \in \mathbb{R}$, with $C(j)$ independent of N. This implies, by the dominated convergence theorem, that, for $t \neq 0$,

$$T_N(t) = -t^{-2}\mathcal{F}^{-1}(U_N^{(2)})(t) \to -t^{-2}\mathcal{F}^{-1}(U^{(2)})(t) = F(t)$$

as $N \to \infty$. Moreover, we have

$$T_N(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}\hat{\chi}(t|\omega|)U_N(\omega)\, d\omega + \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}[1 - \hat{\chi}(t|\omega|)]U_N(\omega)\, d\omega. \tag{3.5}$$

We show that $||T_N(t)||_E \leq C(0)|t|^{-1}$ with $C(0)$ independent of N; in fact, the first summand in (3.5) can be estimated with $C|t|^{-1}$, with C independent of N. Moreover,

$$\begin{aligned} &t^2 \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}[1 - \hat{\chi}(t|\omega|)]U_N(\omega)\, d\omega \\ &\quad = -\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}(\frac{d}{d\omega})^2\{[1 - \hat{\chi}(t|\cdot|)]U_N(\cdot)\}(\omega)\, d\omega \\ &\quad = -\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}[1 - \hat{\chi}(t|\omega|)]U_N^{(2)}(\omega) d\omega \\ &\qquad - \sum_{j=1}^{2}\binom{2}{j}\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}(\frac{d}{d\omega})^j[1 - \hat{\chi}(t|\cdot|)](\omega)U_N^{(2-j)}(\omega)\, d\omega. \end{aligned}$$

We have

$$||\int_{\mathbb{R}} e^{i\omega t}[1 - \hat{\chi}(t|\omega|)]U_N^{(2)}(\omega)\, d\omega||_E \leq C\int_{|\omega| \geq \frac{1}{|t|}} (1 + |\omega|)^{-2}\, d\omega \leq C|t|.$$

Next, for $j = 1, 2$

$$||\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}(\frac{d}{d\omega})^j[1-\hat{\chi}(t|\cdot|)](\omega)U_N^{(2-j)}(\omega)\,d\omega||_E$$
$$\leq C|t|^j \int_{\frac{1}{|t|}}^{\frac{2}{|t|}} (1+|\omega|)^{j-2}\,d\omega \leq C|t|.$$

So (II) is proved in the case $k = 0$. The general case can be treated analogously remarking that for any $k \in \mathbb{N}$ $(-it)^k T^{(k)}(t) = \mathcal{F}^{-1}(\partial^k((i\omega)^k U(\cdot)))(t)$ and $\partial((i\omega)^k U)$ satisfies estimates of the same type of U.

We prove (IV): we have

$$T(\phi(t.)) = \frac{1}{2\pi t}\int_{\mathbb{R}} \hat{\phi}(-\frac{\omega}{t})U(\omega)\,d\omega$$

so that

$$||T(\phi(t.))||_E \leq \frac{1}{2\pi}||U||_{0,\infty,E}||\hat{\phi}||_{0,1}.$$

We prove (V): owing to (III) it clearly suffices to estimate $||\int_{r<|t|<1} F(t)\,dt||_E$ for $0 < r < 1$; fix $\phi \in \mathcal{D}(\mathbb{R})$ such that $\phi(t) = 1$ if $|t| \leq 1$, $\phi(t) = 0$ if $|t| \geq 2$. Then,

$$\int_{r<|t|<1} F(t)\,dt = T(\phi - \phi(\frac{2.}{r})) - \int_{1<|t|<2} \phi(t)F(t)\,dt - \int_{\frac{r}{2}\leq|t|\leq r}[1-\phi(\frac{2t}{r})]F(t)\,dt$$

and the result follows from (IV) and (II).

We prove (VI): we have

$$T(\phi) = \lim_{N\to+\infty}\int_{\mathbb{R}} \phi(t)T_N(t)\,dt = \lim_{N\to+\infty}\int_{\mathbb{R}}[\phi(t)-\phi(0)\psi(t)]T_N(t)\,dt$$
$$+\,\phi(0)\lim_{N\to+\infty} T_N(\psi) = \int_{\mathbb{R}}[\phi(t)-\phi(0)\psi(t)]F(t)\,dt + \phi(0)a,$$

with $a = T(\psi)$ (to pass to the limit one can use the estimate following (3.5)).

To prove (VII) it suffices to use estimate

$$||F(t-s)-F(t)||_E \leq C|\int_t^{t-s} \tau^{-2}\,d\tau| = C|s||t(t-s)|^{-1}.$$

Proposition 3.10. *Let L and G be Banach spaces, $E = \mathcal{L}(L, G)$ and suppose U satisfies the assumptions of* 3.9. *If T and F are the same as in* 3.9 *and T is of the form described in* 3.9(VI), *set, for $\phi \in B^{\theta}_{p,q}(L)$ or $B^{\theta}_{p,q,T}(L)$ with $1 \leq p, q \leq +\infty$, $T > 0$, $\theta > 0$,*

$$(T * \phi)(t) := \int_{\mathbb{R}} F(s)[\phi(t-s) - \psi(s)\phi(t)]\, ds + a(\phi(t)).$$

Then, $\phi \to T * \phi$ is a linear bounded operator from $B^{\theta}_{p,q}(L)$ to $B^{\theta}_{p,q}(G)$ and from $B^{\theta}_{p,q,T}(L)$ to $B^{\theta}_{p,q,T}(L)$.

Proof. It clearly suffices to consider the cases $a = 0$, $q = +\infty$ (owing to interpolation) and, as the operator commutes with translations, $0 < \theta < 1$. We shall indicate with I the set $\mathbb{R}$ in the nonperiodic case, the interval $[0, T]$ in the T-periodic case. For brevity we shall indicate with N the norm of ϕ in $B^{\theta}_{p,\infty}(L)$ in the nonperiodic case, the norm of ϕ in $B^{\theta}_{p,\infty,T}(L)$ in the T-periodic case. We start by proving that in any case the convolution has a meaning at least almost everywhere and it belongs to $L^p(I; G)$; assume first $1 \leq p < +\infty$. Assume for simplicity that $\psi(t) = 1$ if $|t| \leq 1$, $\psi(t) = 0$ if $|t| \geq 2$. From Minkowski's inequality

$$\begin{aligned}
&\Big(\int_I \Big(\int_{\mathbb{R}} ||F(s)||_E ||\phi(t-s) - \psi(s)\phi(t)||_L\, ds\Big)^p dt\Big)^{\frac{1}{p}} \\
&\quad \leq \int_{\mathbb{R}} ||F(s)||_E \Big(\int_I ||\phi(t-s) - \psi(s)\phi(t)||_L^p\, dt\Big)^{\frac{1}{p}} ds \\
&\quad \leq C \int_{|s|\leq 1} |s|^{-1} \Big(\int_I ||\phi(t-s) - \phi(t)||_L^p\, dt\Big)^{\frac{1}{p}} ds + 2||\phi||_{0,p,L} \int_{|s|>1} ||F(s)||_E\, ds
\end{aligned}$$

so that $T * \phi$ is well defined for almost every t and $T * \phi \in W^{0,p}(G)$ (in the periodic case the analogous statement comes from the T-periodicity of $T * \phi$). The case $p = +\infty$ follows the same lines.

Let now $h \in [0, \frac{1}{3}]$; we estimate $||(T * \phi)(\cdot + h) - T * \phi||_{L^p(I;G)}$: we have, for almost every t,

$$\begin{aligned}
(T * \phi)(t+h) - (T * \phi)(t) &= \int_{|t-s|\leq 2h} F(t+h-s)[\phi(s) - \phi(t+h)]\, ds \\
&\quad - \int_{|t-s|\leq 2h} F(t-s)[\phi(s) - \phi(t)]\, ds + \int_{|t-s|>2h} F(t+h-s)[\phi(s) \\
&\quad - \psi(t+h-s)\phi(t+h)]\, ds - \int_{|t-s|>2h} F(t-s)[\phi(s) - \psi(t-s)\phi(t)]\, ds.
\end{aligned}$$

Considering for the sake of brevity only the case $1 \leq p < +\infty$,

$$\begin{aligned}
&(\int_I || \int_{|t-s|\leq 2h} F(t-s)[\phi(s)-\phi(t)]\, ds||_G^p \, dt)^{\frac{1}{p}} \\
&\quad \leq C(\int_I (\int_{|t-s|\leq 2h} |t-s|^{-1} ||\phi(s)-\phi(t)||_L \, ds)^p \, dt)^{\frac{1}{p}} \\
&\quad \leq C \int_{|s|\leq 2h} |s|^{-1} (\int_I ||\phi(t-s)-\phi(t)||_L^p \, ds)^{\frac{1}{p}} \, ds \leq CN 2^{1+\theta} \theta^{-1} h^\theta.
\end{aligned}$$

Analogously, one has

$$\begin{aligned}
&(\int_I || \int_{|t-s|\leq 2h} F(t+h-s)[\phi(s)-\phi(t+h)]\, ds||_G^p \, dt)^{\frac{1}{p}} \\
&\quad \leq C \int_{|s|\leq 3h} |s|^{-1} (\int_I ||\phi(t+h-s)-\phi(t+h)||_F^p \, ds)^{\frac{1}{p}} \, ds \leq C(\theta) N h^\theta.
\end{aligned}$$

Next we have

$$\begin{aligned}
&\int_{|t-s|>2h} F(t+h-s)[\phi(s)-\psi(t+h-s)\phi(t+h)]\, ds \\
&\qquad - \int_{|t-s|>2h} F(t-s)[\phi(s)-\psi(t-s)\phi(t)]\, ds \\
&\quad = \int_{|t-s|>2h} [F(t+h-s)-F(t-s)][\phi(s)-\psi(t+h-s)\phi(t+h)]\, ds \\
&\qquad + \int_{|t-s|>2h} F(t-s)[\psi(t-s)\phi(t)-\psi(t+h-s)\phi(t+h)]\, ds.
\end{aligned}$$

We have

$$\begin{aligned}
&(\int_I \Big\| \int_{|t-s|>2h} [F(t+h-s)-F(t-s)][\phi(s)-\psi(t+h-s)\phi(t+h)] ds \Big\|_G^p dt)^{\frac{1}{p}} \\
&\quad = (\int_I \Big\| \int_{|s|>2h} [F(s+h)-F(s)][\phi(t-s)-\psi(s+h)\phi(t+h)]\, ds \Big\|_G^p \, dt)^{\frac{1}{p}} \\
&\quad \leq Ch(\int_I (\int_{|s|>2h} s^{-2} ||\phi(t-s)-\psi(s+h)\phi(t+h)||_L \, ds)^p \, dt)^{\frac{1}{p}} \\
&\quad \leq Ch \int_{|s|>2h} s^{-2} (\int_I ||\phi(t-s)-\psi(s+h)\phi(t+h)||_L^p \, dt)^{\frac{1}{p}} \, ds.
\end{aligned}$$

In any case

$$(\int_I ||\phi(t-s)-\psi(s+h)\phi(t+h)||_L^p \, dt)^{\frac{1}{p}} \leq 2||\phi||_{0,p,L} (||\phi||_{0,p,T,L})$$

and, if $|s+h| \leq 1$,

$$(\int_I ||\phi(t-s) - \psi(s+h)\phi(t+h)||_L^p \, dt)^{\frac{1}{p}} \leq N|s+h|^\theta$$

so that

$$\begin{aligned} h \int_{|s|>2h} & s^{-2}(\int_I ||\phi(t-s) - \psi(s+h)\phi(t+h)||_L^p \, dt)^{\frac{1}{p}} \, ds \\ & \leq CNh(\int_{2h<|s|<\frac{2}{3}} s^{-2}|s+h|^\theta \, ds + \int_{|s|>\frac{2}{3}} s^{-2}) \, ds \leq CNh^\theta. \end{aligned}$$

Finally,

$$\begin{aligned} \int_{|s|\geq 2h} & F(s)[\psi(s)\phi(t) - \psi(s+h)\phi(t+h)] \, ds \\ & = \int_{2h\leq|s|\leq\frac{2}{3}} F(s) \, ds[\phi(t+h) - \phi(t)] \\ & \quad + \int_{|s|>\frac{2}{3}} F(s)[\psi(s)\phi(t) - \psi(s+h)\phi(t+h)] \, ds. \end{aligned}$$

We have from 3.9(V)

$$\Big\|\int_{2h\leq|s|\leq\frac{2}{3}} F(s) \, ds[\phi(t+h) - \phi(t)]\Big\|_G \leq C||\phi(t+h) - \phi(t)||_L,$$

$$\begin{aligned} \Big\|\int_{|s|>\frac{2}{3}} & F(s)[\psi(s)\phi(t) - \psi(s+h)\phi(t+h)] \, ds\Big\|_G \\ & \leq \int_{|s|>\frac{2}{3}} ||F(s)||_E \big[|\psi(s) - \psi(s+h)|||\phi(t)||_L \\ & \qquad + |\psi(s+h)|||\phi(t) - \phi(t+h)||_L\big] \, ds \\ & \leq C(h||\phi(t)||_L + ||\phi(t) - \phi(t+h)||_L) \end{aligned}$$

so that

$$\begin{aligned} (\int_I || \int_{|t-s|>2h} & F(t-s)[\psi(t-s)\phi(t) - \psi(t-s+h)\phi(t+h)] \, ds||_G^p \, dt)^{\frac{1}{p}} \\ & = (\int_I || \int_{|s|>2h} F(s)[\psi(s)\phi(t) - \psi(s+h)\phi(t+h)] \, ds||_G^p \, dt)^{\frac{1}{p}} \\ & \leq C(\int_I (||\phi(t+h) - \phi(t)||_L + h||\phi(t)||_L)^p \, dt)^{\frac{1}{p}} \leq CN(h^\theta + h). \end{aligned}$$

With this the result is completely proved.

Lemma 3.11. *Assume that the assumptions of* 3.10 *are satisfied. Let* $\hat{\chi} \in \mathcal{D}(\mathbb{R})$ *such that* $\hat{\chi}(\omega) = 1$ *if* $|\omega| \leq 1$, $\hat{\chi}(\omega) = 0$ *if* $|\omega| \geq 2$ *and set* $U_N := \hat{\chi}(\frac{\cdot}{N})U$, $T_N = \mathcal{F}^{-1}U_N$. *Then, if* $\phi \in W^{\theta,p}(L)(W_T^{\theta,p}(L))$ *for some* $T > 0, \theta > 0$, $T_N * \phi \to T * \phi$ *in* $W^{0,p}(G)$ $(W_T^{0,p}(G))$ *as* $N \to +\infty$.

Proof. We continue to indicate with E the space $\mathcal{L}(L,G)$; for simplicity we consider only the nonperiodic case. We have

$$(T_N * \phi)(t) = \int_{\mathbb{R}} T_N(s)[\phi(t-s) - \psi(s)\phi(t)]\, ds + T_N(\psi)\phi(t).$$

From the proof of 3.9 we have that $T_N(\psi) \to T(\psi) = a$ in $\mathcal{L}(L,G)$ as $N \to +\infty$. Moreover, always in the proof of 3.9, we have seen that $T_N(t) \to F(t)$ in $\mathcal{L}(L,G)$ for any $t \neq 0$, $||T_N(t)||_E \leq C|t|^{-1}$ with C independent of N and $||U_N^{(2)}(\omega)||_E \leq C(1+|\omega|)^{-2}$ again with C independent of N and so $||T_N(t)||_E \leq Ct^{-2}$, with C independent of N. Therefore,by Minkowski's inequality, assuming $0 < \theta < 1$,

$$\begin{aligned}
&(\int_{\mathbb{R}} || \int_{\mathbb{R}} [T_N(s) - F(s)][\phi(t-s) - \psi(s)\phi(t)]\, ds||_G^p\, dt)^{\frac{1}{p}} \\
&\quad \leq \int_{|s|\leq 1} ||T_N(s) - F(s)||_E |s|^\theta\, ds + \int_{|s|>1} ||T_N(s) - F(s)||_E\, ds 2||\phi||_{0,p,F}
\end{aligned}$$

and the result follows from the dominated convergence theorem.

We pass now to give one of the main results of the paper:

Theorem 3.12. *Assume that condition* (C) *is satisfied; then for any* $\theta \in \mathbb{R}^+ - \mathbb{N}$, $p \in [1, +\infty]$ (1.2) *is* (θ, p)*-well posed.*

Proof. The uniqueness of a solution has already been proved in 3.3; we want to prove the existence. We start by assuming that $K \in \mathcal{S}(\mathbb{R}; \mathcal{L}(X,Y))$, $H \in \mathcal{S}(\mathbb{R}; \mathcal{L}(X,Z))$. Set $U(\omega) = S(\omega)^{-1}$ and apply 3.8 and 3.10 (in case $p = q$), with $L = Y \times Z$, $G = X$, $\phi(t) = (f(t), g(t))$. Then, $u := T * \phi \in W^{\theta,p}(X)$. We start by showing that u satisfies the first equation in (1.2). To this aim we start by observing that, owing to 3.11, $(\mathcal{A} + K*)(T_N * \phi)$ converges to $(\mathcal{A} + K*)u$ in $W^{0,p}(Y)$; moreover, it is easily seen (using the fact that $T_N \in \mathcal{S}(\mathbb{R}; \mathcal{L}(Y \times Z; X))$) that, if $\mathcal{F}\chi = \hat{\chi}$,

$$(\mathcal{A} + K*)(T_N * \phi) = (\mathcal{A}T_N + K * T_N) * \phi = N\chi(N.) * f,$$

converging to f in $W^{0,p}(Y)$ as N tends to $+\infty$.

Set now $V(\omega) := i\omega\gamma S(\omega)^{-1}$ and apply again 3.8 and 3.10. Put $R := \mathcal{F}^{-1}V$ and remark that $(\gamma T_N)' = \mathcal{F}^{-1}(\hat{\chi}(\frac{\cdot}{N})V))$. From 3.11 $(\gamma T_N)' * \phi \to R * \phi$ in $W^{0,p}(Z)$.

Moreover $R * \phi \in W^{\theta,p}(Z)$. But

$$(\gamma T_N)' * \phi = (\gamma T_N * \phi)' \to (\gamma u)'$$

in $\mathcal{D}'(\mathbb{R}; Z)$. This shows that $(\gamma u)' = R * \phi$ and that $\gamma u \in W^{1+\theta,p}(Z)$. To conclude, we have

$$[\gamma(T_N * \phi)]' - \mathcal{B}[(T_N * \phi)] - H * (T_N * \phi) = N\chi(N.) * g \to g$$

in $W^{0,p}(Z)$ as $N \to +\infty$. So, u satisfies also the second equation in (1.2).

We consider now the general case $K \in L^1(\mathbb{R}; \mathcal{L}(X, Y))$ and $H \in L^1(\mathbb{R}; \mathcal{L}(X, Z))$. We take two sequences $(K_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ respectively in $\mathcal{S}(\mathbb{R}; \mathcal{L}(X, Y))$ and in $\mathcal{S}(\mathbb{R}; \mathcal{L}(X, Z))$, converging to K in $L^1(\mathbb{R}; \mathcal{L}(X, Y))$ and to H in $L^1(\mathbb{R}; \mathcal{L}(X, Z))$; then if we replace K with K_n and H with H_n with n large enough, condition (C) is satisfied by the new system, which is, from the first part of the proof, (θ, p)-well posed. Therefore the general result is a consequence of 3.4.

We are going to give some sufficient conditions implying the well-posedness in case $\theta = 0$; we shall see in 3.16 that in general in such a case (C) is not sufficient; we start with the following result, which is proved in [8] (Theorem 3.4) :

Theorem 3.13. *Let E and H be Banach spaces, $S \in \mathcal{L}(L^p(\mathbb{R}; E); L^p(\mathbb{R}; H))$ for some $p \in [1, +\infty]$; assume that, if $\phi \in L^\infty(\mathbb{R}; E)$ and has compact support, for any $t \in \mathbb{R} - \mathrm{supp}\,(\phi)$*

$$S\phi(t) = \int_{\mathbb{R}} F(t - s)\phi(s)\, ds,$$

with F with values in $\mathcal{L}(E, H)$, locally integrable in $\mathbb{R} - \{0\}$ and such that there exists $C \geq 0$ such that for any $s \in \mathbb{R}$

$$\int_{|t|\geq 2|s|} ||F(t - s) - F(t)||_{\mathcal{L}(E,H)}\, dt \leq C.$$

Then, S can be extended to an element of $\mathcal{L}(L^q(\mathbb{R}; E); L^q(\mathbb{R}; H))$ for every $q \in]1, +\infty[$.

Corollary 3.14. *Assume that (1.2) is $(0, p)$-well posed for some $p \in [1, +\infty)$; then it is $(0, q)$-well posed for every $q \in (1, +\infty)$.*

Proof. By the same argument of 3.12 it is not restrictive to assume that $K \in \mathcal{S}(\mathbb{R}; \mathcal{L}(X, Y))$ and $H \in \mathcal{S}(\mathbb{R}; \mathcal{L}(X, Z))$. Let $f \in W^{\theta,p}(Y)$, $g \in W^{\theta,p}(Z)$, for some $\theta > 0$ both with compact support. Set $\phi(t) := (f(t), g(t))$. Let $t_0 \in \mathbb{R} - \mathrm{supp}\,(\phi)$. Then the solution u has the form

$$u(t) = \int_{\mathbb{R}} F(s)[\phi(t - s) - \psi(s)\phi(t)]\, ds + a(\phi(t))$$

(see 3.9 and 3.10). If U is a neighbourhood of t_0 such that $\phi_{|U} \equiv 0$, we have, for $t \in U$,

$$u(t) = \int_{\mathbb{R}} F(s)\phi(t - s)\, ds.$$

This formula is easily extendable to the case when $\phi \in L^\infty(\mathbb{R}; Y \times Z)$ and $\text{supp}\,(\phi) \cap U = \emptyset$ by a density argument. So, in force of Theorem 3.13 we have that $\phi \to T * \phi$ can be extended to an element of $\mathcal{L}(W^{0,q}(Y \times Z); W^{0,q}(X))$ for every $q \in (1, +\infty)$. The same argument shows that also $\phi \to (\gamma(T * \phi))'$ is extendable to an element of $\mathcal{L}(W^{0,q}(Y \times Z); W^{0,q}(Z))$. So the result is proved.

Proposition 3.15. *Assume that X, Y and Z are Hilbert spaces; then, if condition* (C) *is satisfied,* (1.2) *is* $(0,p)$*-well posed for every* $p \in (1, +\infty)$.

Proof. Owing to 3.14, it suffices to prove the result in case $p = 2$. In this case we have, given $f \in W^{0,2}(Y)$ and $g \in W^{0,2}(Z)$, $u = \mathcal{F}^{-1}(S(\cdot)^{-1}(\hat{f}, \hat{g}))$ so that, by Parseval's theorem,

$$\begin{aligned} ||u||_{0,2,X} &= (2\pi)^{-1/2}||\hat{u}||_{0,2,X} = (2\pi)^{-1/2}||S(\cdot)^{-1}(\hat{f}, \hat{g})||_{0,2,X} \\ &\leq C(||\hat{f}||_{0,2,Y} + ||\hat{g}||_{0,2,Z}) \leq C(||f||_{0,2,Y} + ||g||_{0,2,Z}). \end{aligned}$$

Analogously, one has

$$\mathcal{F}((\gamma u)')(\omega) = i\omega\gamma S(\omega)^{-1}(\hat{f}(\omega), \hat{g}(\omega))$$

and

$$||(\gamma u)'||_{0,2,Z} \leq C(||f||_{0,2,Y} + ||g||_{0,2,Z}).$$

3.16. If $\theta = 0$, condition (C) is not in general sufficient to guarantee the well-posedness of (1.2). To see this, consider the following particular case of (1.2): assume $X \subseteq Z$ (continuous embedding), $Y = \{0\}$, $\mathcal{A} = 0$, $K \equiv 0$; let γ be the embedding of X in Z and assume that the resolvent set $\rho(\mathcal{B})$ of $\mathcal{B}$ (as an operator in Z) contains $\{\omega \in \mathbb{C} : Re(\omega) \geq 0\}$; moreover, for ω in this set $||(\omega - \mathcal{B})^{-1}||_{\mathcal{L}(Z)} \leq C(1 + |\omega|)^{-1}$, with $C > 0$ independent of ω. Then it is easily seen that, for $\theta \in \mathbb{R}^+ - \mathbb{Z}$, $h \in W^{\theta,p}(Z)$, the solution u of

$$u'(t) - \mathcal{B}u(t) = h(t) \tag{3.6}$$

in $W^{\theta+1,p}(Z) \cap W^{\theta,p}(Z)$ is

$$u(t) = \int_{-\infty}^{t} T(t-s)h(s)\, ds,$$

where $\{T(t) : t \geq 0\}$ is the semigroup (possibly not strongly continuous on 0) generated by $\mathcal{B}$ in Z. Assume now that the problem is $(0,p)$-well posed, for some $p \in [1, +\infty)$. Fix $T > 0$, let $g \in L^p([0,T]; Z)$ and consider the Cauchy problem

$$\begin{aligned} &u'(t) = \mathcal{B}u(t) + g(t), \qquad t \in [0,T] \\ &u(0) = 0. \end{aligned}$$

Its mild solution is

$$u(t) = \int_0^t T(t-s)g(s)\,ds. \tag{3.7}$$

Set $g^*(t) = g(t)$ if $0 \leq t \leq T$, $g^*(t) = 0$ if $t \in \mathbb{R} - [0,T]$. Consider (3.6) with $h = g^*$. Then it is clear that the solution in $W^{0,p}(X) \cap W^{1,p}(Z)$ coincides in $[0,T]$ with the solution of the Cauchy problem. This implies that for any $g \in L^p([0,T]; Z)$ the function $t \to \int_0^t T(t-s)g(s)\,ds \in L^p(X) \cap W^{1,p}(Z)$. This does not happen for any $\mathcal{B}$ with the declared properties; for a counterexample see [3].

We conclude this section considering the T-periodic case, for some $T > 0$. We have

Theorem 3.17. *Assume that $\theta \in \mathbb{R}^+ - \mathbb{N}$, $1 \leq p \leq +\infty$; then (1.2) is (θ, p, T)-well posed if and only if (C_T) is satisfied.*

Proof. We have already proved the uniqueness of a solution in 3.6. Consider first the case $K \in \mathcal{S}(\mathbb{R}; \mathcal{L}(X,Y))$, $H \in \mathcal{S}(\mathbb{R}; \mathcal{L}(X,Z))$. If Ω is the set introduced just after 3.4, fix $\rho \in \mathcal{S}(\mathbb{R})$ such that $\hat{\rho} \in \mathcal{D}(\mathbb{R})$, $\hat{\rho}(\omega) = 1$ in an open neighbourhood of $\mathbb{R} - \Omega$, $\hat{\rho}(\omega) = 0$ in an open neighbourhood of $\{\frac{2j\pi}{T} : j \in \mathbb{Z}\}$. Set $U(\omega) := (1 - \hat{\rho}(\omega))S(\omega)^{-1}$. Then the proof follows the lines of the proof of 3.12: it is easily seen that U satisfies all the assumptions of 3.10, with $L = Y \times Z, G = X$. Put $T := \mathcal{F}^{-1}U$, $\phi(t) = (f(t), g(t))$. Then, by 3.10, if $u := T * \phi$, $u \in W_T^{\theta,p}(X)$. Owing to 3.11, $(\mathcal{A} + K*)(T_N * \phi)$ converges to $(\mathcal{A} + K*)u$ in $W_T^{0,p}(Y)$; moreover,

$$(\mathcal{A} + K*)(T_N * \phi) = [N\chi(\frac{\cdot}{N}) - N\chi(\frac{\cdot}{N}) * \rho)] * f = N\chi(\frac{\cdot}{N}) * f,$$

as $\rho * f = 0$. As $Ng(\frac{\cdot}{N}) * f \to f$ in $W_T^{0,p}(Y)$ as $N \to +\infty$, we have that $\mathcal{A}u + K * u = f$. Analogously one can show that $\gamma u \in W_T^{\theta+1,p}(Z)$ and $(\gamma u)' - Bu - H * u = g$.

4. A sufficient condition in case $\theta = 0$. In this section we shall give a result of well-posedness of (1.2) in case $\theta = 0$, inspired by well-known maximal regularity results for functions with values in certain interpolation spaces (see [4]).

Lemma 4.1. *Let $p \in [1, +\infty]$. Assume that condition (C) is satisfied. Moreover, there exists $M > 0$ such that for every $u \in W^{0,p}(X)$ such that $\gamma u \in W^{1,p}(Z)$*

$$||u||_{0,p,X} + ||\gamma u||_{1,p,Z} \leq M(||\mathcal{A}u||_{0,p,Y} + ||(\gamma u)' - \mathcal{B}u||_{0,p,Z} + ||\gamma u||_{0,p,Z}).$$

Then, (1.2) is $(0,p)$-well posed.

Let $T > 0$; assume that condition (C_T) is satisfied. Moreover, there exists $M > 0$ such that for every $u \in W_T^{0,p}(X)$ such that $\gamma u \in W_T^{1,p}(Z)$

$$||u||_{0,p,T,X} + ||\gamma u||_{1,p,T,Z} \leq M(||\mathcal{A}u||_{0,p,T,Y} + ||(\gamma u)' - \mathcal{B}u||_{0,p,T,Z} + ||\gamma u||_{0,p,T,Z}).$$

Then, (1.2) is $(0,p,T)$-well posed.

Proof. We prove the first statement; as usual, owing to 3.4 and 3.7, it suffices to consider the case $K \in \mathcal{S}(\mathbb{R}; \mathcal{L}(X, Y))$, $H \in \mathcal{S}(\mathbb{R}; \mathcal{L}(X, Z))$.

Consider first $f \in W^{\theta,p}(Y), g \in W^{\theta,p}(Z)$ for some $\theta \in (0, 1)$. By Theorem 3.12, (1.2) is (θ, p)-well posed. Let u be the solution of (1.2) with data f and g; we want to show that for some $C > 0$

$$||u||_{0,p,X} + ||\gamma u||_{1,p,Z} \leq C||Su||_{0,p,Y\times Z}.$$

To this aim, from the assumptions we have

$$\begin{aligned}||u||_{0,p,X} + ||\gamma u||_{1,p,Z} \leq M(&||f||_{0,p,Y} + ||g||_{0,p,Z} + ||K * u||_{0,p,Y}\\ &+ ||H * u||_{0,p,Z} + ||\gamma u||_{0,p,Z}).\end{aligned}$$

Set $K_1 = K' + K$; then $K = e * K_1$, with $e(t) = e^{-t}\theta(t)$. It is easily seen that the solution with data $e * f$ and $e * g$ is $e * u$, so that, from 3.12,

$$||e * u||_{\theta,p,X} \leq C(||e * f||_{\theta,p,Y} + ||e * g||_{\theta,p,Z}).$$

So,

$$\begin{aligned}||K * u||_{0,p,Y} &\leq ||K_1 * (e * u)||_{\theta,p,Y} \leq C||e * u||_{\theta,p,X}\\ &\leq C(||e * f||_{\theta,p,Y} + ||e * g||_{\theta,p,Z}) \leq C(||f||_{0,p,Y} + ||g||_{0,p,Z}).\end{aligned}$$

(Remark that $(f, g) \to e * (f, g) \in \mathcal{L}(W^{\xi,p}(Y \times Z); W^{\xi+1,p}(Y \times Z))$ for any $\xi \geq 0$.)

The same argument shows that also

$$||H * u||_{0,p,Z} \leq C(||f||_{0,p,Y} + ||g||_{0,p,Z})$$

for some $C > 0$. It remains to estimate $||\gamma u||_{0,p,Z}$: employing the same basic idea,

$$\begin{aligned}||\gamma u||_{0,p,Z} &= ||(e * \gamma u)' + (e * \gamma u)||_{0,p,Z}\\ &\leq C||e * \gamma u||_{1,p,Z} \leq C||e * \gamma u||_{1+\theta,p,Z}\\ &\leq C(||e * f||_{\theta,p,Y} + ||e * g||_{\theta,p,Z}) \leq C(||f||_{0,p,Y} + ||g||_{0,p,Z}).\end{aligned}$$

So the estimate is proved. The first statement follows easily from this estimate, 3.12 and the fact that $W^{\theta,p}(Y \times Z)$ is dense in $W^{0,p}(Y \times Z)$.

The periodic case can be treated with the same argument.

We set $Z_1 := \gamma(X)$. Clearly Z_1 is a linear subspace of Z. We equip Z_1 with its natural norm: if $z \in Z_1$, $||z||_{Z_1} := \inf\{||x||_X : \gamma x = z\}$. Next, we introduce the new condition (D): if $(\mathcal{A}, \gamma) : X \to Y \times Z_1, (\mathcal{A}, \gamma)x = (\mathcal{A}x, \gamma x)$

$$(\mathcal{A}, \gamma) \text{ is an isomorphism of } X \text{ onto } Y \times Z_1. \tag{D}$$

We shall see in 5.1 that (D) is not a consequence of (C). A simple condition implying (D) is the following:

Proposition 4.2. *Assume that* X *is reflexive, condition* (C′) *is satisfied and* $(\mathcal{A},\gamma)$ *is a Fredholm operator of index 0 between* X *and* $Y \times Z_1$*; then condition* (D) *is satisfied.*

Proof. As $(\mathcal{A},\gamma)$ has index 0, it is sufficient to show that it is onto $Y \times Z_1$. Let $(f,g) \in Y \times Z_1$. Let $x_0 \in X$, such that $\gamma x_0 = g$; subtracting x_0 one is reduced to showing that for any $y \in Y$ there exists $x \in X$ such that $\mathcal{A}x = y, \gamma x = 0$. Let $n \in \mathbb{N}$, so large that $S_0(n)$ is an isomorphism between X and $Y \times Z$. Put $x_n := S_0(n)^{-1}(y,0)$; then $(x_n)_{n\in\mathbb{N}}$ is bounded in X. As X is reflexive, we can assume, perhaps considering a subsequence, that $x_n \to x$, with $x \in X$, in the weak topology of X. So, $\mathcal{A}x = y$, and from $\gamma x_n = (in)^{-1}\mathcal{B}x_n$ we have $\gamma x = 0$.

Assume now that (D) is satisfied; we introduce the operator G:

$$G : Z_1 \to Z, Gz = \mathcal{B}(\mathcal{A},\gamma)^{-1}(0,z)(z \in Z_1).$$

We shall think of G as an operator in Z (possibly unbounded). We have

Lemma 4.3. *Assume that conditions* (C′) *and* (D) *are satisfied; let* $p \in [1,+\infty]$ *and assume moreover that there exists* $M > 0$ *such that for every* $u \in W^{1,p}(Z) \cap W^{0,p}(Z_1)$

$$||u||_{0,p,Z_1} + ||u||_{1,p,Z} \le M(||u' - Gu||_{0,p,Z} + ||u||_{0,p,Z}).$$

Then if condition (C) *is satisfied,* (1.2) *is* $(0,p)$*-well posed.*

Proof. We verify that the assumptions of 4.1 are satisfied.

Let $u \in W^{0,p}(X)$ such that $\gamma u \in W^{1,p}(Z)$; set $v(t) := (\mathcal{A},\gamma)^{-1}(\mathcal{A}u(t),0)$. Then $||v||_{0,p,X} \le C||\mathcal{A}u||_{0,p,Y}$, for some $C > 0$. We have $G\gamma(u-v) = \mathcal{B}(u-v)$, so that

$$[\gamma(u-v)]' - G\gamma(u-v) = (\gamma u)' - \mathcal{B}u + \mathcal{B}v,$$

as $\gamma v \equiv 0$. It follows

$$\begin{aligned}
||u-v||_{0,p,X} + ||\gamma(u-v)||_{1,p,Z} &\le C(||\gamma(u-v)||_{0,p,Z_1} + ||\gamma(u-v)||_{1,p,Z}) \\
&\le C(||(\gamma u)' - \mathcal{B}u||_{0,p,Z} + ||\mathcal{B}v||_{0,p,Z} + ||\gamma(u-v)||_{0,p,Z}) \\
&\le C(||(\gamma u)' - \mathcal{B}u||_{0,p,Z} + ||v||_{0,p,X} + ||\gamma u||_{0,p,Z}) \\
&\le C(||\mathcal{A}u||_{0,p,Y} + ||(\gamma u)' - \mathcal{B}u||_{0,p,Z} + ||\gamma u||_{0,p,Z}).
\end{aligned}$$

Putting together all the estimates, one has, for some $C > 0$,

$$||u||_{0,p,X} + ||\gamma u||_{1,p,Z} \le C(||\mathcal{A}u||_{0,p,Y} + ||(\gamma u)' - \mathcal{B}u||_{0,p,Z} + ||\gamma u||_{0,p,Z}).$$

So the result follows from 4.1.

With the same argument one can also show

Lemma 4.4. *Assume that conditions* (C′) *and* (D) *are satisfied; let* $p \in [1, +\infty]$, $T > 0$ *and assume moreover that there exists* $M > 0$ *such that for every* $u \in W^{1,p}_T(Z) \cap W^{0,p}_T(Z_1)$

$$||u||_{0,p,T,Z_1} + ||u||_{1,p,T,Z} \le M(||u' - Gu||_{0,p,T,Z} + ||u||_{0,p,T,Z}).$$

Then, if condition (C_T) *is satisfied,* (1.2) *is* $(0, p, T)$*-well posed.*

The next theorem is the key result of this section:

Theorem 4.5. *Let* E *be a Banach space,* $\mathcal{D}$ *and* $\mathcal{B}$ *operators in* E, F *a nonempty closed subset of* $\mathbb{R}$. *Assume moreover that:*

(a) $\sigma(\mathcal{D}) \subseteq iF \subseteq \rho(\mathcal{B})$;
(b) *there exists* $C > 0$ *such that for any* $\lambda \in \mathbb{C}$, *if* $\mathrm{dist}\,(\lambda, iF) \ge C$, *one has* $||(\lambda - \mathcal{D})^{-1}||_{\mathcal{L}(E)} \le C(\mathrm{dist}\,(\lambda, iF))^{-1}$;
(c) *if* F *is not upper (lower) bounded, there exists* $\omega_+(\omega_-) \in \mathbb{R}$ *such that*

$$\{i\omega : \omega \in \mathbb{R}, \omega \ge \omega_+(\le \omega_-)\} \subseteq \rho(\mathcal{B}), \text{ and}$$
$$\sup_{\omega \ge \omega_+} |\omega| ||(i\omega - \mathcal{B})^{-1}||_{\mathcal{L}(E)} < +\infty, \quad (\sup_{\omega \le \omega_-} |\omega| ||(i\omega - \mathcal{B})^{-1}||_{\mathcal{L}(E)} < +\infty);$$

(d) *if* $\lambda \in \rho(\mathcal{D}), \mu \in \rho(\mathcal{B})$, $(\lambda - \mathcal{D})^{-1}(\mu - \mathcal{B})^{-1} = (\mu - \mathcal{B})^{-1}(\lambda - \mathcal{D})^{-1}$.

Let $f \in (E, D(\mathcal{B}))_{\theta,p}$, *for* $\theta \in (0, 1), p \in [1, +\infty]$. *Then there exists a unique* $u \in D(\mathcal{D}) \cap D(\mathcal{B})$ *such that* $(\mathcal{D} - \mathcal{B})u = f$. *Moreover,* $\mathcal{B}u$ *and* $\mathcal{D}u$ *belong to* $(E, D(\mathcal{B}))_{\theta,p}$.

Proof. We distinguish some cases: assume first that $F = \mathbb{R}$. From (a) and (c) we have that $i\mathbb{R} \subseteq \rho(\mathcal{B})$ and $||(i\omega - \mathcal{B})^{-1}||_{\mathcal{L}(E)} \le M(1 + |\omega|)^{-1}$, for some $M > 0$ and for every $\omega \in \mathbb{R}$. It follows that there exist α and β positive such that $\{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| \le \alpha + \beta|\mathrm{Im}\,\lambda|\} \subseteq \rho(\mathcal{B})$ and in this set $||(\lambda - \mathcal{B})^{-1}||_{\mathcal{L}(E)} \le C(1 + |\lambda|)^{-1}$ for some $C > 0$. In this case we put $\mathrm{int}\,(\gamma) := \{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| < \alpha + \beta|\mathrm{Im}\,\lambda|\}$ and γ is the counterclockwise oriented boundary of $\mathrm{int}\,(\gamma)$.

Consider now the case that F is a proper subset of $\mathbb{R}$. If F is not upper (lower) bounded, there exists $\beta > 0$ such that $\{\lambda \in \mathbb{C} : \mathrm{Im}\,(\lambda) \ge \omega_+, |\mathrm{Re}\,(\lambda)| \le \beta|\mathrm{Im}\,(\lambda) - \omega_+|\} \subseteq \rho(\mathcal{B})$ $(\{\lambda \in \mathbb{C} : \mathrm{Im}\,(\lambda) \le \omega_-, |Re(\lambda)| \le \beta|\mathrm{Im}\,(\lambda) - \omega_-|\} \subseteq \rho(\mathcal{B}))$ and in this set $||(\lambda - \mathcal{B})^{-1}||_{\mathcal{L}(E)} \le C(1 + |\lambda|)^{-1}$ for some $C > 0$. We put $\mathrm{int}\,(\gamma_\pm) := \{\lambda \in \mathbb{C} : \mathrm{Im}\,(\lambda) > (<)\omega_\pm, |Re(\lambda)| < \beta|\mathrm{Im}\,(\lambda) - \omega_\pm|\}$ and $\gamma_\pm$ is the counterclockwise oriented boundary of $\mathrm{int}\,(\gamma_\pm)$. If $(\omega_-, \omega_+) \cap F \ne \emptyset$ (replacing $\omega_\pm$ with $\pm\infty$ in case one of them or both is not defined), we indicate with $\gamma_1, \dots, \gamma_p$ counterclockwise-oriented simple closed paths contained in $\rho(\mathcal{D}) \cap \rho(\mathcal{B})$ such that the union of their interiors $\mathrm{int}\,(\gamma_1), \dots, \mathrm{int}\,(\gamma_p)$ is contained in $\rho(\mathcal{B})$ and contains $iF \cap (i\omega_-, i\omega_+)$. Finally define $\gamma := \gamma_+ \cup \gamma_- \cup \gamma_1 \cup \dots \cup \gamma_p$ (some of the summands can lack), and $\mathrm{int}\,(\gamma) := \mathrm{int}\,(\gamma_+) \cup \mathrm{int}\,(\gamma_-) \cup \mathrm{int}\,(\gamma_1) \cup \dots \cup \mathrm{int}\,(\gamma_p)$. We assume that $0 \notin \gamma$.

Set now $\int_\gamma f(\lambda)d\lambda := \int_{\gamma_+} f(\lambda)d\lambda + \int_{\gamma_-} f(\lambda)d\lambda + \dots + \int_{\gamma_p} f(\lambda)d\lambda$ whenever each summand has a meaning and $S := \frac{1}{2\pi i}\int_\gamma (\lambda - \mathcal{D})^{-1}(\lambda - \mathcal{B})^{-1}d\lambda$. Assumptions (b) and (c) guarantee that $S \in \mathcal{L}(E)$. Let now $f \in (E, D(\mathcal{B}))_{\theta,\infty}$.We show that $Sf \in D(\mathcal{D}) \cap D(\mathcal{B})$ and $(\mathcal{D} - \mathcal{B})Sf = f$. If $\lambda \in \gamma$ we have $||\mathcal{B}(\lambda - \mathcal{B})^{-1} f|| \leq C(1 + |\lambda|)^{-\theta}$ for some $C > 0$. We have

$$Sf = \frac{1}{2\pi i} \int_\gamma \lambda^{-1}(\lambda - \mathcal{D})^{-1} f d\lambda + \frac{1}{2\pi i} \int_\gamma \lambda^{-1}(\lambda - \mathcal{D})^{-1}\mathcal{B}(\lambda - \mathcal{B})^{-1} f d\lambda.$$

The first integral equals $\delta\mathcal{D}^{-1}f$, with $\delta = 0$ if $0 \in \text{int}\,(\gamma)$, $\delta = 1$ otherwise. If $\lambda \in \gamma$ $||\lambda^{-1}\mathcal{D}(\lambda - \mathcal{D})^{-1}\mathcal{B}(\lambda - \mathcal{B})^{-1} f|| \leq C|\lambda|^{-1-\theta}$ for some $C > 0$. This implies that $Sf \in D(\mathcal{D})$ and

$$\mathcal{D}Sf = \delta f + \int_\gamma \lambda^{-1}\mathcal{D}(\lambda - \mathcal{D})^{-1}\mathcal{B}(\lambda - \mathcal{B})^{-1} f d\lambda.$$

From (d) we have that for any $\lambda \in \rho(\mathcal{D})$ $(\lambda - \mathcal{D})^{-1}(D(\mathcal{B})) \subseteq D(\mathcal{B})$ and $\mathcal{B}(\lambda - \mathcal{D})^{-1}(\lambda - \mathcal{B})^{-1} = (\lambda - \mathcal{D})^{-1}\mathcal{B}(\lambda - \mathcal{B})^{-1}$. Moreover, $||(\lambda - \mathcal{D})^{-1}\mathcal{B}(\lambda - \mathcal{B})^{-1} f||_{\mathcal{L}(E)} \leq C(1 + |\lambda|)^{-1-\theta}$ for any $\lambda \in \gamma, f \in (E, D(\mathcal{B}))_{\theta,\infty}$. It follows that $Sf \in D(\mathcal{B})$ and

$$\mathcal{B}Sf = \frac{1}{2\pi i} \int_\gamma (\lambda - \mathcal{D})^{-1}\mathcal{B}(\lambda - \mathcal{B})^{-1} f d\lambda.$$

So we have

$$\begin{aligned}(\mathcal{D} - \mathcal{B})Sf &= \delta f + \frac{1}{2\pi i} \int_\gamma (\lambda^{-1}\mathcal{D} - 1)(\lambda - \mathcal{D})^{-1}\mathcal{B}(\lambda - \mathcal{B})^{-1} f d\lambda \\ &= \delta f - \frac{1}{2\pi i} \int_\gamma \lambda^{-1}\mathcal{B}(\lambda - \mathcal{B})^{-1} f d\lambda = \delta f - (\delta - 1)f = f.\end{aligned}$$

Now we show that $\mathcal{B}Sf$ (and so also $\mathcal{D}Sf$) belong to $(E, D(\mathcal{B}))_{\theta,\infty}$. We start with the simpler case that γ is bounded. In this case it is easily seen that just assuming $f \in E$ we have $Sf \in D(\mathcal{D}) \cap D(\mathcal{B})$. Moreover, if $f \in D(\mathcal{B})$ and $\lambda_0 \in \rho(\mathcal{B})$ we have

$$\mathcal{B}Sf = (\lambda_0 - \mathcal{B})^{-1}\mathcal{B}S(\lambda_0 - \mathcal{B})f$$

so that $\mathcal{B}Sf \in D(\mathcal{B})$. By interpolation we have that, if $f \in (E, D(\mathcal{B}))_{\theta,\infty}$, $\mathcal{B}Sf \in (E, D(\mathcal{B}))_{\theta,\infty}$. We pass to consider the case that γ is not bounded. Pick $t \in \rho(\mathcal{B}) \cap \text{int}\,(\gamma)$; then

$$(t - \mathcal{B})^{-1}\mathcal{B}Sf = \frac{1}{2\pi i} \int_\gamma (t - \lambda)^{-1}(\lambda - \mathcal{D})^{-1}\mathcal{B}(\lambda - \mathcal{B})^{-1} f d\lambda.$$

Choose now a half-line γ' of maximal decrease of the resolvent of $\mathcal{B}$ contained in int (γ) such that there exists $C > 0$ so that for $\lambda \in \gamma, t \in \gamma'$ one has $|t - \lambda| \geq C(|t| +$

$|\lambda|)$ (for example, if F is not bounded above, we can take $\gamma' = \{i\omega : \omega \in \mathbb{R}, \omega \geq \omega_+\}$). We have, for $t \in \gamma'$,

$$\begin{aligned}\mathcal{B}(t-\mathcal{B})^{-1}\mathcal{B}Sf &= t(t-\mathcal{B})^{-1}\mathcal{B}Sf - \mathcal{B}Sf \\ &= \frac{1}{2\pi i}\int_\gamma \lambda(t-\lambda)^{-1}(\lambda-\mathcal{D})^{-1}\mathcal{B}(\lambda-\mathcal{B})^{-1}f\,d\lambda\end{aligned}$$

implying

$$||\mathcal{B}(t-\mathcal{B})^{-1}\mathcal{B}Sf|| \leq C\int_\gamma (|t|+|\lambda|)^{-1}|\lambda|^{-\theta}d|\lambda| \leq C|t|^{-\theta}.$$

So, if $p = +\infty$, we have that $\mathcal{D}Sf$ and $\mathcal{B}Sf$ belong to $(E, D(\mathcal{B}))_{\theta,\infty}$. By interpolation this can be extended to any $p \in [1, +\infty]$.

It remains to show the uniqueness of the solution; let $u \in D(\mathcal{D}) \cap D(\mathcal{B})$ such that $(\mathcal{D}-\mathcal{B})u = 0$; it is easily seen that $S\mathcal{D}u = \mathcal{D}Su, S\mathcal{B}u = \mathcal{B}Su$. So,

$$0 = S(\mathcal{D}-\mathcal{B})u = (\mathcal{D}-\mathcal{B})Su = u,$$

because $u \in (E, D(\mathcal{B}))_{\theta,p}$ for any $\theta \in (0,1)$.

We set now, for $0 < \theta < 1$, $1 \leq p \leq +\infty$, $Z_{\theta,p} := (Z, Z_1)_{\theta,p}$, $Z_{1+\theta,p} := \{z \in Z_1 : Gz \in Z_{\theta,p}\}$. Remark that, if (C') is satisfied, G is a closed operator in Z and so $Z_{1+\theta,p}$ is a Banach space with its natural norm. We have

Lemma 4.6. *Assume that* (C′) *and* (D) *are satisfied. Let* $\theta \in (0,1), p \in [1,+\infty]$; *then there exists* $M > 0$ *such that for every* $u \in W^{1,p}(Z_{\theta,p}) \cap W^{0,p}(Z_{1+\theta,p})$ *one has*

$$||u||_{1,p,Z_{\theta,p}} + ||Gu||_{0,p,Z_{\theta,p}} \leq M[||u'-Gu||_{0,p,Z_{\theta,p}} + ||u||_{0,p,Z_{\theta,p}}].$$

Moreover, if $T > 0$, *for every* $u \in W^{1,p}_T(Z_{\theta,p}) \cap W^{0,p}_T(Z_{1+\theta,p})$ *one has*

$$||u||_{1,p,T,Z_{\theta,p}} + ||Gu||_{0,p,T,Z_{\theta,p}} \leq M[||u'-Gu||_{0,p,T,Z_{\theta,p}} + ||u||_{0,p,T,Z_{\theta,p}}].$$

Proof. Observe first that our assumptions imply that there exists $m \geq 0$ such that $i((-\infty,-m] \cup [m,+\infty)) \subseteq \rho(G)$ and, for a certain $C > 0$, $||(i\omega - G)^{-1}||_{\mathcal{L}(Z)} \leq C|\omega|^{-1}$. Consider first the nonperiodic case; for $1 \leq p \leq +\infty$, set $E := \{z \in W^{0,p}(Z) : \operatorname{supp}(\hat{z}) \subseteq (-\infty,-m] \cup [m,+\infty)\}$. E is a closed subspace of $W^{0,p}(Z)$. Put $D(\mathcal{D}) = W^{1,p}(Z) \cap E$, $\mathcal{D}z = z'$. $\mathcal{D}$ is clearly an operator in E. It is easily seen that $\mathbb{C} - i\mathbb{R} \subseteq \rho(\mathcal{D})$; moreover, if $\operatorname{Re}\lambda \neq 0, (\lambda-\mathcal{D})^{-1}z = K_\lambda * z$, with $K_\lambda(t) = H(t)e^{\lambda t}$ if $\operatorname{Re}\lambda < 0$, $K_\lambda(t) = -H(-t)e^{\lambda t}$ if $\operatorname{Re}\lambda > 0$. Let now $\omega \in \mathbb{R}, |\omega| < m$. We show that $i\omega \in \rho(\mathcal{D})$: in fact, let $u \in D(\mathcal{D})$ such that $i\omega u - \mathcal{D}u = 0$; this implies that $\operatorname{supp}(\hat{u}) \subseteq \{\omega\}$ which implies $u = 0$. Let $f \in E$; fix $\hat{\psi} \in \mathcal{D}(\mathbb{R})$ such that $\operatorname{supp}(\hat{\psi}) \subseteq [-m,m]$ and $\hat{\psi}(\xi) = -1$ in a neighbourhood of ω. Let $\hat{k}(\xi) = i(\omega-\xi)^{-1}(\hat{\psi}(\xi)+1)$, $k = \mathcal{F}^{-1}\hat{k}$. We verify that $k \in L^1(\mathbb{R})$. We have for any l and j in $\mathbb{N}_0$ $(-it)^l k^{(j)} = \mathcal{F}^{-1}(\partial^{(l)}((i\xi)^j\hat{k})$

and $\partial^l((i\xi)^j\hat{k}) \in L^1(\mathbb{R})$ if $l \geq j+1$. This means that $k_{|\mathbb{R}-\{0\}} \in C^\infty(\mathbb{R}-\{0\})$ and is rapidly decreasing at infinity with all its derivatives. Set now

$$h(t) = -\frac{1}{2\pi it}\int_{\mathbb{R}} e^{it\xi}\hat{k}'(\xi)\,d\xi.$$

As $\int_{\mathbb{R}} \hat{k}'(\xi)\,d\xi = 0$, we have for any $\alpha \in (0,1)$, $t \in \mathbb{R}-\{0\}$

$$|h(t)| \leq |2\pi t|^{-1}\int_{\mathbb{R}} |e^{it\xi}-1||t\xi|^{-\alpha}|t\xi|^{\alpha}|\hat{k}'(\xi)|\,d\xi \leq C|t|^{\alpha-1}$$

which implies that $h = O(|t|^{\alpha-1})$ as t tends to 0 for any $\alpha \in (0,1)$. Observe now that $(-it)h = \mathcal{F}^{-1}(\hat{k}') = (-it)k$ so that $\mathrm{supp}\,(k-h) \subseteq \{0\}$. This implies that $h \in L^1(\mathbb{R})$ and, as $\mathcal{F}(k-h) = o(1)$ as $|\xi| \to +\infty$, that $h = k$. Put now $u = k * f$. Then $u \in E$ and, in the sense of distributions with values in E, $(i\omega - \partial)u = f + \psi * f = f$, where $\mathcal{F}\psi = \hat{\psi}$. This implies also that $u' = i\omega u - f$, so that $u \in D(\mathcal{D})$ and $(i\omega - \mathcal{D})u = f$. If we put $F = (-\infty,-m] \cup [m,+\infty)$, we have $\sigma(\mathcal{D}) \subseteq iF$; thanks to estimate $||K_\lambda||_{0,1} \leq |\mathrm{Re}\,\lambda|^{-1}$ for $\mathrm{Re}\,\lambda \neq 0$ it is easily seen that assumption (b) in 4.5 is satisfied. Set now $D(\mathcal{B}) := W^{0,p}(Z_1) \cap E$, $\mathcal{B}u(t) := Gu(t)$ for any $u \in D(\mathcal{B}), t \in \mathbb{R}$. Then all other assumptions in 4.5 are easy to verify. So we have that for any $f \in (E, D(\mathcal{B}))_{\theta,p}$ there exists a unique $u \in D(\mathcal{D}) \cap D(\mathcal{B})$ such that $(\mathcal{D}-\mathcal{B})u = f$; moreover, $\mathcal{D}u$ and $\mathcal{B}u$ belong to $(E, D(\mathcal{B}))_{\theta,p}$. We try now to characterize the space $(E, D(\mathcal{B}))_{\theta,p}$; using 1.14.2 in [15] it is not difficult to see that, if $1 \leq p < +\infty$, $(E, D(\mathcal{B}))_{\theta,p} = \{f \in L^p(Z_{\theta,p}) : \mathrm{supp}\,(\hat{f}) \subseteq F\}$, while, in case $p = +\infty$, $(E, D(\mathcal{B}))_{\theta,\infty} = \{f \in W^{0,\infty}(Z) : f(t) \in Z_{\theta,\infty}$ and $||f(t)||_{(E,D(G))_{\theta,p}} \leq C$ for some $C > 0$ for almost every $t \in \mathbb{R}\}$.

Let now $u \in W^{1,p}(Z_{\theta,p}) \cap W^{0,p}(Z_{1+\theta,p})$. Fix $\chi \in \mathcal{S}(\mathbb{R})$ such that $\hat{\chi} \in \mathcal{D}(\mathbb{R})$, $\hat{\chi}(\omega) \equiv 1$ if $|\omega| \leq m+1$ and set $u_1 := u - \chi * u, u_2 := \chi * u$. We have that $\mathrm{supp}\,(\mathcal{F}u_1) \subseteq (-\infty,-m] \cup [m,+\infty)$ and $u_1 \in W^{1,p}(Z_{\theta,p}) \cap W^{0,p}(Z_{1+\theta,p})$. It follows that

$$\begin{aligned} &||u_1'||_{0,p,Z_{\theta,p}} = ||\mathcal{D}u_1||_{0,p,Z_{\theta,p}} \leq C||(\mathcal{D}-\mathcal{B})u_1||_{0,p,Z_{\theta,p}} \\ =&C||(u'-Gu) - \chi * (u'-Gu)||_{0,p,Z_{\theta,p}} \leq C||u'-Gu||_{0,p,Z_{\theta,p}}. \end{aligned}$$

Moreover,

$$||u_2'||_{0,p,Z_{\theta,p}} = ||\chi' * u||_{0,p,Z_{\theta,p}} \leq C||u||_{0,p,Z_{\theta,p}}$$

so that, for some $C > 0$,

$$||u'||_{0,p,Z_{\theta,p}} \leq C(||u'-Gu||_{0,p,Z_{\theta,p}} + ||u||_{0,p,Z_{\theta,p}}).$$

By difference one has the same estimate for $||Gu||_{0,p,Z_{\theta,p}}$.

An analogous argument can be applied to the periodic case.

We have therefore that

Theorem 4.7. *Assume that conditions* (C′) *and* (D) *are satisfied,* $\theta \in (0,1)$, $p \in [1,+\infty]$; *suppose that* X_θ *and* Y_θ *are Banach spaces continuously embedded respectively in* X *and* Y, *such that* $\mathcal{A}(X_\theta) \subseteq Y_\theta$, $\gamma(X_\theta) = Z_{1+\theta,p}$, $\mathcal{B}(X_\theta) \subseteq Z_{\theta,p}$; *let* $K \in L^1(\mathbb{R};\mathcal{L}(X_\theta,Y_\theta))$, $H \in L^1(\mathbb{R};\mathcal{L}(X_\theta,Z_{\theta,p}))$. *Assume that conditions* (C) *and* (D) *are satisfied substituting for* X X_θ, *for* Y Y_θ, *for* Z $Z_{\theta,p}$; *then, in the new setting, problem* (1.2) *is* $(0,p)$*-well posed.*

Proof. It follows from 4.3 and 4.6.

Analogously (applying 4.4 and 4.6) one obtains:

Theorem 4.8. *Assume that conditions* (C′) *and* (D) *are satisfied,* $\theta \in]0,1[$, $p \in [1,+\infty]$, $T > 0$*; suppose that* X_θ *and* Y_θ *are Banach spaces continuously embedded respectively in* X *and* Y, *such that* $\mathcal{A}(X_\theta) \subseteq Y_\theta$, $\gamma(X_\theta) = Z_{1+\theta,p}$, $\mathcal{B}(X_\theta) \subseteq Z_{\theta,p}$*; let* $K \in L^1(\mathbb{R};\mathcal{L}(X_\theta,Y_\theta))$, $H \in L^1(\mathbb{R};\mathcal{L}(X_\theta,Z_{\theta,p}))$. *Assume that conditions* (C_T) *and* (D) *are satisfied substituting to* X X_θ, *to* Y Y_θ, *to* Z $Z_{\theta,p}$*; then in the new setting problem* (1.2) *is* $(0,p,T)$*-well posed.*

5. Examples and applications.

5.1. The following simple example shows that assumption (C) does not imply (D): consider the problem

$$\begin{aligned} &\partial_t u(t,x) - \partial_t(\chi *_x u(t,.))(x) - \Delta_x u(t,x) + u(t,x) \\ &\qquad - \int_{\mathbb{R}} a(t-s)\Delta_x u(s,x)\,ds = g(t,x) \end{aligned} \tag{5.1}$$

under the following assumptions (we have indicated with $*_x$ the convolution with respect to the x variable only):

- (h_1) $a \in L^1(\mathbb{R})$ and $\hat{a}(\omega) \neq -1$ for any $\omega \in \mathbb{R}$;
- (h_2) $\chi \in \mathcal{S}(\mathbb{R}^n), \hat{\chi} \in \mathcal{D}(\mathbb{R}^n), \hat{\chi}(\xi) = 1$ if $|\xi| \leq 1$, $\hat{\chi}$ is real-valued;
- (h_3) $M(\omega,\xi) = i\omega[1-\hat{\chi}(\xi)] + [1+\hat{a}(\omega)]|\xi|^2 + 1 \neq 0$ for any $\omega \in \mathbb{R}, \xi \in \mathbb{R}^n$.

We set $X = H^2(\mathbb{R}^n)$, $Y = \{0\}$, $Z = L^2(\mathbb{R}^n)$, $\gamma u = u - \chi * u$ for any $u \in X$, $\mathcal{B} = \Delta - 1$, $H(t) = a(t)\Delta$. If $\omega \in \mathbb{R}, u \in X$, we have $S(\omega)u = i\omega(u - \chi * u) - (1 + \hat{a}(\omega))\Delta u + u$, $S_0(\omega)u = i\omega(u - \chi * u) - \Delta u + u$. Fix $g \in Z$ and consider the equation

$$S(\omega)u = g,$$

looking for a solution $u \in X$. Observe first that there exists $C > 0$ such that for any $\xi \in \mathbb{R}^n, \omega \in \mathbb{R}$, $|M(\omega,\xi)| \geq C[|\omega||1-\hat{\chi}(\xi)| + |\xi|^2 + 1]$. In fact, put

$$M_0(\omega,\xi) = i\omega(1-\hat{\chi}(\xi)) + |\xi|^2 + 1.$$

One has

$$|M(\omega,\xi)| \geq |M_0(\omega,\xi)| - |\hat{a}(\omega)||\xi|^2$$

which gives the desired estimate if $|\omega|$ is sufficiently large, taking into account the fact that $\hat{a}(\omega) \to 0$ as $|\omega|$ tends to $+\infty$. Let now $|\omega| \le C_1$, for some $C_1 \ge 0$. Remark that there exists $C_2 > 0$ such that for any $\omega \in \mathbb{R}$ $|1 + \hat{a}(\omega)| \ge C_2$. So

$$|M(\omega, \xi)| \ge C_2|\xi|^2 - 1 - C_1|1 - \hat{\chi}(\xi)|$$

giving the desired estimate if $|\xi|$ is sufficiently large. The case when both $|\omega| \le C_1$ and $|\xi| \le C_3$ for some $C_3 \ge 0$ is a consequence of (h_3). Therefore, it follows (using Parseval's identity) that condition (C) is satisfied. As a consequence, taking into account that X and Z are Hilbert spaces, we have for example that, if $\theta \ge 0$, $p \in (1, +\infty)$, for any $g \in W^{\theta,p}(Z)$ there exists a unique $u \in W^{\theta,p}(X)$ with $u - \chi *_x u \in W^{1+\theta,p}(Z)$ solving (5.1). Remark that γ is not injective, so that condition (D) is not satisfied.

5.2. Set now, for $u : \mathbb{R} \to \mathbb{C}^2$, $u(x) = (u_1(x), u_2(x))$,

$$\mathcal{B}_0 u(x) := (-u_2'(x), u_1'(x)).$$

Let $\omega \in \mathbb{R}^+$. We consider the following problem:

$$\begin{cases} \partial_t u(t,x) - \mathcal{B}_0 u(t,x) = f(t,x), t \in \mathbb{R}, x \in (0,\omega), \\ u_1(t,0) = u_1(t,\pi) = 0, t \in \mathbb{R}. \end{cases} \tag{5.2}$$

We set $X = \{u \in W^{1,1}((0,\omega);\mathbb{C}^2) : u_1(0) = u_1(\omega) = 0, \int_0^\omega u_2(x)\,dx = 0\}$, $Z = \{f \in L^1((0,\omega);\mathbb{C}^2) : \int_0^\omega f_2(x)\,dx = 0\}$, $Y = \{0\}$, and indicate with γ the embedding of X in Z and with $\mathcal{B}$ the operator of domain X such that $\mathcal{B}u = \mathcal{B}_0 u$ for every $u \in X$. One can verify (see [7]) that $\sigma(\mathcal{B}) = \{\frac{k\pi}{\omega} : k \in \mathbb{Z}\} - \{0\}$ and for any $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that if $\lambda \in \mathbb{C}$ and $|\mathrm{Im}\,(\lambda)| \ge \epsilon$, $||(\lambda - \mathcal{B})^{-1}||_{\mathcal{L}(Z)} \le C(\epsilon)|\mathrm{Im}\,z|^{-1}$. So, condition (C) is satisfied and, by 3.3, given $f \in W^{0,1}(Z)$, there exists at most one solution u of (5.2) belonging to $W^{1,1}(Z) \cap W^{0,1}(X)$; by 3.12, if $f \in W^{\theta,1}(Z)$ for some $\theta \in \mathbb{R}^+ - \mathbb{N}$, such a solution really exists and belong to $W^{\theta,1}(X) \cap W^{1+\theta,1}(Z)$.

We have already observed in the introduction that this problem cannot be reduced by projection to some case treated in [13].

We apply now the foregoing result to a boundary value problem for the Poisson equation in an angle: let $\omega \in (0, 2\pi)$ and $\Omega := \{(y,z) \in \mathbb{R}^2 : y = \rho\cos(x), z = \rho\sin(x), (\rho, x) \in \mathbb{R}^+ \times (0,\omega)\}$; let $h \in L^1(\Omega)$; we consider the problem of the existence and the regularity of a function w such that $\Delta w = h$ in Ω and $w_{|\partial\Omega} = 0$. We set $z(\rho, x) := w(\rho\cos(x), \rho\sin(x))$ $((\rho, x) \in \mathbb{R}^+ \times (0,\omega))$, $v(t,x) := z(e^{-t}, x)$ $((t,x) \in \mathbb{R} \times (0,\omega))$; it can be easily verified that $\Delta v = g$ in $\mathbb{R} \times (0,\omega)$, with $||g||_{L^1(\mathbb{R}\times(0,\omega))} = ||h||_{L^1(\Omega)}$. Moreover,

$$||\partial_t v||_{L^1(\mathbb{R}\times(0,\omega))} = ||\partial_\rho z||_{L^1(\mathbb{R}^+\times(0,\omega))},$$

$$||\partial_x v||_{L^1(\mathbb{R}\times(0,\omega))} = ||\rho^{-1}\partial_x z||_{L^1(\mathbb{R}^+\times(0,\omega))},$$

$$||\partial_t^2 v||_{L^1(\mathbb{R}\times(0,\omega))} = ||\partial_\rho(\rho\partial_\rho z)||_{L^1(\mathbb{R}^+\times(0,\omega))},$$
$$||\partial_{tx}^2 v||_{L^1(\mathbb{R}\times(0,\omega))} = ||\partial_{\rho,x}^2 z)||_{L^1(\mathbb{R}^+\times(0,\omega))},$$
$$||\partial_x^2 v||_{L^1(\mathbb{R}\times(0,\omega))} = ||\rho^{-1}\partial_x^2 z)||_{L^1(\mathbb{R}^+\times(0,\omega))}.$$

So, we shall study the Poisson equation in the stripe $\mathbb{R}\times(0,\omega)$:

Proposition 5.3. *Let $g \in L^1(\mathbb{R}\times(0,\omega))$; there exists at most one function $v \in L^1_{loc}(\mathbb{R}\times(0,\omega)$ such that*

(a) *$\Delta v = g$ in $\mathbb{R}\times(0,\omega)$;*
(b) *the derivatives of order 1 and 2 of v belong to $L^1(\mathbb{R}\times(0,\omega))$;*
(c) *$v(.,0) = v(.,\omega) = 0$ (owing to (b) v can be identified with an element of $C([0,\omega];\mathcal{D}'(\mathbb{R}))$; so it has a sectional trace $v(.,x)$ for any $x \in [0,\omega]$).*

If $g \in W^{\theta,1}(\mathbb{R};L^1((0,\omega))$ such a solution v really exists and its derivatives of order 1 and 2 of v belong to $W^{\theta,1}(\mathbb{R};L^1((0,\omega)))$.

Proof. We start by proving the uniqueness of the solution; assume v satisfies all conditions (a)–(c) with $g = 0$; set $u_1 := \partial_t v, u_2 := \partial_x v$, $u(t,x) := (u_1(t,x), u_2(t,x))$; then $u \in W^{0,1}(X) \cap W^{1,1}(Z)$ and solves (5.2) with $f = 0$; it follows that $u \equiv 0$, which implies that v is constant in $\mathbb{R}\times(0,\omega)$; so the result follows from (c). We show the existence in case $g \in W^{\theta,1}(\mathbb{R};L^1((0,\omega)))$: set $f(t,x) := (g(t,x),0)$ and let u be the solution of (5.2) in $W^{\theta,1}(X) \cap W^{1+\theta,1}(Z)$; we have $\partial_t u_2 = \partial_x u_1$, so that there exists $v_0 \in \mathcal{D}'(\mathbb{R}\times(0,\omega))$ such that $\partial_t v_0 = u_1$ and $\partial_x v_0 = u_2$; (5.2) implies that $\Delta v_0 = g$; from $u_1(.,0) = u_1(.,\omega) = 0$, it follows that $v_0(.,0)$ and $v_0(.,\omega)$ are constant functions; the condition $\int_0^\omega u_2(t,x)\,dx = 0$ for almost every t implies that $v_0(.,0) = v_0(.,\omega) = C$, for some $C \in \mathbb{C}$; setting $v := v_0 - C$, one has the solution with all the desired properties.

5.4. We conclude with problem (1.1). Let Ω be an open bounded subset of $\mathbb{R}^n$ lying on one side of its boundary $\partial\Omega$, which is a compact submanifold of $\mathbb{R}^n$ of class C^∞. Let $A(x,\partial_x)$ be a second-order properly elliptic differential operator with coefficients of class $C^\infty(\overline{\Omega})$. For $x' \in \partial\Omega$ let $B(x',\partial_x)$ be a first-order differential operator with coefficients of class $C^\infty(\partial\Omega)$ such that $\partial\Omega$ is not characteristic for it. We shall often write $A(\partial)$ and $B(\partial)$ instead of $A(x,\partial_x)$ and $B(x',\partial_x)$. We shall indicate with $A^\#(\partial)$ and $B^\#(\partial)$ the principal parts of, respectively, $A(\partial)$ and $B(\partial)$. Let, for $t \in \mathbb{R}$, $K(t,x,\partial_x)$ be a linear differential operator of order not exceeding two whose coefficients are in $L^1(\mathbb{R};C^\infty(\overline{\Omega})) := \bigcap_{k=1} L^1(\mathbb{R};C^k(\overline{\Omega}))$, $H(t,x',\partial_x)$ a linear differential operator of order not exceeding one whose coefficients are in $L^1(\mathbb{R};C^\infty(\partial\Omega)) := \bigcap_{k=1}^\infty L^1(\mathbb{R};C^k(\partial\Omega))$. We shall often write $K(t,\partial)$ instead of $K(t,x,\partial_x)$ and $H(t,\partial)$ instead of $H(t,x',\partial_x)$. We shall indicate with $\hat{K}(\omega,\partial)$ or $\hat{K}(\omega,x,\partial)$ and with $\hat{H}(\omega,\partial)$ or $\hat{H}(\omega,x',\partial)$ the operators obtained taking the Fourier transforms of the coefficients with respect to t, with parts of order respectively two and one, $\hat{K}^\#(\omega,\partial)$ and $\hat{H}^\#(\omega,\partial)$. We assume that the following assumptions are satisfied:

(h1) : for any $\omega \in \mathbb{R}$, $A(\partial) + \hat{K}(\omega,\partial)$ is properly elliptic;

(h2) : there exist $\theta_0 \in (-\pi, \pi], \nu > 0$ such that for any $x \in \overline{\Omega}, \xi \in \mathbb{R}^n, r \in \mathbb{R}$ we have

$$|A^{\#}(x, i\xi) + \hat{K}^{\#}(\omega, x, i\xi) - r^2 e^{i\theta_0}| \geq \nu(|\xi|^2 + r^2).$$

(h3) : Consider the o.d.e. problem

$$\begin{cases} A^{\#}(x', i\xi' + \nu(x')\partial_t)v(t) - r^2 e^{i\theta_0} v(t) = 0, \\ i\omega v(0) - B^{\#}(x', i\xi' + \nu(x')\partial_t)v(0) = 0. \end{cases}$$

If $x' \in \partial\Omega, \xi' \in T_{x'}(\partial\Omega), r \in \mathbb{R}, \omega \in \mathbb{R}$ and $(\xi', r) \neq (0, 0)$, the unique solution bounded in $\mathbb{R}^+$ is the trivial one.

(h4) : Consider the o.d.e. problem

$$\begin{cases} A^{\#}(x', i\xi' + \nu(x')\partial_t)v(t) + \hat{K}^{\#}(\omega, x', i\xi' + \nu(x')\partial_t)v(t) - r^2 e^{i\theta_0} v(t) = 0, \\ B^{\#}(x', i\xi' + \nu(x')\partial_t)v(0) + \hat{H}^{\#}(\omega, x', i\xi' + \nu(x')\partial_t)v(0) = 0. \end{cases}$$

If $x' \in \partial\Omega, \xi' \in T_{x'}(\partial\Omega), r \in \mathbb{R}, \omega \in \mathbb{R}$ and $(\xi', r) \neq (0, 0)$, the unique solution bounded in $\mathbb{R}^+$ is the trivial one.

We want to study problem (1.1) under the assumptions (h1)–(h4); we recall (see the introduction) that problems of this type are of interest in viscoelasticity, in the case of viscous boundary conditions (for related results in the Hilbert space case see [1]).

We fix $p \in (1, +\infty), s \in \mathbb{R}, s > \frac{1}{p} - 1$. We indicate with γ_Ω the trace operator on $\partial\Omega$. We start with the following

Lemma 5.5. *Consider the problem*

$$\begin{cases} A(\partial)u - r^2 e^{i\theta_0} u = f \ \text{in } \Omega, \\ \gamma_\Omega u = g. \end{cases} \tag{5.3}$$

Then there exists $r_0 \geq 0$ such that (5.3) has a unique solution $u \in W^{s+2,p}(\Omega)$ for any $r \geq r_0, f \in W^{s,p}(\Omega), g \in W^{s+2-\frac{1}{p},p}(\partial\Omega)$.

Proof. From (h2) we have (letting $(\omega \to +\infty)$ that $|A^{\#}(x, i\xi) - r^2 e^{i\theta_0}| \geq \nu(|\xi|^2 + r^2)$ for any $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$, $r \in \mathbb{R}$, $(\xi, r) \neq (0, 0)$. So the proof for $s \notin \mathbb{Z}$ can be obtained with the same method as [10], Theorem 2.17, where a similar result is shown in the strongly elliptic case. In the same order of ideas one can prove the result for $s \in \mathbb{Z}$ (see [14], paragraph 3.8).

We put now $V = \mathbb{R} \times \Omega$, with generic element (t, x), and introduce the operators $L := e^{i\theta_0}\partial_t^2 + A(\partial_x)$ and, for any $\omega \in \mathbb{R}$, $L_\omega := L + \hat{K}(\omega, \partial_x)$. Also, we indicate with γ_V the trace operator on $\partial V = \mathbb{R} \times \partial\Omega$.

Lemma 5.6. *There exists $C > 0$ such that for any $\omega \in \mathbb{R}$, for any $v \in W^{s+2,p}(V)$*

$$||v||_{s+2,p,V} \leq C[||Lv||_{s,p,V} + ||\gamma_V(i\omega v - B(\partial_x)v)||_{s+1-\frac{1}{p},\partial V} + ||v||_{s,p,V}].$$

Proof. By the usual methods of partitions of unity, compositions with diffeomorphisms and Korn's artifice, one can reduce oneself to the case $\Omega = \mathbb{R}^n_+, A(\partial) = A(\partial)^\#$, $B(\partial) = B(\partial)^\#$, both with constant coefficients. We start by considering the case $Lv - e^{i\theta_0}v = 0$. Consider the o.d.e. problem:

$$\begin{cases} -e^{i\theta_0}(\tau^2+1)v(x_n) + A(i\xi', \partial_{x_n})v(x_n) = 0, \\ i\omega v(0) - B(i\xi', \partial_{x_n})v(0) = 1. \end{cases}$$

Owing to (h3) such a problem has a unique solution $x_n \to \Omega(\omega, \tau, \xi', x_n)$ exponentially decaying in $\mathbb{R}^+$ for every $\omega \in \mathbb{R}, \tau \in \mathbb{R}, \xi' \in \mathbb{R}^{n-1}$. If $g = \gamma_V(i\omega v - B(\partial_x)v)$, we have

$$\mathcal{F}(\gamma_V v)(\tau, \xi') = \Omega(\omega, \tau, \xi', 0)\hat{g}(\tau, \xi').$$

It is easily seen that $\Omega(\omega, \tau, \xi', 0) = [i\omega - B(i\xi', \lambda(\tau, \xi')]^{-1}$, where $\lambda(\tau, \xi')$ is the unique solution with negative real part of the algebraic equation

$$A(i\xi', \lambda) - e^{i\theta_0}(\tau^2 + 1) = 0.$$

If $(\tau, \rho, \xi') \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n-1}$, $(\tau, \rho, \xi') \neq (0, 0, 0)$, we indicate with $\lambda(\tau, \rho, \xi')$ the unique solution with negative real part of the algebraic equation

$$A(i\xi', \lambda) - e^{i\theta_0}(\tau^2 + \rho^2) = 0$$

so that $\lambda(\tau, \xi') = \lambda(\tau, 1, \xi')$. Remark that λ is positively homogeneous of order one and of class C^∞ in $(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}) - \{(0, 0, 0)\}$. Now we show that for any $\beta \in \mathbb{N}^n_0$ there exists $C(\beta) \geq 0$ such that for any $(\tau, \xi') \in \mathbb{R} \times \mathbb{R}^{n-1}$, for any $\omega \in \mathbb{R}$

$$|\partial^\beta_{\tau,\xi'}\Omega(\omega, \tau, \xi', 0)| \leq C(\beta)(1 + |\tau| + |\xi'|)^{-|\beta|-1}.$$

To this aim, observe first that this is true if $|\beta| = 0$, because $\Omega(\omega, \tau, \xi', 0)^{-1} = i\omega - B(i\xi', \lambda(\tau, 1, \xi'))$ and $S(\omega, \tau, \rho, \xi') := i\omega - B(i\xi', \lambda(\tau, \rho, \xi'))$ is a continuous, positively homogeneous function of order one in $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}$ (putting $\lambda(0, 0, 0) = 0$), vanishing only in $(0, 0, 0, 0)$ (owing to (h3)).

Remark also that it is of class C^∞ outside $\{(\omega, 0, 0, 0) : \omega \in \mathbb{R}\}$. Consider now the case $|\beta| > 0$, assuming the estimate true for any multi-index δ such that $|\delta| < |\beta|$. One has

$$\begin{aligned} &\partial^\beta_{\tau,\xi'}\Omega(\omega, \tau, \xi', 0) \\ &= -\Omega(\omega, \tau, \xi', 0) \sum_{\delta<\beta} \binom{\beta}{\delta} \partial^{\beta-\delta}_{(\tau,\xi')} S(\omega, \tau, 1, \xi') \partial^\delta_{(\tau,\xi')}\Omega(\omega, \tau, \xi', 0). \end{aligned}$$

We have that, as $|\beta - \delta| \geq 1$,

$$\partial^{\beta-\delta}_{(\tau,\xi')} S(\omega, \tau, 1, \xi') = \partial^{\beta-\delta}_{(\tau,\xi')} T(\tau, 1, \xi')$$

with $T(\tau, \rho, \xi') = B(i\xi', \lambda(\tau, \rho, \xi'))$. T is positively homogeneous of degree one in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}$ and of class C^∞ outside $(0, 0, 0)$, and this implies that

$$|\partial^{\beta-\delta}_{(\tau,\xi')} T(\tau, 1, \xi')| \leq C(\beta - \delta)(1 + |\tau| + |\xi'|)^{1-|\beta|+|\delta|},$$

and from this the estimate that we want follows easily. So we can apply Mikhlin's multiplier theorem, assuring that there exists $C > 0$ independent of ω such that

$$||\gamma_V v||_{s+2-\frac{1}{p},p,\partial V} \leq C||g||_{s+1-\frac{1}{p},p,\partial V}.$$

As L is properly elliptic, it follows that

$$||v||_{s+2,p,V} \leq C[||\gamma_V v||_{s+2-\frac{1}{p},p,\partial V} + ||v||_{s,p,V}] \leq C[||g||_{s+1-\frac{1}{p},p,\partial V} + ||v||_{s,p,V}].$$

Assume now that $Lv = f$. Then, $Lv - e^{i\theta_0}v = f - e^{i\theta_0}v$. Let $\tilde{v}$ be the solution in $W^{s+2,p}(V)$ of the problem

$$\begin{cases} (L - e^{i\theta_0})\tilde{v} = f - e^{i\theta_0}v, \\ \gamma_V \tilde{v} = 0. \end{cases}$$

The existence and uniqueness of the solution can be proved following the method of Lemma 2.8 and Proposition 2.3 in [10]. We have

$$\begin{aligned} &(L - e^{i\theta_0})(v - \tilde{v}) = 0, \\ &\gamma_V((i\omega - B(\partial_x))(v - \tilde{v})) = g + \gamma_V B(\partial_x)\tilde{v}, \end{aligned}$$

so that

$$\begin{aligned} &||v||_{s+2,p,V} \leq ||\tilde{v}||_{s+2,p,V} + ||v - \tilde{v}||_{s+2,p,V}, \\ &||\tilde{v}||_{s+2,p,V} \leq C(||f||_{s,p,V} + ||v||_{s,p,V}), \end{aligned}$$

and

$$||v - \tilde{v}||_{s+2,p,V} \leq C(||g||_{s+1-\frac{1}{p},p,\partial V} + ||\tilde{v}||_{s+2,p,V} + ||v||_{s,p,V}.$$

Lemma 5.7. *There exists $C > 0$ such that for any $v \in W^{s+2,p}(V)$, for any $\omega \in \mathbb{R}$*

$$\begin{aligned} &||v||_{s+2,p,V} \\ &\leq C(||L_\omega v||_{s,p,V} + ||\gamma_V(i\omega - B(\partial_x) - \hat{H}(\omega, \partial_x)v)||_{s+1-\frac{1}{p},p,\partial V} + ||v||_{s,p,V}). \end{aligned}$$

Proof. If $|\omega|$ is sufficiently large it follows from 5.6, by a simple perturbation argument, remarking that the coefficients of L_ω and $\hat{H}(\omega, \partial_x)$ converge respectively to the corresponding coefficients of L in $C^\infty(\overline{\Omega})$ and to 0 in $C^\infty(\partial\Omega)$ as $|\omega|$ tends to $+\infty$. The case $|\omega| \leq \omega_0$ for a fixed positive ω_0 is a consequence of assumption (h4) (see for example Proposition 2.9 in [10]).

Lemma 5.8. *Consider the problem*

$$r^2 e^{i\theta_0} u - A(\partial)u - \hat{K}(\omega,\partial)u = f \ \textit{in}\ \Omega,$$
$$\gamma_\Omega(i\omega u - B(\partial)u - \hat{H}(\omega,\partial)u) = g.$$

Then there exists $r_1 \geq 0$ *such that, if* $r \geq r_1$*, this problem has a unique solution* $u \in W^{s+2,p}(\Omega)$ *for any* $\omega \in \mathbb{R}, f \in W^{s,p}(\Omega), g \in W^{s+1-\frac{1}{p},p}(\partial\Omega)$*.*

Moreover, one has

$$||u||_{s+2,p,\Omega} + |\omega|||\gamma_\Omega u||_{s+1-\frac{1}{p},p,\partial\Omega} \leq C(r)[||f||_{s,p,\Omega} + ||g||_{s+1-\frac{1}{p},p,\partial\Omega}]. \tag{5.4}$$

Proof. We start by looking for estimates of the solution. Let $u \in W^{s+2,p}(\Omega)$ solve the problem; fix $\zeta \in \mathcal{D}(\mathbb{R})$ such that $\zeta(t) = 1$ if $|t| \leq 1$ and put $v(t,x) := e^{irt}\zeta(t)u(x)$. Using a well-known method due to Agmon, one applies the estimate obtained in 5.7 which is uniform in ω, to obtain that there exists $r_1 \geq 0$ independent of ω such that, if $r \geq r_1$,

$$||u||_{s+2,p,\Omega} \leq C(r)[||f||_{s,p,\Omega} + ||g||_{s+1-\frac{1}{p},p,\partial\Omega}],$$

where $C(\cdot)$ is a locally bounded function of r. (One can find the details of this estimate for example in [10], Section 2.) This implies the uniqueness of the solution. Using the second equation and the foregoing estimate, one obtains immediately

$$|\omega|||\gamma_\Omega u||_{s+1-\frac{1}{p},p,\partial\Omega} \leq C(r)[||f||_{s,p,\Omega} + ||g||_{s+1-\frac{1}{p},p,\partial\Omega}].$$

Next, for a fixed ω, one has with the same method of Theorem 2.17 in [10] that, if r is large enough, one has also the existence of a solution; so, applying the local boundedness of $C(\cdot)$ and the continuation method, one obtains the result.

We are now in position to state and prove the following

Theorem 5.9. *Let* $1 < p < +\infty, s \in \mathbb{N}_0$; *let* $\theta \geq 0$; *if* r *is large enough, problem* (1.1) *has for any* $f \in W^{\theta,p}(W^{s,p}(\Omega)), g \in W^{\theta,p}(W^{1+s-\frac{1}{p},p}(\partial\Omega))$ *a unique solution* $u \in W^{\theta,p}(W^{s+2,p}(\Omega))$ *such that* $\gamma_\Omega u \in W^{1+\theta,p}(W^{1+s-\frac{1}{p},p}(\partial\Omega))$*. Moreover, for every* $T > 0$*, for any* $f \in W_T^{\theta,p}(W^{s,p}(\Omega)), g \in W_T^{\theta,p}(W^{1+s-\frac{1}{p},p}(\partial\Omega))$ (1.1) *has a unique solution* $u \in W_T^{\theta,p}(W^{s+2,p}(\Omega))$ *such that* $\gamma_\Omega u \in W_T^{1+\theta,p}(W^{1+s-\frac{1}{p},p}(\partial\Omega))$*.*

Proof. Consider first the case $\theta \notin \mathbb{Z}$; set $X = W^{s+2,p}(\Omega)$, $Y = W^{s,p}(\Omega)$, $Z = W^{s+1-\frac{1}{p},p}(\partial\Omega)$, $\mathcal{A} = A(\partial) - r^2 e^{i\theta_0}$, $K(t) = K(t,\partial)$, $\gamma = \gamma_\Omega$, $H(t) = \gamma_\Omega \circ H(t,\partial)$. Lemma 5.8 assures that, if r is large enough, condition (C) is satisfied, so that the result follows from 3.12 and 3.17.

Consider now the case $\theta \in \mathbb{Z}$; fix $s' < s$, such that $s' > \frac{1}{p} - 1$, $s' + 1 > s$ and $s' - \frac{1}{p} \notin \mathbb{Z}$; set $X = W^{s'+2,p}(\Omega)$, $Y = W^{s',p}(\Omega)$, $Z = W^{s'+1-\frac{1}{p},p}(\partial\Omega)$, $Z_1 =$

$W^{s'+2-\frac{1}{p},p}(\partial\Omega)$. Then, conditions (C) and (D) are satisfied (the second by 5.5). We have (following the notations of the fourth section) $Z_{s-s',p} = W^{s+1-\frac{1}{p},p}(\partial\Omega)$ (see [15]). So, if we set $X_{s-s'} = W^{2+s,p}(\Omega)$ and $Y_{s-s'} = W^{s,p}(\Omega)$, all assumptions of 4.7 are satisfied.

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