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STABILITY CRITERIA FOR LINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract. Necessary and sufficient conditions are given for the stability and asymptotic stability of nonantonomous linear delay differential systems under the assumption that the coefficient matrix or the delay is an L^p-function.

1. Introduction. Our aim in this paper is to present explicit necessary and sufficient conditions for the stability and asymptotic stability of the zero solution of linear nonautonomous delay differential systems. The proofs of the stability theorems are based on previous qualitative results showing that if the coefficient or the delay is "small," then every solution of the delay differential system is asymptotic to some member of an n-parameter family of special solutions. (Results of this type were initiated by Ryabov ([11]). Further developments were made by Driver ([2, 3]), Jarnik and Kurzweil ([8]) and the authors ([4, 5, 10]).) The special solutions form a fundamental system of an ordinary differential equation which is asymptotically equivalent to the delay differential equation; that is, it has the same stability properties. In the recent paper [5] the first author made an attempt to describe this ordinary differential equation in the case of linear systems with constant coefficients and time-dependent delay. The main point of the present paper is that by an appropriately defined recursive sequence we are able to describe the above-mentioned ordinary differential equation in the general case which results in sharp stability theorems.

Applying our general results to the equation

$$\dot{x}(t) = \frac{\sin t}{t^{\alpha}}\, x(t-r), \quad t \geq 1 \tag{1.1}$$

we may observe the following interesting phenomenon. If $\alpha > 1/2$, then equation (1.1) has the same stability properties as the equation without delay

$$\dot{x}(t) = \frac{\sin t}{t^{\alpha}}\, x(t);$$

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that is, the zero solution of (1.1) is stable independent of the delay r. If $1/3 < \alpha \leq 1/2$ then the approximating ordinary differential equation has the form

$$\dot{x}(t) = \frac{\sin t}{t^{\alpha}} \big(1 - \int_{t-r}^{t} \frac{\sin s}{s^{\alpha}}\, ds\big)\, x(t). \tag{1.2}$$

We will see (cf. Corollary 3.2) that the stability and instability regions of equation (1.2) and hence of equation (1.1) alternate periodically depending on the delay r.

The paper is organized as follows. In Section 2 we introduce some notations and definitions. The stability theorems are discussed in Section 3. In Section 4, we summarize some earlier results on the asymptotic characterization of the solutions by certain special solutions. Finally, in Section 5, we present the proofs of the stability theorems.

2. Notations and definitions. Let R^n and M_n denote the n-dimensional Euclidean space and the space of n-by-n matrices with real entries, respectively. If $|\cdot|$ is any norm in R^n then the induced norm of a matrix $A \in M_n$ is given by

$$|A| = \sup\{|Ax| \colon x \in R^n, |x| \leq 1\}.$$

The *logarithmic norm* of a matrix A is defined by

$$\mu(A) = \lim_{h \to 0^+} \frac{|I + h\,A| - 1}{h},$$

where I denotes the n-by-n unit matrix (see [1]).

We shall deal with the system of linear delay differential equations

$$\dot{x}(t) = A(t)x(t - \tau(t)), \tag{2.1}$$

where $A : [t_0, \infty) \to M_n$ and $\tau : [t_0, \infty) \to [0, r]$ $(0 < r = \text{const.})$ are continuous functions.

Let $t_1 \geq t_0$. A function x is said to be a *solution of* (2.1) *on* $[t_1 - r, \infty)$ if it is defined and continuous on $[t_1 - r, \infty)$, differentiable on $[t_1, \infty)$ and satisfies (2.1) for $t \geq t_1$. By a *solution* of (2.1) we mean a solution on $[t_1 - r, \infty)$ for some $t_1 \geq t_0$.

It is well-known ([7]) that the zero solution of (2.1) is *stable* (*asymptotically stable*) if and only if every solution of (2.1) is bounded (tends to zero at ∞).

3. The stability results and their discussion. The stability conditions will be formulated in terms of the following recursive sequence. Put

$$B_1(t, s) = A(s) \quad \text{for} \quad t_0 \leq s \leq t,$$

and

$$B_{k+1}(t, s) = -A(s) \int_{s-\tau(s)}^{t} B_k(t, u)\, du$$

for $k = 1, 2, \dots$ and $t_0 + (k-1)r \leq s \leq t$.

Theorem 3.1. *Assume that*

$$\int^{\infty} |A(t)|^{k+1}\, dt < \infty \tag{3.1}$$

for some positive integer k. Then the following statements are valid.

(A) *If*

$$\limsup_{t\to\infty} \int^{t} \mu\big(\sum_{i=1}^{k} B_i(s,s)\big)\, ds < \infty, \tag{3.2}$$

then the zero solution of (2.1) *is stable.*

(B) *If*

$$\lim_{t\to\infty} \int^{t} \mu\big(\sum_{i=1}^{k} B_i(s,s)\big)\, ds = -\infty, \tag{3.3}$$

then the zero solution of (2.1) *is asymptotically stable.*

Theorem 3.2. *Assume that*

$$A(t) \text{ is bounded,} \tag{3.4}$$

$$\tau(t) \to 0 \quad \text{as} \quad t \to \infty, \tag{3.5}$$

$$\begin{gathered}\tau \text{ is Lipschitzian in a neighborhood of } +\infty\text{; i.e.,}\\ \text{there exist constants } T \geq t_0 \text{ and } L > 0 \text{ such that}\\ |\tau(t) - \tau(s)| \leq L|t-s| \text{ for all } t, s \geq T,\end{gathered} \tag{3.6}$$

and, finally,

$$\int^{\infty} |A(t)|\tau^{k}(t)\, dt < \infty \tag{3.7}$$

for some positive integer k. Then statements (A) *and* (B) *of Theorem* (3.1) *are valid.*

In the scalar case the above results are sharp; namely,

Theorem 3.3. *Assume that $n = 1$ and let condition* (3.1) *be fulfilled. Then the following statements are valid.*

(A) *The zero solution of* (2.1) *is stable if and only if*

$$\limsup_{t\to\infty} \int^{t} \sum_{i=1}^{k} B_i(s,s)\, ds < \infty. \tag{3.8}$$

(B) *The zero solution of* (2.1) *is asymptotically stable if and only if*

$$\lim_{t\to\infty} \int^{t} \sum_{i=1}^{k} B_i(s,s)\, ds = -\infty. \tag{3.9}$$

Theorem 3.4. *Assume that $n = 1$ and let conditions* (3.4)–(3.7) *be fulfilled. Then statements* (A) *and* (B) *of Theorem* 3.3 *are valid.*

For $k = 1$ and $k = 2$ conditions (3.8) and (3.9) have the form

$$\limsup_{t\to\infty} \int^t A(s)\,ds < \infty \tag{3.8$_1$}$$

$$\limsup_{t\to\infty} \int^t A(s)\left(1 - \int_{s-\tau(s)}^s A(u)\,du\right) ds < \infty \tag{3.8$_2$}$$

and

$$\lim_{t\to\infty} \int^t A(s)\,ds = -\infty, \tag{3.9$_1$}$$

$$\lim_{t\to\infty} \int^t A(s)\left(1 - \int_{s-\tau(s)}^s A(u)\,du\right) = -\infty, \tag{3.9$_2$}$$

respectively. Thus, from Theorem 3.3 we obtain

Corollary 3.1. *If $n = 1$ and $\int^\infty |A(t)|^2\,dt < \infty$, then the zero solution of* (2.1) *is stable* (*asymptotically stable*) *if and only if* $(3.8)_1$ ($(3.9)_1$) *holds.*

If $n = 1$ and $\int^\infty |A(t)|^3\,dt < \infty$, then the zero solution of (2.1) *is stable* (*asymptotically stable*) *if and only if* $(3.8)_2$ ($(3.9)_2$) *holds.*

It is known (cf. Lemma 5.1) that if $\int^\infty |A(t)|^p\,dt < \infty$ for some $1 \le p < \infty$, then for every $r > 0$, $\int_{t-r}^t |A(s)|\,ds \to 0$ as $t \to \infty$. This means that the integrands in $(3.8)_1$ and $(3.8)_2$ "slightly" differ from each other. Therefore the following question arises: can the delay change the stability properties of equation (2.1) assuming that $\int^\infty |A(t)|^3\,dt < \infty$? The following corollary shows that the answer is yes.

Corollary 3.2. *Consider the equation*

$$\dot{x}(t) = \frac{\sin t}{t^\alpha}\,x(t-r), \quad t \ge 1, \tag{3.10}$$

where α, r are nonnegative constants. If $\alpha > 1/2$ then for every $r \ge 0$ the zero solution of (3.10) *is stable, but it is not asymptotically stable.*

If $1/3 < \alpha \le 1/2$ then the following statements are valid.

(A) *The zero solution of* (3.10) *is stable if and only if*

$$r \in \bigcup_{k\ge 0 \text{ integer}} [2k\pi, (2k+1)\,\pi]. \tag{3.11}$$

(B) *The zero solution of* (3.10) *is asymptotically stable if and only if*

$$r \in \bigcup_{k\ge 0 \text{ integer}} (2k\pi, (2k+1)\,\pi). \tag{3.12}$$

Proof. If $\alpha > 1/2$ then $\int_1^\infty |(\sin t)/\, t^\alpha|^2\, dt < \infty$. Since, according to the Dirichlet criterion for improper integrals, $\lim_{t\to\infty} \int_1^t (\sin s)/s^\alpha\, ds$ is finite, the first statement follows from the first part of Corollary 3.1.

Now assume that $1/3 < \alpha \le 1/2$. Then $\int_1^\infty\ |(\sin t)/t^\alpha|^3\, dt < \infty$. Define

$$f(t) = \int_1^t \frac{\sin s}{s^\alpha} \int_{s-r}^s \frac{\sin u}{u^\alpha}\, du\, ds$$

and

$$g(t) = \int_1^t \frac{\sin s}{s^\alpha} \int_{s-r}^s \big(\frac{\sin u}{u^\alpha} - \frac{\sin u}{s^\alpha}\big)\, du\, ds$$

for $t \ge 1$. We claim that

(i) $\lim_{t\to\infty} g(t)$ is finite;

(ii) if

$$r \in \{k\pi : k \ge 0 \text{ integer}\} \tag{3.13}$$

then $\lim_{t\to\infty}(f(t) - g(t))$ is finite;

(iii) if

$$r \in \bigcup_{k\ge 0 \text{ integer}} ((2k+1)\,\pi, (2k+2)\,\pi) \tag{3.14}$$

then $\lim_{t\to\infty}(f(t) - g(t)) = -\infty$;

(iv) if (3.12) is satisfied then $\lim_{t\to\infty}(f(t) - g(t)) = \infty$.

Proof of (i). Obviously

$$\begin{aligned} &\Big|\frac{\sin s}{s^\alpha} \int_{s-r}^s \big(\frac{\sin u}{u^\alpha} - \frac{\sin u}{s^\alpha}\big)\, du\Big| \\ &\quad \le \int_{s-r}^s \big(\frac{1}{u^\alpha} - \frac{1}{s^\alpha}\big)\, du \le r\big(\frac{1}{(s-r)^\alpha} - \frac{1}{s^\alpha}\big) \\ &\quad = r\frac{1 - \left(1 - \frac{r}{s}\right)^\alpha}{(s-r)^\alpha} = O\big(\frac{1}{s^{1+\alpha}}\big) \quad \text{as} \quad s \to \infty, \end{aligned}$$

the last equality being a consequence of the fact that $(1+x)^\alpha = 1 + O(x)$ as $x \to 0$. Since $\int_1^\infty 1/s^{1+\alpha}\, ds < \infty$, the proof of (i) is complete.

Proof of (ii)–(iv). We have

$$\begin{aligned} f(t) - g(t) &= \int_1^t \frac{\sin s}{s^{2\alpha}} \int_{s-r}^s \sin u\, du\, ds = \int_1^t \frac{\sin s}{s^{2\alpha}}\, (\cos(s-r) - \cos s)\, ds \\ &= 2\sin\frac{r}{2} \int_1^t \frac{\sin s\, \sin\left(s - \frac{r}{2}\right)}{s^{2\alpha}}\, ds \\ &= \sin\frac{r}{2}\, \cos\frac{r}{2} \int_1^t \frac{ds}{s^{2\alpha}} - \sin\frac{r}{2} \int_1^t \frac{\cos\left(2s - \frac{r}{2}\right)}{s^{2\alpha}}\, ds \\ &= \frac{1}{2}\sin r \int_1^t \frac{ds}{s^{2\alpha}} - \sin\frac{r}{2} \int_1^t \frac{\cos\left(2s - \frac{r}{2}\right)}{s^{2\alpha}}\, ds, \end{aligned}$$

where we used the following well-known rules from trigonometry:

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2},$$
$$\sin\alpha\,\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)),$$
$$2\,\sin\alpha\,\cos\alpha = \sin 2\alpha.$$

Evidently, $\lim_{t\to\infty}\int_1^t 1/s^{2\alpha}\,ds = +\infty$ and by the Dirichlet criterion we see that $\lim_{t\to\infty}\int_1^t \cos(2s - r/2)/s^{2\alpha}\,ds$ is finite. From this Claims (ii)–(iv) immediately follow.

Claims (i)–(iv) imply

(v) if (3.13) is satisfied then $\lim_{t\to\infty} f(t)$ is finite;
(vi) if (3.14) is satisfied then $\lim_{t\to\infty} f(t) = -\infty$;
(vii) if (3.12) is satisfied then $\lim_{t\to\infty} f(t) = \infty$.

Since

$$\int_1^t \frac{\sin s}{s^\alpha}\Big(1 - \int_{s-r}^s \frac{\sin u}{u^\alpha}\,du\Big)\,ds = \int_1^t \frac{\sin s}{s^\alpha}\,ds - f(t)$$

and $\lim_{t\to\infty}\int_1^t \sin s/s^\alpha\,ds$ is finite, statements (A) and (B) follow from (v)–(vii) by the second part of Corollary 3.1.

4. Asymptotic characterization of the solutions. The following result extracted from [10] will play a central role in the proof of our stability theorems. It shows that under a certain smallness condition the solutions of (2.1) corresponding to constant initial functions satisfy an ordinary differential equation which is asymptotically equivalent to equation (2.1).

Lemma 4.1. ([10, Corollary 7.6 and Lemma 5.1]) *Assume that*

$$\sup_{t\ge t_1}\int_{\max\{t_1, t-\tau(t)\}}^t |A(s)|\,ds < \frac{1}{e} \tag{4.1}$$

for some $t_1 \ge t_0$. Let $U(t)$ be the solution of the matrix equation

$$\dot U(t) = A(t)U(t-\tau(t)) \tag{4.2}$$

on $[t_1 - r, \infty)$ with initial condition

$$U(t) = I \quad \text{for} \quad t_1 - r \le t \le t_1. \tag{4.3}$$

Then the following statements are valid.

(A) *For every $t \ge t_1 - r$, $\det U(t) \ne 0$ and*

$$|U(s)\,U^{-1}(t)| \le \exp\Big(e\int_s^t |A(u)|\,du\Big) \tag{4.4}$$

for all $t_1 \le s \le t$.

(B) *For every solution* $x(t)$ *of* (2.1) *on* $[t_1 - r, \infty)$ *the limit*

$$\xi = \lim_{t\to\infty} U^{-1}(t)\, x(t) \in R^n$$

exists, and

$$\lim_{t\to\infty} |x(t) - U(t)\xi| = 0.$$

The matrix solution $U(t)$ is a fundamental system of an ordinary differential equation described in the next lemma.

Lemma 4.2. *Let* (4.1) *be fulfilled. Define* $C_1(t,s) = A(s)U(s-\tau(s))U^{-1}(t)$ *for* $t_1 \le s \le t$ *and*

$$C_{k+1}(t,s) = -A(s) \int_{s-\tau(s)}^{t} C_k(t,u)\, du$$

for $k = 1, 2, \dots$ *and* $t_1 + (k-1)\, r \le s \le t$. *If* $B_k(t,s)$ *has the meaning from Section* 3 *then*

$$\dot U(t) = \Big(\sum_{i=1}^{k} B_i(t,t) + C_{k+1}(t,t)\Big)\, U(t) \tag{4.5}$$

for all $k = 1, 2, \dots$ *and* $t \ge t_1 + kr$.

Proof. We shall show by induction that

$$B_k(t,s) + C_{k+1}(t,s) = C_k\,(t,s) \tag{4.6}$$

for all $k = 1, 2, \dots$ and $t_1 + kr \le s \le t$. For $t_1 + r \le s \le t$, we have

$$\begin{aligned} B_1(t,s) + C_2(t,s) &= A(s) - A(s) \int_{s-\tau(s)}^{t} A(u)U(u-\tau(u))U^{-1}(t)\, du \\ &= A(s) - A(s) \int_{s-\tau(s)}^{t} \dot U(u) U^{-1}(t)\, du \\ &= A(s) - A(s)(U(t) - U(s-\tau(s)))\, U^{-1}(t) \\ &= A(s)\, U(s-\tau(s))\, U^{-1}(t) = C_1(t,s); \end{aligned}$$

that is, (4.6) holds for $k = 1$.

Assume that (4.6) holds for some k. Then for $t_1 + (k+1)r \le s \le t$,

$$\begin{aligned} B_{k+1}(t,s) + C_{k+2}(t,s) &= -A(s) \int_{s-\tau(s)}^{t} B_k(t,u)\, du - A(s) \int_{s-\tau(s)}^{t} C_{k+1}(t,u)\, du \\ &= -A(s) \int_{s-\tau(s)}^{t} (B_k(t,u) + C_{k+1}(t,u))\, du \\ &= -A(s) \int_{s-\tau(s)}^{t} C_k(t,u)\, du = C_{k+1}(t,s). \end{aligned}$$

Thus, (4.6) is confirmed.

Since according to (4.6) $B_k(t,t) + C_{k+1}(t,t) = C_k(t,t)$ for $k = 1,2,\dots$ and $t \geq t_1 + kr$, it suffices to verify that (4.5) holds for $k = 1$. For $t \geq t_1 + r$, we have

$$\begin{aligned}(B_1(t,t) + C_2(t,t))\,U(t) &= A(t)\,U(t) - A(t)\int_{t-\tau(t)}^{t} A(u)\,U(u-\tau(u))\,du \\ &= A(t)\,U(t) - A(t)\int_{t-\tau(t)}^{t} \dot{U}(u)\,du \\ &= A(t)\,U(t) - A(t)(U(t) - U(t-\tau(t))) \\ &= A(t)\,U(t-\tau(t)) = \dot{U}(t).\end{aligned}$$

The proof of the lemma is complete.

5. Proofs of the theorems. In the proof of Theorem 3.1 we shall need the following lemmas on L^p-functions. (A function $f(t)$ is said to be in $L^p(t_0,\infty)$ for some $1 \leq p < \infty$ if $\int_{t_0}^{\infty} |f(t)|^p\,dt < \infty$. We write $f(t) \in L^p(t_0,\infty)$.)

Lemma 5.1. ([6, Lemma 2.1]) *If $f(t) \in L^p(t_0,\infty)$ for some $1 \leq p < \infty$, then for every $r > 0$,*

$$\int_{t-r}^{t} |f(s)|\,ds \in L^p(t_0 + r,\infty)$$

and

$$\int_{t-r}^{t} |f(s)|\,ds \to 0 \quad \text{as} \quad t \to \infty.$$

Lemma 5.2. *If $f_i(t) \in L^p(t_0,\infty)$ for some $1 \leq p < \infty$, $i = 1,2,\dots,k$, then*

$$\prod_{i=1}^{k} f_i(t) \in L^{p/k}(t_0,\infty).$$

Proof. The proof follows by induction, the induction step being a consequence of the fact that if $f(t) \in L^p(t_0,\infty)$ and $g(t) \in L^{p/k}(t_0,\infty)$, then $f(t)\,g(t) \in L^{p/(k+1)}(t_0,\infty)$. Indeed, since $k+1$ and $(k+1)/k$ are complementary exponents, according to Hölder's inequality,

$$\begin{aligned}\int_{t_0}^{\infty} |f(t)\,g(t)|^{\frac{p}{k+1}}\,dt &\leq \Big(\int_{t_0}^{\infty} \big(|f(t)|^{\frac{p}{k+1}}\big)^{k+1} dt\Big)^{\frac{1}{k+1}} \Big(\int_{t_0}^{\infty} \big(|g(t)|^{\frac{p}{k+1}}\big)^{\frac{k+1}{k}} dt\Big)^{\frac{k}{k+1}} \\ &= \Big(\int_{t_0}^{\infty} |f(t)|^p dt\Big)^{\frac{1}{k+1}} \Big(\int_{t_0}^{\infty} |g(t)|^{\frac{p}{k}}\,dt\Big)^{\frac{k}{k+1}} < \infty.\end{aligned}$$

Proof of Theorem 3.1. By Lemma 5.1, (3.1) implies that if $t_1 \geq t_0$ is sufficiently large then condition (4.1) of Lemma 4.1 is fulfilled. It follows from the definition of $C_k(t,s)$ by easy induction that

$$|C_{k+1}(t,s_0)| \leq |A(s_0)| \int_{s_0-\tau(s_0)}^{t} |A(s_1)| \int_{s_1-\tau(s_1)}^{t} |A(s_2)| \cdots \cdots \int_{s_{k-2}-\tau(s_{k-2})}^{t} |A(s_{k-1})| \int_{s_{k-1}-\tau(s_{k-1})}^{t} |C_1(t,s_k)|\, ds_k\, ds_{k-1} \cdots ds_1$$

for all $k = 1, 2, \ldots$ and $t_1 + (k-1)r \leq s_0 \leq t$. Especially,

$$|C_{k+1}(t,t)| \leq |A(t)| \int_{t-\tau(t)}^{t} |A(s_1)| \int_{s_1-\tau(s_1)}^{t} |A(s_2)| \cdots \cdots \int_{s_{k-2}-\tau(s_{k-2})}^{t} |A(s_{k-1})| \int_{s_{k-1}-\tau(s_{k-1})}^{t} |C_1(t,s_k)|\, ds_k\, ds_{k-1} \cdots ds_1 \tag{5.1}$$

for $k = 1, 2, \ldots$ and $t \geq t_1 + (k-1)r$. Hence

$$|C_{k+1}(t,t)| \leq |A(t)| \int_{t-r}^{t} |A(s)|\, ds \int_{t-2r}^{t} |A(s)|\, ds \cdots \cdots \int_{t-(k-1)r}^{t} |A(s)|\, ds \int_{t-kr}^{t} |C_1(t,s)|\, ds \tag{5.2}$$

for $k = 1, 2, \ldots$ and $t \geq t_1 + (k-1)r$. In view of (4.4) we have for $t - kr \leq s \leq t$,

$$|C_1(t,s)| \leq |A(s)|\, |U(s-\tau(s))U^{-1}(t)| \leq |A(s)| \exp\Big(e \int_{t-kr}^{t} |A(u)|\, du\Big).$$

According to Lemma 5.1, (3.1) implies that

$$\int_{t-kr}^{t} |A(u)|\, du \to 0 \quad \text{as} \quad t \to \infty.$$

Consequently, there exists a constant $K > 0$ such that

$$|C_1(t,s)| \leq K\, |A(s)| \tag{5.3}$$

for all $t \geq t_1$ and $t - kr \leq s \leq t$. From (5.2) and (5.3), we obtain

$$|C_{k+1}(t,t)| \leq K\, |A(t)| \int_{t-r}^{t} |A(s)|\, ds \int_{t-2r}^{t} |A(s)|\, ds \cdots \int_{t-kr}^{t} |A(s)|\, ds$$

for $t \geq t_1 - kr$, which, according to Lemmas 5.1 and 5.2, implies that

$$\int^{\infty} |C_{k+1}(t,t)|\, dt < \infty. \tag{5.4}$$

From (5.4), by [1, Theorem 3, page 58] it follows that

$$\begin{aligned} |U(t)| &\leq |U(t_1+kr)| \exp\Big(\int_{t_1+kr}^{t} \mu\Big(\sum_{i=1}^{k} B_i(s,s) + C_{k+1}(s,s)\Big)\, ds\Big) \\ &\leq |U(t_1+kr)| \exp\Big(\int_{t_1+kr}^{\infty} |C_{k+1}(s,s)|\, ds\Big) \exp\Big(\int_{t_1+kr}^{t} \mu\Big(\sum_{i=1}^{k} B_i(s,s)\Big)\, ds\Big) \end{aligned}$$

for $t \geq t_1 + kr$, the last inequality being a consequence of (5.4) and the following properties of the logarithmic norm (see [1]):

$$\begin{aligned} \mu(A+B) \leq \mu(A) + \mu(B), \quad & A, B \in M_n \\ \mu(A) \leq |A|, \quad & A \in M_n. \end{aligned}$$

The estimate of $|U(t)|$ shows that if (3.2) ((3.3)) is satisfied then $|U(t)|$ is bounded (tends to zero at ∞). Since, according to Lemma 4.1 (B), all solutions of (2.1) on $[t_1 - r, \infty)$ have the same properties, the proof of the theorem is complete.

Proof of Theorem 3.2. Clearly, assumptions (3.4) and (3.5) imply that for all t_1 large enough (4.1) is fulfilled. Now we show that if $t - \tau(t) \leq s_1 \leq t$ and $s_i - \tau(s_i) \leq s_{i+1} \leq t$ for $i = 1, 2, \ldots$, then

$$t - (s_i - \tau(s_i)) \leq (2+L)^i\, \tau(t) \tag{5.5}$$

for $i = 1, 2, \ldots$. In view of (3.6),

$$|\tau(s_1) - \tau(t)| \leq L(t - s_1) \leq L\, \tau(t),$$

hence

$$\begin{aligned} &\tau(s_1) = \tau(t) + \tau(s_1) - \tau(t) \leq (1+L)\, \tau(t), \\ &t - (s_1 - \tau(s_1)) \leq t - s_1 + \tau(s_1) \leq (2+L)\, \tau(t); \end{aligned}$$

that is, (5.5) holds for $i = 1$. Assume for induction that (5.5) holds for some i. We have

$$|\tau(s_{i+1}) - \tau(t)| \leq L\,(t - s_{i+1}) \leq L\,(t - (s_i - \tau(s_i))),$$

hence

$$\tau(s_{i+1}) = \tau(t) + \tau(s_{i+1}) - \tau(t) \leq \tau(t) + L\,(t - (s_i - \tau(s_i))).$$

Consequently,

$$\begin{aligned} &t - (s_{i+1} - \tau(s_{i+1})) = t - s_{i+1} + \tau(s_{i+1}) \\ &\le t - (s_i - \tau(s_i)) + \tau(s_{i+1}) \le \tau(t) + (1+L)(t - (s_i - \tau(s_i))) \\ &\le \tau(t) + (1+L)(2+L)^i \tau(t) \le (2+L)^{i+1}\tau(t). \end{aligned}$$

Thus, (5.5) is confirmed.

Referring to (5.1) in the proof of Theorem 3.1 and taking into account (5.5), we obtain for $t \ge t_1 + k$,

$$|C_{k+1}(t,t)| \le \alpha^{k-1}\,\kappa^{(k-1)\,k/2}\,|A(t)|\,\tau^k(t) \sup_{t-kr\le s\le t} |C_1(t,s)|, \tag{5.6}$$

where $\alpha = \sup_{t\ge t_0} |A(t)|$ and $\kappa = 2 + L$. By virtue of (4.4),

$$\sup_{t-kr\le s\le t} |C_1(t,s)| \le \alpha\, \exp(e\,\alpha\,k\,r)$$

which, together with (5.6) and (3.7), implies that (5.4) is satisfied. The proof now can be completed by using the same argument as in the proof of Theorem 3.1.

Proof of Theorems 3.3 and 3.4. The “if” parts follow from Theorems 3.1 and 3.2, respectively.

To prove the “only if” parts, observe that in the scalar case equation (4.5) can be solved explicitly; namely,

$$U(t) = U(t_1 + kr)\, \exp\Big(\int_{t_1+kr}^{t} \Big(\sum_{i=1}^{k} B_i(s,s) + C_{k+1}(s,s)\Big)\, ds\Big)$$

for $t \ge t_1 + kr$. Taking into account (5.4) and the positivity of $U(t)$ (cf. Lemma 4.1 (A)), we have

$$U(t) \ge U(t_1 + kr)\, \exp\Big(-\int_{t_1+kr}^{\infty} |C_{k+1}(s,s)|\, ds\Big) \exp\Big(\int_{t_1+kr}^{t} \sum_{i=1}^{k} B_i\,(s,s)\, ds\Big)$$

for $t \ge t_1$. The last estimate shows that if (3.8) ((3.9)) does not hold, then the solution $U(t)$ is unbounded (has no zero limit at ∞) which completes the proof.

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