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SIMILARITY SOLUTIONS FOR A CLASS OF HYPERBOLIC INTEGRODIFFERENTIAL EQUATIONS

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Abstract. For a class of hyperbolic partial integrodifferential equations of the form $u_{tt} - u_{xx} + a * u_{xx} = 0$, fundamental solutions are found that depend on the similarity variable $\xi = x(t - |x|)^{-\alpha}$, where $\alpha \in (0, 1)$ and the integral kernel a behaves like $t^{-\alpha}$ near $t = 0$. The asymptotic behavior of these solutions in various scaling limits and their regularity is discussed. Applications to solutions of general initial-value problems of such equations are given.

1. Introduction. In this note, fundamental solutions of a class of linear partial integrodifferential equations of the form

$$u_{tt}(x,t) - u_{xx}(x,t) + \int_0^t a(t-s)u_{xx}(x,s)\,ds = 0 \tag{1.1}$$

will be constructed, that is, distributional solutions u and v on $\mathbf{R} \times (0, \infty)$ that satisfy the initial conditions

$$u(\cdot, 0) = \delta_0, \quad u_t(\cdot, 0) = 0 \tag{1.2}$$

$$v(\cdot, 0) = 0, \quad v_t(\cdot, 0) = \delta_0 \tag{1.3}$$

in a suitable sense. Here δ_0 is Dirac's delta distribution with unit mass at $x = 0$, and subscripts are abbreviations for partial derivatives. Such equations govern shear motions of certain viscoelastic materials and occur in linearizations of the equations of motion for other materials; see [12].

If $a = 0$, then (1.1) is the wave equation, and one obtains $u(\cdot, t) = \frac{1}{2}\big(\delta_t + \delta_{-t}\big) = p_2(x/t)$, $v(x,t) = \frac{1}{2}1_{[-t,t]}(x)$ as solutions of (1.1), (1.2), (1.3), where $p_2(\cdot) = \frac{1}{2}\big(\delta_0(\cdot - 1) + \delta_0(\cdot + 1)\big)$. Thus u depends on the *similarity variable* $\zeta = x/t$. Here ζ is homogeneous in x and t; note however that one could also write u as a distribution that depends on the similarity variable $\xi = (|x| - t)/x$, which is homogeneous in x and $|x| - t$. It is instructive to think of the wave equation as the special case $\alpha = 2$ of the partial integrodifferential equation that can be written somewhat imprecisely as

$$u(\cdot, t) = \delta_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha - 1} u_{xx}(\cdot, s)\,ds \tag{1.4}$$

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where Γ is the usual gamma function. For $\alpha = 1$, one obtains the heat equation, with the solution $u(x,t) = t^{-1/2}p_1(|\zeta|)$, $\zeta = xt^{-1/2}$, $p_1(\zeta) = (4\pi)^{-1/2}e^{-\zeta^2/4}$. Here u depends on the similarity variable $\zeta = xt^{-1/2}$.

It turns out that for all other values of α between 0 and 2, there is a solution u of (1.4) that depends on $\zeta = xt^{-\alpha/2}$:

$$u(x,t) = t^{-\alpha/2}p_\alpha(|\zeta|) \tag{1.6}$$

where $\frac{1}{2}p_\alpha$ is a probability density function that is supported on $[0,\infty)$. These solutions were derived formally in [16] and are discussed in detail in [4] and [19]. For $\alpha \in (1,2)$, the equation (1.4) is equivalent to the first-order partial integrodifferential equation

$$u_t(\cdot,t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}u_{xx}(\cdot,s)\,ds \tag{1.7}$$

together with the initial condition (1.2_1). Equation (1.7) can be written as $D_t^\alpha u(\cdot,t) = u_{xx}(\cdot,t)$, where D_t^α denotes Riemann-Liouville fractional order differentiation; see [21]. Such equations result from phenomenological models for viscoelastic materials that have recently been proposed; see e.g. [6] and [10]. They have also been discussed as "fractional diffusion" equations in [22]. The equations share some properties with the heat equation. In particular, the speed of propagation is infinite (compactly supported nonnegative initial data at $t = 0$ result in solutions that are everywhere positive for all positive times), and solutions of initial-value problems gain two spatial derivatives over the regularity of the initial data. Moreover, there are estimates that are strong enough to guarantee "maximal regularity" properties also for solutions of inhomogeneous problems; see [19].

In contrast to the "parabolic" character of (1.7), the problem (1.1) has more properties in common with the wave equation. In particular, for reasonable kernels (locally integrable with some smoothness), the speed of propagation is always equal to 1. Yet, conditions are known under which the solution still gains two spatial derivatives over the regularity of the initial data, as soon as $t > 0$. This happens if the kernel a behaves like $ct^{-\alpha}$ near $t = 0$ with $c > 0$ and $\alpha \in (0,1)$. Explicit examples are given in [14], [20], [11]; there also kernels with weaker singularities are discussed. A general abstract result in a Hilbert space setting implies that the ratio $\|u_{xx}(\cdot,t)\|/\|u(\cdot,0)\|$ is at most $O(t^{-2/\alpha})$ as $t \to 0$, where the L^2-norm is used ([1]). However, it has not been clear if such an estimate is sharp and whether it carries over to other norms such as L^p-norms or Hölder-norms. In my view, a principal obstacle has been a lack of examples that are understood well and that might serve as test cases for conjectures, illustrations for the typical behavior, and building blocks for more general problems. The main purpose of this paper is to present and discuss such a class of examples.

One might attempt to construct solutions of (1.1), (1.2) by fixing a particular type of kernel and then look for a solution of the form $u(x,t) = \beta(t)u(\zeta)$ with $\zeta = x\gamma(t)$. Then the only possibility is essentially $\zeta = x/t$, $\beta(t) = t^{-1}$, since otherwise the speed of propagation could not be constant equal to 1. However, attempts to find such solutions quickly run into difficulties, and I therefore conjecture that no such solution of (1.1), (1.2) can exist. Instead, the kernel will be chosen together with the solution u, and a different type of similarity variable will be permitted. This can be done for any desired behavior $a(t) \sim t^{-\alpha}$, $\alpha \in (0,1)$. The solution u then is found to depend essentially only on the variable $\xi = x(t-|x|)^{-\alpha}$. This construction is begun in Section 2 of the paper, using an approach that relies on the solution of the Rayleigh problem for (1.1) which has been exploited systematically in [18], together with a natural factorization of the second-order equation (1.1). The factorization idea was also used in [14] to construct the solution v of (1.1), (1.3). In Section 3, the construction of u and v is completed, a number of equivalent representations for these functions are derived, and some of their elementary properties together with consequences for solutions of general initial-value problems for (1.1) are given. In Section 4, the behavior of u and v in the limits as $t \to 0$ and as $t \to \infty$ is studied. It turns out that near $t = 0$, the solutions u looks like the corresponding fundamental solution of the wave equation, while for t near ∞, u behaves like a solution of (1.4) for a suitable index α. The behavior as $t \to \infty$ is closely related to the behavior of the kernel a for large t. In Section 5, various regularity properties of u and v are given, including asymptotic expansions for the behavior across the interfaces $\pm x = t$, the exact behavior of u and all its spatial derivatives at $x = 0$, and the behavior of the variation of $u(\cdot,t)$ and $v(\cdot,t)$ and of all their spatial derivatives for t near 0. The results in Sections 4 and 5 are again applied to derive properties of solutions of general initial-value problems for (1.1). The kernel a and also the solutions of (1.7) can be expressed in terms of Mittag-Leffler functions, and the solutions that are constructed here are related to density functions for P. Lévy's extremal stable probability laws, and also to E.M. Wright's generalized Bessel functions. Properties of these objects are collected in an appendix.

The special equations that are studied here have the form of linearized equations for shear motions of viscoelastic liquids, yet in all likelihood they do not describe any real material. In fact, for $\alpha \in (0, \frac{1}{2})$, the equations studied here would lead to nonphysical properties, such as positive shear strains resulting in negative shear stresses for large times. However, I do expect that solutions of general linear hyperbolic equations for viscoelastic materials of integral type with singular kernels will have some essential features in common with the solutions that are presented here. The results in this paper can be used to discuss such problems with more general kernels in higher space dimensions; this will be done elsewhere.

I shall employ the usual notation $k * l(t) = \int_0^t k(t-s)l(s)\,ds$ for the convolution of two locally integrable functions that are supported on $[0,\infty)$. This notation will also be used if these functions depend additionally on other variables. The letter C denotes constants that can be estimated in terms of previously determined quantities and that may change from occurrence to occurrence.

2. Solution of a Rayleigh problem. Let $\alpha \in (0,1)$ be a fixed given number. In this section I construct a scalar integral kernel $a = a_\alpha \in L^1(0,\infty,\mathbf{R})$ and a function $U = U_\alpha : [0,\infty) \times [0,\infty) \to \mathbf{R}$ that satisfy the partial integrodifferential equation

$$U_{tt}(x,t) - U_{xx}(x,t) + \int_0^t a(t-s) U_{xx}(x,s)\,ds = 0 \quad (x>0,\, t>0) \tag{2.1a}$$

with initial and boundary data

$$U(x,0) = 0 \quad (x>0), \quad U(0,t) = 1 \quad (t>0). \tag{2.1b}$$

The kernel a behaves like $ct^{-\alpha}$, $c>0$ as $t \to 0$, and U has the special form

$$U(x,t) = \Phi_\alpha\big(\frac{t-x}{x^{1/\alpha}}\big) \tag{2.2}$$

with $\Phi = \Phi_\alpha$ an increasing, C^∞-smooth function that is supported on $[0,\infty)$, and for which $\lim_{r\to\infty} \Phi(r) = 1$. Thus Φ is a cumulative probability distribution function; it turns out to be the distribution function of an extremal stable probability law. Therefore U is C^∞-smooth in the first quadrant of the (x,t)-plane except at the origin and is supported on $\{(x,t) : 0 \le x \le t\}$.

For the rest of the section, the dependence of U, Φ, a etc. on α will no longer be explicitly indicated, once it has been introduced. Let us look for the kernel a in the form

$$a = 2b - b * b, \tag{2.3}$$

where $b = b_\alpha \in L^1(0,\infty,\mathbf{R})$. Then it is sufficient to solve the first-order integrodifferential equation

$$V = U_t + U_x - b * U_x = 0 \tag{2.4}$$

which results from factoring (2.1a), since then also

$$U_{tt} - U_{xx} + a * U_{xx} = V_t - \big(V_x - b * V_x\big) = 0. \tag{2.5}$$

The kernel b in turn is found by specifying its resolvent kernel ([7]), i.e., the locally integrable kernel $c = c_\alpha$ with the property

$$c = b + b * c. \tag{2.6}$$

Suppose c is given; then, (2.4) is equivalent to

$$U_t + c * U_t + U_x = 0. \tag{2.7}$$

Taking the Laplace transform in the t-direction and employing the usual notation

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

one sees that U and c must satisfy

$$s\hat{U}(x,s) + s\hat{c}(s)\hat{U}(x,s) + \hat{U}_x(x,s) = 0 \quad (x > 0,\ s > 0) \tag{2.8a}$$

together with

$$\hat{U}(0,s) = \frac{1}{s} \quad (s > 0) \tag{2.8b}$$

which results from taking the transform of the boundary condition in (2.1b). The solution of this family of ordinary differential equations is

$$\hat{U}(x,s) = \frac{1}{s}e^{-(s+s\hat{c}(s))x}. \tag{2.9}$$

Let us now choose

$$c(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = \frac{\sin\alpha\pi}{\pi}\Gamma(\alpha)t^{-\alpha}, \quad \hat{c}(s) = s^{\alpha-1}. \tag{2.10}$$

Then the Laplace transform of U must satisfy

$$\hat{U}(x,s) = \frac{1}{s}e^{-(s+s^{\alpha})x}. \tag{2.11}$$

Suppose a function $\phi = \phi_\alpha : [0,\infty) \to \mathbf{R}$ with Laplace transform

$$\hat{\phi}(s) = e^{-s^{\alpha}} \tag{2.12}$$

can be found. Then the inverse Laplace transform of $s \mapsto e^{-s^{\alpha}x}$ is

$$t \mapsto x^{-1/\alpha}\phi(tx^{-1/\alpha}),$$

the inverse Laplace transform of $s \mapsto s^{-1}e^{-s^{\alpha}x}$ is

$$t \mapsto \int_0^t x^{-1/\alpha}\phi(sx^{-1/\alpha})\,ds = \Phi(tx^{-1/\alpha}),$$

where $\Phi' = \phi$, and thus

$$U(x,t) = \Phi\Big(\frac{t-x}{x^{1/\alpha}}\Big). \tag{2.13}$$

A function ϕ with the Laplace transform (2.12) can indeed be found by expanding the right-hand side into a power series in s^{α} and inverting the Laplace transform for each term, using Hankel's formula ([9]). Apparently, this was first done in [17]. The result is

$$\begin{aligned} \phi(r) &= \sum_{k=1}^{\infty}(-1)^{k-1}\frac{\Gamma(1+\alpha k)}{k!}\frac{\sin\,\alpha k\pi}{\pi}r^{-\alpha k-1} \\ &= \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{\Gamma(1+k)\Gamma(-\alpha k)}r^{-\alpha k-1}. \end{aligned} \tag{2.14}$$

The function ϕ turns out to be the probability density function of an extremal stable probability distribution with index α; see the appendix. It is infinitely often differentiable and supported on $[0,\infty)$. Thus Φ is the cumulative distribution function, also supported on $[0,\infty)$, with $\Phi(r)\to 1$ as $r\to\infty$, and U is indeed smooth in the first quadrant of the (x,t)-plane. Also, U has the boundary behavior (2.1b) and vanishes for $x>t$.

It remains to specify the kernel a such that the construction can be justified. It follows from general properties of Volterra integral equations ([7]) that (2.10), (2.6), and (2.3) suffice to determine $a\in L^1_{\mathrm{loc}}(0,\infty,\mathbf{R})$ uniquely and that

$$a(t)=\frac{2}{\Gamma(1-\alpha)}t^{-\alpha}+o(t^{-\alpha})$$

as $t\to 0$. It is then a routine exercise in Laplace transforms to check that U and a indeed satisfy (2.4) and therefore also (2.1a) everywhere in $(0,\infty)\times(0,\infty)$.

Additional properties of the kernels a and b can be derived as follows. From (2.10), (2.6) and (2.3) one obtains that

$$\hat{b}(s)=\frac{s^{\alpha-1}}{1+s^{\alpha-1}}=\frac{1}{1+s^{1-\alpha}},\quad \hat{a}(s)=1-\frac{1}{(1+s^{\alpha-1})^2}. \tag{2.15}$$

This shows that a and b are in $L^1(0,\infty,\mathbf{R})$ and $\int_0^\infty a(t)\,dt=\int_0^\infty b(t)\,dt=1$. Expanding $\hat{b}$ as a geometric series in $s^{\alpha-1}$ (valid for complex arguments s outside the unit disk and off the negative real axis), inverting the Laplace transform term by term, and using Hankel's formula, one obtains

$$b(t)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\Gamma((1-\alpha)k)}t^{-1+(1-\alpha)k}. \tag{2.16}$$

The series converges uniformly on bounded subsets of $(0,\infty)$. Formula (2.16) can be written compactly as

$$b(t)=(1-\alpha)t^{-\alpha}E'_{1-\alpha}(-t^{1-\alpha})=-t^{-1}E_{1-\alpha,0}(-t^{1-\alpha}), \tag{2.17}$$

where $E_\sigma=E_{\sigma,1}$ and $E_{\sigma,0}$ are Mittag-Leffler functions (see the appendix). The "Rabotnov kernel" b has been proposed elsewhere as a creep kernel in viscoelasticity ([14]). In the same manner, an infinite series can be found for the kernel a; it converges uniformly on any bounded subset of $(0,\infty)$:

$$a(t)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(k+1)}{\Gamma((1-\alpha)k)}t^{-1+(1-\alpha)k}. \tag{2.18}$$

This kernel can also be expressed in terms of Mittag-Leffler functions, namely

$$\begin{aligned}a(t)&=2(1-\alpha)t^{-\alpha}E'_{1-\alpha}(-t^{1-\alpha})+(\alpha-1)t^{1-2\alpha}E''_{1-\alpha}(-t^{1-\alpha})\\&=\frac{1}{\alpha-1}\big(t\frac{d^2}{dt^2}E_{1-\alpha}(-t^{1-\alpha})+(2-\alpha)\frac{d}{dt}E_{1-\alpha}(-t^{1-\alpha})\big)\\&=t^{-\alpha}E'_{\alpha,0}(-t^{1-\alpha})-t^{-1}E_{\alpha,0}(-t^{1-\alpha}).\end{aligned} \tag{2.19}$$

Due to (A.4), this implies that

$$a(t) = \sum_{j=2}^{N-1} \frac{(j-1)(-1)^{j-1}}{\Gamma((\alpha-1)j)} t^{(\alpha-1)j-1} + O(|t|^{(\alpha-1)N-1}) \tag{2.20}$$

as $t \to \infty$, for any $N \geq 3$. Finally, using the real inversion formula for the Laplace transform ([9]), one obtains

$$a(t) = \frac{1}{\pi} \int_0^\infty \frac{2\sigma^{2-2\alpha}(\sigma^{1-\alpha} - \cos\alpha\pi)\sin\alpha\pi}{(1 - 2\sigma^{1-\alpha}\cos\alpha\pi + \sigma^{2-2\alpha})^2} e^{-\sigma t}\, d\sigma. \tag{2.21}$$

The kernel therefore is completely monotonic if and only if $\frac{1}{2} \leq \alpha < 1$. Notice that at $\alpha = \frac{1}{2}$, the leading term in (2.20) vanishes. For $0 < \alpha < \frac{1}{2}$, a has at least one zero. For a viscoelastic material that is described by (2.1), the stress response Σ to a step shear strain of unit magnitude is given by

$$\Sigma(t) = 1 - \int_0^t a(\tau)\, d\tau,$$

and one sees that for $\alpha < \frac{1}{2}$, unphysical negative stresses occur for large times t.

The creep kernel d that is associated with the relaxation kernel a is defined by the equation $d = a + a * d$. Using Laplace transforms and (2.15), one obtains

$$\begin{aligned} \hat{d}(s) &= 2s^{\alpha-1} + s^{2\alpha-2}, \\ d(t) &= \frac{2t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{t^{1-2\alpha}}{\Gamma(2-2\alpha)} = 2\frac{\sin\alpha\pi}{\pi}\Gamma(\alpha)t^{-\alpha} + \frac{\sin 2\alpha\pi\Gamma(2\alpha)}{\pi(1-2\alpha)} t^{1-2\alpha}. \end{aligned} \tag{2.22}$$

For a viscoelastic material that is described by (2.1), the creep response C to a unit step shear stress is given by

$$C(t) = 1 + \int_0^t d(\tau)\, d\tau = 1 + \frac{2t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{t^{2-2\alpha}}{\Gamma(3-2\alpha)}. \tag{2.23}$$

For $\alpha < \frac{1}{2}$, the shear rate goes to infinity for a constant shear stress, and the shear viscosity vanishes.

In the case $\alpha = \frac{1}{2}$, U and a can be expressed in terms of elementary functions. In this case, the associated stable probability law is the Lévy distribution with density

$$\phi(r) = \begin{cases} 0 & (r \leq 0) \\ \frac{1}{2\sqrt{\pi}} r^{-3/2} e^{-\frac{1}{4r}} & (r > 0) \end{cases} \tag{2.24}$$

with cumulative distribution function

$$\Phi(r) = \operatorname{erfc}\left(\frac{1}{2\sqrt{r}}\right) \quad (r > 0), \tag{2.25}$$

and therefore

$$U(x,t) = \operatorname{erfc}\big(\frac{x}{2\sqrt{t-x}}\big) \quad (0 \le x < t < \infty), \tag{2.26}$$

where

$$\operatorname{erfc}(r) = \frac{2}{\sqrt{\pi}} \int_r^\infty e^{-s^2}\, ds$$

is the complementary error function. This special solution is given in [19]. Then the kernel a can be written as

$$a(t) = \frac{2}{\sqrt{\pi t}} + 2\sqrt{\frac{t}{\pi}} - (3+2t)e^t \operatorname{erfc}(\sqrt{t}) \tag{2.27}$$

due to (2.19) and since $E_{1/2}(-z) = e^{z^2}\operatorname{erfc}(z)$ ([2]). Thus $a(t)$ behaves like $\frac{3}{2\sqrt{\pi}}t^{-5/2}$ as $t \to \infty$.

3. Solutions of initial boundary value problems. With the notation of the previous section, let us define

$$u(x,t) = -\frac{1}{2}U_x(|x|,t) \quad (x \ne 0,\, t > 0) \tag{3.1}$$

$$v(x,t) = \int_0^t u(x,s)\, ds \quad (x \ne 0,\, t > 0) \tag{3.2}$$

$$w(x,t) = U_t(x,t) \quad (x > 0, t > 0). \tag{3.3}$$

These functions are well-defined, and it is clear that they are infinitely often differentiable. Let us give explicit expressions for them. Firstly,

$$u(x,t) = \begin{cases} \phi\big(\frac{t-|x|}{|x|^{1/\alpha}}\big)\frac{t-(1-\alpha)|x|}{2\alpha|x|^{1/\alpha+1}} & (0 < |x| < t) \\ 0 \qquad (|x| \ge t > 0). \end{cases} \tag{3.4}$$

The formula shows that $u(x,t) > 0$ if $0 < |x| < t$. Then (2.14) implies that for $0 < |x| < t$

$$u(x,t) = \frac{1}{2}\big(\frac{1}{\alpha|x|} + \frac{1}{t-|x|}\big)\sum_{k=1}^\infty (-1)^{k+1}\frac{\Gamma(1+\alpha k)}{k!}\frac{\sin \alpha\pi k}{\pi}\frac{|x|^k}{(t-|x|)^{\alpha k}}. \tag{3.5}$$

The series converges uniformly with all derivatives in any set $\{(x,t) : |x| + \epsilon|x|^{1/\alpha} \le t\}$ with $\epsilon > 0$ arbitrary. We read off that u can be extended continuously to the half line $\{(0,t) : t > 0\}$ and that

$$u(0,t) = \frac{\Gamma(\alpha)\sin\alpha\pi}{2\pi t^\alpha} = \frac{1}{2}c(t), \tag{3.6}$$

with c defined in (2.10). Also,

$$\lim_{x\to 0+} u_x(x,t) = \frac{1}{2}\big(2\frac{\sin\alpha\pi}{\pi}\Gamma(1+\alpha)t^{-\alpha-1} - \frac{\sin 2\alpha\pi}{\pi}\Gamma(2\alpha)t^{-2\alpha}\big)$$
$$= -\frac{1}{2}d'(t), \tag{3.7}$$

with d being the creep kernel defined in (2.22). If $\alpha \geq \frac{1}{2}$, then $u_x(0+,t) > 0$ for all t. If $\alpha < \frac{1}{2}$, then $u_x(0+,t) > 0$ if and only if

$$t < \big(\frac{\Gamma(1+\alpha)}{\cos\alpha\pi\Gamma(2\alpha)}\big)^{\frac{1}{1-\alpha}}.$$

The relation between $u_x(0+,t)$ and the creep kernel is no accident and was proved for general creep kernels in [14]. The argument is based on the observation that

$$\hat{u}_x(x,s) = -\frac{s}{2}(1+\hat{c}(s))^2 e^{-(s+s\hat{c}(s))x} = -\frac{s}{2}(1+\hat{d}(s))e^{-(s+s\hat{c}(s))x}.$$

Inverting the Laplace transform and sending x to zero results in $u_x(0+,t) = -d'(t)/2$. By the same argument, the formula $u(0,t) = \frac{1}{2}c(t)$ holds for solutions of more general problems. In particular, the curious relationship follows

$$\frac{d}{dt}\big(2u(0,t) + 2\int_0^t u(0,t-s)u(0,s)\,ds\big) + u_x(0+,t) = 0 \quad (0 < t < \infty)$$

which can be expected to hold for the case of general kernels.

Let us next derive the following size estimates:

$$\|u(\cdot,t)\|_{L^q} \leq Ct^{\frac{1-q}{\alpha q}} \tag{3.8}$$

for $0 < t \leq T$, $1 \leq q \leq \infty$, with $C = C(\alpha,T)$ independent of q. Indeed, for $q = 1$ this estimate just follows from the construction of u. For $q = \infty$, one estimates as follows in (3.4): If $\frac{t}{2} \leq |x| < t$, then

$$u(x,t) \leq \sup_r \phi(r)\frac{t-(1-\alpha)t/2}{2\alpha(t/2)^{1/\alpha+1}} = Ct^{-1/\alpha}.$$

If $0 < |x| \leq \frac{t}{2}$, then by (A.13)

$$\phi\big(\frac{t-|x|}{|x|^{1/\alpha}}\big) \leq C\big(\frac{t-|x|}{|x|^{1/\alpha}}\big)^{-1-\alpha} \leq C\frac{|x|^{1/\alpha+1}}{t^{1+\alpha}},$$

and therefore

$$u(x,t) \leq C\frac{t}{|x|^{1/\alpha+1}}\frac{|x|^{1/\alpha+1}}{t^{1+\alpha}} \leq Ct^{-\alpha} \leq Ct^{-1/\alpha}.$$

This proves (3.8) for $q = \infty$. Hölder's inequality implies the estimate for all other q. Set $x(t) = t - t^{1/\alpha}$; then,

$$\lim_{t\to 0} t^{1/\alpha} u(x(t), t) = \frac{\phi(1)}{2},$$

and thus (3.8) is sharp for $q = \infty$. In the next section it will become clear that (3.8) is also sharp for all intermediate q.

Recall that the Laplace transform of u in the t-direction is

$$\hat{u}(x, s) = \frac{1}{2}\big(1 + s^{\alpha-1}\big)e^{-sx-s^{\alpha}x}. \tag{3.9}$$

This implies the formula

$$u(x,t) = \frac{1}{2|x|^{1/\alpha}}\Big(\phi\big(\frac{t-|x|}{|x|^{1/\alpha}}\big) + \int_{|x|}^{t} \frac{1}{\Gamma(1-\alpha)(t-s)^{\alpha}}\phi\big(\frac{s-|x|}{|x|^{1/\alpha}}\big)\,ds\Big) \tag{3.10}$$

for the same range as the first case of (3.4).

Set $V(x,t) = \int_0^t U(x,s)\,ds$ for $x > 0$, $t > 0$. Then $\hat{V}(x,s) = s^{-2}e^{-sx-s^{\alpha}x}$, and therefore

$$\hat{v}(x,s) = \frac{1}{2s}\big(1 + s^{\alpha-1}\big)e^{-sx-s^{\alpha}x}. \tag{3.11}$$

This implies the formula

$$v(x,t) = \frac{1}{2}\Big(\Phi\big(\frac{t-|x|}{|x|^{1/\alpha}}\big) + \int_{|x|}^{t} \frac{1}{\Gamma(1-\alpha)(t-s)^{\alpha}}\Phi\big(\frac{s-|x|}{|x|^{1/\alpha}}\big)\,ds\Big). \tag{3.12}$$

The function v was constructed in this form in [14]. Differentiating and using (3.4) one sees that

$$\begin{aligned} v_x(x,t) &= -\text{sign}(x)\Big(u(x,t) + \int_{|x|}^{t} \frac{1}{\Gamma(1-\alpha)(t-s)^{\alpha}}u(x,s)\,ds\Big) \\ &= -\text{sign}(x)\Big(u(x,t) + \int_0^t c(t-s)u(x,s)\,ds\Big) \\ &= -\text{sign}(x)\frac{1}{2|x|^{1/\alpha}}\Big(\phi\big(\frac{t-|x|}{|x|^{1/\alpha}}\big) + \int_0^t d(t-s)\phi\big(\frac{s-|x|}{|x|^{1/\alpha}}\big)\,ds\Big). \end{aligned} \tag{3.13}$$

Alternatively, by setting $G(r) = \int_0^r \Phi(s)\,ds$, one can write

$$V(x,t) = x^{1/\alpha}G\big(\frac{t-x}{x^{1/\alpha}}\big) \quad (0 < x < t), \tag{3.14}$$

and therefore

$$v(x,t) = -\frac{1}{2}\Big(\frac{1}{\alpha|x|^{1/\alpha-1}}G\big(\frac{t-|x|}{|x|^{1/\alpha}}\big) + \frac{(1-\alpha)|x| - t}{\alpha|x|}\Phi\big(\frac{t-|x|}{|x|^{1/\alpha}}\big)\Big). \tag{3.15}$$

For later use, let us set

$$\tilde{\Phi}(r) = \frac{1}{r}\int_0^r s\phi(s)\,ds = \Phi(r) - \frac{1}{r}G(r). \tag{3.16}$$

Note that $1 - \Phi(r) = O(r^{-\alpha})$ and $\tilde{\Phi}(r) = O(r^{-\alpha})$ by (A.13). One then obtains the formula

$$v(x,t) = \frac{1}{2}\Big(\frac{t-|x|}{\alpha|x|}\tilde{\Phi}\Big(\frac{t-|x|}{|x|^{1/\alpha}}\Big) + \Phi\Big(\frac{t-|x|}{|x|^{1/\alpha}}\Big)\Big). \tag{3.17}$$

From (3.2) and the positivity of u, one infers that $v(x,t) > 0$ if $|x| \le t$ and that

$$\|v(\cdot,t)\|_{L^1} = t. \tag{3.18}$$

From (3.12) one deduces on the other hand that

$$v(x,t) \le \frac{1}{2}\Big(1 + \int_0^t \frac{ds}{\Gamma(1-\alpha)s^\alpha}\Big) = \frac{1}{2} + \frac{t^{1-\alpha}}{2\Gamma(2-\alpha)} \tag{3.19}$$

with equality for $x = 0$. By Hölder's inequality,

$$\|v(\cdot,t)\|_{L^q} \le Ct^{1/q} \tag{3.20}$$

for $0 \le t \le T$ with $C = C(T)$ and

$$\|v(\cdot,t)\|_{L^q} \le C' t^{1+\alpha(1-q)/q} \tag{3.21}$$

for $0 < T \le t < \infty$ with $C' = C'(T)$.

Formula (3.13) shows that $x \mapsto v(x,t)$ is decreasing for $x \ge 0$. It also suggests that a size estimate like (3.8) holds for v_x. Indeed, by a direct integration one sees that

$$\|v_x(\cdot,t)\|_{L^1} = 1 + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.22}$$

for all $t > 0$. An L^∞-estimate for finite t, say, $0 < t \le T$, can be established as follows: For positive x,

$$|v_x(x,t)| = u(x,t) + \int_0^{t/2} c(t-s)u(x,s)\,ds + \int_{t/2}^t c(t-s)u(x,s)\,ds. \tag{3.23}$$

The first term can be estimated by $Ct^{-1/\alpha}$ by (3.8). The second term is bounded by

$$\sup_{t/2\le s\le t} c(t-s)\int_0^{t/2} u(x,s)\,ds = Ct^{-\alpha}v(x,t/2) \le C(T)t^{-\alpha} \tag{3.24}$$

by the positivity of u, the definition of v, and (3.19). The third term is bounded by

$$\int_{t/2}^{t} c(t-s)\,ds \sup_{t/2\le s\le t} u(x,s) \le C(T)t^{-1/\alpha} \tag{3.25}$$

by (3.8). Combining these estimates and noting again that $t^{-\alpha} \le Ct^{-1/\alpha}$ on $[0,T]$, one obtains

$$\|v_x(\cdot,t)\|_{L^\infty} \le Ct^{-1/\alpha} \tag{3.26}$$

for $0 < t \le T$. Using the L^1-estimate (3.22) and Hölder's inequality results in

$$\|v_x(\cdot,t)\|_{L^q} \le Ct^{\frac{1-q}{\alpha q}} \tag{3.27}$$

for $0 < t \le T$, $1 \le q \le \infty$. Finally of course

$$\hat{w}(x,s) = e^{-sx-s^\alpha x}, \qquad w(x,t) = x^{-1/\alpha}\phi\Big(\frac{t-x}{x^{1/\alpha}}\Big). \tag{3.28}$$

The function $t \mapsto w(x,t)$ describes the displacement of a viscoelastic material that is governed by (2.1a) at the station $x > 0$ in response to a pulse shear at the boundary $x = 0$ at time $t = 0$. The pulse is not felt until the time $t = x$ and it has the shape of the scaled probability density function ϕ. In particular, the pulse is unimodal (it has exactly one maximum). Unimodulaity in viscoelastic pulse propagation was studied in [8]. The class of kernels a studied in that paper does not overlap with the set of kernels that is used here.

From the construction in the previous section, it follows that the function

$$\bar{U} : (x,t) \;\mapsto\; \frac{1}{2}\big(1 - U(|x|,t)\big) \cdot \operatorname{sign}(x)$$

solves the integrodifferential equation (1.1) in the sense of distributions on $\mathbf{R} \times (0,\infty)$; that is, with $\bar{Z}(x,t) = U(x,t) - \int_0^t a(t-s)\bar{U}(x,s)\,ds$, the equation $\bar{U}_{tt} - \bar{Z}_{xx} = 0$ holds in $\mathcal{D}'(\mathbf{R} \times (0,\infty))$. Then u and v solve (1.1) in the same sense, and w solves the equation in the sense of distributions on $(0,\infty) \times (0,\infty)$.

Since $u \ge 0$, the support of $u(\cdot,t)$ is $[-t,t]$, and $\int_{-t}^{t} u(x,t)\,dx = 1$ for all t, one obtains next

$$u(\cdot,t) \to \delta_0 \quad \text{as} \quad t \to 0, \tag{3.29}$$

weak-* in the space of measures, the dual of the space $BC^0(\mathbf{R},\mathbf{R})$ of bounded continuous functions on $\mathbf{R}$. Let us further show that

$$u_t(\cdot,t) \to 0 \quad \text{as} \quad t \to 0 \tag{3.30}$$

weak-* in the dual space of $BC^1(\mathbf{R},\mathbf{R}) = \{\varphi \in BC^0(\mathbf{R},\mathbf{R}) : \varphi' \in BC^0(\mathbf{R},\mathbf{R})\}$. Thus let $\varphi \in BC^1(\mathbf{R},\mathbf{R})$ and set

$$k(t) = \int_{\mathbf{R}} u(y,t)\varphi(y)\,dy = \int_0^\infty u(y,t)\big(\varphi(y) + \varphi(-y) - 2\varphi(0)\big)\,dy + 2\varphi(0). \tag{3.31}$$

Recall that $u_t = -u_x + b * u_x$ everywhere in $(0,\infty) \times (0,\infty)$by construction, with b given by (2.16). Also, (3.6) implies that u_t and u_x are smooth and bounded for $x \geq 0$, $\epsilon \leq t \leq T$, for any ϵ, T, with bounds depending on ϵ and T. Thus one can integrate by parts in (3.31) and obtain that k is differentiable on $(0,\infty)$. Setting $z(x,t) = u(x,t) - \int_0^t b(t-s)u(x,s)\,ds$, one obtains

$$\begin{aligned} k'(t) &= \int_0^\infty u_t(y,t)\big(\varphi(y) + \varphi(-y) - 2\varphi(0)\big)\,dy \\ &= \int_0^\infty z_y(y,t)\big(\varphi(y) + \varphi(-y) - 2\varphi(0)\big)\,dy \\ &= \int_0^\infty z(y,t)\big(\varphi'(-y) - \varphi'(y)\big)\,dy. \end{aligned}$$

By (3.29), $k'(t) \to 0$ as $t \to 0$, proving (3.31).

As to the function v, it follows e.g. from (3.12) that $v(\cdot,t) \to 0$ uniformly away from $x = 0$ and in any $L^p(\mathbf{R})$ with $p < \infty$. Also, of course, (3.29) implies that $v_t(\cdot,t) = u(\cdot,t) \to \delta_0$. Similarly, (3.29) implies immediately that

$$w(x,\cdot) \to \delta_0 \quad \text{as} \quad t \to 0 \tag{3.32}$$

weak-* in $BC^0([0,\infty),\mathbf{R})$.

These considerations allow one to construct solutions z of the homogeneous integrodifferential equation (1.1) in $\mathbf{R} \times (0,\infty)$ with initial data

$$z(\cdot,0) = \varphi_0, \quad z_t(\cdot,0) = \varphi_1 \quad (x \in \mathbf{R}). \tag{3.33}$$

Namely, set

$$z(x,t) = \int_{\mathbf{R}} \big(u(x-y,t)\varphi_0(y) + v(x-y,t)\varphi_1(y)\big)\,dy. \tag{3.34}$$

Then z solves (1.1) and satisfies (3.33) if $\varphi_0 \in BC^1(\mathbf{R},\mathbf{R})$ and $\varphi_1 \in BC^0(\mathbf{R},\mathbf{R})$. If for example only φ_0, φ_0', $\varphi_1 \in L^1_{\text{loc}}(\mathbf{R})$, then (3.34) still defines a function z with $z(\cdot,t)$, $z_t(\cdot,t) \in L^1_{\text{loc}}(\mathbf{R})$ for $t > 0$, that solves (1.1) in the sense of distributions and for which $z(\cdot,t) \to \varphi_0$, $z_t(\cdot,t) \to \varphi_1$ in $L^1_{\text{loc}}(\mathbf{R})$. In this case, one can also define weak solutions z of (1.1) and (3.33) by requiring that with $\bar{z}(x,t) = z(x,t) - \int_0^t a(t-s)z(x,s)\,ds$, for all test functions $\psi \in C_0^\infty(\mathbf{R} \times [0,\infty),\mathbf{R})$ the equation holds

$$\int_0^\infty \int_{\mathbf{R}} \big(z \cdot \psi_{tt} - \bar{z} \cdot \psi_{xx}\big)\,dx\,dt + \int_{\mathbf{R}} \big(\varphi_0\psi_t(\cdot,0) - \varphi_1\psi(\cdot,0)\big)\,dx = 0. \tag{3.35}$$

Such a weak solution can also be represented in the form (3.34). This is clear for smooth initial data; for locally integrable data it follows by an approximation argument. However, weak solutions can be defined for even less regular data, e.g.

for measures or weak derivatives of locally integrable functions. It is easy to see that even in these cases (3.34), suitably interpreted, still furnishes a weak solution.

Similarly, a solution z of the homogeneous equation (1.1) in $(0,\infty)\times(0,\infty)$ with initial data $z(\cdot,0)=z_t(\cdot,0)=0$ and boundary data $z(0,t)=\varphi_2(t)\,(t>0)$ is given by

$$z(x,t)=\int_0^t w(x,t-s)\varphi_2(s)\,ds. \tag{3.36}$$

This gives a smooth solution if e.g. φ_2 is smooth and $\varphi_2(0)=\varphi_2'(0)=0$; weak solutions are defined in the usual way and are still given by (3.36). Solutions in $(0,\infty)\times(0,\infty)$ with general initial data and zero boundary data can be found with the well-known trick of extending the initial data as odd functions on $\mathbf{R}$, solving the resulting problem on $\mathbf{R}\times(0,\infty)$, and restricting the solution to $(0,\infty)\times(0,\infty)$. Solutions with general initial and boundary data are found by superposition.

Finally, solutions z of inhomogeneous equations

$$z_{tt}(x,t)-z_{xx}(x,t)+\int_0^t a(t-s)z_{xx}(x,s)\,ds=f(x,t)\quad(x\in\mathbf{R},\,t>0) \tag{3.37}$$

with zero data at $t=0$ are found with DUHAMEL's principle:

$$z(x,t)=\int_0^t\int_{\mathbf{R}} v(x-y,t-s)f(y,s)\,dy\,ds. \tag{3.38}$$

This formal representation can be justified for smooth functions by means of (3.30); for less smooth functions, it follows by approximation arguments. Weak solutions can be represented in the same way, if the integral is interpreted suitably. Details can be found in any textbook on partial differential equations and are omitted.

The section closes with results that summarize some of these considerations in a functional-analytic setting.

Theorem 3.1. *Let $\varphi_0\in L^p(\mathbf{R})$, $\varphi_1\in L^1(\mathbf{R})$, $f\in L^1(0,T;L^1(\mathbf{R}))$ be given. Then there exists a unique function $z:\mathbf{R}\times[0,T]\to\mathbf{R}$ that is a distributional solution of* (3.37)*, satisfying* (3.33) *in the sense of* (3.35)*, and is continuous with values in $L^p(\mathbf{R})$.*

Sketch of proof. Define z as the sum of the two right-hand sides of (3.34) and (3.38); then z is a distributional solution of (3.37) with initial data (3.33). By the properties of u and v that have been listed above, z is L^p-valued and continuous with values in that space. The uniqueness of z is shown by a standard adjoint argument: It suffices to show that solutions with the regularity that is asserted in the theorem must vanish if their data are zero. Thus choose an arbitrary function $\chi\in C_0^\infty(\mathbf{R}\times(0,T))$ and define the C^∞-function ψ by

$$\psi(x,t)=\int_0^{T-t}\int_{\mathbf{R}} v(x-y,T-t-s)\chi(y,T-s)\,dy\,ds.$$

Let z be a solution with $\varphi_0 = \varphi_1 = 0$ and $f = 0$. Using (3.35), one obtains with the notation in that formula

$$0 = \int_0^T \int_{\mathbf{R}} \big(z \cdot \psi_{tt} - \bar{z} \cdot \psi_{xx}\big)\, dx\, dt = \int_0^T \int_{\mathbf{R}} z(x,t)\chi(x,t)\, dx\, dt.$$

Therefore, $z = 0$ as asserted.

Theorem 3.2. *Let z be the solution that has been constructed in Theorem* 3.1.

a) *If $\varphi_1 = 0$, $f = 0$, then $z(\cdot,t) \in L^\rho(\mathbf{R})$ for all $p \le \rho \le \infty$ for $t > 0$, and the estimate holds*

$$\|z(\cdot,t)\|_{L^\rho} \le Ct^{\frac{1}{\alpha}\left(\frac{1}{\rho}-\frac{1}{p}\right)}\|\varphi_0\|_{L^p}. \tag{3.39}$$

b) *If $\varphi_0 = 0$, $\varphi_1 \in L^p(\mathbf{R}), f = 0$, then $z_x(\cdot,t) \in L^\rho(\mathbf{R})$ for all $p \le \rho \le \infty$ for $t > 0$, and the estimate holds*

$$\|z_x(\cdot,t)\|_{L^\rho} \le Ct^{\frac{1}{\alpha}\left(\frac{1}{\rho}-\frac{1}{p}\right)}\|\varphi_1\|_{L^p}. \tag{3.40}$$

c) *If $\varphi_0 = \varphi_1 = 0$, $f \in L^s(0,T; L^r(\mathbf{R}))$ and $\alpha p \le 1$, then $z_x \in L^\sigma(0,T;\, L^\rho(\mathbf{R}))$ for all $\rho \in [p, \frac{p}{1-\alpha p})$, $\sigma < \big(\frac{1}{\alpha}\big(\frac{1}{p} - \frac{1}{\rho}\big) + \frac{1}{s} - 1\big)^{-1}$, and the estimate holds*

$$\|z_x\|_{L^\sigma(0,T;L^\rho)} \le C\|f\|_{L^s(0,T;L^r)}. \tag{3.41}$$

If $\alpha p > 1$, then the same is true for all $\rho \in [p,\infty]$ and all corresponding σ.

Sketch of Proof. To prove a), one uses the representation formula (3.34) and Young's inequality together with the estimate (3.8) with $\frac{1}{q} = 1 + \frac{1}{\rho} - \frac{1}{p}$. For the proof of b), one uses (3.34) in the same manner together with Young's inequality and estimate (3.27). For the proof of part c), one also employs Young's inequality and (3.27), applied first to the convolution in the x-direction in (3.38), with q the same as in the previous arguments. The restriction of ρ then implies that $t \mapsto \|v_x(\cdot,t)\|_{L^\rho}$ is in $L^\beta(0,T)$ for any $\beta < \alpha\big(\frac{1}{q} - \frac{1}{\rho}\big)^{-1}$. Applying now Young's inequality also to the convolution in the t-direction in (3.38), the assertion follows.

Theorem 3.1 is also true for the wave equation in one space dimension and for solutions of integrodifferential equations of the form (3.37) with sufficiently smooth kernels a. Theorem 3.2 is a quantitative expression for well-known smoothing effects that can only occur for equations with singular kernels. It is known that in fact solutions of such problems gain two spatial derivatives over the regularity of their data. Below, a quantitative version of this property will be given, and it will be shown what the obstacle to further smoothing is.

4. Behavior near $t = 0$ and $t = \infty$. Let us first consider the behavior of u near $x = t = 0$. Let $\epsilon > 0$ be given. For the scaled function

$$(x,t) \mapsto \epsilon^\alpha u(\epsilon x, \epsilon t) \tag{4.1}$$

one then obtains

$$\epsilon^\alpha u(\epsilon x, \epsilon t) = \epsilon^{\alpha-1/\alpha}\phi\big(\epsilon^{1-1/\alpha}\frac{t-|x|}{|x|^{1/\alpha}}\big)\frac{t-(1-\alpha)|x|}{2\alpha|x|^{1+1/\alpha}}.$$

Fix $0<\rho<1$, $0<T$, and consider the region $R_{\rho,T} = \{(x,t) : |x| \le \rho t,\ t \le T\}$. By (A.13) one obtains that

$$\epsilon^\alpha u(\epsilon x, \epsilon t) \to \frac{\Gamma(\alpha)\sin\alpha\pi}{2\pi}\frac{t-(1-\alpha)|x|}{(t-|x|)^{1+\alpha}} = \frac{1}{2\Gamma(1-\alpha)}\frac{t-(1-\alpha)|x|}{(t-|x|)^{1+\alpha}} \tag{4.2}$$

uniformly in $R_{\rho,T}$ as $\epsilon \to 0$. In fact, from (3.5) one obtains a convergent power series in $\epsilon^{1-\alpha}$ for $\epsilon^\alpha u(\epsilon x, \epsilon t)$. Each coefficient is a sum of two functions that are homogeneous in $|x|$ and $t-|x|$; the right-hand side of (4.2) is the lowest-order term. Changing the scale, let us set $u^\epsilon(x,t) = \epsilon u(\epsilon x, \epsilon t)$. Then $\int_{-t}^t u^\epsilon(x,t)\,dx = 1$ and from (4.2)

$$|u^\epsilon(x,t)| \le \epsilon^{1-\alpha}C(\alpha)\frac{1}{(1-\rho)^{1+\alpha}t^\alpha}$$

in any $R_{\rho,T}$. Therefore $\int_{-\rho t}^{\rho t} u^\epsilon(x,t)\,dx \le C(\rho)(\epsilon t)^{1-\alpha}$, and

$$u^\epsilon(x,t) \to \frac{1}{2}\big(\delta_0(\cdot+t) + \delta_0(\cdot - t)\big) \tag{4.3}$$

as $\epsilon \to 0$, in the sense of measures. One can similarly define

$$v^\epsilon(x,t) = v(\epsilon x, \epsilon t) = \int_0^t u^\epsilon(x,s)\,ds. \tag{4.4}$$

Then $v^\epsilon(x,t) \to \frac{1}{2}1_{[-t,t]}(x)$, pointwise and locally uniformly away from the rays $|x| = t$. Details are left to the reader. These properties show that u and v behave like corresponding distributional solutions of the wave equation near $x=t=0$, and (4.2) describes the effects of additional terms inside the wave cone where $|x| \le t$.

Let us next look more closely at the behavior of u and v near the rays $|x| = t$. By symmetry, it is sufficient to consider the case $x = t$. Consider then the function

$$\xi \mapsto t^{1/\alpha}u(t - t^{1/\alpha}\xi, t) = \phi\big(\frac{\xi}{(1-t^{1/\alpha-1}\xi)^{1/\alpha}}\big)\frac{\alpha+(1-\alpha)t^{1/\alpha-1}\xi}{2\alpha(1-t^{1/\alpha-1}\xi)^{1+1/\alpha}}$$

for $\xi \le t^{1-1/\alpha}$. As $t \to 0$, one sees that

$$t^{1/\alpha}u(t-t^{1/\alpha}\xi, t) \to \frac{1}{2}\phi(\xi). \tag{4.5}$$

This shows that immediately behind the line $x = t$, the function u looks like a suitably scaled version of the probability density ϕ. It also implies that

$$t^{(q-1)/\alpha}\int_{t-t^{1/\alpha}}^t |u(x,t)|^q\,dx \to \frac{1}{2^q}\int_0^1 \phi^q(x)\,dx$$

as $t \to 0$, i.e., that estimate (3.8) is sharp for all q. Similarly, one obtains from (3.12)

$$v(t - t^{1/\alpha}\xi, t) \to \frac{1}{2}\Phi(\xi) \tag{4.6}$$

as $t \to 0$, indicating that the shape of $v(\cdot, t)$ becomes that of the cumulative distribution function Φ, up to a specific scaling.

Let us now turn to the behavior of u and v as $t \to \infty$. Then, still a different scaling is appropriate. First consider the function $x \mapsto t^\alpha u(t^\alpha x, t)$. Then due to (3.4), one obtains

$$t^\alpha u(t^\alpha x, t) \to \frac{1}{2\alpha}|x|^{-1-1/\alpha}\phi_\alpha\big(|x|^{-1/\alpha}\big) = \frac{1}{2}\psi_{1/\alpha}(|x|) \quad \text{as} \quad t \to \infty, \tag{4.7}$$

where $\psi_{1/\alpha}$ is the density of a trans-stable probability distribution with index $1/\alpha \in (1, \infty)$ (see the appendix). This shows that for large t, u behaves like the function

$$\tilde{u}(x, t) = \frac{1}{2}t^{-\alpha}\psi_{1/\alpha}(|x|t^{-\alpha}). \tag{4.8}$$

By (A.20), the function $\tilde{u}$ has the Fourier transform

$$\mathcal{F}\big(\tilde{u}(\cdot, t)\big)(\xi) = \frac{1}{2}\big(E_{\alpha,1}(i\xi t^\alpha) + E_{\alpha,1}(-i\xi t^\alpha)\big).$$

Property (A.5) then implies that $p(\xi, t) = \mathcal{F}\big(\tilde{u}(\cdot, t)\big)(\xi)$ satisfies

$$p(\xi, t) + \frac{\xi^2}{\Gamma(2\alpha)}\int_0^t (t - s)^{2\alpha-1}p(\xi, s)\, ds = 1. \tag{4.9}$$

Inverting the Fourier transform, one sees that $\tilde{u}$ solves the integrodifferential equation

$$\tilde{u}(x, t) = \delta_0(x) + \frac{1}{\Gamma(2\alpha)}\int_0^t (t - s)^{2\alpha-1}\tilde{u}_{xx}(x, s)\, ds \tag{4.10}$$

in the sense of distributions, which is just (1.4) with α replaced by 2α. For $\alpha = 1/2$ in particular, $\tilde{u}$ is the fundamental solution of the heat equation. In this special case, the solution u shows "diffusive" behavior as $t \to \infty$. In all other cases, u behaves like the solution of the fractional diffusion equation (1.4) for large times. Such problems have been studied in [16], [22], [4] and [19]; see also [3]. The large-time behavior of u is tied to the properties of the kernel a; by (2.20) this kernel behaves like $-\frac{d^2}{dt^2}\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}$ as $t \to \infty$.

The asymptotic behavior of v in the limit as $t \to \infty$ can also be obtained; namely,

$$\begin{aligned} t^{\alpha-1}v(t^\alpha x, t) &= \frac{1 - |x|t^{\alpha-1}}{2\alpha|x|}\tilde{\Phi}\Big(\frac{1 - |x|t^{\alpha-1}}{|x|^{1/\alpha}}\Big) + t^{\alpha-1}\Phi\Big(\frac{1 - |x|t^{\alpha-1}}{|x|^{1/\alpha}}\Big) \\ &\to \frac{1}{2\alpha|x|}\tilde{\Phi}\big(|x|^{-1/\alpha}\big). \end{aligned} \tag{4.11}$$

Thus for large t, $v(x,t)$ behaves like the simpler function

$$\tilde{v}(x,t) = \frac{t^{1-\alpha}}{2\alpha|x|}\tilde{\Phi}\big(t|x|^{-1/\alpha}\big) = \int_0^t \tilde{u}(x,s)\,ds \tag{4.12}$$

which solves an integrodifferential equation that results from integrating (4.10).

These results have consequences for solutions of general initial-value problems as constructed in the previous section.

Theorem 4.1. *Let φ_0, φ_1, f, and the solution z of the initial-value problem* (3.33), (3.37) *be as in Theorem* 3.1.

a) *If $\int_{\mathbf{R}} \varphi_0(x)\,dx = A$, $\varphi_1 = 0$, $f = 0$, then as $t \to \infty$*

$$t^\alpha z(t^\alpha x, t) \to \frac{A}{2}\psi_{1/\alpha}(|x|).$$

b) *If $\varphi_0 = 0$, $\int_{\mathbf{R}} \varphi_1(x)\,dx = B$, $\int_0^\infty \int_{\mathbf{R}} f(x,t)\,dx\,dt = C$, then as $t \to \infty$*

$$t^{\alpha-1} z(t^\alpha x, t) \to \frac{B+C}{2\alpha|x|}\tilde{\Phi}(|x|^{-1/\alpha}).$$

Sketch of proof. To prove part a), let us use (3.34) and write

$$t^\alpha z(t^\alpha x, t) = \int_{\mathbf{R}} t^\alpha u(t^\alpha(x - t^{-\alpha}y), t)\varphi_0(y)\,dy$$

and use (4.7) and the dominated convergence theorem. For part b), the same argument together with (4.11) applies, if $f = 0$. If $f \neq 0$, one notes that (4.11) generalizes to

$$(t+\tau_1)^{\alpha-1} v((t+\tau_2)^\alpha x + y, t+\tau_3) \to \frac{1}{2\alpha|x|}\tilde{\Phi}\big(|x|^{-1/\alpha}\big)$$

as $t \to \infty$, for any fixed τ_1, τ_2, τ_3, y. Formula (3.38) and the dominated convergence theorem then imply the full result of part b).

5. Regularity properties. In this section, I study regularity properties of the special solutions u and v across the interfaces $|x| = t$ and at $x = 0$. First recall (3.5) and the definition of Wright's function $W_{-\alpha,0}$ in the appendix. This allows one to write

$$u(x,t) = \frac{1}{2}\Big(\frac{1}{\alpha|x|} + \frac{1}{t-|x|}\Big) W_{-\alpha,0}\Big(\frac{-|x|}{(t-|x|)^\alpha}\Big). \tag{5.1}$$

Then property (A.9) implies that as $x \to t$, with $y = \frac{|x|}{(t-|x|)^\alpha}$, the expansion holds

$$u(x,t) = \frac{1}{2}\Big(\frac{1}{\alpha|x|} + \frac{1}{t-|x|}\Big) R_m(y), \tag{5.2}$$

$$R_m(y) = \sqrt{By^{1/(1-\alpha)}}e^{-By^{1/(1-\alpha)}}\Big(\sum_{k=0}^{m-1} C_k y^{-k/(1-\alpha)} + O(y^{-m/(1-\alpha)})\Big),$$

for any $m \geq 1$, with certain constants B, C_0, $C_1, \dots$. A result for v follows by integrating; it can also be found in [14].

Let us turn to the behavior of u near $x = 0$. Setting

$$c_k = \frac{\sin \alpha\pi k \Gamma(1+\alpha k)}{\pi k!} = \frac{1}{\Gamma(1+k)\Gamma(-\alpha k)}$$

for $k \geq 0$, (3.5) can be written as

$$u(x,t) = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\Big(\frac{c_{k+1}}{\alpha(t-|x|)^{\alpha k+\alpha}} - \frac{c_k}{(t-|x|)^{\alpha k+1}}\Big)\cdot |x|^k. \tag{5.3}$$

Using the POCHHAMMER symbol $(z)_r = z(z-1)(z-2)\cdots(z-r+1)$, $(z)_0 = 1$, one obtains

$$\begin{aligned}\frac{\partial^k}{\partial x^k}u(0+,t) = \frac{1}{2}\sum_{l=0}^{k}(-1)^l(k)_l\Big(&\frac{c_{l+1}}{\alpha}(-\alpha l-\alpha)_{k-l}t^{-\alpha} \\ &- c_l(-\alpha l-1)_{k-l}t^{-1}\Big)t^{-k+(1-\alpha)l} = p_k(t).\end{aligned} \tag{5.4}$$

For $k = 1$, one recovers in particular the relation (3.7) between $u_x(0+,t)$ and the creep kernel d. For higher odd values of $k = 2l-1$, $\frac{\partial^k}{\partial x^k}u(0+,t)$ is the k-th derivative of the function $\sum_{j=1}^{l}\binom{l}{j}d^{j*}$, where d^{j*} is the j-fold convolution product with itself.

Set $\tilde{u}_k(x,t) = \frac{\partial^k}{\partial x^k}u(x,t)$ for $x \neq 0$. Since u is an even function of x, (5.4) implies that for all $k \geq 1$

$$\frac{\partial^k}{\partial x^k}u(x,t) = \tilde{u}_k(x,t) + 2\sum_{0\leq j\leq [k/2]-1} p_{2j+1}(t)\delta_0^{(k-2-2j)} \tag{5.5}$$

in the sense of distributions, where $\delta_0^{(r)}$ is the r-th distributional derivative of δ_0. Similarly, one obtains from (3.15) various expansions of v which imply versions of (5.4) and (5.5) also for this function; namely,

$$\begin{aligned}\frac{\partial^k}{\partial x^k}v(0+,t) = \frac{1}{2}\sum_{l=0}^{k-1}(-1)^l(k)_l\Big(&\frac{c_{l+1}}{\alpha}(-\alpha l-\alpha)_{k-l-1}t^{-\alpha} \\ &- c_l(-\alpha l-1)_{k-l-1}t^{-1}\Big)t^{-k+(1-\alpha)l} \\ &+ \frac{(-1)^k k!}{2}\Big(\frac{c_{k+1}}{\alpha(1-\alpha-\alpha k)}t^{1-\alpha-\alpha k} + \frac{c_k}{\alpha k}t^{-\alpha k}\Big) = q_k(t)\end{aligned} \tag{5.4$'$}$$

and

$$\frac{\partial^k}{\partial x^k}v(x,t) = \tilde{v}_k(x,t) + 2\sum_{0\leq j\leq [k/2]-1} q_{2j+1}(t)\delta_0^{(k-2-2j)}. \tag{5.5$'$}$$

For $k = 2$, (5.5) implies that $u_{xx}(\cdot, t)$ is a measure of the form

$$u_{xx}(\cdot, t) = -d'(t)\delta_0 + \tilde{u}_2(\cdot, t)$$

with an integrable function $\tilde{u}_2(\cdot, t)$. One sees that solutions of initial-value problems as constructed in Theorem 3.1 cannot gain more than two spatial derivatives over their initial data φ_0, unless the creep kernel d becomes constant for some $t > 0$ (which does not happen in the case at hand). Also, since $d'(t) = O(t^{-1-\alpha})$ as $t \to 0$, the operator that maps the initial data φ_0 to the second derivative of the solution for fixed $t > 0$ has a norm that is at least $O(t^{-1-\alpha})$ as $t \to 0$. However, the behavior of the second term, $\tilde{u}_2$, turns out to be even worse. A general statement is the following, which will be proved for all $\tilde{u}_k(\cdot, t)$, $0 < t \leq T$, $k \geq 1$:

$$\|\tilde{u}_k(\cdot, t)\|_{L^1(0,\infty)} \leq C(\alpha, k, T)t^{-k/\alpha}. \tag{5.6}$$

Similarly, for $\tilde{v}_k(\cdot, t) = \frac{\partial^k}{\partial x^k} v(\cdot, t)$, one obtains for the same range of k, t:

$$\|\tilde{v}_k(\cdot, t)\|_{L^1(0,\infty)} \leq C(\alpha, k, T)t^{(1-k)/\alpha}. \tag{5.7}$$

To prove (5.6), let us write

$$\|\tilde{u}_k(\cdot, t)\|_{L^1(0,\infty)} = \int_0^{t/2} |\partial_x^k u(x, t)| \, dx + \int_{t/2}^t |\partial_x^k u(x, t)| \, dx.$$

The first integral can be estimated by differentiating the series (5.3) term by term, estimating, and integrating the result. After some straightforward computations, the result is

$$\int_0^{t/2} |\partial_x^k u(x, t)| \, dx \leq C(k, \alpha, T)t^{-\alpha-k+1} \tag{5.8}$$

for $0 < t \leq T$. To estimate the second integral, one substitutes $\xi = \frac{x}{t} \in [\frac{1}{2}, 1]$ for $\frac{t}{2} \leq x \leq t$ and write

$$w(\xi, t) = u(t\xi, t) = t^{-1/\alpha} p'(\xi)\phi(t^{1-1/\alpha} p(\xi)), \tag{5.9}$$

with $p(\xi) = (1 - \xi)\xi^{-1/\alpha}$. Note that p is positive and monotonically decreasing on $[\frac{1}{2}, 1]$ and $p(1) = 0$. Then

$$\int_{1/2}^1 |\partial_\xi^k w(\xi, t)| \, d\xi = t^{k-1} \int_{t/2}^t |\partial_x^k u(x, t)| \, dx. \tag{5.10}$$

The function $\partial_\xi^k w(\xi, t)$ has the form

$$\partial_\xi^k w(\xi, t) = \sum_{j=0}^k t^{-1/\alpha + j(1-1/\alpha)} g_{k,j}(\xi)\phi^{(j)}(t^{1-1/\alpha} p(\xi)) \tag{5.11}$$

with suitable functions $g_{k,j}$ which are polynomials in p and its derivatives. Thus all $g_{k,j}$ are uniformly bounded on $[\frac{1}{2}, 1]$, with bounds depending on k. Substituting now $\rho = t^{1-1/\alpha}p(\xi)$ and setting $M(t) = t^{1-1/\alpha}p(\frac{1}{2})$, one obtains

$$\int_{1/2}^{1} |\partial_\xi^k w(\xi,t)|\, d\xi \le C \sum_{j=0}^{k} t^{-1/\alpha+(j-1)(1-1/\alpha)} \int_0^{M(t)} |\phi^{(j)}(\rho)|\, d\rho \qquad (5.12)$$

$$\le C \sum_{j=0}^{k} t^{-1/\alpha+(j-1)(1-1/\alpha)} \int_0^{\infty} |\phi^{(j)}(\rho)|\, d\rho \le Ct^{-1/\alpha+(k-1)(1-1/\alpha)}.$$

Here the series expansion (A.11) was used to guarantee that all derivatives $\phi^{(j)}$ are Lebesgue-integrable on $[0, \infty)$. Then

$$\int_{t/2}^{t} |\partial_x^k u(x,t)|\, dx \le Ct^{-1/\alpha+(k-1)(1-1/\alpha)+1-k} = Ct^{-k/\alpha}. \qquad (5.13)$$

Since $t^{-\alpha-k+1} \le Ct^{-k/\alpha}$ for $0 < t \le T$, the desired estimate (5.6) follows by combining (5.8) and (5.13).

Note that the leading-order term in (5.11) is

$$t^{-1/\alpha+k(1-1/\alpha)}\big(p'(\xi)\big)^{k+1}\phi^{(k)}(t^{1-1/\alpha}p(\xi)),$$

resulting in a corresponding leading-order term in the estimate (5.12). Since $M(t) \to \infty$ as $t \to 0$, this suggests that in fact

$$\|\tilde{u}_k(\cdot,t)\|_{L^1(0,\infty)} = C^* t^{-k/\alpha} + o(t^{-k/\alpha}) \qquad (5.14)$$

as $t \to 0$, with $C^* > 0$.

The estimate (5.7) for v is proved similarly. The integral defining the L^1-norm of $\tilde{v}_k(\cdot,t)$ is split into two parts as before. For $\frac{t}{2} \le x \le t$, one obtains from (3.17)

$$v(x,t) = v(t\xi,t) = z(\xi,t) = \frac{\xi-1}{2\alpha\xi}\tilde{\Phi}(t^{1-1/\alpha}p(\xi)) + \Phi(t^{1-1/\alpha}p(\xi)). \qquad (5.15)$$

One now proceeds as in the proof of (5.12), noting the absence of the factor $t^{-1/\alpha}$. All derivatives of Φ and $\tilde{\Phi}$ are integrable on $[0, \infty)$, while

$$|\tilde{\Phi}(t^{1-1/\alpha}p(\xi))| = O\big((t^{1-1/\alpha}p(\xi))^{-\alpha}\big) = O(t^{1-\alpha}),$$

uniformly in $\xi \in [\frac{1}{2}, 1]$, as $t \to 0$. An analogue of (5.12) then follows also for z, resulting in the estimate

$$\int_{t/2}^{t} |\partial_x^k v(x,t)|\, dx \le Ct^{(1-k)/\alpha}. \qquad (5.16)$$

For $0 \le x \le \frac{t}{2}$, one differentiates the series for v directly, estimates, and integrates, obtaining terms of lower order than (5.16). Details are left to the reader.

With these results, one can give quantitative results for the regularity of solutions of the initial-value problem (1.1), (3.33) (with $f = 0$).

Theorem 5.1. *Let* $1 \le p \le \infty$, φ_0, $\varphi_1 \in L^p(\mathbf{R})$, $f = 0$, *and let* z *be the solution of the initial-value problem* (3.33), (3.37), *as constructed in Theorem* 3.1.

a) *If* $\varphi_1 = 0$, *then* $z(\cdot, t) \in W^{2,p}(\mathbf{R})$ *and* $z_{xx}(\cdot, t) + d'(t)\varphi_0(\cdot) \in W^{2,p}(\mathbf{R})$ *for* $t > 0$, *and the estimates hold*

$$\|z_{xx}(\cdot, t)\|_{L^p} \le C(\alpha, T)t^{-2/\alpha}\|\varphi_0\|_{L^p}$$
$$\|(z_{xx}(\cdot, t) + d'(t)\varphi_0(\cdot))_{xx}\|_{L^p} \le C(\alpha, T)t^{-4/\alpha}\|\varphi_0\|_{L^p}.$$

b) *If* $\varphi_0 = 0$, *then* $z(\cdot, t) \in W^{2,p}(\mathbf{R})$ *and* $z_{xx}(\cdot, t) + d(t)\varphi_1(\cdot) \in W^{2,p}(\mathbf{R})$ *for* $t > 0$, *and the estimates hold*

$$\|z_{xx}(\cdot, t)\|_{L^p} \le C(\alpha, T)t^{-1/\alpha}\|\varphi_1\|_{L^p}$$
$$\|(z_{xx}(\cdot, t) + d(t)\varphi_1(\cdot))_{xx}\|_{L^p} \le C(\alpha, T)t^{-3/\alpha}\|\varphi_1\|_{L^p}.$$

Sketch of proof. The proof uses (3.34). In the case of part a), one obtains

$$z_{xx}(x, t) = \int_{\mathbf{R}} \tilde{u}_2(x - y, t)\varphi_0(y)\, dy - d'(t)\varphi_0(x, t)$$

and

$$\big(z_{xx}(x, t) + d'(t)\varphi_0(x, t)\big)_{xx} = \int_{\mathbf{R}} \tilde{u}_4(x - y, t)\varphi_0(y)\, dy + p_3(t)\varphi_0(x, t)$$

by (5.5). Using (5.4) and (5.6) for $k = 2$ and $k = 4$, the result follows. Essentially the same argument applies in case b): Using (3.34), one obtains

$$z_{xx}(x, t) = \int_{\mathbf{R}} \tilde{v}_2(x - y, t)\varphi_1(y)\, dy - d(t)\varphi_1(x, t)$$

and

$$\big(z_{xx}(x, t) + d(t)\varphi_1(x, t)\big)_{xx} = \int_{\mathbf{R}} \tilde{v}_4(x - y, t)\varphi_1(y)\, dy + q_3(t)\varphi_1(x, t).$$

One then uses (5.4′), (5.5′) and (5.7) to obtain the desired estimates.

Appendix.

Mittag-Leffler functions. The entire functions

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)} \tag{A.1}$$

for α, $\beta > 0$ are known as *generalized Mittag-Leffler functions* (see [2]). Mittag-Leffler originally studied $E_\alpha = E_{\alpha,1}$ in 1903 ([15]). Special cases include

$$E_{1,1}(z) = e^z, \quad E_{2,1}(z^2) = \cosh z, \quad E_{1/2,1}(z) = e^{z^2} erfc(-z), \tag{A.2}$$

where $erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-s^2}\, ds$ is the complementary error function. The function $x \mapsto E_{\alpha,1}(-x)$ is known to be completely monotonic if $0 < \alpha \leq 1$. For $\alpha < 2$, there is an asymptotic expansion

$$E_{\alpha,\beta}(z) = -\sum_{n=1}^{N} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-N-1}) \quad \text{as} \quad z \to -\infty \tag{A.4}$$

which is valid in a sector about the negative real axis and can therefore be differentiated. As usual, the reciprocal Γ-function is extended as an entire function, such that it vanishes at the nonpositive integers. The function $u_\lambda(t) = E_{\alpha,1}(-\lambda t^\alpha)$ is the solution of the integral equations

$$\begin{aligned} u_\lambda(t) &= 1 - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u_\lambda(s)\, ds, \\ u_\lambda(t) &= 1 - \frac{\lambda t^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda^2}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} u_\lambda(s)\, ds, \end{aligned} \tag{A.5}$$

for any $\alpha > 0$, $\lambda \in \mathbf{C}$. The first equation can be checked by a direct computation; the second equation is a simple consequence. That is, u_λ solves the fractional-order differential equation (if α is nonintegral) $D^\alpha u_\lambda + \lambda u_\lambda = 0$, $u_\lambda(0) = 1$, $u'_\lambda(0) = \cdots = u_\lambda^{([\alpha]-1)} = 0$, where D^α denotes Riemann–Liouville fractional differentiation ([21]).

Wright's generalized Bessel functions. For $\gamma > -1$ and $\beta \in \mathbf{R}$, consider the entire function

$$W_{\gamma,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k)\Gamma(\beta+\gamma k)}. \tag{A.6}$$

If $\beta = 0$ and $\gamma = -\alpha \in (-1, 0)$, one can also write

$$W_{-\alpha,0}(z) = \sum_{k=1}^{\infty} \frac{\sin \alpha\pi k}{\pi} \frac{\Gamma(1+\alpha k)}{k!} z^k$$

by the reciprocity relation for the Γ-function. This function was studied by E.M. Wright in 1940 ([23]). The Laplace transform of $W_{\gamma,\beta}$ is related to a Mittag-Leffler function with the same parameters by the formula

$$\int_0^\infty e^{-st} W_{\gamma,\beta}(t)\, dt = s^{-1} E_{\gamma,\beta}(s^{-1}) \tag{A.7}$$

which is valid for $\gamma > 1$, $\beta > 0$ ([2]). Also,

$$\left(\frac{z}{2}\right)^\nu W_{1,1+\nu}\left(-\frac{z^2}{4}\right) = J_\nu(z), \tag{A.8}$$

the usual Bessel function. Wright showed in particular that in the case $\gamma = -\alpha \in (-1, 0)$ there is the following asymptotic expansion, valid in a suitable sector about the negative real axis:

$$W_{-\alpha,\beta}(z) = Y^{\frac{1}{2}-\beta} e^{-Y} \left(\sum_{m=0}^{M-1} A_m Y^{-m} + O(|Y|^{-M}) \right) \quad \text{as} \quad z \to -\infty \tag{A.9}$$

with $Y = (1-\alpha)\big(-\alpha^\alpha z\big)^{1/(1-\alpha)}$, where the A_m are certain real numbers.

Stable probability distributions. A probability distribution $\mathcal{F}$ is called *stable* if for any two independent random variables X_1, $X_2 \sim \mathcal{F}$ and any b_1, $b_2 > 0$ there exist numbers $b > 0$, $c \in \mathbf{R}$ such that

$$b(b_1 X_1 + b_2 X_2 - c) \sim \mathcal{F}.$$

The name goes back to P. Lévy who was the first to study these distributions systematically in 1925 ([13]). After possibly rescaling X by replacing it with $AX + B$ for suitable constants $A > 0$, $B \in \mathbf{R}$, the characteristic function of a stable distribution either has the form

$$E(e^{itX}) = \exp\big(- |t|^\alpha \cdot (-i)^{K(\alpha)\beta sign(t)}\big) \tag{A.10}$$

with $0 < \alpha \leq 2$, $\alpha \neq 1$, $-1 \leq \beta \leq 1$, $K(\alpha) = \alpha - 1 - sign(\alpha - 1)$, or it has an exceptional form, corresponding to $\alpha = 1$, that can also be parametrized by $\beta \in [-1, 1]$. Distributions with $\beta = 1$ or $\beta = -1$ are called extremal.

For extremal distributions with $\beta = 1$ and $0 < \alpha < 1$, the probability density function $\phi = \phi_\alpha$ can be expressed in terms of WRIGHT's generalized Bessel function; namely,

$$\phi(r) = \frac{1}{r} W_{-\alpha,0}(-r^{-\alpha}) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(1+\alpha k)}{k!} \frac{\sin \alpha\pi k}{\pi} r^{-\alpha k - 1} \quad (0 < r < \infty). \tag{A.11}$$

For $1 < \alpha \leq 2$ and $\beta = -1$, the formula holds

$$\phi(r) = -\frac{1}{r} W_{-1/\alpha,0}(-r) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(1+k/\alpha)}{k!} \frac{\sin(\pi k/\alpha)}{\pi} r^{k-1} \quad (-\infty < r < \infty). \tag{A.12}$$

These probability densities are infinitely often differentiable. The behavior of ϕ near $r = 0$ for $0 < \alpha < 1$ follows from (A.9). For $1 < \alpha \leq 2$, (A.9) can be used to describe the behavior of ϕ as $r \to \infty$, resulting in $\phi(r) = o(e^{-Cr})$ for all $C > 0$. Special cases are $\alpha = 2$ (and arbitrary β) and $\alpha = \frac{1}{2}$, namely

$$\phi_2(r) = \frac{1}{2\sqrt{\pi}} e^{-r^2/4}, \qquad \phi_{1/2}(r) = \frac{1}{2\sqrt{\pi} r^{3/2}} e^{-1/4r} \quad (r > 0).$$

The first distribution is a normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 2$. The second one is known as the Lévy distribution. The series (A.11) shows that for $0 < \alpha < 1$, as $r \to \infty$,

$$\phi(r) = \frac{\sin \alpha\pi}{\pi} \Gamma(1+\alpha) r^{-\alpha-1} + O(r^{-2\alpha-1}), \tag{A.13}$$

and therefore

$$\Phi(r) = 1 - \frac{\sin \alpha\pi}{\pi} \Gamma(\alpha) r^{-\alpha} + O(r^{-2\alpha}) \tag{A.14}$$

for the cumulative distribution function Φ. This also shows that the moments $E(X^s)$ exist if and only if $s < \alpha$ for this range of α. Indeed, it is known that

$$E(X^s) = \frac{\Gamma(1 - \frac{s}{\alpha})}{\Gamma(1-s)}, \quad -\infty < s < \alpha.$$

For $1 < \alpha \leq 2$, one obtains $\phi(r) = O(|r|^{-\alpha-1})$ as $r \to -\infty$. Formulae (A.11) and (A.12) have the simple consequence

$$r^{-1/\alpha - 1} \phi_\alpha(r^{-1/\alpha}) = \phi_{1/\alpha}(r) \tag{A.15}$$

for $\frac{1}{2} \leq \alpha < 1$, $r > 0$. This a special case of a more general *duality law* for stable probability distributions ([24]). It also shows that the function

$$\psi_{1/\alpha}(r) = \frac{1}{\alpha} \phi_{1/\alpha}(r) \cdot 1_{(0,\infty)}(r) \tag{A.16}$$

is a probability density function. The corresponding probability distributions are called *trans-stable* and can be defined by (A.16) for any index $1/\alpha \in (1,\infty)$. Their density can be obtained directly from (A.11):

$$\psi_{1/\alpha}(r) = -\frac{1}{\alpha r} W_{-\alpha,0}(-r) \quad (r > 0). \tag{A.17}$$

The Laplace transforms of extremal stable distributions with index $\alpha \in (0,1)$ are

$$\hat{\phi}(s) = \int_0^\infty \phi_\alpha(r) e^{-sr}\, dr = e^{-s^\alpha} \quad (\Re s \geq 0). \tag{A.18}$$

This implies that ϕ satisfies the integral equation

$$r\phi(r) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^r (r-\tau)^{-\alpha} \phi(\tau)\, d\tau. \tag{A.19}$$

The Laplace transforms of the densities of trans-stable distributions are *Mittag-Leffler* functions; namely,

$$\hat{\psi}_{1/\alpha}(s) = E_{\alpha,1}(-s), \tag{A.20}$$

if $0 < \alpha < 1$, $\Re s \geq 0$.

It is known that all stable distributions (extremal or not) are unimodal and indeed *bell-shaped:* The k-th derivative of the probability density function has exactly k zeroes ([5]). Stable one-dimensional probability distributions are well studied; the principal reference is [24] where most of the material presented here was found.

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Note added in Proof. After this paper was completed, the author became aware of the derivation and discussion of (1.7) in [25] and in other work cited there.