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GLOBAL STABILITY IN NONAUTONOMOUS LOTKA–VOLTERRA SYSTEMS OF "PURE-DELAY TYPE"

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Abstract. In this paper, nonautonomous Lotka–Volterra systems of "pure-delay type" are considered and some sufficient conditions on the global asymptotic stability are obtained. As a corollary, we show that, under the conditions of Theorem 2.1 in Kuang [11], the system remains globally asymptotically stable provided the delays are sufficiently small. Both finite and infinite delays are allowed in the systems. Our results give an affirmative answer to the two open problems due to Kuang. The results are established by constructing suitable Lyapunov functionals.

1. Introduction. We will be concerned with the solutions of the following nonautonomous "pure-delay-type" Lotka–Volterra system with both distributed and discrete, finite and infinite delays:

$$\begin{aligned} x_i'(t) =& b_i(x_i(t))\big[r_i(t) - a_i(t)x_i(t-\tau_i) - c_i(t)\int_{-\infty}^{0} x_i(t+s)h_i(s)\,ds \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}} b_{ijl}(t)x_j(t-\tau_{ijl}) + \sum_{j=1}^{n}\int_{-\tau_{ij}}^{0} x_j(t+s)k_{ij}(t,s)\,ds \\ &+ \sum_{j=1}^{n} d_{ij}(t)\int_{-\infty}^{0} x_j(t+s)h_{ij}(s)\,ds\big], \qquad i = 1,\dots,n \end{aligned} \tag{1.1}$$

associated with initial condition

$$x_i(s) = \phi(s), \qquad s < 0, \qquad \phi_i(0) > 0, \qquad i = 1,\dots,n, \tag{1.2}$$

where $\phi_i \in C((-\infty,0],R_+)$ are bounded and $R_+ = [0,\infty)$. System (1.1) is called "pure-delay type" since it is not necessary for (1.1) to have delay-independent intraspecific and interspecific effects (see Kolmanovskii et al.

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[9] and Gopalsamy and He [3, 8]). We assume that for $i, j = 1, \dots, n$, $l = 1, \dots, l_{ij}$:

(H$_1$) $b_i(\cdot)$ is continuously differentiable such that $b_i(0) = 0$, $b_i'(x) > 0$ for $x \geq 0$;

(H$_2$) $r_i(t)$, $a_i(t)$, $c_i(t)$ and $b_{ijl}(t)$ are bounded continuous functions for $t > 0$; $k_{ij}(t,s)$ are integrable with respect to $s \in [-\tau_{ij}, 0]$ and continuous and bounded with respect to t; $h_i(s)(\geq 0)$, $|h_{ij}(s)|$, $|s|h_i(s)$, $|sh_{ij}(s)|$ are integrable with respect to $s \in (-\infty, 0]$; τ_i, τ_{ij}, τ_{ijl} are nonnegative constants.

For $i, j = 1, \dots, n, l = 1, \dots, l_{ij}$, we set

$$\begin{aligned}
&a_i = \liminf_{t\to\infty} a_i(t), \quad A_i = \limsup_{t\to\infty} a_i(t), \quad c_i = \liminf_{t\to\infty} c_i(t),\\
&C_i = \limsup_{t\to\infty} c_i(t), \quad D_{ij} = \limsup_{t\to\infty} |d_{ij}(t)|, \quad B_{ijl} = \limsup_{t\to\infty} |b_{ijl}(t)|,\\
&R_i = \limsup_{t\to\infty} |r_i(t)|, \quad K_{ij}(s) = \limsup_{t\to\infty} |k_{ij}(t,s)|, \quad K_{ij} = \int_{-\tau_{ij}}^{0} K_{ij}(s)\, ds,
\end{aligned}$$

$\tau = \max\{\tau_i, \tau_{ij}, \tau_{ijl}; i, j = 1, \dots, n, l = 1, \dots, l_{ij}\}$. It is well known that (1.1) and (1.2) have a unique positive solution $x(t) = (x_1(t), \dots, x_n(t))$ for $t > 0$. As in Kuang [11], we say solutions of (1.1) and (1.2) are eventually uniformly bounded above and below by positive constant vectors $\Delta = (\Delta_1, \dots, \Delta_n)$ and $\delta = (\delta_1, \dots, \delta_n)$, respectively if, for any solution $x(t) = (x_1(t), \dots, x_n(t))$ of (1.1) and (1.2), we have, for all large t, $\delta_i \leq x_i(t) \leq \Delta_i$ and $i = 1, \dots, n$. We say system (1.1) is globally asymptotically stable if for any two solutions $x(t)$, $y(t)$ of (1.1) and (1.2), we have

$$\lim_{t\to\infty} (x(t) - y(t)) = 0.$$

The global asymptotic behaviours of the solutions of system (1.1) have been studied extensively, particularly for certain special cases of (1.1). The model systems considered are basically of two classes; some models contain an instantaneous (without delay) negative stabilizing density-dependent feedback which dominates other intraspecific and interspecific interaction effects with and without delays (we call this class as no-pure-delay type); in the other category such an instantaneous stabilizing negative feedback mechanism is not present (we refer this class as pure-delay type). By using either the Lyapunov functional method or Razumikhin-type function argument, various results on the global asymptotic stability of both types of systems have been obtained. For autonomous systems, we refer to Gopalsamy [2] and He and Gopalsamy [8] for the competition systems and He [5, 6] for predator-prey systems. When the systems are nonautonomous, we refer to Gopalsamy and He [3] and He [4, 7] for pure-delay-type competition systems and Kuang [10, 11] and Kuang and Smith [12] for more general systems. It is noticed that the systems studied in Kuang and Smith [12] are required to have a saturated equilibrium. Recently, Kuang ([11]) dropped this unnatural requirement and studied the no-pure-delay-type system (1.1) with $\tau_i = 0$, $c_i(t) = 0$ and $d_{ij}(t) = 0$ for $i, j = 1, \ldots, n$ and $t \in R$. In this case, the corresponding initial condition has the form

$$x_i(s) = \phi(s), \quad s \in [-\tau, 0], \quad \phi_i(0) > 0, \quad i = 1, \ldots, n, \tag{1.3}$$

with $\phi_i \in C([-\tau, 0], R_+)$ and $R_+ = [0, \infty)$. Kuang ([11]) derived the following result (Theorem 2.1, Kuang [11]) on the global asymptotic stability of the solutions.

Theorem A. *Assume system* (1.1) *with* $\tau_i = 0$, $c_i(t) = 0$ *and* $d_{ij}(t) = 0$ $(i, j = 1, \ldots, n)$ *satisfies* (H_1), (H_2) *and*

(H_3) *solutions of* (1.1) *and* (1.3) *are eventually uniformly bounded above and below by positive constant vectors* Δ *and* δ*, respectively;*

(H_4) *the matrix* $M = (m_{ij})_{n \times n}$ *with*

$$m_{ii} = a_i - \sum_{l=1}^{l_{ii}} B_{iil} - K_{ii}, \qquad m_{ij} = -\sum_{l=1}^{l_{ij}} B_{ijl} - K_{ij}, \quad i \neq j$$

is an M-matrix.

Then system (1.1) *is globally asymptotically stable.*

The purpose of this paper is to generalize the above Theorem A to the "pure-delay-type" system (1.1) with both finite and infinite delays. Some sufficient conditions on the global asymptotic stability of (1.1) are obtained. As a corollary, we show that, for the pure-delay-type system (1.1) with finite delays ($\tau > 0$, $c_i(t) = 0$ and $d_{ij} = 0\,(i, j = 1, \ldots, n)$), if the conditions of Theorem A are satisfied and the τ_i are sufficiently small, then system (1.1) with (1.3) remains globally asymptotically stable. We also show that similar results can be obtained for systems (1.1) with infinite delays. Consequently, an affirmative answer is given to the two open problems due to Kuang ([11]). All the results are established by constructing suitable Lyapunov functionals. It is shown that the Lyapunov functionals method is a very effective tool in the study of the global asymptotic stability of (1.1).

2. Finite delay case. In this section, we consider system (1.1) with finite delays, that is, system (1.1) with $c_i(t) = 0$ and $d_{ij}(t) = 0$ for $i, j = 1, \ldots, n$ and $t \in R$.

Theorem 2.1. *Assume system* (1.1) *with* $c_i(t) = 0$ *and* $d_{ij}(t) = 0$ $(i, j = 1, \dots, n,\ t \in R)$ *satisfies* (H_1), (H_2), (H_3) *and*

(H_5) *the matrix* $M = (m_{ij})_{n\times n}$ *with*

$$m_{ii} = a_i - \sum_{l=1}^{l_{ii}} B_{iil} - K_{ii} - A_i\tau_i\big\{R_iB_i + b_i(\Delta_i)\big[A_i + \sum_{l=1}^{l_{ii}} B_{iil} + K_{ii}\big]$$

$$+ B_i\big[A_i\Delta_i + \sum_{j=1}^{n} \Delta_j\big(\sum_{l=1}^{l_{ij}} B_{ijl} + K_{ij}\big)\big]\big\}$$

$$m_{ij} = -[1 + A_i\tau_ib_i(\Delta_i)]\big[\sum_{l=1}^{l_{ij}} B_{ijl} + K_{ij}\big], \quad i \neq j \tag{2.1}$$

is an M-matrix, where $B_i = \max_{\delta_i \le s \le \Delta_i} |b_i'(s)|$.

Then system (1.1) *is globally asymptotically stable.*

Remark 2.2. It is noticed that condition (H_5) depends on τ_i and the ultimate upper bound Δ of the solutions of (1.1), from which one can derive an estimate on the allowed size of delays for the global asymptotic stability of (1.1). If $\tau_i = 0$ for $i = 1, \dots, n$, then condition (H_5) is reduced to (H_4). Thus Theorem 2.1 is a generalization of Theorem A. When the τ_i are sufficiently small, we have the following corollary, which gives an affirmative answer to the open problem 1 due to Kuang ([11]).

Corollary 2.3. *For system* (1.1) *with* $c_i(t) = 0$ *and* $d_{ij}(t) = 0$ (*for* $i, j = 1, \dots, n$ *and* $t \in R$), *if the conditions of Theorem* A *are satisfied and the* τ_i *are sufficiently small, then system* (1.1) *remains globally asymptotically stable.*

Proof of Theorem 2.1. Since M is an M-matrix, we know that (see He and Gopalsamy [8] and Kuang [10]) there exist $\alpha_i > 0$ and $\epsilon > 0$ such that

$$\alpha_i(m_{ii} - \epsilon) > \sum_{j=1, j\neq i}^{n} \alpha_j(|m_{ji}| + \epsilon), \quad i = 1, \dots, n. \tag{2.2}$$

Clearly there is an $\epsilon_1 \in (0, \epsilon)$ such that

$$m_{ii} - \epsilon < m_{ii}(\epsilon_1), \quad m_{ij} - \epsilon < m_{ij}(\epsilon_1), \quad i, j = 1, \dots, n, \tag{2.3}$$

where

$$m_{ii}(\epsilon_1) = a_i - \epsilon_1 - \sum_{l=1}^{l_{ii}}(B_{iil}+\epsilon_1) - (K_{ii}+\epsilon_1) - \tau_i(A_i+\epsilon_1)\big[(R_i+\epsilon_1)B_i + b_i(\Delta_i)\big((A_i+\epsilon_1) + \sum_{l=1}^{l_{ii}}(B_{iil}+\epsilon_1) + (K_{ii}+\epsilon_1)\big) + B_i\big(\Delta_i(A_i+\epsilon_1) + \sum_{j=1}^{n}\Delta_j\big(\sum_{l=1}^{l_{ij}}(B_{ijl}+\epsilon_1) + (K_{ij}+\epsilon_1)\big)\big)\big] \tag{2.4}$$

$$m_{ij}(\epsilon_1) = -[1+\tau_i b_i(\Delta_i)(A_i+\epsilon_1)]\big(\sum_{l=1}^{l_{ij}}(B_{ijl}+\epsilon_1) + (K_{ij}+\epsilon_1)\big),\ i \neq j.$$

Denote

$$v_i(p,q) = \int_q^p \frac{ds}{b_i(s)}, \quad i = 1,\dots,n.$$

For $i = 1, 2$, let $x^{(i)}(t) = (x_1^{(i)}(t),\dots,x_n^{(i)}(t))$ be any two solutions of (1.1) and (1.3). Then, for $t \geq \tau$,

$$\begin{aligned}\frac{d}{dt}v_i(x_i^{(1)}(t),x_i^{(2)}(t)) &= \frac{1}{b_i(x_i^{(1)}(t))}\frac{d}{dt}x_i^{(1)}(t) - \frac{1}{b_i(x_i^{(2)}(t))}\frac{d}{dt}x_i^{(2)}(t)\\ &= -a_i(t)[x_i^{(1)}(t-\tau_i) - x_i^{(2)}(t-\tau_i)]\\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}} b_{ijl}(t)[x_i^{(1)}(t-\tau_{ijl}) - x_i^{(2)}(t-\tau_{ijl})]\\ &\quad + \sum_{j=1}^{n}\int_{-\tau_{ij}}^{0}[x_j^{(1)}(t+s) - x_j^{(2)}(t+s)]k_{ij}(t,s)\,ds\end{aligned} \tag{2.5}$$

$$\begin{aligned}&= -a_i(t)[x_i^{(1)}(t) - x_i^{(2)}(t)] + a_i(t)w_i(x^{(1)}(t),x^{(2)}(t))\\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}} b_{ijl}(t)[x_j^{(1)}(t-\tau_{ijl}) - x_j^{(2)}(t-\tau_{ijl})]\\ &\quad + \sum_{j=1}^{n}\int_{-\tau_{ij}}^{0}[x_j^{(1)}(t+s) - x_j^{(2)}(t+s)]k_{ij}(t,s)\,ds,\end{aligned}$$

where

$$w_i(x^{(1)}(t),x^{(2)}(t)) = [x_i^{(1)}(t) - x_i^{(2)}(t)] - [x_i^{(1)}(t-\tau_i) - x_i^{(2)}(t-\tau_i)], \tag{2.6}$$

which can be written as follows:

$$
\begin{aligned}
w_i(x^{(1)}(t),x^{(2)}(t)) &= \int_{t-\tau_i}^{t} \frac{d}{du}[x_i^{(1)}(u)-x_i^{(2)}(u)]\,du \\
&= \int_{t-\tau_i}^{t} \big\{r_i(u)[b_i(x_i^{(1)}(u)-b_i(x_i^{(2)}(u)] \\
&- a_i(t)\big[b_i(x_i^{(1)}(u))[x_i^{(1)}(u-\tau_i)-x_i^{(2)}(u-\tau_i)] \\
&+ x_i^{(2)}(u-\tau_i)[b_i(x_i^{(1)}(u))-b_i(x_i^{(2)}(u))]\big] \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}} b_{ijl}(u)\big[b_i(x_i^{(1)}(u))[x_j^{(1)}(u-\tau_{ijl})-x_j^{(2)}(u-\tau_{ijl})] \\
&+ x_j^{(2)}(u-\tau_{ijl})[b_i(x_i^{(1)}(u))-b_i(x_i^{(2)}(u))]\big] \\
&+ \sum_{j=1}^{n}\int_{-\tau_{ij}}^{0} \big[b_i(x_i^{(1)}(u))[x_j^{(1)}(u+s)-x_j^{(2)}(u+s)] \\
&+ x_j^{(2)}(u+s)[b_i(x_i^{(1)}(u))-b_i(x_i^{(2)}(u))]\big]k_{ij}(t,s)\,ds\big\}\,du.
\end{aligned} \tag{2.7}
$$

From (H_1)–(H_3), we know that there exists a large $T_1 > \tau$ such that, for $t \ge T_1$, $\delta_i \le x_i^{(1)}(t), x_i^{(2)}(t) \le \Delta_i$, $b_i(s) \le b_i(\Delta_i)$ and $|b_i'(s)| \le B_i$ for $s \in [\delta_i, \Delta_i]$, and $a_i - \epsilon_1 \le a_i(t) \le A_i + \epsilon_1$, $|r_i(t)| \le R_i + \epsilon_1$, $|b_{ijl}(t)| \le B_{ijl} + \epsilon_1$ and $|k_{ij}(t,s)| \le K_{ij}(s) + \epsilon_1/(1+\tau_{ij})$. Let

$$
V_{i1}(x_i^{(1)}, x_i^{(2)}) = |v_i(x_i^{(1)}, x_i^{(2)})|. \tag{2.8}
$$

Then it follows from (2.5)–(2.8) that, for $t \ge T_2 \ge T_1 + 2\tau$, the upper-right derivative of V_{i1} along the solutions of (1.1) is governed by

$$
\begin{aligned}
&\frac{d}{dt}V_{i1}(x_i^{(1)}(t), x_i^{(2)}(t)) \\
&\quad \le -(a_i-\epsilon_1)|x_i^{(1)}(t)-x_i^{(2)}(t)| + (A_i+\epsilon_1)|w_i(x_i^{(1)}(t), x_i^{(2)}(t)| \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}} (B_{ijl}+\epsilon_1)|x_j^{(1)}(t-\tau_{ijl})-x_j^{(2)}(t-\tau_{ijl})| \\
&\quad + \sum_{j=1}^{n}\int_{-\tau_{ij}}^{0} |x_j^{(1)}(t+s)-x_j^{(2)}(t+s)|(K_{ij}(s)+\frac{\epsilon_1}{1+\tau_{ij}})\,ds,
\end{aligned} \tag{2.9}
$$

where

$$
\begin{aligned}
|w_i(x^{(1)}(t), x^{(2)}(t))| &\le \int_{t-\tau_i}^{t} \big\{(R_i+\epsilon_1)B_i|x_i^{(1)}(u)-x_i^{(2)}(u)| \\
&+ (A_i+\epsilon_1)\big[b_i(\Delta_i)|x_i^{(1)}(u-\tau_i)-x_i^{(2)}(u-\tau_i)| + \Delta_i B_i|x_i^{(1)}(u)-x_i^{(2)}(u)|\big] \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}(B_{ijl}+\epsilon_1)\big[b_i(\Delta_i)|x_j^{(1)}(u-\tau_{ijl})-x_j^{(2)}(u-\tau_{ijl})| \qquad (2.10)\\
&+ \Delta_j B_i|x_i^{(1)}(u))-x_i^{(2)}(u)|\big] \\
&+ \sum_{j=1}^{n}\int_{-\tau_{ij}}^{0} \big[b_i(\Delta_i)|x_j^{(1)}(u+s)-x_j^{(2)}(u+s)| \\
&+ \Delta_j B_i|x_i^{(1)}(u)-x_i^{(2)}(u)|\big](K_{ij}(s)+\frac{\epsilon_1}{1+\tau_{ij}})\,ds\big\}\,du \overset{\text{def}}{=} \int_{t-\tau_i}^{t} W_{i1}(u)\,du,
\end{aligned}
$$

in which the mean-value theorem is used. Let $y_i(t) = x_i^{(1)}(t) - x_i^{(2)}(t)$, $i = 1, \dots, n$, and

$$
\begin{aligned}
&V_{i2}(x^{(1)}(t), x^{(2)}(t)) \\
&= (A_i+\epsilon_1)\int_{t-\tau_i}^{t}\int_{v}^{t} W_{i1}(u)\,du\,dv + (A_i+\epsilon_1)^2 b_i(\Delta_i)\tau_i \int_{t-\tau_i}^{t}|y_i(u)|\,du \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}(B_{ijl}+\epsilon_1)[b_i(\Delta_i)\tau_i(A_i+\epsilon_1)+1]\int_{t-\tau_{ijl}}^{t}|y_j(u)|\,du \qquad (2.11)\\
&\quad + \sum_{j=1}^{n}[1+(A_i+\epsilon_1)b_i(\Delta_i)\tau_i]\int_{-\tau_{ij}}^{0}\int_{t+s}^{t}|y_j(u)|\,du(K_{ij}(s)+\frac{\epsilon_1}{1+\tau_{ij}})\,ds
\end{aligned}
$$

and

$$V_i(x^{(1)}(t), x^{(2)}(t)) = (V_{i1}+V_{i2})(x^{(1)}(t), x^{(2)}(t)). \qquad (2.12)$$

Then $V_i \ge 0$ and V_i is differentiable. For $t \ge T_3 \ge T_2 + 2\tau$, one has from (2.9)–(2.12) and (2.3), (2.4) that

$$
\begin{aligned}
&\frac{d}{dt}V_i(x^{(1)}(t), x^{(2)}(t)) \\
&\quad \le -(a_i-\epsilon_1)|y_i(t)| + \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}(B_{ijl}+\epsilon_1)|y_j(t)| + \sum_{j=1}^{n}(K_{ij}+\epsilon_1)|y_j(t)| \\
&\quad + \tau_i(A_i+\epsilon_1)\big\{(R_i+\epsilon_1)B_i|y_i(t)| + (A_i+\epsilon_1)[b_i(\Delta_i)+\Delta_i B_i]|y_i(t)|
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}(B_{ijl}+\epsilon_1)[b_i(\Delta_i)|y_j(t)|+\Delta_j B_i|y_i(t)|]\\
&+\sum_{j=1}^{n}[b_i(\Delta_i)|y_j(t)|+\Delta_j B_i|y_i(t)|][K_{ij}+\epsilon_1]\big\}\\
&=\big\{-(a_i-\epsilon_1)+\sum_{l=1}^{l_{ii}}(B_{iil}+\epsilon_1)+(K_{ii}+\epsilon_1)\\
&+\tau_i(A_i+\epsilon_1)\big[(R_i+\epsilon_1)B_i+(A_i+\epsilon_1)[b_i(\Delta_i)+\Delta_i B_i]\\
&+b_i(\Delta_i)\sum_{l=1}^{l_{ii}}(B_{iil}+\epsilon_1)+B_i\sum_{j=1}^{n}\Delta_j\sum_{l=1}^{l_{ij}}(B_{ijl}+\epsilon_1) \qquad (2.13)\\
&+b_i(\Delta_i)(K_{ii}+\epsilon_1)+B_i\sum_{j=1}^{n}\Delta_j(K_{ij}+\epsilon_1)\big]\big\}|y_i(t)|\\
&+\sum_{j=1,j\neq i}^{n}\big\{\sum_{l=1}^{l_{ij}}(B_{ijl}+\epsilon_1)+(K_{ij}+\epsilon_1)\\
&+\tau_i(A_i+\epsilon_1)b_i(\Delta_i)\big[\sum_{l=1}^{l_{ij}}(B_{ijl}+\epsilon_1)+(K_{ij}+\epsilon_1)\big]\big\}|y_j(t)|\\
&=-\sum_{j=1}^{n}m_{ij}(\epsilon_1)|y_j(t)|\le-(m_{ii}-\epsilon)|y_i(t)|+\sum_{j=1,j\neq i}^{n}(|m_{ij}|+\epsilon)|y_j(t)|.
\end{aligned}
$$

Now we define the Lyapunov functional

$$
V(x^{(1)},x^{(2)})=\sum_{i=1}^{n}\alpha_i V_i(x^{(1)},x^{(2)}). \qquad (2.14)
$$

Then V is well defined and, for $t\ge T_3$, one can see from (2.13) and (2.14) that

$$
\begin{aligned}
\frac{d}{dt}V(x^{(1)},x^{(2)})(t)&\le-\sum_{i=1}^{n}\alpha_i\big[-(m_{ii}-\epsilon)|y_i(t)|+\sum_{j=1,j\neq i}^{n}(|m_{ij}|+\epsilon)|y_j(t)|\big]\\
&=-\sum_{i=1}^{n}\big(\alpha_i(m_{ii}-\epsilon)-\sum_{j=1,j\neq i}^{n}\alpha_j(|m_{ji}|+\epsilon)\big)|y_i(t)|. \qquad (2.15)
\end{aligned}
$$

Let

$$
\beta_i=\alpha_i(m_{ii}-\epsilon)-\sum_{j=1}^{n}\alpha_j(|m_{ji}|+\epsilon).
$$

Then it follows from (2.2) that $\beta_i > 0$. Integrating (2.15) from T_3 to $t > T_3$, we have

$$\sum_{i=1}^{n} \beta_i \int_{T_3}^{t} |y_i(s)| ds = \sum_{i=1}^{n} \beta_i \int_{T_3}^{t} |x_i^{(1)}(s) - x_i^{(2)}(s)| ds \le V(x^{(1)}, x^{(2)})(T_3) < \infty,$$

which implies that $|x_i^{(1)}(t) - x_i^{(2)}(t)| \in L_1[T_3, \infty)$. Also, from (H_3), $|\frac{d}{dt}[x_i^{(1)}(t) - x_i^{(2)}(t)]|$ exists and is uniformly bounded for all $t \ge T_3$. Thus, we have

$$\lim_{t\to\infty} |x_i^{(1)}(t) - x_i^{(2)}(t)| = 0, \qquad i = 1, \dots, n.$$

This completes the proof.

Remark 2.4. Following the proof of our Theorem 2.1 and the argument used for Theorem 3.1 in Kuang [11], one can extend our Theorem 2.1 to a more general nonautonomous Kolmogorov-type delay systems; we refer to Kuang [11] for details.

3. Infinite-delay case. In this section, we generalize our discussion in Section 2 and consider the system (1.1) with both finite and infinite delays. Consequently, we solve the open problem 2 in Kuang [11]. We will first consider a special case of system (1.1) with $c_i(t) = 0$ for $i = 1, \dots, n$.

Theorem 3.1. *Let* $c_i(t) = 0\,(i = 1, \dots, n)$ *in* (1.1). *Suppose that*

$$\int_{-\infty}^{0} |h_{ij}(s)|\, ds = 1, \quad \int_{-\infty}^{0} |s h_{ij}(s)|\, ds < \infty \quad (i, j = 1, \dots, n).$$

If system (1.1) *satisfies* (H_1), (H_2), (H_3) *and*

(H_6) *the matrix* $P = (p_{ij})_{n\times n}$ *with*

$$\begin{cases} p_{ii} = a_i - \sum_{l=1}^{l_{ii}} B_{iil} - K_{ii} - D_{ii} \\ \quad -A_i\tau_i\big\{R_iB_i + b_i(\Delta_i)\big[A_i + \sum_{l=1}^{l_{ii}} B_{iil} + K_{ii} + D_{ii}\big] \\ \quad +B_i\big[\Delta_iA_i + \sum_{j=1}^{n} \Delta_j\big(\sum_{l=1}^{l_{ij}} B_{ijl} + K_{ij} + D_{ij}\big)\big]\big\} \\ p_{ij} = -[1 + A_i\tau_ib_i(\Delta_i)]\big[\sum_{l=1}^{l_{ij}} B_{ijl} + K_{ij} + D_{ij}\big], \quad i \ne j \end{cases} \tag{3.1}$$

is an M-matrix, where $B_i = \max_{\delta_i \le s \le \Delta_i} |b_i'(s)|$;

(H_7) *There exists a solution* $\tilde{x}(t) = (\tilde{x}_1(t), \dots, \tilde{x}_n(t))$ *of* (1.1) *and* (1.2) *with* $\delta_i \le \tilde{x}_i(t) \le \Delta_i$ *for all* $t \in R$ *and* $i = 1, \dots, n$, *then system* (1.1) *is globally asymptotically stable.*

Remark 3.2. Condition (H_7) is needed since infinite delays are allowed in system (1.1). In the case the system (1.1) has a positive saturated steady state, (H_7) is obviously satisfied. In the case when the systems are periodic or almost periodic Lotka–Volterra-type systems, condition (H_7) can also be removed. In fact, for the almost-periodic system (1.1), following the argument used for Lemma 2 in Murakami [14], one can prove that, under condition (H_3), system (1.1) has a solution $\tilde{x}(t)$ satisfying (H_7). Then, employing the argument used in Fink [1], Gopalsamy and He [3], He [7], Murakami [13] and Seifert [14], one can derive the existence of a positive almost-periodic solution of (1.1). We refer to Gopalsamy and He [3] and He [4, 7] for the related discussion on pure-delay-type competition systems, where we constructed different Lyapunov functionals and obtained some different results.

Condition (H_6) depends only on the finite delays τ_i. When τ_i $(i = 1, \ldots, n)$ are sufficiently small, we have following corollary.

Corollary 3.3. *If condition* (H_6) *in Theorem* 3.1 *is replaced by the condition that the matrix* $\tilde{P} = (\tilde{p})_{n\times n}$ *with*

$$\tilde{p}_{ii} = a_i - \sum_{l=1}^{l_{ii}} B_{iil} - K_{ii} - D_{ii}, \quad \tilde{p}_{ij} = -\big[\sum_{l=1}^{l_{ij}} B_{ijl} + K_{ij} + D_{ij}\big] \quad (i \neq j)$$

is an M-matrix, then the conclusion of Theorem 3.1 *remains true provided delays* τ_i $(i = 1, \ldots, n)$ *are sufficiently small.*

Proof of Theorem 3.1. The proof of this theorem is similar to that of Theorem 2.1 and we indicate it briefly. Let $x^{(1)}(t)$ be any solution of (1.1) and (1.2) and $x^{(2)}(t)$ be the solution which satisfies condition (H_7); that is, $\delta_i \le x_i^{(2)}(t) \le \Delta_i$ for all $t \in R$ and $i = 1, \ldots, n$. Then it is sufficient to prove that

$$\lim_{t\to\infty} (x_i^{(1)}(t) - x_i^{(2)}(t)) = 0.$$

Let $v_i(p, q)$ be defined as in Section 2, and denote

$$\begin{aligned} v_i(x_i(t)) &= v_i(x_i^{(1)}(t), x_i^{(2)}(t)), \quad y_i(t) = x_i^{(1)}(t) - x_i^{(2)}(t), \\ B_i(y_i(t)) &= b_i(x_i^{(1)}(t)) - b_i(x_i^{(2)}(t)), \quad i = 1, \ldots, n. \end{aligned} \tag{3.2}$$

Then, from (1.1) and (3.2) and for $t \geq \tau$,

$$\begin{aligned}\frac{d}{dt}v_i(x_i(t)) = &- a_i(t)y_i(t) + \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}} b_{ijl}(t)y_i(t-\tau_{ijl}) \\ &+ \sum_{j=1}^{n}\int_{-\tau_{ij}}^{0} y_j(t+s)k_{ij}(t,s)\,ds \\ &+ \sum_{j=1}^{n} d_{ij}(t)\int_{-\infty}^{0} y_j(t+s)h_{ij}(s)\,ds + a_i(t)w_i(y_i(t))\end{aligned} \tag{3.3}$$

with

$$w_i(y_i(t)) = \int_{t-\tau_i}^{t} y_i'(u)\,du \stackrel{\text{def}}{=} \int_{t-\tau_i}^{t} W_{i2}(u)\,du \tag{3.4}$$

and

$$\begin{aligned}W_{i2}(u) = {}& r_i(u)B_i(y_i(u)) - a_i(t)[b_i(x_i^{(1)}(u))y_i(u-\tau_i) + x_i^{(2)}(u-\tau_i)B_i(y_i(u))] \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}} b_{ijl}(u)[b_i(x_i^{(1)}(u))y_j(u-\tau_{ijl}) + x_j^{(2)}(u-\tau_{ijl})B_i(y_i(u))] \\ &+ \sum_{j=1}^{n}\int_{-\tau_{ij}}^{0}[b_i(x_i^{(1)}(u))y_j(u+s) + x_j^{(2)}(u+s)B_i(y_i(u))]k_{ij}(t,s)\,ds \\ &+ \sum_{j=1}^{n} d_{ij}(u)\int_{-\infty}^{0}[b_i(x_i^{(1)}(u))y_j(u+s) + x_j^{(2)}(u+s)B_i(y_i(u))]h_{ij}(s)\,ds.\end{aligned} \tag{3.5}$$

Let V_{i1} be defined by (2.8); then, it follows from (3.3)–(3.5) that, for all large t,

$$\begin{aligned}\frac{d}{dt}V_{i1}(t) \leq {}& -(a_i - \epsilon_1)|y_i^{(1)}(t)| + \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}(B_{ijl} + \epsilon_1)|y_j(t-\tau_{ijl})| \\ &+ \sum_{j=1}^{n}\int_{-\tau_{ij}}^{0}|y_j(t+s)|(K_{ij}(s) + \frac{\epsilon_1}{1+\tau_{ij}})\,ds \\ &+ \sum_{j=1}^{n}(D_{ij} + \epsilon_1)\int_{-\infty}^{0}|y_j(t+s)||h_{ij}(s)|\,ds + (A_i + \epsilon_1)\int_{t-\tau_i}^{t}|W_{i2}(u)|\,du.\end{aligned} \tag{3.6}$$

Define

$$V_{i3}(y_i(t)) = \int_{t-\tau_i}^{t}\int_{v}^{t}|W_{i2}(u)|\,du\,dv. \tag{3.7}$$

Noting that (from (H_3)), for all large t, $x_i^{(1)}(t), x_i^{(2)}(t) \le \Delta_i$, $b_i(x_i^{(1)}(t)) \le b_i(\Delta_i)$ and $|B_i(y_i(t))| \le B_i|y_i(t)|$ $(i = 1, \ldots, n)$. From (3.5)–(3.7) and (H_3), one can see that, for all large t,

$$
\begin{aligned}
&\frac{d}{dt}(V_{i1} + V_{i3})(y_i(t)) = \frac{d}{dt}V_{i1}(y_i(t)) + (A_i + \epsilon_1)\big[-\int_{t-\tau_i}^{t}|W_{i2}(u)|du + \tau_i|W_{i2}(t)|\big] \\
&\le -(a_i - \epsilon_1)|y_i^{(1)}(t)| + \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}(B_{ijl} + \epsilon_1)|y_j(t - \tau_{ijl})| \\
&+ \sum_{j=1}^{n}\int_{-\tau_{ij}}^{0}|y_j(t+s)|(K_{ij}(s) + \frac{\epsilon_1}{1+\tau_{ij}})\,ds \\
&+ \sum_{j=1}^{n}(D_{ij} + \epsilon_1)\int_{-\infty}^{0}|y_j(t+s)||h_{ij}(s)|\,ds \\
&+ (A_i + \epsilon_1)\tau_i\big\{(R_i + \epsilon_1)B_i|y_i(t)| + (A_i + \epsilon_1)[b_i(\Delta_i)|y_i(t - \tau_i)| + \Delta_i B_i|y_i(t)|] \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}(B_{ijl}(t) + \epsilon_1)[b_i(\Delta_i)|y_j(t - \tau_{ijl})| + \Delta_j B_i|y_i(t)|] \\
&+ \sum_{j=1}^{n}\int_{-\tau_{ij}}^{0}[b_i(\Delta_i)|y_j(t+s)| + \Delta_j B_i|y_i(t)|](K_{ij}(s) + \frac{\epsilon_1}{1+\tau_{ij}})\,ds \qquad (3.8) \\
&+ \sum_{j=1}^{n}(D_{ij} + \epsilon_1)\int_{-\infty}^{0}[b_i(\Delta_i)|y_j(u+s)| + x_j^{(2)}(t+s)B_i|y_i(t)|]|h_{ij}(s)|\,ds\big\}.
\end{aligned}
$$

For the last term in (3.8), we use (H_7) and have $x_j^{(2)}(t+s) \le \Delta_j$ for $t+s \in R$. Note that $\int_{-\infty}^{0}|h_{ij}(s)|\,ds = 1$ $(i, j = 1, \ldots, n)$. Thus, for all large t,

$$
\begin{aligned}
&\frac{d}{dt}(V_{i1} + V_{i3})(y_i(t)) \le -(a_i - \epsilon_1)|y_i^{(1)}(t)| + (A_i + \epsilon_1)\tau_i\big\{(R_i + \epsilon_1)B_i|y_i(t)| \\
&+ (A_i + \epsilon_1)[b_i(\Delta_i)|y_i(t - \tau_i)| + \Delta_i B_i|y_i(t)|] \qquad (3.9) \\
&+ B_i\sum_{j=1}^{n}\Delta_j\big[\sum_{l=1}^{l_{ij}}(B_{ijl} + \epsilon_1) + (K_{ij} + \epsilon_1) + (D_{ij} + \epsilon)\big]|y_i(t)|\big\} \\
&+ [1 + (A_i + \epsilon_1)\tau_i b_i(\Delta_i)]\sum_{j=1}^{n}\big[\sum_{l=1}^{l_{ij}}(B_{ijl} + \epsilon_1)|y_j(t - \tau_{ijl})| \\
&+ \int_{-\tau_{ij}}^{0}|y_j(t+s)|(K_{ij}(s) + \frac{\epsilon_1}{1+\tau_{ij}})ds + (D_{ij} + \epsilon_1)\int_{-\infty}^{0}|y_j(t+s)||h_{ij}(s)|ds\big].
\end{aligned}
$$

Let

$$V_i(y_i(t)) = (V_{i1} + V_{i3} + V_{i4})(y(t)) \tag{3.10}$$

with

$$\begin{aligned} V_{i4}(y(t)) &= (A_i + \epsilon_1)^2 \tau_i b_i(\Delta_i) \int_{t-\tau_i}^{t} |y_i(u)|\, du \\ &\quad + [1 + (A_i + \epsilon_1)\tau_i b_i(\Delta_i)] \sum_{j=1}^{n} \big[\sum_{l=1}^{l_{ij}} (B_{ijl} + \epsilon_1) \int_{t-\tau_{ijl}}^{t} |y_j(u)|\, du \\ &\quad + \int_{-\tau_{ij}}^{0} \big(\int_{t+s}^{t} |y_j(u)|\, du \big) (K_{ij}(s) + \frac{\epsilon_1}{1+\tau_{ij}})\, ds \\ &\quad + (D_{ij} + \epsilon_1) \int_{-\infty}^{0} \big(\int_{t+s}^{t} |y_j(u)|\, du \big) |h_{ij}(s)|\, ds \big]. \end{aligned} \tag{3.11}$$

Then, for all large t,

$$\begin{aligned} \frac{d}{dt} V_i(y(t)) &\le -(a_i - \epsilon_1)|y_i^{(1)}(t)| \\ &\quad + (A_i + \epsilon_1)\tau_i \big\{ \big[(R_i + \epsilon_1) B_i + (A_i + \epsilon_1)[b_i(\Delta)_i + \Delta_i B_i] \big] \\ &\quad + B_i \sum_{j=1}^{n} \Delta_j \big[\sum_{l=1}^{l_{ij}} (B_{ijl} + \epsilon_1) + (K_{ij} + \epsilon_1) + (D_{ij} + \epsilon) \big] \big\} |y_i(t)| \\ &\quad + [1 + (A_i + \epsilon_1)\tau_i b_i(\Delta_i)] \sum_{j=1}^{n} \big[\sum_{l=1}^{l_{ij}} (B_{ijl} + \epsilon_1) + (K_{ij} + \epsilon_1) + (D_{ij} + \epsilon_1) \big] |y_j(t)|. \end{aligned} \tag{3.12}$$

Now, let

$$V(y)(t) = \sum_{i=1}^{n} \alpha_i V_i(y(t)).$$

Then, in a way similar to the last part of the proof of Theorem 2.1, we can derive the conclusion. This completes the proof. □

Note that condition (H_6) depends only on the finite delays τ_i, not the infinite delays. Therefore, the terms with infinite delays are dominated by the terms with finite delays τ_i. One can see from the following Theorem 3.4 that this is no longer the case when there are no such dominated terms with finite delays.

Theorem 3.4. *Let* $a_i(t) = 0$ $(i = 1, 2, \dots, n)$ *in system* (1.1)*. Assume that* $h_i(s) \geq 0$ *for* $s \in (-\infty, 0]$ *and, for* $i = 1, \dots, n$,

$$\int_{-\infty}^{0} h_i(s)\,ds = 1, \quad \sigma_i = \int_{-\infty}^{0} |s| h_i(s)\,ds < \infty, \quad \int_{-\infty}^{0} s^2 h_i(s)\,ds < \infty.$$

If system (1.1) *satisfies* (H_1)–(H_3), (H_7) *and*

(H_8) *the matrix* $Q = (q_{ij})_{n\times n}$ *with*

$$\begin{cases} q_{ii} = c_i - \sum_{l=1}^{l_{ii}} B_{iil} - K_{ii} - D_{ii} \\ \qquad -C_i\sigma_i\big[R_iB_i + b_i(\Delta_i)\big(C_i + \sum_{l=1}^{l_{ii}} B_{iil} + K_{ii} + D_{ii}\big) \\ \qquad +B_i\big(C_i\Delta_i + \sum_{j=1}^{n} \Delta_j\big[\sum_{l=1}^{l_{ij}} B_{ijl} + (K_{ij} + D_{ij})\big]\big)\big]; \\ q_{ij} = -[1 + C_i\sigma_i b_i(\Delta_i)]\big[\sum_{l=1}^{l_{ij}} B_{ijl} + K_{ij} + D_{ij}\big], \quad i \neq j \end{cases}$$

is an M-matrix, where $B_i = \max_{\delta_i \leq s \leq \Delta_i} |b_i'(s)|$, *then system* (1.1) *is globally asymptotically stable.*

Proof. In a way similar to the proof of Theorem 3.1, one can derive from (1.1) and (3.2) that

$$\begin{aligned} \frac{d}{dt} v_i(x_i(t)) &= -c_i(t)\big[1 - \int_{-\infty}^{-t} h_i(s)\,ds\big] y_i(t) + \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}} b_{ijl}(t) y_j(t - \tau_{ijl}) \\ &+ \sum_{j=1}^{n} \int_{-\tau_{ij}}^{0} y_j(t+s) k_{ij}(t,s)\,ds + \sum_{j=1}^{n} d_{ij}(t) \int_{-\infty}^{0} y_j(t+s) h_{ij}(s)\,ds \qquad (3.13) \\ &+ c_i(t) \int_{-t}^{0} \big(\int_{t+v}^{t} y_i'(u)\,du\big) h_i(v)\,dv - c_i(t) \int_{-\infty}^{-t} y_i(t+s) h_i(s)\,ds \end{aligned}$$

and

$$\int_{-t}^{0} \big(\int_{t+v}^{t} y_i'(u)\,du\big) h_i(v)\,dv = \int_{-t}^{0} \big(\int_{t+v}^{t} W_{i3}(u)\,du\big) h_i(v)\,dv$$

with

$$\begin{aligned} W_{i3}(u) &= r_i(u) B_i(y_i(u)) \qquad (3.14) \\ &- c_i(u) \int_{-\infty}^{0} [b_i(x_i^{(1)}(u)) y_i(u+s) + x_i^{(2)}(u+s) B_i(y_i(u))] h_i(s)\,ds \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}} b_{ijl}(u)[b_i(x_i^{(1)}(u)) y_j(u - \tau_{ijl}) + x_j^{(2)}(u - \tau_{ijl}) B_i(y_i(u))] \\ &+ \sum_{j=1}^{n} \int_{-\tau_{ij}}^{0} [b_i(x_i^{(1)}(u)) y_j(u+s) + x_j^{(2)}(u+s) B_i(y_i(u))] k_{ij}(u,s)\,ds \end{aligned}$$

$$+\sum_{j=1}^{n} d_{ij}(u)\int_{-\infty}^{0}[b_i(x_i^{(1)}(u))y_j(u+s)+x_j^{(2)}(u+s)B_i(y_i(u))]h_{ij}(s)\,ds.$$

Note that

$$\lim_{t\to\infty}\int_{-\infty}^{-t} h_i(s)\,ds = 0$$

and

$$\Big|c_i(t)\int_{-\infty}^{-t} y_i(t+s)h_i(s)\,ds\Big| \le (C_i+\epsilon_1)\sup_{s\in(-\infty,0]}|y_i(s)|\int_{-\infty}^{-t}h_i(s)\,ds \overset{\text{def}}{=} J_i(t).$$

Define the V_{i1} as in (2.8); we have from (3.13) and (3.14) that, for all large t,

$$\begin{aligned}\frac{d}{dt}V_{i1}(y_i(t)) \le{}& -(c_i-\epsilon_1)(1-\epsilon_1)|y_i(t)| + \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}(B_{ijl}+\epsilon_1)|y_j(t-\tau_{ijl})|\\ &+\sum_{j=1}^{n}\int_{-\tau_{ij}}^{0}|y_j(t+s)|(K_{ij}(s)+\frac{\epsilon_1}{1+\tau_{ij}})\,ds\\ &+\sum_{j=1}^{n}(D_{ij}+\epsilon_1)\int_{-\infty}^{0}|y_j(t+s)||h_{ij}(s)|\,ds \qquad (3.15)\\ &+(C_i+\epsilon_1)\int_{-\infty}^{0}\Big(\int_{t+v}^{t}|W_{i3}(u)\,du\Big)h_i(v)\,dv + J_i(t).\end{aligned}$$

From (H_3), we know that, for all large t, $x_i^{(1)}(t), x_i^{(2)}(t) \le \Delta_i, b_i(x_i^{(1)}(t)) \le b_i(\Delta_i)$ and $|B_i(y_i(t))| \le B_i|y_i(t)|\,(i=1,\dots,n)$. Then, using ($H_7$), one can verify that, for all large t,

$$\begin{aligned}|W_{i3}(t)| \le{}&(R_i+\epsilon_1)B_i|y_i(t)|\\ &+(C_i+\epsilon_1)\int_{-\infty}^{0}[b_i(\Delta_i)|y_i(t+s)|+\Delta_iB_i|y_i(t)|]h_i(s)\,ds\\ &+\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}(B_{ijl}+\epsilon_1)[b_i(\Delta_i)|y_j(t-\tau_{ijl})|+\Delta_jB_i|y_i(t)|]\\ &+\sum_{j=1}^{n}\int_{-\tau_{ij}}^{0}[b_i(\Delta_i)|y_j(t+s)|+\Delta_jB_i|y_i(t)|]\big(K_{ij}(s)+\frac{\epsilon_1}{1+\tau_{ij}}\big)\,ds\\ &+\sum_{j=1}^{n}(D_{ij}+\epsilon_1)\int_{-\infty}^{0}[b_i(\Delta_i)|y_j(t+s)|+\Delta_jB_i|y_i(t)|]|h_{ij}(s)|\,ds.\end{aligned}$$

Let $V_i(y)(t) = (V_{i1} + V_{i5})(y)(t)$ with

$$\begin{aligned} V_{i5}(y(t)) = [1 + (C_i + \epsilon_1)\sigma_i b_i(\Delta_i)]\big\{ \sum_{j=1}^{n} \sum_{l=1}^{l_{ij}} (B_{ijl} + \epsilon_1) \int_{t-\tau_{ijl}}^{t} |y_j(u)| \, du \\ + \sum_{j=1}^{n} \int_{-\tau_{ij}}^{0} \big(\int_{t+s}^{t} |y_j(u)| \, du \big) (K_{ij}(s) + \frac{\epsilon_1}{1+\tau_{ij}}) \, ds \\ + \sum_{j=1}^{n} (D_{ij} + \epsilon_1) \int_{-\infty}^{0} \big(\int_{t+s}^{t} |y_j(u)| \, du \big) |h_{ij}(s)| \, ds \big\} \\ + (C_i + \epsilon_1)^2 \sigma_i b_i(\Delta) \int_{-\infty}^{0} \big(\int_{t+s}^{t} |y_i(u)| \big) h_i(s) \, ds \\ + (C_i + \epsilon_1) \int_{-\infty}^{0} \big(\int_{t+v}^{t} \int_{w}^{t} |W_{i3}(u)| \, du \, dw \big) h_i(v) \, dv. \end{aligned} \tag{3.16}$$

Then we have from (3.16) that, for all large t,

$$\begin{aligned} \frac{d}{dt} V_i(y)(t) \le \big\{ -(c_i - \epsilon_1)(1-\epsilon_1) + \sum_{l=1}^{l_{ii}} (B_{iil} + \epsilon_1) + (K_{ii} + \epsilon_1) + (D_{ii} + \epsilon_1) \\ + (C_i + \epsilon_1)\sigma_i \big[(R_i + \epsilon_1) B_i + b_i(\Delta_i) \big((C_i + \epsilon_1) \\ + \sum_{l=1}^{l_{ii}} (B_{iil} + \epsilon_1) + (K_{ii} + \epsilon_1) + (D_{ii} + \epsilon_1) \big) + B_i \big(\Delta_i (C_i + \epsilon_1) \\ + \sum_{j=1}^{n} \Delta_j \sum_{l=1}^{l_{ij}} [(B_{ijl} + \epsilon_1) + (K_{ij} + \epsilon_1) + (D_{ij} + \epsilon_1)] \big) \big] \big\} |y_i(t)| \\ + \sum_{j=1, j\neq i}^{n} [1 + (C_i + \epsilon_1)\sigma_i b_i(\Delta_i)] \big[\sum_{l=1}^{l_{ij}} [(B_{iil} + \epsilon_1) + (K_{ij} + \epsilon_1) \\ + (D_{ij} + \epsilon_1)] \big] |y_j(t)| + J_i(t). \end{aligned} \tag{3.17}$$

Note that

$$\int_0^t \int_{-\infty}^{-t} h_i(u) \, du \, ds \le \int_{-\infty}^{0} |s| h_i(s) \, ds < \infty.$$

Then, from (3.17), one can use (H_8) and the same argument as in Theorems 2.1 and 3.1 to show that the system (1.1) is globally asymptotically stable. □

As indicated in Remark 3.2, condition (H_7) in Theorem 3.4 can be removed when (1.1) is an almost-periodic system. One can see that condition

(H_8) depends only on the $\sigma_i = \int_{-\infty}^{0} |s| h_i(s)\, ds\, (i = 1, \dots, n)$, which measure the size of the infinite delays. When they are sufficiently small, we have the following corollary.

Corollary 3.5. *If condition* (H_8) *in Theorem* 3.4 *is replaced by the condition that the matrix* $\tilde{Q} = (\tilde{q})_{n\times n}$ *with*

$$\tilde{q}_{ii} = c_i - \sum_{l=1}^{l_{ii}} B_{iil} - K_{ii} - D_{ii}, \quad \tilde{q}_{ij} = -\big[\sum_{l=1}^{l_{ij}} B_{ijl} + K_{ij} + D_{ij}\big] \quad (i \neq j)$$

is an M-matrix, then the conclusion of Theorem 3.4 *remains true provided delays* $\sigma_i\,(i = 1, \dots, n)$ *are sufficiently small.*

4. Discussion. By constructing suitable Lyapunov functionals in this paper, we are able to generalize the results in Kuang [11] to the pure-delay-type system (1.1) and establish the global asymptotic stability of (1.1). Consequently, we give an affirmative answer to the two open problems in Kuang [11]. As in Kuang [11], we can also apply our results to some nonautonomous pure-delay-type Lotka–Volterra competition and predator-prey systems. Our conditions are different from those obtained in Kuang [11] on the no-pure-delay-type systems. In Kuang [11], the conditions are generally delay independent, which characterizes the systems of no-pure-delay type, while for the pure-delay-type systems, the conditions are delay dependent. The size of the allowed delays to maintain the global asymptotic stability of the systems relies on the estimates of the ultimate upper bounds of the solutions. For Lotka–Volterra competition and prey-predator systems of pure-delay type, some estimates on the ultimate upper bound of the solution can be done explicitly; see Gopalsamy [2], Gopalsamy and He [3], He [4, 5] and Kuang [10]. However, for Lotka–Volterra cooperation systems of pure-delay type, proving the uniform persistence and finding the ultimate upper bounds are still unsolved problems. With the estimation on the upper and lower bounds of the solutions of the systems, we could locate the attractive region of the solutions and, when the systems are periodic or almost periodic, indicate the region for the attractive periodic and almost-periodic solutions. We leave this as our further study.

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