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EXPONENTIAL ASYMPTOTIC STABILITY IN LINEAR DELAY-DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS*

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(Submitted by: Roger Nussbaum)

Dedicated to Professor Junji Kato on his 60th birthday

Abstract. In this paper we give some new necessary and sufficient conditions under which the zero solution of the linear delay-differential equations with variable coefficients

$$x'(t) = A(t)x(t-\tau) \tag{L}$$

is exponentially asymptotically stable. For example, in the case

$$A(t) = -\rho(t)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $\rho(t) > 0$, $\lim_{t\to\infty}\int_{t-\tau}^{t}\rho(s)\,ds = q > 0$ and $|\theta| < \frac{\pi}{2}$, the zero solution of (L) is exponentially asymptotically stable if and only if $q < \frac{\pi}{2} - |\theta|$.

1. Introduction. The asymptotic stability problem for (L) has not been solved completely even for a scalar linear delay-differential equation

$$x'(t) = -a(t)x(t-\tau). \tag{1.1}$$

In [7] and [8], it was shown that the well-known $\pi/2$ stability condition for a scalar equation with a constant coefficient is not generally valid for (1.1); instead, a $3/2$ stability condition is best possible for (1.1). At first we intended to obtain some necessary and sufficient asymptotic stability condition for (1.1) although attaching some restricted assumption. After a precise, elegant result (Corollary 4.1) was obtained for (1.1), we tried to extend our method to the 2-dimensional linear delay-differential equations with variable coefficients and obtained some necessary and sufficient asymptotic stability conditions.

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Recently, in [4], the first author showed that the zero solution of the linear delay-differential equation with constant coefficients,

$$x'(t) = -\rho R(\theta) x(t-\tau),$$

where ρ is a constant and $R(\theta)$ represents a 2×2 matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $|\theta| < \frac{\pi}{2}$, is exponentially asymptotically stable if and only if

$$0 < \rho\tau < \frac{\pi}{2} - |\theta|. \tag{1.2}$$

In this paper we mostly consider the linear delay-differential equations

$$x'(t) = -\rho(t) R(\theta) x(t-\tau), \tag{1.3}$$

where $\rho(t) > 0$ is a continuous function. In our investigations, we study the following two cases:

(I) The case in which there exists $\lim_{t\to\infty} \int_{t-\tau}^{t} \rho(s)\, ds$.

(II) The case in which there doesn't exist $\lim_{t\to\infty} \int_{t-\tau}^{t} \rho(s)\, ds$.

For the case (I), we have

Theorem A. *Suppose that there exists a positive constant q such that*

$$q \equiv \lim_{t\to\infty} \int_{t-\tau}^{t} \rho(s)\, ds.$$

Then the zero solution of (1.3) *is exponentially asymptotically stable if and only if*

$$q < \frac{\pi}{2} - |\theta|. \tag{1.4}$$

If $\rho(t)$ is a constant number, the condition (1.4) coincides with (1.2). For the case (II), we have

Theorem B. *Suppose*

(i) $\limsup_{t\to\infty} \int_{t-\tau}^{t} \rho(s)\, ds < \cos\theta$, *and*

(ii) $\liminf_{t\to\infty} \int_{t}^{t+\Delta} \rho(s)\, ds > 0$ *for some* $\Delta > 0$

are satisfied. Then the zero solution of (1.3) *is exponentially asymptotically stable.*

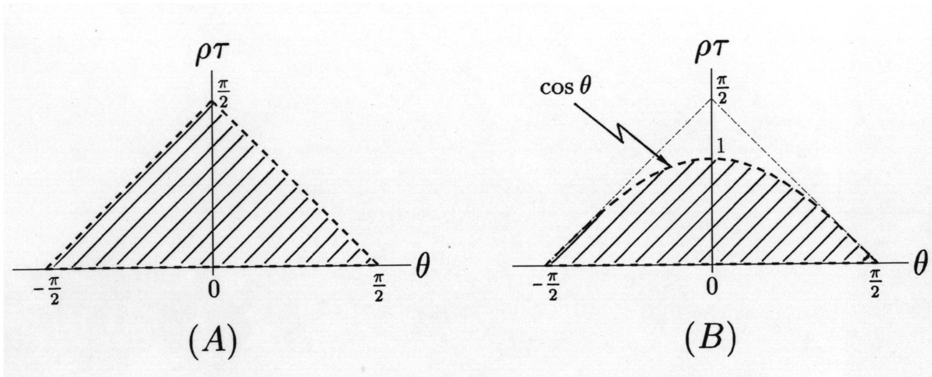

Figure 1.

Theorem C. *Suppose*

(i) $\limsup_{t\to\infty}\int_{t-\tau}^{t}\rho(s)\,ds<\cos\theta$, *and*

(ii) $\int_0^\infty \rho(s)\,ds=\infty$

are satisfied. Then the zero solution of (1.3) *is uniformly stable and attractive.*

Let $\rho(t)\equiv\rho$. Then Figures 1-(A) and 1-(B) show the regions given by the condition (1.4) in Theorem A and the conditions (i) and (ii) in Theorem B respectively.

For the case (II), we have not found the best possible condition which corresponds to a 3/2 stability condition for (1.1). We feel this problem is fairly difficult.

2. Notations and definitions. We consider the functional differential equations with bounded delay

$$x'(t)=F(t,x_t), \tag{2.1}$$

where $F:\mathbf{R}^+\times C\to\mathbf{R}^n$ is continuous and $F(t,0)=0$ for all $t\in\mathbf{R}^+$. Here $\mathbf{R}^n$ is the Euclidean n-space with a suitable norm $|\cdot|$, $\mathbf{R}^+$ is the set of nonnegative reals and C is the space of continuous functions that map the interval $[-r,0]$ for some $r>0$ into $\mathbf{R}^n$ with the norm $\|\phi\|=\sup_{-r\le s\le 0}|\phi(s)|$. For any continuous mapping $x:[-r,0]\to\mathbf{R}^n$ and $t\ge 0$, $x_t\in C$ is defined by $x_t(s)=x(t+s)$ for $-r\le s\le 0$. For $t_0\in\mathbf{R}^+$ and $\phi\in C$, we say $x(t_0,\phi)$ is a solution of (2.1) through (t_0,ϕ), if $x(t_0,\phi):[t_0-r,t_0+\alpha)\to\mathbf{R}^n$, which is continuous for some $\alpha>0$, satisfies (2.1) for $t\in[t_0,t_0+\alpha)$ and $x_{t_0}(t_0,\phi)=\phi$. Throughout this paper we denote $x(t_0,\phi)(t)$ by $x(t,t_0,\phi)$. From $F(t,0)\equiv 0$, $x(t)\equiv 0$ is a solution of (2.1), and it is called the zero solution of (2.1).

Definition 2.1. The zero solution of (2.1) is stable (S), if for each $\varepsilon > 0$ and each $t_0 \geq 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\| \phi \| < \delta$ implies

$$|x(t, t_0, \phi)| \leq \varepsilon \quad \text{for all } t \geq t_0 \geq 0.$$

The zero solution of (2.1) is uniformly stable (US), if the above δ can be chosen independent of t_0.

Definition 2.2. The zero solution of (2.1) is attractive (Att), if for each $t_0 \geq 0$, there exists $\delta_0 = \delta_0(t_0) > 0$ such that $\| \phi \| < \delta_0$ implies that

$$x(t, t_0, \phi) \to 0 \;\text{ as }\; t \to \infty.$$

The zero solution of (2.1) is asymptotically stable (AS), if it is stable and attractive.

Definition 2.3. The zero solution of (2.1) is uniformly attractive ($U.Att$), if there exists $\delta_0 > 0$ and for each $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\| \phi \| < \delta_0$ imply

$$|x(t, t_0, \phi)| \leq \varepsilon \quad \text{for all } \; t \geq t_0 + T.$$

The zero solution of (2.1) is uniformly asymptotically stable (UAS), if it is uniformly stable and uniformly attractive.

Definition 2.4. The zero solution of (2.1) is exponentially asymptotically stable ($Ex.AS$), if there exists $\lambda > 0$, and for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\| \phi \| < \delta$ imply

$$|x(t, t_0, \phi)| \leq \varepsilon e^{-\lambda(t - t_0)} \quad \text{for all } t \geq t_0 \geq 0.$$

3. Preliminaries. Consider the following n-dimensional linear equations

$$x'(t) = Ax(t - \sigma(t)) \tag{3.1}$$

and

$$y'(t) = Ay(t - \tau), \tag{3.2}$$

where A is a constant $n \times n$ matrix, τ is a positive constant and a continuous function $\sigma(t)$ satisfies $0 \leq \sigma(t) \leq h$ for some h on $[0, \infty)$.

Theorem 3.1. *Suppose*

$$\lim_{t \to \infty} \int_t^{t+1} |\sigma(s) - \tau| \, ds = 0$$

is satisfied. Then the zero solution of (3.1) *is* $Ex.AS$ *if and only if the zero solution of* (3.2) *is* $Ex.AS$.

Proof. Let r be $\max\{h, \tau\}$. Suppose the zero solution of (3.1) is $Ex.AS$; that is, there exists $\lambda > 0$ and for each $\varepsilon > 0$, there exists $\tilde{\delta} > 0$ with $\tilde{\delta} \leq \frac{\varepsilon}{3}$ such that any $t_1 \geq 0$ and any $\| \phi \| < \tilde{\delta}$ imply

$$|x(t, t_1, \phi)| \leq \frac{\varepsilon}{3} e^{-\lambda(t-t_1)} \quad \text{for } t \geq t_1 \geq 0. \tag{3.3}$$

Letting $W(t,s)$ represent the fundamental matrix solution of (3.1), from Lemma 5.3 in [3, pp. 185–186], there exists $K > 0$ such that

$$|W(t,s)| \leq K e^{-\lambda(t-s)} \quad \text{for } t \geq s \geq 0. \tag{3.4}$$

It is well known that the condition of this theorem can be rewritten as

$$\lim_{t\to\infty} e^{-\lambda t} \int_0^t e^{\lambda s} |\sigma(s) - \tau| \, ds = 0.$$

Then using K defined in (3.4), we find that there exists $T_1 \geq r$ such that

$$K|A|^2 e^{-\lambda t} \int_0^t e^{\lambda s} |\sigma(s) - \tau| \, ds \leq \frac{1}{2} \quad \text{for} \quad t \geq T_1. \tag{3.5}$$

Here, let $y(t) = y(t, t_0, \psi)$ be any solution of (3.2) with initial time $t_0 \geq 0$ and initial function $\psi \in C$. Then, by comparison theorem,

$$|y(t)| \leq \| \psi \| \, e^{\int_{t_0}^t |A| \, ds} \quad \text{for } t \geq t_0$$

holds. Letting $\delta = \frac{\tilde{\delta}}{e^{|A|T_1}}$ and $t_1 = t_0 + T_1$, if $\| \psi \| < \delta$, then we have

$$|y(t)| = |y(t, t_0, \psi)| \leq \| \psi \| \, e^{\int_{t_0}^t |A| \, ds} = \| \psi \| \, e^{|A|T_1} < \tilde{\delta} \text{ for } t_0 \leq t \leq t_1;$$

that is,

$$\max_{t_0 - r \leq s \leq t_1} |y(s)| < \tilde{\delta}. \tag{3.6}$$

Next, let $x(t)$ be a solution of (3.1) through (t_1, y_{t_1}). If we define $z(t) = y(t) - x(t)$, it follows that $z(t) = 0$ for $t \in [t_1 - r, t_1]$ and

$$\begin{aligned} z'(t) - Az(t - \sigma(t)) &= y'(t) - x'(t) - A(y(t - \sigma(t)) - x(t - \sigma(t))) \\ &= A(y(t - \tau) - y(t - \sigma(t))). \end{aligned}$$

Since $W(t,s)$ is the fundamental matrix solution of the equation

$$z'(t) = Az(t - \sigma(t)),$$

using the variation-of-constants formula (cf. [3, page 173]) to above equation, we have

$$z(t) = \int_{t_1}^{t} W(t,s)A(y(s-\tau) - y(s-\sigma(s)))\, ds \quad \text{for } t \geq t_1,$$

and by (3.4) we also have

$$|z(t)| \leq Ke^{-\lambda t} \int_{t_1}^{t} e^{\lambda s} |A||y(s-\tau) - y(s-\sigma(s))|\, ds. \tag{3.7}$$

Noticing that $s-\tau$ and $s-\sigma(s)$ are in the interval $[t_0, t]$ for $s \in [t_1, t]$, from (3.2), we have

$$|y(s-\tau) - y(s-\sigma(s))| \leq |\sigma(s) - \tau||A| \max_{\eta \in J(s)} |y(\eta - \tau)|,$$

where $J(s)$ is the interval between $s - \sigma(s)$ and $s - \tau$, which is included in the interval $[t_0, t]$ for all $s \in [t_1, t]$. Therefore using the above inequality, we can rewrite estimate (3.7) as

$$\begin{aligned} |z(t)| &\leq Ke^{-\lambda t} \int_{t_1}^{t} e^{\lambda s} |A||\sigma(s) - \tau||A| \max_{\eta \in J(s)} |y(\eta - \tau)|\, ds \\ &\leq K|A|^2 \max_{t_0 - r \leq s \leq t} |y(s)| e^{-\lambda t} \int_{t_1}^{t} e^{\lambda s} |\sigma(s) - \tau|\, ds. \end{aligned} \tag{3.8}$$

Since $\max_{t_0 - r \leq s \leq t_1} |y(s)| < \frac{\varepsilon}{3}$ by (3.6), and from (3.5), we have for $t \geq t_1$

$$\begin{aligned} |y(t)| &< |x(t)| + K|A|^2 (\frac{\varepsilon}{3} + \max_{t_1 \leq s \leq t} |y(s)|) e^{-\lambda t} \int_{t_1}^{t} e^{\lambda s} |\sigma(s) - \tau|\, ds \\ &\leq |x(t)| + \frac{1}{2} (\frac{\varepsilon}{3} + \max_{t_1 \leq s \leq t} |y(s)|). \end{aligned}$$

Here, we can choose $t_2 (t_1 \leq t_2 \leq t)$ with $|y(t_2)| = \max_{t_1 \leq s \leq t} |y(s)|$ and

$$|y(t_2)| < |x(t_2)| + \frac{1}{2} (\frac{\varepsilon}{3} + \max_{t_1 \leq s \leq t_2} |y(s)|) < \max_{t_1 \leq s \leq t} |x(s)| + \frac{1}{2} (\frac{\varepsilon}{3} + \max_{t_1 \leq s \leq t} |y(s)|);$$

then we have

$$\max_{t_1 \le s \le t} |y(s)| < 2 \max_{t_1 \le s \le t} |x(s)| + \frac{\varepsilon}{3}.$$

If $\| \psi \| < \delta$ we have $\| x_{t_1} \| = \| y_{t_1} \| < \tilde{\delta}$ from (3.6), so under the assumption that the zero solution of (3.1) is $Ex.AS$, from (3.3), we have

$$|x(t)| \le \frac{\varepsilon}{3} e^{-\lambda(t-t_1)} \le \frac{\varepsilon}{3} \quad \text{for } t \ge t_1.$$

Consequently,

$$|y(t)| < \frac{2}{3}\varepsilon + \frac{\varepsilon}{3} = \varepsilon \quad \text{for } t \ge t_1$$

holds. Therefore, from (3.6) and this, the zero solution of (3.2) is US. So, there exists $\delta_0 > 0$ such that $|y(t, t_0, \psi)| < 1$ for any $t \ge t_0$ and $\| \psi \| < \delta_0$. Using this, we have

$$|y(t)| < |x(t)| + K|A|^2 e^{-\lambda t} \int_0^t e^{\lambda s} |\sigma(s) - \tau| \, ds \quad \text{for } t \ge t_0$$

which corresponds to (3.8). Here for any $\varepsilon > 0$ we can choose $T \ge r$ such that

$$K|A|^2 e^{-\lambda t} \int_0^t e^{\lambda s} |\sigma(s) - \tau| \, ds < \frac{2}{3}\varepsilon \quad \text{for } t \ge T.$$

Then from (3.3) and the above two inequalities,

$$|y(t, t_0, \psi)| < \varepsilon \quad \text{for } t \ge t_0 + T$$

holds; that is, the zero solution of (3.2) is $U.Att$. From the above mentioned, the zero solution of (3.2) is UAS and is $Ex.AS$ (cf. [3, page 185]). The proof of the sufficiency is quite similar to that of the necessity. The proof of Theorem 3.1 is completed.

Remark 3.1. In [2], Györi considers similar problems under the assumption that

$$|A| \limsup_{t \to \infty} \sigma(t) < \frac{1}{e} \quad \text{and} \quad \int_0^\infty |\sigma(t) - \tau| \, dt < \infty.$$

Remark 3.2. In [6], a similar result to Theorem 3.1 for a scalar equation is obtained under the assumption that $\lim_{t \to \infty} \sigma(t) = \tau$. Our theorem is inspired by this result.

Remark 3.3. In [4], a necessary and sufficient condition is given explicitly for the zero solution of (3.2) to be $Ex.AS$.

4. Main theorems. We consider the 2-dimensional linear delay differential equations with variable coefficients

$$x'(t) = -\rho(t)R(\theta)x(t-\tau), \tag{4.1}$$

where $\tau > 0$ is a constant, $\rho(t) > 0$ is continuous on $[-\tau, \infty)$ and $R(\theta)$ represents $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $|\theta| < \frac{\pi}{2}$.

Our main result is stated in the following:

Theorem 4.1. *Suppose that there exists a constant q such that*

$$q \equiv \lim_{t\to\infty} \int_{t-\tau}^{t} \rho(s)\,ds.$$

Then the zero solution of (4.1) *is Ex.AS if* $0 < q < \frac{\pi}{2} - |\theta|$ *and only if* $0 \le q < \frac{\pi}{2} - |\theta|$.

We notice that Theorem A is obtained as a corollary of Theorem 4.1 in the case $q > 0$.

For the proof of Theorem 4.1, we need the following proposition and lemmas. Consider 2-dimensional linear delay differential equations with constant coefficients

$$x'(t) = -\rho R(\theta)x(t-\tau), \tag{4.2}$$

where ρ and $\tau > 0$ are constant numbers.

Proposition 1. *The zero solution of* (4.2) *is Ex.AS if and only if*

$$0 < \rho\tau < \frac{\pi}{2} - |\theta|.$$

This proposition is shown in [4], so the proof is omitted.

Next consider the 2-dimensional linear ordinary differential equations

$$y' = -\rho(t)R(\theta)y, \tag{4.3}$$

where $\rho(t)$ is a positive-valued continuous function. Then we have the first lemma.

Lemma 4.1. *Suppose*

$$M = \int_{0}^{\infty} \rho(s)\,ds < \infty.$$

Then every solution of (4.3) *except the zero solution converges to some point except zero.*

Proof of Lemma 4.1. Let $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ be a solution of (4.3). Having made the change of variables $y_1 = r\cos\phi\ y_2 = r\sin\phi$, we have

$$r(t) = r(t_0)\exp(-\cos\theta\int_{t_0}^{t}\rho(s)\,ds)$$

and

$$\phi(t) = \phi(t_0) - \sin\theta\int_{t_0}^{t}\rho(s)\,ds.$$

So from the condition, if we set $r(t_0) \neq 0$, we have

$$\lim_{t\to\infty} r(t) = r(t_0)e^{-M\cos\theta} \neq 0 \quad \text{and} \quad \lim_{t\to\infty}\phi(t) = \phi(t_0) - M\sin\theta.$$

Then the statement of Lemma 4.1 holds.

We consider again the equation (4.1). We will show the second lemma.

Lemma 4.2. *Suppose*

$$M \equiv \int_0^{\infty}\rho(s)\,ds < \infty.$$

Then the zero solution of (4.1) *is not Att.*

Proof of Lemma 4.2. Suppose, for the sake of contradiction, that the zero solution of (4.1) is *Att*. Then the fundamental matrix solution $X(t,s)$ of (4.1) satisfies that for each fixed s

$$|X(t,s)| \to 0 \quad \text{as } t\to\infty. \tag{4.4}$$

Let $x(t) \equiv x(t,t_0,\phi)$ be a solution of (4.1) with $x(t_0) \neq 0$. Note that $x(t)\to 0$ as $t\to\infty$. Let $y(t)$ be a function which is a solution of (4.3) on $t \geq t_0$ and $y_{t_0} = \phi$. Then $y(t)$ does not go to zero by Lemma 4.1.

Here we define $z(t) \equiv x(t) - y(t)$ for $t \geq t_0$, then

$$\begin{aligned} z'(t) &= -\rho(t)R(\theta)x(t-\tau) + \rho(t)R(\theta)y(t) \\ &= -\rho(t)R(\theta)z(t-\tau) + \rho(t)R(\theta)(y(t) - y(t-\tau)). \end{aligned}$$

By the variation-of-constants formula,

$$z(t) = \int_{t_0}^{t} X(t,u)\rho(u)R(\theta)(y(u) - y(u-\tau))\,du$$

and then

$$|z(t)| \leq \int_{t_0}^{t} |X(t,u)|\rho(u)|y(u) - y(u-\tau)|\, du.$$

Now since $y(t)$ is bounded from Lemma 4.1, we can set $|y(u)-y(u-\tau)| \leq K$. Therefore

$$|z(t)| \leq K \int_{t_0}^{t} |X(t,u)|\rho(u)\, du. \tag{4.5}$$

Now from the definition of $z(t)$, if we can prove $\lim_{t\to\infty} z(t) = 0$, we have $\lim_{t\to\infty} y(t) = 0$ which contradicts Lemma 4.1 and can conclude the proof of this lemma. By the comparison theorem (cf. Lemma 1 in [1]) we have

$$|x(t)| \leq e^{\int_{t_0}^{t} \rho(s)\, ds} \parallel \phi \parallel \leq \parallel \phi \parallel e^{M} \quad \text{for } t \geq t_0.$$

Thus, we may set

$$|X(t,u)| \leq e^{M} \quad \text{for } t \geq u \geq 0. \tag{4.6}$$

From the assumption of Lemma 4.2, for $\varepsilon > 0$ there exists $T_1 > 0$ such that

$$\int_{T_1}^{\infty} \rho(s)\, ds < \frac{\varepsilon}{2Ke^{M}}.$$

Now using the above inequality, we can estimate (4.5) as follows:

$$\begin{aligned} |z(t)| &\leq K \int_{t_0}^{T_1} |X(t,u)|\rho(u)\, du + K \int_{T_1}^{t} |X(t,u)|\rho(u)\, du \\ &\leq K \int_{t_0}^{T_1} |X(t,u)|\rho(u)\, du + Ke^{M} \int_{T_1}^{\infty} \rho(u)\, du \\ &\leq K \int_{t_0}^{T_1} |X(t,u)|\rho(u)\, du + \frac{\varepsilon}{2}. \end{aligned} \tag{4.7}$$

Next let's settle the first term of the right-hand side of (4.7). Let $L = \max_{0\leq u\leq T_1} \rho(u)$. From (4.4), for any $u \in [0, T_1]$ and any $\varepsilon > 0$ there exists $T_2 = T_2(u,\varepsilon) > T_1$ and we have

$$|X(t,u)| < \frac{\varepsilon}{2KLT_1} \quad \text{for } t \geq T_2. \tag{4.8}$$

Hence, if T_2 is independent of $u \in [0, T_1]$, we obtain

$$\begin{aligned} K \int_{t_0}^{T_1} |X(t,u)|\rho(u)\, du &\leq KL \int_{0}^{T_1} |X(t,u)|\, du \\ &\leq KL \frac{\varepsilon}{2KLT_1} T_1 = \frac{\varepsilon}{2} \quad \text{for } t \geq T_2. \end{aligned}$$

Therefore, considering (4.7), for any $\varepsilon > 0$ we have $|z(t)| < \varepsilon$ for large t; that is, $\lim_{t\to\infty} z(t) = 0$.

Therefore finally we will show that T_2 is independent of $u \in [0, T_1]$. Consequently we have only to show that for $\eta > 0$ there exists $T > T_1$ such that $u \in [0, T_1]$ and $t \geq T$ imply $|X(t, u)| < \eta$.

Suppose, for the sake of contradiction, that there exist $\eta_0 > 0$, a sequence $\{u_n\} \subset [0, T_1]$ and $\{t_n\} \uparrow \infty$ such that

$$|X(t_n, u_n)| \geq \eta_0. \tag{4.9}$$

But now, by the Bolzano-Weierstrass Theorem applied to $\{u_n\}$, there exists a convergent subsequence $\{u_{n_k}\}$. Say, we can denote again $\{u_{n_k}\}$ and $\{t_{n_k}\}$ by $\{u_n\}$ and $\{t_n\}$ with $\lim_{n\to\infty} u_n = u_* \in [0, T_1]$. (4.9) also holds for these subsequences. It holds that

$$\begin{aligned} &X(t, u_n) - X(t, u_*) \\ &= -\int_{u_n}^{t} \rho(s)R(\theta)X(s-\tau, u_n)\,ds + \int_{u_*}^{t} \rho(s)R(\theta)X(s-\tau, u_*)\,ds \\ &= \int_{u_*}^{t} \rho(s)R(\theta)\{X(s-\tau, u_*) - X(s-\tau, u_n)\}\,ds \\ &\quad + \int_{u_n}^{u_*} \rho(s)R(\theta)X(s-\tau, u_n)\,ds. \end{aligned}$$

Then for $t \geq \max(u_*, u_n)$,

$$\begin{aligned} &|X(t, u_n) - X(t, u_*)| \\ &\leq \int_{u_*}^{t} \rho(s)|X(s-\tau, u_n) - X(s-\tau, u_*)|\,ds + \left|\int_{u_n}^{u_*} \rho(s)|X(s-\tau, u_n)|\,ds\right| \\ &\leq \int_{u_*-\tau}^{t-\tau} \rho(s+\tau)|X(s, u_n) - X(s, u_*)|\,ds + e^M\left|\int_{u_n}^{u_*} \rho(s)\,ds\right| \\ &\leq \int_{u_*-\tau}^{t} \rho(s+\tau)|X(s, u_n) - X(s, u_*)|\,ds + e^M\left|\int_{u_n}^{u_*} \rho(s)\,ds\right|. \end{aligned}$$

By Gronwall's inequality,

$$|X(t, u_n) - X(t, u_*)| \leq e^M\left|\int_{u_n}^{u_*} \rho(s)\,ds\right| \exp\left(\int_{u_*-\tau}^{t} \rho(s+\tau)\,ds\right)$$

for $t \geq \max(u_*, u_n)$, so

$$\begin{aligned} |X(t, u_n)| &\leq |X(t, u_*)| + e^M\left|\int_{u_n}^{u_*} \rho(s)\,ds\right| \exp\left(\int_{0}^{\infty} \rho(s)\,ds\right) \\ &= |X(t, u_*)| + e^{2M}\left|\int_{u_n}^{u_*} \rho(s)\,ds\right| \quad \text{for } t \geq \max(u_*, u_n). \end{aligned}$$

By (4.8), there exists $T_2(u_*, \frac{\eta_0}{2}) > T_1$ and for any $t > T_2(u_*, \frac{\eta_0}{2})$ the first term in the right-hand side of the above inequality satisfies that

$$|X(t, u_*)| < \frac{\eta_0}{2}.$$

Moreover, applying $\lim_{n\to\infty} u_n = u_*$ to the second term, there exists $N(\frac{\eta_0}{2}) > 0$, and for any $n > N(\frac{\eta_0}{2})$

$$e^{2M}|\int_{u_n}^{u_*} \rho(s)\,ds| < \frac{\eta_0}{2} \quad \text{and} \quad |u_n - u_*| < 1.$$

That is,

$$|X(t_n, u_n)| < \frac{\eta_0}{2} + \frac{\eta_0}{2} = \eta_0 \quad \text{for } t_n > \max\{T_2(u_*, \frac{\eta_0}{2}), t_{N(\frac{\eta_0}{2})}, u_* + 1\}$$

which contradicts (4.9). This completes the proof of Lemma 4.2.

Now, we are in a position to prove Theorem 4.1.

Proof of Theorem 4.1. Notice that $\int_0^\infty \rho(s)\,ds = \infty$ holds in both proofs of necessity and sufficiency. In fact, if we suppose the zero solution of (4.1) is $Ex.AS$, because it is also Att, then using Lemma 4.2 we have

$$\int_0^\infty \rho(s)\,ds = \infty.$$

Needless to say, if we suppose $q > 0$, then the above equation is also satisfied. Therefore we can consider the transformation

$$u = \sigma(t) \equiv \int_0^t \rho(s)\,ds \quad \text{for } t \geq -\tau,$$

in both proofs of necessity and sufficiency.

And we find that $\lim_{t\to\infty} u(t) = \infty$, σ^{-1} exists, $\lim_{u\to\infty} \sigma^{-1}(u) = \infty$ and

$$\sigma(t-\tau) = \int_0^{t-\tau} \rho(s)\,ds = \int_0^t \rho(s)\,ds - \int_{t-\tau}^t \rho(s)\,ds = u - \int_{\sigma^{-1}(u)-\tau}^{\sigma^{-1}(u)} \rho(s)\,ds.$$

Thus, we have $t - \tau = \sigma^{-1}(u - \int_{\sigma^{-1}(u)-\tau}^{\sigma^{-1}(u)} \rho(s)\,ds)$. Then the transformation

$$z(u) = x(\sigma^{-1}(u))$$

reduces (4.1) to

$$z'(u) + R(\theta)z(u - \int_{\sigma^{-1}(u)-\tau}^{\sigma^{-1}(u)} \rho(s)\, ds) = 0. \tag{4.10}$$

In view of the assumption in Theorem 4.1, there exists a positive constant h such that $\int_{t-\tau}^{t} \rho(s)\, ds \leq h$ for $t \geq 0$ and

$$\lim_{t\to\infty} \int_{t}^{t+1} | \int_{\sigma^{-1}(s)-\tau}^{\sigma^{-1}(s)} \rho(v) dv - q|\, ds = 0,$$

so the hypotheses of Theorem 3.1 are all satisfied for equation (4.10). And therefore considering the equation

$$y'(u) + R(\theta)y(u - q) = 0 \tag{4.11}$$

which corresponds to (3.2), by Theorem 3.1 we realize that the zero solution of (4.10) is $Ex.AS$ if and only if the zero solution of (4.11) is $Ex.AS$. In the case $q = 0$, the zero solution of (4.11) is $Ex.AS$, since $\cos\theta > 0$. And by Proposition 1, the zero solution of (4.11) is $Ex.AS$ if and only if $q < \frac{\pi}{2} - |\theta|$.

Now we use the following claim which will be proved later.

Claim 1. If there exists q such that $q \equiv \lim_{t\to\infty} \int_{t-\tau}^{t} \rho(s)\, ds$, then there exists $\alpha > 0$ such that

$$u - u_0 \leq \alpha(t - t_0 + \tau),$$

and, still more, if $q > 0$ then there exists $\beta > 0$ such that

$$u - u_0 \geq \beta(t - t_0 - \tau),$$

where $u_0 = \sigma(t_0)$.

First suppose that the zero solution of (4.1) is $Ex.AS$; that is, there exists $\lambda_1 > 0$ and for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\| \phi \| < \delta$ imply

$$|x(t, t_0, \phi)| < \varepsilon e^{-\lambda_1(t-t_0)} \quad \text{for } t \geq t_0. \tag{4.12}$$

For any initial time u_0 of (4.10) there exists a one-to-one correspondence $t_0 = \sigma^{-1}(u_0) \geq 0$, and $t_0 - \tau \leq t \leq t_0$ is equivalent to

$$u_0 - \int_{\sigma^{-1}(u_0)-\tau}^{\sigma^{-1}(u_0)} \rho(s)\, ds \leq u \leq u_0.$$

For any initial function ψ of (4.10), we set $\phi(t) = \psi(\sigma(t))$ on $[-\tau, 0]$. It is clear that if $|\psi(u)| < \delta$ on $[-h, 0]$, then $|\phi(t)| < \delta$ on $[-\tau, 0]$. Hence, setting $\lambda_2 = \frac{\lambda_1}{\alpha}$,

$$\begin{aligned}|z(u, u_0, \psi)| &= |x(t, t_0, \phi)| < \varepsilon e^{-\lambda_1(t-t_0)} \\ &= \varepsilon e^{-\lambda_1(t-t_0+\tau)} e^{\lambda_1\tau} \le \varepsilon e^{-\frac{\lambda_1}{\alpha}(u-u_0)} e^{\lambda_1\tau} \equiv \varepsilon e^{-\lambda_2(u-u_0)} e^{\lambda_1\tau};\end{aligned}$$

that is, there exists $\lambda_2 > 0$ and for each $\varepsilon > 0$, there exists $\tilde{\delta} = \delta(\varepsilon e^{\lambda_1\tau}) > 0$ such that $u_0 \ge 0$ and $\| \psi \| < \tilde{\delta}$ imply

$$|z(u, u_0, \psi)| < \varepsilon e^{-\lambda_2(u-u_0)} \quad \text{for } u \ge u_0.$$

Then the zero solution of (4.10) is $Ex.AS$. Thus we have $0 \le q < \frac{\pi}{2} - |\theta|$.

Next, suppose that the inequality $0 < q < \frac{\pi}{2} - |\theta|$ holds. Then the zero solution of (4.10) is $Ex.AS$; that is, the above inequality $|z(u, u_0, \psi)| < \varepsilon e^{-\lambda_2(u-u_0)}$ holds. In this case, because $q > 0$, we can use the latter part of the above claim. In a similar way to the above proof, for any initial time t_0 there exists a one-to-one correspondence $u_0 = \sigma(t_0)$. And for any initial function $\| \phi \| < \delta$, let $\psi(u) = \phi(\sigma^{-1}(u))$. Then $\| \psi \| < \tilde{\delta}$. Setting $\lambda_1 = \beta\lambda_2$, we have

$$\begin{aligned}|x(t, t_0, \phi)| &= |z(u, u_0, \psi)| < \varepsilon e^{-\lambda_2(u-u_0)} \\ &\le \varepsilon e^{-\beta\lambda_2(t-t_0-\tau)} \equiv \varepsilon e^{-\lambda_1(t-t_0)} e^{\lambda_1\tau},\end{aligned}$$

which shows (4.12). Thus, the zero solution of (4.1) is $Ex.AS$. This completes the proof of Theorem 4.1.

Proof of Claim 1. From the assumption of Claim 1, there exists $h > 0$ such that

$$\int_{t-\tau}^{t} \rho(s)\, ds \le h \quad \text{for all } t \ge 0.$$

For any $a \in \mathbf{R}^+$, $[a]$ represents the maximal integer which doesn't exceed a.

Then, from the definition of u,

$$u - u_0 = \int_{t_0}^{t} \rho(s)\, ds \le ([\frac{t-t_0}{\tau}] + 1)h \le (\frac{t-t_0}{\tau} + 1)h = (t - t_0 + \tau)\frac{h}{\tau},$$

so there exists $\alpha = \frac{h}{\tau} > 0$. Moreover, when $q > 0$, there exist m as $q \ge m > 0$ and $t_2 \ge 0$ as

$$\int_{t-\tau}^{t} \rho(s)\, ds \ge m > 0 \quad \text{for } t \ge t_2.$$

Let $m_1 = \min\{\min_{0\le t\le t_2} \int_{t-\tau}^{t} \rho(s)\,ds, m\} > 0$. Then we have

$$u - u_0 \ge [\frac{t-t_0}{\tau}]m_1 \ge (\frac{t-t_0}{\tau} - 1)m_1 = (t - t_0 - \tau)\frac{m_1}{\tau}$$

and there exists $\beta = \frac{m_1}{\tau} > 0$. The facts mentioned above complete the proof of Claim 1.

Our next result of this section asserts a necessary and sufficient condition for the zero solution of the scalar equation

$$x'(t) = -a(t)x(t-\tau), \tag{4.13}$$

where τ is a positive constant and $a(t)$ is a positive continuous function, to be $Ex.AS$. We have this result as a corollary of Theorem 4.1 by setting $\theta = 0$ in (4.1).

Corollary 4.1. *Suppose there exists a positive constant q such that*

$$q = \lim_{t\to\infty} \int_{t-\tau}^{t} a(s)\,ds;$$

then the zero solution of (4.13) *is $Ex.AS$ if and only if $q < \frac{\pi}{2}$.*

Our final theorem of this section asserts a necessary and sufficient condition for the zero solution of the 2-dimensional equations

$$x'(t) = -A(t)x(t-\tau) \tag{4.14}$$

to be $Ex.AS$. Here τ is a positive constant number and $A(t)$ represents

$$\begin{pmatrix} a_1(t) & b(t) \\ 0 & a_2(t) \end{pmatrix}$$

where $a_1(t)$, $a_2(t)$ are positive continuous and $b(t)$ is a continuous function and bounded.

Corollary 4.2. *Suppose there exist a positive constant q_k such that*

$$q_k = \lim_{t\to\infty} \int_{t-\tau}^{t} a_k(s)\,ds \qquad (k = 1, 2).$$

Then the zero solution of (4.14) *is $Ex.AS$ if and only if $q_k < \frac{\pi}{2}$ $(k = 1, 2)$.*

We can prove this result by use of Corollary 4.1 and the variation-of-constants formula.

5. Appendix. In Section 4, we studied the $Ex.AS$ of the zero solution of (4.1) in the case that $\lim_{t\to\infty} \int_{t-\tau}^{t} \rho(s)\,ds$ exists. In this section we shall study the case $\lim_{t\to\infty} \int_{t-\tau}^{t} \rho(s)\,ds$ does not exist.

Consider again the equations

$$x'(t) = -\rho(t)R(\theta)x(t-\tau). \tag{4.1}$$

Our results are stated in the following:

Theorem 5.1 (Theorem B). *Suppose*

(i) $\limsup_{t\to\infty} \int_{t-\tau}^{t} \rho(s)\,ds < \cos\theta$, *and*

(ii) $\liminf_{t\to\infty} \int_{t}^{t+\Delta} \rho(s)\,ds > 0$ *for some* $\Delta > 0$

are satisfied. Then the zero solution of (4.1) *is* $Ex.AS$.

Theorem 5.2 (Theorem C). *Suppose*

(i) $\limsup_{t\to\infty} \int_{t-\tau}^{t} \rho(s)\,ds < \cos\theta$, *and*

(ii) $\int_{0}^{\infty} \rho(s)\,ds = \infty$

are satisfied. Then the zero solution of (4.1) *is* US *and* Att.

We can obtain these theorems by setting

$$a(t) = \rho(t), \quad A = -\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad B = \frac{1}{2\cos\theta} I,$$

$\mu^2 = \lambda^2 = \frac{1}{2\cos\theta}$ and $|BA^2| = \frac{1}{2\cos\theta}$ in Theorem 3.4 of [5].

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