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MULTIPLE SOLUTIONS OF NONHOMOGENEOUS ELLIPTIC EQUATION WITH CRITICAL NONLINEARITY

Daomin Cao*
Young Scientist Laboratory of Mathematical Physics
Wuhan Institute of Mathematical Sciences
The Chinese Academy of Science, PO Box 71007, Wuhan 430071, P.R. China

J. Chabrowski
The University of Queensland, Department of Mathematics, St. Lucia 4072, Queensland, Australia

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1. Introduction. The main purpose of this work is to investigate the existence of multiple positive solutions of the following problem:

$$\begin{cases} -\Delta u = Q(x)|u|^{p-2}u + \epsilon h(x) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1_n}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 3)$, $p = \frac{2N}{N-2}$ is a critical Sobolev exponent, $h \in L^2(\Omega)$, with $h \geq 0$, $\not\equiv 0$ on Ω, $Q \in C(\bar{\Omega})$ is positive and $\epsilon > 0$ is a parameter.

In recent years several authors have studied problems of this nature (see for example [3], [4], [16], [14], [15]. In particular, in the case where $Q(x) \equiv 1$ on Ω, Tarantello ([17]) proved the existence of at least two positive distinct solutions for $\epsilon > 0$ small. This result has been extended by Rey ([16]) who proved that problem (1_n) has at least $\operatorname{cat}\Omega + 1$ positive distinct solutions for $\epsilon > 0$ small.

In this paper we are concerned with the effect of the shape of the graph of Q on the number of positive solutions. Throughout this paper we assume the hypothesis

(Q) $Q \in C(\bar{\Omega})$, $Q > 0$ on $\bar{\Omega}$ and there exist points $a_1, \dots, a_k \in \Omega$ where Q takes on strict local maxima; i.e., $Q(a_j) = \max_{x\in\Omega} Q(x)$ and $Q(x) < Q(a_j)$ for x in a neighbourhood U_j of a_j, $j = 1, \dots, k$, and moreover for $x \in U_j$

$$Q(x) - Q(a_j) = o\big(|x - a_j|^{\frac{N-2}{2}}\big).$$

In what follows we use the notation $Q_M = \max_{x\in\Omega} Q(x)$. The main results of this paper are the following:

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Theorem 1. *Suppose that Q satisfies (Q). Then there exists $\epsilon_\circ > 0$ such that for $\epsilon \in (0, \epsilon_\circ]$, problem (1_n) has, besides a minimum positive solution which tends to 0 in $H^1_\circ(\Omega)$ as $\epsilon \to 0$, at least k positive distinct solutions $u_{i,\epsilon}$ satisfying*

$$|\nabla u_{i,\epsilon}|^2 \to Q_M^{-\frac{N-2}{2}} S^{\frac{N}{2}} \delta_{a_i} \quad \text{and} \quad |u_{i,\epsilon}|^p \to Q_M^{-\frac{N}{2}} S^{\frac{N}{2}} \delta_{a_i}$$

in the sense of measure as $\epsilon \to 0$, where δ_{a_i} is the Dirac measure assigned to a_i and S is the best Sobolev constant for a continuous embedding of $H^1_\circ(\Omega)$ into $L^p(\Omega)$.

The second result is concerned with the Dirichlet problem with nonzero boundary data

$$\begin{cases} -\Delta u = Q(x)|u|^{p-2}u & \text{in } \Omega \\ u(x) = \epsilon g(x) & \text{on } \partial\Omega, \end{cases} \tag{1_d}$$

where $g \in H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$, $g \geq 0$, $\not\equiv 0$ on $\partial\Omega$ and $\epsilon > 0$ is a small parameter and $\partial\Omega$ is smooth.

Theorem 2. *Suppose that condition* (Q) *holds. Then there exists $\epsilon_\circ > 0$ such that for each $\epsilon \in (0, \epsilon_\circ]$, problem (1_d) has, besides a minimum positive solution which tends to 0 in $H^1(\Omega)$ as $\epsilon \to 0$, at least k distinct positive solutions v^i_ϵ, $i = 1, \dots, k$, in $H^1(\Omega)$ satisfying*

$$|\nabla v^i_\epsilon|^2 \to Q_M^{-\frac{N-2}{2}} S^{\frac{N}{2}} \delta_{a_i} \quad \text{and} \quad |v^i_\epsilon|^p \to Q_M^{-\frac{N}{2}} S^{\frac{N}{2}} \delta_{a_i}$$

in the sense of measure as $\epsilon \to 0$.

We mention here some earlier work dealing with the effect of the shape of Q on the number of solutions. Cao and Noussair ([7]), under assumptions similar to those of Theorem 1, have established the existence of at least k positive and k nodal solutions for the homogeneous problem

$$\begin{cases} -\Delta u = Q(x)|u|^{p-2}u + \epsilon u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Their result generalizes to some extent the work of Escobar ([12]), who proved the existence of at least one positive solution for each $\epsilon \in (0, \lambda_1)$, where λ_1 is the smallest eigenvalue of $-\Delta$ with zero Dirichlet boundary values. A similar result has also been established for the problem

$$\begin{cases} -\Delta u + \lambda u = Q(x)|u|^{l-2}u & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

with $2 < l < \frac{2N}{N-2}$ in [6] for $\lambda > 0$ large or for fixed $\lambda > 0$ and l close to $\frac{2N}{N-2}$ in [9].

The paper is organized as follows. In Section 2 we reduce problem (1_n) to problem (4). Problem (4) will be solved by constrained minimization subject to artificial constraints. However, in order to obtain the existence of k critical points we minimize a variational functional for problem (4) on suitably chosen subsets of artificial constraints which can be constructed using the assumption that Q has k strict local maxima on Ω. The main point is to show that infima of these localized minimizing problems belong to the range of the level sets of the variational functional for problem (4) for which the Palais–Smale condition holds (see Sections 3 and 4). Applying the Ekeland variational principle, we construct in Section 5 minimizing sequences satisfying the Palais–Smale condition. In Section 6 we briefly show how results of Sections 2–5 can be used to obtain k distinct positive solutions of the Dirichlet problem (1_d) with nonzero boundary conditions.

2. Preliminaries. In this paper we use standard notation and terminology. We denote by $H^1_\circ(\Omega)$ a Sobolev space defined as a completion of $C^\infty_\circ(\Omega)$ with respect to the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^2\,dx\Big)^{\frac{1}{2}}.$$

Its dual space is denoted by $H^{-1}(\Omega)$. The norm in the space $L^q(\Omega)$, $1 \le q < \infty$, is denoted by

$$\|u\|_q = \Big(\int_\Omega |u|^q\,dx\Big)^{\frac{1}{q}}.$$

We always denote in a given Banach space X a weak convergence by "$\rightharpoonup$" and a strong convergence by "$\to$". The duality pairing between X and its dual X^* is denoted by $\langle\cdot,\cdot\rangle$.

We say that a C^1 functional $F : X \to \mathbb{R}$ satisfies the Palais–Smale condition at level c (the $(PS)_c$ condition for short) if each sequence $\{u_m\} \subset X$ such that $F(u_m) \to c$ and $F'(u_m) \to 0$ in X^* is relatively compact in X.

For $r > 0$ and $x_\circ \in \mathbb{R}^N$ we let $B(x_\circ, r) = \{x \in \mathbb{R}^N : |x - x_\circ| < r\}$ and $S(x_\circ, r) = \{x \in \mathbb{R}^N : |x - x_\circ| = r\}$.

According to results in [8] (see Lemma 3 there) there exists $\epsilon_\circ > 0$ (small) such that for each $0 < \epsilon \le \epsilon_\circ$ problem (1_n) has a minimum positive solution u_ϵ which satisfies

$$\int_\Omega |\nabla u|^2\,dx \ge (p-1)\mu \int_\Omega Q(x) u_\epsilon^{p-2} u^2\,dx \tag{2}$$

for all $u \in H^1_\circ(\Omega)$ and some constant $\mu > 1$. Furthermore, u_ϵ satisfies the estimate

$$\|u_\epsilon\| \le C\epsilon \|h\|_2 \tag{3}$$

for some constant $C > 0$.

One of the main objectives of this paper is to look for solutions of (1_n) which are of the form $u = v + u_\epsilon$, where v is a positive solution of the following problem:

$$\begin{cases} -\Delta v = Q(x)\big((v+u_\epsilon)^{p-1} - u_\epsilon^{p-1}\big) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

A similar idea will be used in Section 6 to solve the Dirichlet problem (1_d). We set

$$f_\epsilon(x,t) = \begin{cases} Q(x)\big((t+u_\epsilon)^{p-1} - u_\epsilon^{p-1}\big) & \text{for } x \in \Omega, \quad t \geq 0 \\ Q(x)\big(-(-t+u_\epsilon)^{p-1} + u_\epsilon^{p-1}\big) & \text{for } x \in \Omega, \quad t < 0; \end{cases}$$

that is, $f_\epsilon(x,t) = -f_\epsilon(x,-t)$. Solutions to problem (4) will be found as critical points of a variational functional $I_\epsilon : H^1_\circ(\Omega) \to \mathbb{R}$ given by

$$I_\epsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F_\epsilon(x,u)\,dx,$$

where

$$F_\epsilon(x,t) = \int_0^t f_\epsilon(x,s)\,ds = \frac{1}{p}Q(x)\big((|t|+u_\epsilon)^p - u_\epsilon^p - pu_\epsilon^{p-1}|t|\big).$$

From now on it is assumed that $\delta > 0$ is chosen so that $Q(x) < Q(a_i)$ for $x \in B(a_i,\delta) - \{a_i\} \subset U_i$, $i = 1, \ldots, k$ (see assumption (Q)). Let

$$M_\epsilon = \{u \in H^1_\circ(\Omega) - \{0\} : \langle I'_\epsilon(u), u\rangle = 0\}.$$

Lemma 1. *Let $l = \min\{p-2, \frac{p}{3}\}$. Then there exists a constant $C_{\epsilon_\circ} > 0$ such that*

$$I_\epsilon(u) \geq \frac{l}{2p}\big(1 - \frac{1}{\mu}\big)\int_\Omega |\nabla u|^2\,dx > C_{\epsilon_\circ} \tag{5}$$

for all $u \in M_\epsilon$ and $\epsilon \in (0, \epsilon_\circ]$.

Proof. Let $u \in M_\epsilon$. Then by inequality (A.1) in the Appendix and (2) we have

$$\begin{aligned} I_\epsilon(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p}\int_\Omega Q(x)\big((|u|+u_\epsilon)^p - u_\epsilon^p - pu_\epsilon^{p-1}|u|\big)\,dx \\ &= \frac{1}{2p}\int_\Omega Q(x)\big\{p\big((|u|+u_\epsilon)^{p-1} - u_\epsilon^{p-1}\big)|u| \\ &\quad - 2\big((|u|+u_\epsilon)^p - u_\epsilon^p - pu_\epsilon^{p-1}|u|\big)\big\}\,dx \\ &\geq \frac{l}{2p}\int_\Omega Q(x)\big\{\big((|u|+u_\epsilon)^{p-1} - u_\epsilon^{p-1}\big)|u| - (p-1)u_\epsilon^{p-2}|u|^2\big\}\,dx \\ &= \frac{l}{2p}\int_\Omega \big(|\nabla u|^2 - (p-1)Q(x)u_\epsilon^{p-2}u^2\big)\,dx \geq \frac{l}{2p}\big(1-\frac{1}{\mu}\big)\int_\Omega |\nabla u|^2\,dx. \end{aligned} \tag{6}$$

Since $u \in M_\epsilon$, we have by (A.2) and the Sobolev inequality that

$$\begin{aligned} \int_\Omega |\nabla u|^2\,dx &= \int_\Omega Q(x)|u|^p\,dx + \int_\Omega Q(x)\big((|u|+u_\epsilon)^{p-1} - u_\epsilon^{p-1} - |u|^{p-1}\big)|u|\,dx \\ &\leq \int_\Omega Q(x)|u|^p\,dx + C\int_\Omega Q(x)\big(|u|^p + u_\epsilon^{p-2}u^2\big)\,dx \\ &\leq (C+1)Q_M S^{-\frac{p}{2}}\big(\int_\Omega |\nabla u|^2\,dx\big)^{\frac{p}{2}} + CQ_M\|u_\epsilon\|^{p-2}\int_\Omega |\nabla u|^2\,dx \end{aligned} \tag{7}$$

for some constant $C > 0$ independent of ϵ. We may now assume, taking $\epsilon_\circ$ smaller if necessary, that $\left(1 - CQ_M S^{-\frac{p}{2}}\|u_\epsilon\|^{p-2}\right) \geq \frac{1}{2}$ for all $0 < \epsilon \leq \epsilon_\circ$. This combined with (7) yields

$$\int_\Omega |\nabla u|^2\, dx \geq C_1 \tag{8}$$

for all $u \in M_\epsilon$ and some constant $C_1 > 0$. The assertion follows from (6) and (8).

3. Localization of constraints. We minimize the functional I_ϵ on some subsets of constraints M_ϵ. For this we need a functional $\beta : H^1_\circ(\Omega) - \{0\} \to \mathbb{R}^N$ defined by

$$\beta(u) = \frac{\int_\Omega x|\nabla u|^2\, dx}{\int_\Omega |\nabla u|^2\, dx}.$$

We set $M_{\epsilon,j} = \{u \in M_\epsilon : \beta(u) \in B(a_j, \delta)\},\ j = 1, \dots, k$. It is easy to see that $M_{\epsilon,j}$ is nonempty for each j. Indeed, let $u_\circ \in C^2(B(a_j,\delta))$ be such that supp $u_\circ \subset B(0, \frac{\delta}{2})$ and $u_\circ \not\equiv 0$. Since $\langle {I_\epsilon}'(tu_\circ), tu_\circ\rangle > 0$ for $t > 0$ small and $\langle {I_\epsilon}'(tu_\circ), tu_\circ\rangle < 0$ for $t > 0$ large we can find $t_\circ > 0$ such that $t_\circ u_\circ \in M_\epsilon$. It is easy to show that $\beta(t_\circ u_\circ) \in B(a_j, \delta)$. We now consider the following variational problems:

$$m_{\epsilon,i} = \inf\{I_\epsilon(u) : u \in M_{\epsilon,i}\} \ \text{ and } \ \bar{m}_{\epsilon,i} = \inf\{I_\epsilon(u) : u \in M_\epsilon,\ \beta(u) \in S(a_i,\delta)\},$$

$i = 1, \dots, k$.

Lemma 2. *Suppose that condition* (Q) *holds. Then there exists a constant $\nu > 0$ such that for all $\epsilon \in (0, \epsilon_\circ]$ we have*

$$\bar{m}_{\epsilon,i} > \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}} + \nu, \tag{9}$$

where $\epsilon_\circ$ is a constant from Lemma 1.

Proof. In the contrary case there exists a sequence $\{\epsilon_m\} \subset (0, \epsilon_\circ]$, such that $\epsilon_m \to 0$ and

$$\bar{m}_{\epsilon_m,i} \to c \leq \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}$$

for some $1 \leq i \leq k$. Hence we can find a sequence $\{u_m\} \subset H^1_\circ(\Omega)$ with $u_m \in M_{\epsilon_m}$ satisfying

$$I_{\epsilon_m}(u_m) \to c \leq \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}$$

as $m \to \infty$ and $\beta(u_m) \in S(a_i, \delta)$ for each m. This means that

$$\int_\Omega |\nabla u_m|^2 dx - \int_\Omega Q(x)\left((|u_m| + u_{\epsilon_m})^{p-1} - u_{\epsilon_m}^{p-1}\right)|u_m|\, dx = 0 \tag{10}$$

and

$$\frac{1}{2}\int_\Omega |\nabla u_m|^2\,dx - \frac{1}{p}\int_\Omega Q(x)\big((|u_m| + u_{\epsilon_m})^p - u_{\epsilon_m}^p - pu_{\epsilon_m}^{p-1}|u_m|\big)\,dx = c + o(1), \quad (11)$$

where $o(1)$ always denotes a quantity satisfying $o(1) \to 0$ as $m \to \infty$. It follows from (10) and (11) that

$$\begin{aligned}&\frac{1}{2}\int_\Omega Q(x)\big((|u_m| + u_{\epsilon_m})^{p-1} - u_{\epsilon_m}^{p-1}\big)|u_m|\,dx \\ &\quad - \frac{1}{p}\int_\Omega Q(x)\big((|u_m| + u_{\epsilon_m})^p - u_{\epsilon_m}^p - pu_{\epsilon_m}^{p-1}|u_m|\big)\,dx = c + o(1),\end{aligned}$$

which implies

$$\begin{aligned}&\frac{1}{N}\int_\Omega Q(x)(|u_m| + u_{\epsilon_m})^p dx + \frac{1}{2}\int_\Omega Q(x)u_{\epsilon_m}^{p-1}|u_m|\,dx \\ &\quad = c + \frac{1}{p}\int_\Omega Q(x)u_{\epsilon_m}^p dx + \frac{1}{2}\int_\Omega Q(x)(|u_m| + u_{\epsilon_m})^{p-1}u_{\epsilon_m}\,dx + o(1).\end{aligned} \quad (12)$$

An application of the Hölder inequality to the last integral on the right side of (12) gives

$$\int_\Omega Q(x)(|u_m| + u_{\epsilon_m})^p\,dx \le C \quad (13)$$

for some constant $C > 0$ independent of m. Combining (10) and (12) we obtain

$$\int_\Omega |\nabla u_m(x)|^2\,dx \le C \quad (14)$$

for some constant $C > 0$ independent of m. As in the proof of Lemma 1 we show that there exists a constant $L > 0$ such that

$$\int_\Omega Q(x)|u_m|^p\,dx \ge L \ \text{ and } \ \int_\Omega |\nabla u_m|^2\,dx \ge L \quad (15)$$

for all m. Let

$$t_m = \Big(\frac{\int_\Omega |\nabla u_m|^2\,dx}{\int_\Omega Q_M|u_m|^p\,dx}\Big)^{\frac{1}{p-2}}$$

and set $v_m = t_m u_m$. Then v_m satisfies

$$\int_\Omega |\nabla v_m|^2\,dx = \int_\Omega Q_M|v_m|^p\,dx.$$

On the other hand it follows from (10) and (A.2) that

$$\begin{aligned}\int_\Omega |\nabla u_m|^2\,dx &= \int_\Omega Q(x)\big((|u_m|+u_{\epsilon_m})^{p-1}-u_{\epsilon_m}^{p-1}\big)|u_m|\,dx\\ &\le \int_\Omega Q_M|u_m|^p\,dx + C\int_\Omega \big(u_{\epsilon_m}|u_m|^{p-1}+u_{\epsilon_m}^{p-1}|u_m|\big)\,dx\end{aligned}$$

for some constant $C>0$ independent of m, which implies

$$\frac{\int_\Omega |\nabla u_m|^2\,dx}{\int_\Omega Q_M|u_m|^p\,dx} \le 1+\frac{CS^{-\frac{p}{2}}\big(\|u_{\epsilon_m}\|\|u_m\|^{p-1}+\|u_{\epsilon_m}\|^{p-1}\|u_m\|\big)}{\int_\Omega Q_M|u_m|^p\,dx}.$$

Taking into account the definition of t_m and (15) we derive from the last estimate, by choosing a subsequence, if necessary, that

$$\lim_{m\to\infty} t_m = t_\circ \le 1, \quad t_\circ>0. \tag{16}$$

We now show that $t_\circ = 1$. Since $\|u_{\epsilon_m}\|\to 0$ we deduce from (10), taking a subsequence if necessary, that

$$\lim_{m\to\infty}\int_\Omega |\nabla u_m|^2\,dx = \lim_{m\to\infty}\int_\Omega Q(x)|u_m|^p\,dx.$$

Hence

$$\begin{aligned} c &= \lim_{m\to\infty} I_{\epsilon_m}(u_m)\\ &= \lim_{m\to\infty}\Big\{\frac12\int_\Omega |\nabla u_m|^2\,dx - \frac1p\int_\Omega Q(x)\big((|u_m|+u_{\epsilon_m})^p - u_{\epsilon_m}^p - pu_{\epsilon_m}^{p-1}|u_m|\big)\,dx\Big\}\\ &= \frac1N \lim_{m\to\infty}\int_\Omega |\nabla u_m|^2\,dx \le \frac{S^{\frac N2}}{NQ_M^{\frac{N-2}{2}}}.\end{aligned}$$

Thus

$$\lim_{m\to\infty}\int_\Omega |\nabla u_m|^2\,dx = \lim_{m\to\infty}\int_\Omega Q(x)|u_m|^p\,dx \le \frac{S^{\frac N2}}{Q_M^{\frac{N-2}{2}}}. \tag{17}$$

Let $w_m = \frac{v_m}{\|v_m\|_p}$. Then $\|w_m\|_p = 1$, and moreover

$$\begin{aligned}\int_\Omega |\nabla w_m|^2\,dx &= \frac{\int_\Omega |\nabla v_m|^2\,dx}{\|v_m\|_p^2} = \frac{\int_\Omega |\nabla u_m|^2\,dx}{\big(\int_\Omega |u_m|^p\,dx\big)^{\frac2p}} = Q_M^{\frac2p}\frac{\int_\Omega |\nabla u_m|^2\,dx}{\big(\int_\Omega Q_M|u_m|^p\,dx\big)^{\frac2p}}\\ &= Q_M^{\frac2p}\big(\int_\Omega |\nabla u_m|^2\,dx\big)^{1-\frac2p}\big(\frac{\int_\Omega |\nabla u_m|^2\,dx}{\int_\Omega Q_M|u_m|^p\,dx}\big)^{\frac2p}\\ &= Q_M^{\frac2p}\big(\int_\Omega |\nabla u_m|^2\,dx\big)^{1-\frac2p} t_m^{\frac{2(p-2)}{p}}.\end{aligned}$$

Using (17) we deduce from this that $\lim_{m\to\infty}\int_\Omega |\nabla w_m|^2\,dx \le t_\circ^{\frac{4}{N}} S$. Since S is the best Sobolev constant we necessarily have $t_\circ = 1$, and $\{w_m\}$ is a minimizing sequence for S. It follows from [14] that there exists $x_\circ \in \bar{\Omega}$ such that

$$|\nabla w_m|^2 \to S\delta_{x_\circ} \text{ and } |w_m|^p \to \delta_{x_\circ} \tag{18}$$

in the sense of measure, where $\delta_{x_\circ}$ is a Dirac measure assigned to $x_\circ$. Since $\beta(u_m) \in S(a_i, \delta)$, we have

$$\beta(u_m) = \frac{\int_\Omega x|\nabla u_m|^2\,dx}{\int_\Omega |\nabla u_m|^2\,dx} = \frac{\int_\Omega x|\nabla w_m|^2\,dx}{\int_\Omega |\nabla w_m|^2\,dx} \to x_\circ.$$

Hence $x_\circ \in S(a_i, \delta)$. Since $t_\circ = 1$ we obtain by (17) and (18)

$$\begin{aligned}\frac{S^{\frac{N}{2}}}{Q_M^{\frac{N-2}{2}}} &= \lim_{m\to\infty}\int_\Omega |\nabla u_m|^2\,dx \\ &= \lim_{m\to\infty} t_m^{-p}\|v_m\|_p^p \int_\Omega Q(x)|w_m|^p\,dx = Q_M^{-1}\frac{S^{\frac{N}{2}}}{Q_M^{\frac{N-2}{2}}}Q(x_\circ);\end{aligned}$$

that is, $Q_M = Q(x_\circ)$, which is impossible, and this completes the proof. $\square$

We now estimate $m_{\epsilon,i}$. We follow the method from the paper [3]. Let ϕ be a $C^2(\mathbb{R}^N)$ radial function such that $\phi(x) = 1$ for $x \in B(0, \frac{\delta}{2})$, $\phi(x) = 0$ for $x \in \mathbb{R}^N - B(0,\delta)$, $0 \le \phi(x) \le 1$ on $\mathbb{R}^N$ and $|\nabla\phi(x)| \le \frac{4}{\delta}$ on $\mathbb{R}^N$. As in [3] we set

$$U_{\lambda,a_j}(x) = (\lambda + |x-a_j|^2)^{-\frac{N-2}{2}}, \quad \lambda > 0$$

and

$$v_{\lambda,a_j}(x) = \frac{U_{\lambda,a_j}(x)\phi(x-a_j)}{\|U_{\lambda,a_j}(\cdot)\phi(\cdot - a_j)\|_p}.$$

In Lemma 3 below we use the following asymptotic properties of v_{λ,a_j} which are taken from the paper [3]:

$$\int_{\mathbb{R}^N} |\nabla v_{\lambda,a_j}|^2\,dx = S + O(\lambda^{\frac{N-2}{2}}) \text{ for } \lambda \text{ small,} \tag{19}$$

$$\int_\Omega v_{\lambda,a_j}^2\,dx = \begin{cases} A\lambda + o(\lambda) & N \ge 5 \\ A\lambda|\log\lambda| + o(\lambda) & N = 4 \\ A\sqrt{\lambda} + O(\lambda) & N = 3 \end{cases} \tag{20}$$

for λ small, where $A = \frac{K_2}{K_3}$, $K_2 = \left(\int_{\mathbb{R}^N}\frac{dx}{(1+|x|^2)^N}\right)^{\frac{N-2}{N}}$, $K_3 = \int_{\mathbb{R}^N}\frac{dx}{(1+|x|^2)^{N-2}}$,

$$\int_\Omega v_{\lambda,a_j}^{\frac{N}{N-2}}\,dx = O\big(\lambda^{\frac{N}{4}}|\log\lambda|\big) \text{ for } \lambda \text{ small,} \tag{21}$$

$$\int_\Omega v_{\lambda,a_j}^{p-1}\,dx = \lambda^{\frac{N-2}{4}}\big(K + O(1)\big) \text{ for } \lambda \text{ small ,} \tag{22}$$

where

$$K = \int_{\mathbb{R}^N} \frac{dx}{(1+|x|^2)^{\frac{N+2}{2}}} \Big(\int_{\mathbb{R}^N} \frac{dx}{(1+|x|^2)^N} \Big).^{-\frac{N+2}{2N}}$$

Finally, with the aid of condition (Q) we derive the following asymptotic relation:

$$\begin{aligned} \int_\Omega Q(x) v_{\lambda,a_j}^p \, dx &= Q_M \int_\Omega v_{\lambda,a_j}^p \, dx - \int_\Omega \big(Q_M - Q(x)\big) v_{\lambda,a_j}^p \, dx \\ &= Q_M + o\big(\lambda^{\frac{N-2}{4}}\big). \end{aligned} \tag{23}$$

Lemma 3. *Suppose that condition* (Q) *holds. Then for every fixed* $\epsilon \in (0, \epsilon_\circ)$ *we have*

$$0 < m_{\epsilon,i} < \frac{S^{\frac{N}{2}}}{N Q_M^{\frac{N-2}{2}}}, \quad i = 1, \dots, k. \tag{24}$$

Proof. For $t \geq 0$ we write

$$\begin{aligned} I_\epsilon(t v_{\lambda,a_j}) &= \frac{t^2}{2} \int_\Omega |\nabla v_{\lambda,a_j}|^2 \, dx - \frac{t^p}{p} \int_\Omega Q(x) v_{\lambda,a_j}^p \, dx \\ &- \frac{1}{p} \int_\Omega Q(x) \big((t v_{\lambda,a_j} + u_\epsilon)^p - u_\epsilon^p - (t v_{\lambda,a_j})^p - p t u_\epsilon^{p-1} v_{\lambda,a_j} \big) \, dx. \end{aligned}$$

Since for $\lambda > 0$ small $\langle {I_\epsilon}'(t v_{\lambda,a_j}), t v_{\lambda,a_j} \rangle < 0$ for $t > 0$ large and $\langle {I_\epsilon}'(t v_{\lambda,a_j}), t v_{\lambda,a_j} \rangle > 0$ for $t > 0$ small, it is easy to see that for $\lambda > 0$ small enough we can find $t_{\lambda,a_j} > 0$ such that $t_{\lambda,a_j} v_{\lambda,a_j} \in M_\epsilon$ and $\beta(t_{\lambda,a_j} v_{\lambda,a_j}) \in B(a_j, \delta)$. Therefore to prove (24) it is sufficient to show that

$$\max_{t \geq 0} I_\epsilon(t v_{\lambda,a_j}) < \frac{S^{\frac{N}{2}}}{N Q_M^{\frac{N-2}{2}}} \tag{25}$$

for $\lambda > 0$ small. Arguing as in [3] we can find $\tilde{t}_\lambda > 0$ such that $\max_{t \geq 0} I_\epsilon(t v_{\lambda,a_j}) = I_\epsilon(\tilde{t}_\lambda v_{\lambda,a_j})$, and moreover $\tilde{t}_\lambda \in [t_1, t_2]$ for some $0 < t_1 < t_2$ independent of λ (for $\lambda > 0$ small). Hence we have

$$\begin{aligned} \max_{t \geq 0} I_\epsilon(t v_{\lambda,a_j}) &\leq \max_{t \geq 0} \Big\{ \frac{t^2}{2} \int_\Omega |\nabla v_{\lambda,a_j}|^2 \, dx - \frac{t^p}{p} \int_\Omega Q(x) v_{\lambda,a_j}^p \, dx \Big\} \\ - \min_{t \in [t_1, t_2]} &\Big\{ \frac{1}{p} \int_\Omega Q(x) \big((t v_{\lambda,a_j} + u_\epsilon)^p - u_\epsilon^p - (t v_{\lambda,a_j})^p - p t u_\epsilon^{p-1} v_{\lambda,a_j} \big) dx \Big\}, \end{aligned} \tag{26}$$

and by (19) and (23)

$$\begin{aligned} &\max_{t \geq 0} \Big\{ \frac{t^2}{2} \int_\Omega |\nabla v_{\lambda,a_j}|^2 \, dx - \frac{t^p}{p} \int_\Omega Q(x) v_{\lambda,a_j}^p \, dx \Big\} \\ &\quad = \frac{1}{N} \int_\Omega |\nabla v_{\lambda,a_j}|^2 \, dx \Big(\frac{\int_\Omega |\nabla v_{\lambda,a_j}|^2 \, dx}{\int_\Omega Q(x) v_{\lambda,a_j}^p \, dx} \Big)^{\frac{2}{p-2}} \\ &\quad = \frac{1}{N} \big(S + O(\lambda^{\frac{N-2}{2}}) \big) \Big(\frac{S + O(\lambda^{\frac{N-2}{2}})}{Q_M + o(\lambda^{\frac{N-2}{4}})} \Big)^{\frac{N-2}{2}} = \frac{S^{\frac{N}{2}}}{N Q_M^{\frac{N-2}{2}}} + o(\lambda^{\frac{N-2}{4}}). \end{aligned} \tag{27}$$

On the other hand it follows from [2] that

$$\begin{aligned}&\int_\Omega Q(x)\big((tv_{\lambda,a_j}+u_\epsilon)^p-u_\epsilon^p-(tv_{\lambda,a_j})^p-ptv_{\lambda,a_j}u_\epsilon^{p-1}\big)\,dx\\&\qquad\geq p\int_\Omega Q(x)u_\epsilon(tv_{\lambda,a_j})^{p-1}\,dx-C\int_\Omega Q_M u_\epsilon^{\frac{N}{N-2}}(tv_{\lambda,a_j})^{\frac{N}{N-2}}\,dx,\end{aligned}$$

where $C>0$ is a constant independent of λ and t. Using asymptotic relations (21) and (22) we deduce from the last inequality that

$$\begin{aligned}&\min_{t\in[t_1,t_2]}\int_\Omega Q(x)\big((tv_{\lambda,a_j}+u_\epsilon)^p-u_\epsilon^p-(tv_{\lambda,a_j})^p-ptv_{\lambda,a_j}u_\epsilon^{p-1}\big)\,dx\\&\quad\geq pt_1^{p-1}\int_\Omega Q(x)u_\epsilon v_{\lambda,a_j}^{p-1}\,dx-Ct_2^{\frac{N}{N-2}}Q_M\|u_\epsilon\|_\infty^{\frac{N}{N-2}}\int_\Omega v_{\lambda,a_j}^{\frac{N}{N-2}}\,dx\\&\quad\geq c\lambda^{\frac{N-2}{4}}+O\big(\lambda^{\frac{N}{4}}|\log\lambda|\big)\geq\tilde{c}\lambda^{\frac{N-2}{4}}\end{aligned}\tag{28}$$

for some constants $c>0$ and $\tilde{c}>0$. By virtue of (26), (27) and (28) we get

$$\max_{t\geq 0}I_\epsilon(tv_{\lambda,a_j})\leq\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}-\tilde{c}\lambda^{\frac{N-2}{4}}+O\big(\lambda^{\frac{N-2}{2}}\big)+o(\lambda^{\frac{N-2}{4}})<\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}-\frac{\tilde{c}}{2}\lambda^{\frac{N-2}{4}}$$

for $\lambda>0$ small, and this completes the proof.

4. Palais–Smale condition. In Lemma 4 below we determine the range of energy levels of the functional I_ϵ for which the Palais–Smale condition holds.

Lemma 4. *Suppose that Q satisfies* (Q)*. Then I_ϵ satisfies the* $(\mathrm{PS})_c$ *condition for* $c\in\big(-\infty,\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}\big)$.

The proof of this lemma is similar to that of Theorem 2.1 in [3] and therefore is omitted.

The following lemma is needed to examine the minimizing sequences for $m_{\epsilon,i}$.

Lemma 5. *Suppose that Q satisfies condition* (Q)*. Let $1\leq j\leq k$ be fixed. Then for every $u\in M_{\epsilon,j}$ there exists $\lambda>0$ and a differentiable function $t=t(w)$ defined for $w\in H_\circ^1(\Omega)$ with $\|w\|\leq\lambda$ such that $t(0)=1$. Moreover $v=t(w)(u-w)\in M_{\epsilon,j}$, and for $\psi\in H_\circ^1(\Omega)$ we have*

$$\langle t'(0),\psi\rangle=\frac{2\int_\Omega\nabla u\nabla\psi\,dx-G(u,\psi)}{\int_\Omega\big(|\nabla u|^2-(p-1)Q(x)u_\epsilon^{p-2}u^2\big)\,dx},\tag{29}$$

where

$$\begin{aligned}G(u,\psi)=(p-1)&\int_\Omega Q(x)(|u|+u_\epsilon)^{p-2}u\psi\,dx\\&-\int_\Omega Q(x)\big((|u|+u_\epsilon)^{p-1}-u_\epsilon^{p-1}\big)\frac{u}{|u|}\psi\,dx.\end{aligned}$$

Proof. We follow some ideas from the paper [17] (Lemma 2.4). However, the proof of Lemma 5 is more involved. We define a function $F : [0,\infty) \times H^1_\circ(\Omega) \to \mathbb{R}$ by

$$F(t,w) = t\int_\Omega |\nabla(u-w)|^2\,dx - \int_\Omega Q(x)\big((u_\epsilon + t|u-w|)^{p-1} - u_\epsilon^{p-1}\big)|u-w|\,dx.$$

Since $u \in M_\epsilon$, we have $F(1,0)=0$, and moreover

$$\begin{aligned}
&\frac{d}{dt}F(1,0) = \int_\Omega \big(|\nabla u|^2 - (p-1)Q(x)(u_\epsilon + |u|)^{p-2}u^2\big)\,dx\\
&= \int_\Omega Q(x)\big((|u|+u_\epsilon)^{p-1} - u_\epsilon^{p-1}\big)|u|\,dx - (p-1)\int_\Omega Q(x)(u_\epsilon+|u|)^{p-2}u^2\,dx\\
&= \int_\Omega Q(x)\big[\big((|u|+u_\epsilon)^{p-1} - u_\epsilon^{p-1}\big)|u| - (p-1)u_\epsilon^{p-2}u^2\big]\,dx\\
&- (p-1)\int_\Omega Q(x)\big((|u|+u_\epsilon)^{p-2}u^2 - u_\epsilon^{p-2}u^2\big)\,dx\\
&= -(p-2)\int_\Omega Q(x)\big((|u|+u_\epsilon)^{p-1} - u_\epsilon^{p-1} - (p-1)u_\epsilon^{p-2}|u|\big)|u|\,dx\\
&+ (p-1)\int_\Omega Q(x)\big((|u|+u_\epsilon)^{p-2}u_\epsilon^2 - u_\epsilon^{p-1} - (p-2)u_\epsilon^{p-2}|u|\big)|u|\,dx\\
&\le -\sigma\int_\Omega Q(x)\big((|u|+u_\epsilon)^{p-1} - u_\epsilon^{p-1} - (p-1)u_\epsilon^{p-2}|u|\big)|u|\,dx
\end{aligned}\tag{30}$$

for some $\sigma \in (0, \min\{1, p-2\})$. In the last step we have used inequality (A.3) in the Appendix. We claim that there exists a constant $L>0$ such that

$$\int_\Omega |\nabla u|^2\,dx \ge L \text{ and } \int_\Omega \big(|\nabla u|^2 - (p-1)Q(x)u_\epsilon^{p-2}u^2\big)dx \ge L \tag{31}$$

for all $u \in M_\epsilon$. Indeed, it follows from (2) and inequality (A.4) that

$$\begin{aligned}
\big(1-\frac{1}{\mu}\big)\int_\Omega |\nabla u|^2\,dx &\le \int_\Omega \big(|\nabla u|^2 - (p-1)Q(x)u_\epsilon^{p-2}u^2\big)\,dx\\
&= \int_\Omega Q(x)\big((|u|+u_\epsilon)^{p-1} - u_\epsilon^{p-1} - (p-1)u_\epsilon^{p-2}|u|\big)|u|\,dx\\
&\le C\int_\Omega Q(x)\big(|u|^p + u_\epsilon^{p-2-\theta}|u|^{2+\theta}\big)\,dx\\
&\le C\big[\big(\int_\Omega |\nabla u|^2\,dx\big)^{\frac{p}{2}} + \|u_\epsilon\|^{p-2-\theta}\big(\int_\Omega |\nabla u|^2\,dx\big)^{\frac{2+\theta}{2}}\big]
\end{aligned}\tag{32}$$

for some $C>0$, $\theta \in (0, p-2)$. The claim (31) easily follows from (32). Combining (30), (31) and (32) we obtain

$$\frac{d}{dt}F(1,0) \le -\sigma\int_\Omega \big(|\nabla u|^2 - (p-1)Q(x)u_\epsilon^{p-2}u^2\big)\,dx \le -\sigma L\big(1-\frac{1}{\mu}\big) < 0.$$

Therefore by the implicit function theorem there exists a function t defined on a ball $\{w \in H^1_\circ(\Omega) : \|w\| < \lambda\}$, where $\lambda > 0$ is small, such that $t(0) = 1$, $t(w)(u-w) \in M_\epsilon$. By the continuity of β, taking λ smaller if necessary, we have $\beta(t(w)(u-w)) \in B(a_j, \delta)$. Finally, the implicit differentiation of $F(t(w), w) = 0$ gives formula (29).

5. Proof of Theorem 1. In order to prove Theorem 1 we consider a minimizing sequence $\{u^j_m\}$ for $m_{\epsilon,j}$. A starting point is to show that a minimizing sequence $\{u^j_m\}$ can be chosen so that $I'_\epsilon(u^j_m) \to 0$ in $H^{-1}(\Omega)$. This will be shown using the Ekeland variational principle ([11]) and Lemma 5.

Proposition 1. *Suppose that condition* (Q) *holds. Then for each* $\epsilon \in (0, \epsilon_\circ]$ *there exist sequences* $\{v^j_m\} \subset M_{\epsilon,j}$, $j = 1, \dots, k$, *such that* $v^j_m \geq 0$ *and* $I_\epsilon(v^j_m) \to m_{\epsilon,j}$ *and* $I'_\epsilon(v^j_m) \to 0$ *in* $H^{-1}(\Omega)$ *as* $m \to \infty$.

Proof. Let Γ_j denote the boundary of the set $M_{\epsilon,j}$; that is, $\Gamma_j = \{u \in M_\epsilon : \beta(u) \in S(a_j, \delta)\}$. Then $\bar{M}_{\epsilon,j} = M_{\epsilon,j} \cup \Gamma_j$. According to Lemmas 2 and 3 we have

$$m_{\epsilon,j} = \inf\{I_\epsilon(u) : u \in \bar{M}_{\epsilon,j}\} < \inf\{I_\epsilon(u); u \in \Gamma_j\}. \tag{33}$$

Let $\{v^j_m\} \subset \bar{M}_{\epsilon,j}$ be a minimizing sequence for $m_{\epsilon,j}$. Replacing v^j_m by $|v^j_m|$, if necessary, we may assume that $v^j_m \geq 0$ in Ω. By virtue of Ekeland's variational principle, for each $1 \leq j \leq k$ we can find a sequence $\{u^j_m\} \subset \bar{M}_{\epsilon,j}$ such that

$$\begin{cases} I_\epsilon(u^j_m) \leq I_\epsilon(v^j_m) < m_{\epsilon,j} + \frac{1}{m}, & \|u^j_m - v^j_m\| \leq \frac{1}{m} \\ I_\epsilon(u^j_m) < I_\epsilon(w) + \frac{1}{m}\|w - u^j_m\| \end{cases} \tag{34}$$

for every $w \in \bar{M}_{\epsilon,j}$ and $w \neq u^j_m$. For simplicity we suppress the superscript j in u^j_m; i.e., we write $u_m = u^j_m$. According to Lemma 5, for each u_m there exist a constant $\theta_m > 0$ and a function $t_m(w)$ defined in a ball $\{\|w\| < \theta_m\} \subset H^1_\circ(\Omega)$ such that $t_m(w)(u_m - w) \in M_{\epsilon,j}$. Let $0 < \theta < \theta_m$, $u \in H^1_\circ(\Omega)$ with $u \not\equiv 0$, and set $w_\theta = \frac{\theta u}{\|u\|}$. For a fixed m we consider $v_\theta = t_m(w_\theta)(u_m - w_\theta)$. Since $v_\theta \in M_{\epsilon,j}$, by (34) we have

$$I_\epsilon(v_\theta) - I_\epsilon(u_m) \geq -\frac{1}{m}\|v_\theta - u_m\|.$$

Applying the Taylor expansion we get

$$\langle I'_\epsilon(u_m), v_\theta - u_m\rangle + o(\|v_\theta - u_m\|) \geq -\frac{1}{m}\|v_\theta - u_m\|.$$

Consequently we have

$$\langle I'_\epsilon(u_m), -w_\theta + (t_m(w_\theta) - 1)(u_m - w_\theta)\rangle \geq -\frac{1}{m}\|v_\theta - u_m\| + o(\|v_\theta - u_m\|),$$

and hence

$$\begin{aligned}\langle I'_\epsilon(u_m), -w_\theta\rangle + (t_m(w_\theta)-1)\langle I'_\epsilon(u_m), u_m - w_\theta\rangle \\ \geq -\frac{1}{m}\|v_\theta - u_m\| + o(\|v_\theta - u_m\|).\end{aligned} \tag{35}$$

Since $\langle I'_\epsilon(v_\theta), t_m(w_\theta)(u_m - w_\theta)\rangle = 0$, we deduce from (35) that

$$\begin{aligned}\langle I'_\epsilon(u_m), -w_\theta\rangle + \frac{t_m(w_\theta)-1}{t_m(w_\theta)}\langle I'_\epsilon(u_m) - I'_\epsilon(v_\theta), t_m(w_\theta)(u_m - w_\theta)\rangle \\ \geq -\frac{1}{m}\|v_\theta - u_m\| + o(\|v_\theta - u_m\|).\end{aligned}$$

This estimate can be rewritten in the form

$$\begin{aligned}\langle I'_\epsilon(u_m), \frac{u}{\|u\|}\rangle \leq \frac{1}{m\theta}\|v_\theta - u_m\| + o\big(\frac{\|v_\theta - u_m\|}{\theta}\big) \\ + \frac{t_m(w_\theta)-1}{\theta}\langle I'_\epsilon(u_m) - I'_\epsilon(v_\theta), u_m - w_\theta\rangle.\end{aligned} \tag{36}$$

Since $\|v_\theta - u_m\| \leq \theta + C|t_m(w_\theta) - 1|$ and by (29) and (32)

$$\lim_{\theta\to 0}\frac{|t_m(w_\theta)-1|}{\theta} \leq \|t'_m(0)\| + \tilde{C}$$

for some constants $C > 0$ and $\tilde{C} > 0$ independent of θ and m, we deduce from (36), letting $\theta \to 0$, that

$$\langle I'_\epsilon(u_m), \frac{u_m}{\|u_m\|}\rangle \leq \frac{C}{m}. \tag{37}$$

Here we have used the fact that $v_\theta \to u_m$ in $H^1_\circ(\Omega)$ and $I'_\epsilon(v_\theta) \to I'_\epsilon(u_m)$ as $\theta \to 0$. Inequality (37) yields that $\|I'_\epsilon(u_m)\|_{H^{-1}(\Omega)} \to 0$ as $m \to \infty$. Since $\|v^j_m - u^j_m\| \to 0$ as $m \to \infty$, we see that $\|{I_\epsilon}'(v^j_m)\|_{H^{-1}(\Omega)} \to 0$ as $m \to \infty$ which completes the proof.

Proof of Theorem 1. According to Proposition 1, for each j, $j = 1, \ldots, k$, there exists a minimizing sequence $\{u^j_m\} \subset M_{\epsilon,j}$ such that $I_\epsilon(u^j_m) \to m_{\epsilon,j}$ and $I'_\epsilon(u^j_m) \to 0$ in $H^{-1}(\Omega)$ as $m \to \infty$. By virtue of Lemmas 2 and 3 we have

$$m_{\epsilon,j} < \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}} < \bar{m}_{\epsilon,j},\ j = 1, \ldots, k.$$

Therefore by Lemma 4 $\{u^j_m\}$ satisfies the Palais–Smale condition and we may assume that $u^j_m \to u^j_\epsilon$ in $H^1_\circ(\Omega)$ and $u^j_\epsilon \geq 0$ in Ω. Hence $I_\epsilon(u^j_\epsilon) = m_{\epsilon,j}$ and $\beta(u^j_\epsilon) \in B(a_j, \delta)$. Since $I'_\epsilon(u^j_\epsilon) = 0$, $j = 1, \ldots, k$, u^j_ϵ are weak solutions of problem (4) and $\tilde{u}^j_\epsilon = u^j_\epsilon + u_\epsilon$, $j = 1, \ldots, k$, are distinct positive solutions of problem (1_n). To show the second part of our assertion it is sufficient to prove that

$$|\nabla u^j_\epsilon|^2 \to Q_M^{-\frac{N-2}{2}} S^{\frac{N}{2}}\delta_{a_j} \ \text{ and } \ |u^j_\epsilon|^p \to Q_M^{-\frac{N}{2}} S^{\frac{N}{2}}\delta_{a_j} \tag{38}$$

as $\epsilon \to 0$ in the sense of measure. Since $u_\epsilon \to 0$ in $H^1_\circ(\Omega)$ as $\epsilon \to 0$, the assertion will follow from (38). We follow the argument used in the proof of Lemma 2. Since $I_\epsilon(u^j_\epsilon) = m_{\epsilon,j}$ and $\langle I'_\epsilon(u^j_\epsilon), u^j_\epsilon\rangle = 0$, we can show that

$$\int_\Omega Q(x)|u^j_\epsilon|^p\,dx \geq L \text{ and } \int_\Omega |\nabla u^j_\epsilon|^2\,dx \geq L \tag{39}$$

for some constant $L > 0$ and all $\epsilon \in (0, \epsilon_\circ]$. Let

$$t^j_\epsilon = \Big(\frac{\int_\Omega |\nabla u^j_\epsilon|^2\,dx}{\int_\Omega Q_M(u^j_\epsilon)^p\,dx}\Big)^{\frac{1}{p-2}},$$

and we set $v^j_\epsilon = t^j_\epsilon u^j_\epsilon$. Then v^j_ϵ satisfies

$$\int_\Omega |\nabla v^j_\epsilon|^2\,dx = \int_\Omega Q_M(v^j_\epsilon)^p\,dx.$$

As in the proof of Lemma 2 we show that for some sequence $\{\epsilon_m\} \subset (0, \epsilon_\circ)$ with $\epsilon_m \to 0$, $t^j_{\epsilon_m} \to t^j_\circ$ and $0 < t^j_\circ \leq 1$. Also, as in the proof of Lemma 2 we can derive the following relation:

$$\lim_{m\to\infty}\int_\Omega |\nabla u^j_{\epsilon_m}|^2\,dx = \lim_{m\to\infty}\int_\Omega Q(x)(u^j_{\epsilon_m})^p\,dx = \frac{S^{\frac{N}{2}}}{Q_M^{\frac{N-2}{2}}}. \tag{40}$$

Finally, we let $z^j_m = \frac{v^j_{\epsilon_m}}{\|v^j_{\epsilon_m}\|_p}$. Then

$$\int_\Omega |\nabla z^j_m|^2\,dx = Q_M^{\frac{2}{p}}\frac{\int_\Omega |\nabla u^j_{\epsilon_m}|^2\,dx}{\big(\int_\Omega Q_M(u^j_{\epsilon_m})^p\,dx\big)^{\frac{2}{p}}} = Q_M^{\frac{2}{p}}\Big(\int_\Omega |\nabla u^j_{\epsilon_m}|^2\,dx\Big)^{1-\frac{2}{p}}\big(t^j_{\epsilon_m}\big)^{\frac{2(p-2)}{p}},$$

and by (40) we have

$$\lim_{m\to\infty}\int_\Omega |\nabla z^j_m|^2\,dx = t_\circ^{\frac{4}{N}} S.$$

We necessarily have $t_\circ = 1$. Therefore by the result of P.L. Lions ([14], Theorem 4.1—see also Coron [10], Théorème 2) $|\nabla z^j_m|^2 \to S\delta_{x_{\circ,j}}$ and $|z^j_m|^p \to \delta_{x_{\circ,j}}$ in the sense of measure for some $x_{\circ,j} \in B(a_j, \delta)$ because

$$\beta(z^j_m) = \frac{\int_\Omega x|\nabla z^j_m|^2\,dx}{\int_\Omega |\nabla z^j_m|^2\,dx} \to x_{\circ,j}.$$

We now observe that by (40) we have

$$\begin{aligned}\frac{S^{\frac{N}{2}}}{Q_M^{\frac{N-2}{2}}} &= \lim_{m\to\infty}\int_\Omega |\nabla u^j_{\epsilon_m}|^2\,dx \\ &= \lim_{m\to\infty}\big(t^j_{\epsilon_m}\big)^{-p}\|v^j_{\epsilon_m}\|^p_p\int_\Omega Q(x)|z^j_m|^p\,dx = Q_M^{-1}\frac{S^{\frac{N}{2}}}{Q_M^{\frac{N-2}{2}}}Q(x_{\circ,j})\end{aligned}$$

which necessarily implies that $x_{\circ,j} = a_j$.

6. Proof of Theorem 2. The method used to prove Theorem 1 can also be employed to show the existence of k distinct positive solutions of problem (1_d). We commence by showing that problem (1_d) has a minimum solution.

Lemma 6. *Suppose that condition* (Q) *holds. Then problem* (1_d) *has a minimum solution* u_ϵ *satisfying*

$$\int_\Omega |\nabla u_\epsilon|^2\,dx \le \epsilon C$$

for some constant $C > 0$ *depending on* g, *and moreover there exists a constant* $\mu > 1$ *such that*

$$\int_\Omega |\nabla v|^2\,dx \ge \mu \frac{N+2}{N-2} \int_\Omega Q(x) u_\epsilon^{\frac{4}{N-2}} v^2\,dx$$

for all $v \in H_\circ^1(\Omega)$.

Proof. Let $v_\circ$ be a solution of the problem

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega \\ v = g & \text{on } \partial\Omega. \end{cases} \tag{41}$$

It is known that $v_\circ$ satisfies

$$\int_\Omega |\nabla v_\circ|^2\,dx \le C\|g\|_{H^{\frac{1}{2}}}$$

for some $C > 0$. We define a functional $I : H_\circ^1(\Omega) \to \mathbb{R}$ by

$$I(w) = \frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \frac{1}{p}\int_\Omega Q(x)(w + \epsilon v_\circ)^p\,dx.$$

It is easy to see that the following minimization problem:

$$\inf\{I(w) : \|w\| < \rho\} < I(0) = -\frac{\epsilon^p}{p+1}\int_\Omega Q(x) v_\circ^p\,dx, \quad \rho > 0,$$

has a solution $w_\epsilon \in H_\circ^1(\Omega)$. The minimizer w_ϵ is a positive solution of the problem

$$\begin{cases} -\Delta w = Q(x)(w^+ + \epsilon v_\circ)^{p-1} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence we have

$$\int_\Omega |\nabla w_\epsilon|^2\,dx = \int_\Omega Q(x)(w_\epsilon + \epsilon v_\circ)^{p-1} w_\epsilon\,dx. \tag{42}$$

On the other hand

$$I(w_\epsilon)=\frac{1}{2}\int_\Omega |\nabla w_\epsilon|^2dx-\frac{1}{p}\int_\Omega Q(x)(w_\epsilon+\epsilon v_\circ)^p dx\le \frac{1}{p}\epsilon^p\int_\Omega Q(x)v_\circ^p dx,$$

and this combined with (42) and (A.2) gives

$$\begin{aligned}\frac{1}{N}\int_\Omega Q(x)(w_\epsilon+v_\circ)^{p-1}w_\epsilon\,dx &\le \frac{\epsilon}{p}\int_\Omega Q(x)\big((w_\epsilon+\epsilon v_\circ)^{p-1}-\epsilon^{p-1}v_\circ^{p-1}\big)v_\circ\,dx\\ &\le C\epsilon\int_\Omega Q(x)\big(\epsilon w_\epsilon^{p-2}v_\circ+\epsilon^{p-1}v_\circ^p\big)\,dx\end{aligned}$$

which implies using the Young inequality

$$\int_\Omega w_\epsilon^p\le C\epsilon^p\int_\Omega v_\circ^p\,dx$$

for some constant $C>0$. Let $\tilde u_\epsilon=w_\epsilon+\epsilon v_\circ$. Then $\tilde u_\epsilon$ is a positive solution of problem (1_d). As in [1] we can show using the method of sub- and supersolutions ([13]) that problem (1_d) has a minimum positive solution u_ϵ such that $u_\epsilon\le\tilde u_\epsilon$ on Ω. Using this fact, as in [1], we can derive the second assertion of this lemma. □

Since problem (1_d) has a minimum solution with properties stated in Lemma 6 we can show, repeating the arguments used in the proof of Theorem 1, that problem (1_d) has k positive solutions.

7. Appendix. We list here some algebraic inequalities used in this paper. We always assume that $a\ge 0$, $b\ge 0$ and $2<p<\infty$.

$$\begin{aligned}&p\big((a+b)^{p-1}-b^{p-1}\big)a-2\big((a+b)^p-b^p-pb^{p-1}a\big)\\ &\quad\ge l\big((a+b)^{p-1}-b^{p-1}-(p-1)b^{p-2}a\big)a,\end{aligned}\tag{A.1}$$

where $l=\min\{p-2,\frac{p}{3}\}$,

$$0\le (a+b)^{p-1}-a^{p-1}-b^{p-1}\le C(a^{p-2}b+b^{p-1})\tag{A.2}$$

for some constant $C>0$,

$$\begin{aligned}&(p-2)\big((a+b)^{p-1}-a^{p-1}-(p-1)a^{p-2}b\big)-(p-1)\big((a+b)^{p-2}a\\ &\quad-a^{p-1}-(p-2)a^{p-2}b\big)\ge\sigma\big((a+b)^{p-1}-a^{p-1}-(p-1)a^{p-2}b\big),\end{aligned}\tag{A.3}$$

where $\sigma\in(0,\min\{1,p-2\})$,

$$\big((a+b)^{p-1}-a^{p-1}-b^{p-1}-(p-1)a^{p-2}b\big)\le C\big(b^{p-1}+a^{p-2-\theta}b^{1+\theta}\big)\tag{A.4}$$

for some $C > 0$ and $\theta \in (0, p-2)$.

Proof. We only prove (A.1) and (A.3). If $b = 0$ then (A.1) holds. If $b > 0$ then (A.1) is equivalent to

$$p\big((1+s)^{p-1} - 1\big)s - 2\big(1+s)^p - 1 - ps\big) \geq l\big((1+s)^{p-1} - 1 - (p-1)s\big)s$$

for $s \geq 0$. To show this we set

$$K(s) = p\big((1+s)^{p-1} - 1\big)s - 2\big((1+s)^p - 1 - ps\big) - l\big((1+s)^{p-1} - 1 - (p-1)s\big)s.$$

We check that $K(0) = 0$, $K'(0) = 0$ and

$$K''(s) = (p-1)\big\{(p-2)(p-l)(1+s)^{p-3}s + 2l - 2l(1+s)^{p-2}\big\}.$$

If $p \geq 3$, then

$$\begin{aligned}&(p-2)(p-l)(1+s)^{p-3}s - 2l\big((1+s)^{p-2} - 1\big)\\ &\quad = (p-2)(p-l)(1+s)^{p-3}s - 2l(p-2)(1+\theta s)^{p-2}s \geq 0\end{aligned}$$

provided $2l \leq p - l$; that is, $l \leq \frac{p}{3}$.
If $p < 3$, then

$$\begin{aligned}&(p-2)(p-l)(1-s)^{p-3}s - 2l(1+s)^{p-2} + 2l\\ &\quad = (1+s)^{p-3}\big((p-2)(p-l)s - 2l(1+s)\big) + 2l\\ &\quad = (1+s)^{p-3}\big((p-2)(p-l)s - 2ls\big) - 2l(1+s)^{p-3} + 2l\\ &\quad = (1+s)^{p-3}s\big((p-2)(p-l) - 2l\big) + 2l\big(1 - (1+s)^{p-3}\big) \geq 0\end{aligned}$$

provided $2l \leq (p-2)(p-l)$; that is, $l \leq p-2$. Hence, if $l \leq \min\{\frac{p}{3}, p-2)\}$, then $K(s) \geq 0$ for $s \geq 0$ and inequality (A.1) follows. To establish (A.3) it is sufficient to consider the function

$$\begin{aligned}H(s) = (p-2-\sigma)&[(p-1)(1+s)^{p-2} - 1 - (p-1)s]\\ &- (p-1)[(1+s)^{p-2} - 1 - (p-2)s]\end{aligned}$$

for $s \geq 0$ and show that $H(s) \geq 0$ for $s \geq 0$.

REFERENCES

[1] Jun Ai and Xi-Ping Zhu, *Positive solutions of inhomogeneous elliptic boundary value problem in half space,* Commun. in PDE, 15(10), 1990, 1421–1446.

[2] A. Bahri and Y.Y. Li, *On a min-max procedure for the existence of a positive solution for certain scalar field equations in* $\mathbb{R}^n$, Rev. Mat. Iberoamericana, 6 (1990), 1–15.

[3] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents,* Commun. Pure Appl. Anal., 36 (1983), 437–477.

[4] H. Brezis and L. Nirenberg, *A minimization problem with critical exponent and nonzero data,* "Symmetry in Nature," Scuola Norm. Sup. Pisa, 1989, 129–140.

[5] L. Caffarelli and J. Spruck, *Variational problems with critical growth and positive data,* Indiana Univ. Maths J., 39 (1990), 1–18.

[6] Daomin Cao and E.S. Noussair, *Multiple positive and nodal solutions for semilinear elliptic problems in* $\mathbb{R}^N$, Ann. I.H. Poincaré, Analyse nonlinéaire, 13(5) (1986), 567–588.

[7] Daomin Cao and E.S. Noussair, *Multiple positive nodal solutions for semilinear elliptic problems with critical exponents*, Indiana Univ. Math. J., 44 (1995), 1249–1271.

[8] Daomin Cao and Huansong Zhou, *On the existence of multiple solutions of nonhomogeneous elliptic equations involving critical Sobolev exponents,* Z. angew, Math. Phys. (ZAMP), 47 (1995), 89–96.

[9] Daomin Cao, E.S. Noussair, and Shusen Yan, *The effect of the coefficient on the multiplicity of solutions of a semilinear elliptic problem,* preprint.

[10] J.M. Coron, *Topologie et cas limite des injections de Sobolev,* C.R. Acad. Sc. Paris, 299($n°$) (1984), 209–212.

[11] I. Ekeland, *On the variational principle,* J. Math. Anal. Appl., 17 (1974), 324–353.

[12] J. Escobar, *Positive solutions for some semilinear elliptic equations with critical Sobolev exponent,* Commun. Pure Appl. Math., 5 (1987), 623–657.

[13] J. Graham-Eagle, *Monotone methods for semilinear elliptic equations in unbounded domains,* J. Math. Anal. Appl., 137 (1989), 122–131.

[14] P.L. Lions, *The concentration–compactness principle in the calculus of variations: the limit case,* Rev. Mat. Iberoamericana, 1 (1985), 142–201.

[15] D. Passaseo, *Multiplicity of positive solutions of nonlinear elliptic equations with critical exponents in some contractible domains,* Manuscripta Math., 65 (1989), 147–166.

[16] O. Rey, *Concentration of solutions to elliptic equations with critical nonlinearity,* Ann. I.H. Poincaré Analyse non linéaire, 9 (1990), 201–218.

[17] G. Tarantello, *On nonhomogeneous elliptic equations involving critical Sobolev exponent,* Ann. Inst. H. Poincaré, 9(3) (1992), 281–304.