Differential and Integral Equations Volume 11, Number 3, May 1998, pp. 521–550.

GLOBAL SOBOLEV SOLUTIONS OF QUASILINEAR PARABOLIC EQUATIONS

Kevin McLeod and Albert Milani
Department of Mathematics, University of Wisconsin, Milwaukee, WI

(Submitted by: L.A. Peletier)

Abstract. We prove that the quasilinear parabolic initial-boundary value problem (1.1) below is globally well-posed in a class of high order Sobolev solutions.

1. Introduction.

1.1. In this paper we continue our investigation of the existence of global Sobolev solutions to the quasilinear initial-boundary value parabolic problem

$$u_t - \sum_{i,j=1}^{n} a_{ij}(\nabla u)\partial_i\partial_j u = f(x,t), \quad u(x,0) = u_0(x), \qquad u\big|_{\partial\Omega} = 0, \quad (1.1)$$

where $u = u(x,t) \in \mathbb{R}$, $x \in \Omega \subset \mathbb{R}^n$, Ω a bounded open domain with sufficiently smooth boundary $\partial\Omega$, $t \geq 0$, $u_t = \partial u/\partial t$, $\partial_i u = \partial u/\partial x_i$, and $\nabla = \{\partial_1, \ldots \partial_n\}$.

Under suitable assumptions on f and the coefficients a_{ij}, Ladyzenskaya, Solonnikov and Ural'tseva ([7]–[10]) have established a series of well known global existence, uniqueness and regularity results for classical solutions of (1.1) (see also Krylov [6], and, for equations in divergence form, Guangchang, [2]); these solutions are classical, in the sense that, typically, they take values in some Hölder space $C^{k,\gamma}(\bar{\Omega})$. Here we are interested in solutions that take values in some Sobolev space $H^m(\Omega)$, with m so high as to ensure that such solutions, which we call "Sobolev solutions", are also classical ones, via the imbedding $H^{m+k}(\Omega) \subset C^{2+k,\gamma}(\bar{\Omega})$, valid for $m > \frac{n}{2} + 2$. We propose to show that an analogous global existence result holds for Sobolev solutions of (1.1), corresponding to data f and u_0 that are in suitable Sobolev spaces, with norm of arbitrary size (which is the essential problem; if the data are sufficiently small, the existence of a global Sobolev solution can be established by an argument similar to that followed by Matsumura, [12], for

Accepted for publication March 1997.
AMS Subject Classifications: 35B45, 35B60, 35B65, 35K55, 35K60.

quasilinear dissipative wave equations). We would also like to investigate the higher regularity and strong continuity of such solutions with respect to the data in the highest Sobolev norm considered, and their asymptotic behavior as $t \to +\infty$. A fairly complete set of results in this direction was obtained in [14] in the case $n \leq 3$. Here, we are able to remove the restriction on the space dimension, and present a global existence result for Sobolev solutions to (1.1) for arbitrary n; furthermore, we have two improvements with respect to [14]: one, eliminating the asymmetry of the spaces in which the solution is sought (which in [14] depended on the parity of m), and the other eliminating a somewhat unnatural extra regularity assumption on the source term f, which in [14] we imposed in order to obtain some extra smoothness of the classical solutions of (1.1). Although we did not pursue an extension of the regularity and asymptotic behavior results of [14] to arbitrary n at this time, we expect that such results should hold as well: in fact, in [14] the existence of an attractor was obtained in a standard way as the ω-limit set of an absorbing ball, so that we only needed to extend the local time estimates to uniform ones, and a regularity result to deduce compactness of the orbits. In this regard, we note that both the estimates on the classical solutions and the regularity result obtained in [14] hold for arbitrary n; however, since they were obtained under the above mentioned extra assumption on f, we would have to make sure that this assumption is no longer essential, which we believe based on the evidence of the result obtained here. At any rate, even if more regularity of f were actually needed, we can report another improvement with respect to [14] concerning the existence of absorbing balls for the classical solutions of (1.1); indeed, referring to Theorem 2 of Chapter 5, Section 5.5 of Krylov, [6], we can deduce that all the required uniform bounds on the classical solutions can be established independently of the initial value u_0, for sufficiently large t (thus, assumption (2.16) of [14] is no longer required). We will address these questions in a separate paper.

1.2. In our previous paper [13], we were able to prove the existence of global Sobolev solutions to (1.1) in the homogeneous case $f \equiv 0$, both for bounded Ω and for $\Omega = \mathbb{R}^n$, for any n. This result was obtained by patching together a local Sobolev solution, defined on some interval $[0, 2\tau]$, with the global classical one, which was known to be more regular, and in fact so regular as to be a Sobolev one, for $t > 0$ (thus, in particular, for $t \geq \tau$). Following the same procedure in the nonhomogeneous case is prevented by the fact that we do not assume f to be so regular as to immediately yield higher regularity of the classical solutions for $t > 0$; thus, we are forced to bootstrap from the minimal amount of extra regularity that we can deduce from f by means of an induction process starting from the classical solution and based on rather delicate estimates on its H^m norms. To achieve this, in [14], we followed a rather standard "energy" method, coupled with an induction argument on m. Since the corresponding estimates make use of the so-called calculus inequalities and Sobolev's product formulas, we were forced to keep the space dimension low, in order to use the global bounds on the classical solution (Theorem 3) as the starting point of the induction. To be able to proceed in the same way for arbitrary n, we thus need to circumvent the restrictions that the Sobolev inequalities impose on the space dimension in relation to the order of regularity of the functions involved; we are able to achieve this by resorting to several specialized versions of the Gagliardo-Nirenberg inequalities which are at the basis of the Sobolev inequalities. These versions are valid on cylindrical domains, so that we can make full use of the assumed regularity of the solution in time, which we did not previously exploit. This crucial part of our procedure is described in Propositions 2.1 to 2.5.

Finally, we remark that we could consider an explicit dependence of the coefficients a_{ij} on x, t and u, as well as a nonlinear lower order term $f = f(x, t, u, \nabla u)$, subject to suitable growth conditions; for simplicity, however, we do not pursue these extensions any further.

2. Notations and results.

2.1. To describe the classical solutions of (1.1), we recall the definitions of the Hölder type spaces $C^{\lambda,\lambda/2}$ we consider in the classical theory. We set $Q = \Omega\times]0,T[$ and, if $u = u(x,t)$ is a smooth function on $\bar{Q}$, we define the following norms of u:

$$|u|_{C_x^\alpha(\bar{Q})} \doteq \sup_{(x,t),(x',t)\in Q; x\neq x'} \frac{|u(x,t)-u(x',t)|}{|x-x'|^\alpha}, |u|_{C_t^\alpha(\bar{Q})} \doteq \sup_{(x,t),(x,t')\in Q; t\neq t'} \frac{|u(x,t)-u(x,t')|}{|t-t'|^\alpha},$$

where $\alpha \in]0,1[$; for $\lambda = j+\alpha$, j a nonnegative integer, we set

$$\|u\|_{C^{\lambda,\lambda/2}(\bar{Q})} \doteq \sum_{2r+|\beta|\leq j} \|\partial_t^r \partial_x^\beta u\|_{C^0(\bar{Q})} + \sum_{2r+|\beta|=j} |\partial_t^r \partial_x^\beta u|_{C_x^\alpha(\bar{Q})} + \sum_{j-1\leq 2r+|\beta|\leq j} |\partial_t^r \partial_x^\beta u|_{C_t^{(\lambda-2r-|\beta|)/2}(\bar{Q})}:$$

the spaces $C_x^\alpha(\bar{Q})$, $C_t^\alpha(\bar{Q})$ and $C^{\lambda,\lambda/2}(\bar{Q})$ are the spaces of appropriately continuously differentiable functions for which the corresponding norms are finite. Similarly, to describe the Sobolev solutions, for $m \in \mathbb{N}$ we set $H^m =$

$H^m(\Omega)$, $H^m_* = H^m \cap H^1_0(\Omega)$ if $m \geq 1$, $H^0 = H^0_* = L^2(\Omega) = L^2$, $H^{-1} = H^{-1}_* = H^{-1}(\Omega)$, and denote by $\|\,.\,\|_m$ the norm in H^m, $\|.\|$ the norm in $L^2(\Omega)$, and $|\,.\,|_p$ the norm in $L^p(Q)$ or $L^p(\Omega)$, $1 \leq p \leq \infty$. Next, for $m \in \mathbb{N}$ we set $m' = [\frac{m+1}{2}]$ and

$$X_m(T) = \{\, u \in L^2(0,T;H^m) \mid \partial_t^{m'} u \in L^2(0,T;H^{m-2m'})\,\},$$

which is a Hilbert space with respect to the norm

$$\|u\|^2_{X_m(T)} = \int_0^T \|u\|^2_m \, dt + \int_0^T \|\partial_t^{m'} u\|^2_{m-2m'} \, dt; \tag{2.1}$$

in particular, if $u \in X_m(T)$, then $\partial_t^{k+1} u \in L^2(0,T;H^{-1})$ if $m = 2k+1$, and in general, by the interpolation results of Lions-Magenes, [11] (Theorems 2.3, 3.1, 9.6 and 12.4 of Chapter I),

$$\partial_t^r u \in L^2(0,T;H^{m-2r}) \quad \text{for } 0 \leq 2r \leq m, \tag{2.2}$$

$$\partial_t^r u \in C([0,T];H^{m-1-2r}) \quad \text{for } 0 \leq 2r \leq m-1; \tag{2.3}$$

in fact,

$$X_m(T) = \cap_{r=0}^{m'} H^r(0,T;H^{m-2r}(\Omega)), \tag{2.4}$$

with equivalent norms.

We assume that the boundary $\partial\Omega$ and the coefficients a_{ij} in (1.1) are at least of class C^s, with integer $s \geq [\frac{n}{2}] + 2$; moreover, in accord with [7]–[10], we also assume that the a_{ij}'s are symmetric, and satisfy the uniform boundedness, strong ellipticity and decay conditions

$$\exists \mu, \nu > 0, \; : \forall p, q \in \mathbb{R}^n, \quad \nu |q|^2 \leq \sum a_{ij}(p) q^i q^j \leq \mu |q|^2, \tag{2.5}$$

$$\exists \mu_1 > 0, \; : \forall p \in \mathbb{R}^n, \quad \Big|\frac{\partial a_{ij}}{\partial p_k}(p)\Big| \leq \mu_1 (1 + |p|^2)^{-1/2}; \tag{2.6}$$

in the sequel, we shall abbreviate

$$\sum_{i,j=1}^n a_{ij}(\nabla u)\partial_i \partial_j u \doteq a(\nabla u)\partial^2 u.$$

We prescribe arbitrary $T > 0$ and (integer) $s \geq [\frac{n}{2}] + 2$, and assume that the data of (1.1) satisfy

$$u_0 \in H^{s+1}_*, \quad f \in X_s(T), \tag{2.7}$$

and the natural compatibility conditions at $\partial\Omega$ for $t=0$; that is, in the usual interpretation,

$$u_k \doteq (\partial_t^k u)(0) = 0 \quad \text{on} : \partial\Omega, \quad 0 \le k \le [\frac{s}{2}]. \tag{2.8}$$

More precisely, in (2.8) the functions u_k are generated recursively from u_0 by means of formal differentiation of equation (1.1) at $t=0$; for example,

$$u_1 \doteq f(.,0) + a(\nabla u_0)\partial^2 u_0,$$
$$u_2 \doteq f_t(.,0) + a(\nabla u_0)\partial^2 u_1 + a'(\nabla u_0)\cdot\nabla u_1 \partial^2 u_0,$$

etc. Note that from (2.7) we can deduce, as in [13], that $u_k \in H^{s+1-2k}$ for $0 \le k \le s' = [\frac{s+1}{2}]$, so that (2.8) makes sense in $H^{1/2}(\partial\Omega)$; in fact, since $f \in X_s(T)$, by (2.3) we have that

$$\partial_t^r f \in C([0,T]; H^{s-1-2r}) \quad \text{for} : 0 \le 2r \le s-1,$$

and (2.8) can be reformulated as the requirement that $u_k \in H_*^{s+1-2k}$ for $0 \le 2k \le s$.

The following local existence result for (1.1) follows immediately from Kato's general theory on quasilinear evolution equations, as e.g. in [5].

Theorem 1. *There exists* $\tau \in]0, \frac{1}{2}T]$, *and a unique* $u \in X_{s+2}(2\tau)$, *solution of* (1.1) *in* $[0, 2\tau]$.

Proof. Uniqueness of solutions of (1.1) in any $X_m(T)$, with $m > \frac{n}{2}+1$, is immediate. Existence of a local solution $u \in \cap_{k=0}^{s'} C^k([0,2\tau]; H^{s+1-2k})$ is a direct consequence of Theorem 8.1 of Kato, [5]; that $u \in X_{s+2}(2\tau)$ follows then from standard energy estimates, similar to the ones we shall establish in the proof of Theorem 4 and in Section 6. □

We remark that the compatibility conditions (2.8) are necessary, because if $u \in X_{s+2}(T')$ for some $T' > 0$ is a solution of (1.1), then by (2.3) we deduce that $\partial_t^k u \in C([0,T']; H^{s+1-2k})$ for $0 \le 2k \le s+1$; hence, equation (1.1) can be differentiated $[\frac{s}{2}]-1$ times in t, and we find that relations (2.8) do hold. In fact, we find that $u \in \cap_{k=0}^{s'} C^k([0,T']; H_*^{s+1-2k})$; in particular, since $u(\cdot,0) = u_0 \in H_*^{s+1} \hookrightarrow C^{2+\alpha}(\overline{\Omega})$ and $u_t(\cdot,0) = u_1 \in H_*^{s-1} \hookrightarrow C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$, the boundary condition of (1.1) and the compatibility condition of order 1 at $\partial\Omega$ are also satisfied in the classical sense.

Our goal is now to extend the local solution provided by Theorem 1 to a global one: the main result of this paper is

Theorem 2. *There exists a unique* $u \in X_{s+2}(T)$*, solution of* (1.1).

2.2. We shall prove Theorem 2 by means of *a priori* estimates on the local Sobolev solution of (1.1) provided by Theorem 1; more precisely, we assume that (1.1) has a solution $u \in X_{s+2}(T')$, with $T' \in (0,T)$, and establish estimates on the norm of u in $X_{s+2}(T')$ independent of T' (although in general they will explicitly depend on T). We achieve this in two steps: first we invoke Ladyzenskaya and Ural'tseva's results, to show directly that (1.1) has a global classical solution in a Hölder space $C^{2+\gamma,1+\gamma/2}(\overline{Q})$ for some $\gamma \in (0,1)$, and then we use this global classical solution as the starting point of a bootstrapping argument to obtain the desired estimates on higher order Sobolev norms of the solution.

Our first step is based on

Theorem 3. *There exist* $\gamma \in (0,1)$ *and a unique* $\bar{u} \in C^{2+\gamma,1+\gamma/2}(\overline{Q})$*, solution of* (1.1).

This result was proven in [14]; note that, as we have remarked above, conditions (2.8) imply that, in particular, the necessary compatibility conditions of order 1 at $\partial\Omega$ are satisfied in the classical sense. Moreover, since by imbedding a Sobolev solution is also a classical one, and uniqueness holds for both kinds of solutions in any interval $[0,\tau]$ where they are defined, Theorem 2 can be also considered as a global regularity statement on the classical solution.

Our second step is based on the *a priori* estimates on higher order Sobolev norms of $u = \bar{u}$, provided by

Theorem 4. *For each* m , $2 \le m \le s+1$, *there exist* $C_m > 0$ *and* $D_m > 0$ *such that for any* $T' \in (0,T)$ *for which* (1.1) *has a solution* $u \in X_{s+2}(T')$, *the estimates*

$$\int_0^{T'} \|u\|^2_{m+1} dt \le C_m, \tag{2.9}$$

$$\int_0^{T'} \|\partial_t^{k+1} u\|^2_{m-2k-1} dt \le D_m \tag{2.10}$$

hold, with $k = [\frac{m}{2}]$ *and* C_m*,* D_m *independent of* T'.

Remark. In general, C_m and D_m will depend explicitly on T, $\|u_0\|_{s+1}$ and $\|f\|_{X_s(T)}$. In the sequel, by the expression "$C \in \Gamma$" we shall mean that C is any general positive constant with such dependence; in particular, $C \in \Gamma$ means that C is independent of T'.

Assume that Theorem 4 holds: then

$$u \in X_{s+2}(T') \cap X_{m+1}(T), \tag{2.11}$$

so that Theorem 2 follows from (2.11) with $m = s + 1$.

We shall prove Theorem 4 by induction on m (in brief: m–induction). In the sequel, it is essential to recall that

$$\|u\|_{C([0,T];C^2(\bar{\Omega}))} + \|u_t\|_{C(\bar{Q})} \in \Gamma. \tag{2.12}$$

When $m = 2$, we establish (2.9) and (2.10) as follows: first, by an energy estimate obtained by differentiating equation (1.1) once with respect to time, and multiplying by $2u_t$ in $L^2(\Omega)$, we deduce that $u_t \in L^2(0,T;H^1)$; from this, we easily deduce (2.9) by ellipticity (see (2.24) below). Next, we deduce (2.10) from the identity

$$u_{tt} = f_t + a'(\nabla u)\nabla u_t \partial^2 u + a(\nabla u)\partial^2 u_t,$$

in which the first term of the right side is in $L^2(0,T;H^{-1})$, the second is in $L^2(0,T;L^2)$ (as we know from our previous step and (2.12)), and the third is estimated in $L^2(0,T;H^{-1})$ as follows: given any $\phi \in L^2(0,T;H_0^1)$, we compute that

$$\int_0^T (a(\nabla u)\partial^2 u_t, \phi)dt = -\int_0^T (a(\nabla u)\partial u_t, \partial\phi)dt + \int_0^T (a'(\nabla u)\partial\nabla u \partial u_t, \phi)dt,$$

and we conclude by recalling that both $a(\nabla u)\partial u_t$ and $a'(\nabla u)\partial\nabla u\partial u_t$ are in $L^2(0,T;L^2)$ (again by our previous step and (2.12)).

Thus, we shall assume that (2.11) holds for some m, $2 \le m \le s$, and proceed directly to prove that it also holds for $m + 1$.

We conclude by mentioning that we believe that extension of our results to the case $\Omega = \mathbb{R}^n$, as treated in [13], should be possible; however, at this point this extension is prevented by the fact that the bound implied by (2.12) increases with the diameter of Ω.

2.3. In this section we report the main technical results we shall use in the sequel; namely, some specialized versions of the Gagliardo-Nirenberg interpolation inequalities, some elliptic estimates and so-called calculus inequalities. We recall the standard inequality (Nirenberg, [17], [18]):

Proposition 2.1. *Let m be a positive integer, and $1 \le p, r \le +\infty$. Given $j = 0, \dots m$, define q, $s \in [1, +\infty]$ by*

$$\frac{1}{q} = \frac{j}{m}\Big(\frac{1}{p} - \frac{1}{r}\Big) + \frac{1}{r}, \quad s = \max(p, r). \tag{2.13}$$

Let $\Omega \subseteq \mathbb{R}^n$ be an open domain satisfying the cone property, and assume that $u \in W^{m,p}(\Omega) \cap L^r(\Omega)$: then for all multi-indices α, with $|\alpha| = j$, $D^\alpha u \in L^q(\Omega)$, and

$$\|D^\alpha u\|_{L^q(\Omega)} \le C \sum_{|\mu|=m} \|D^\mu u\|^{j/m}_{L^p(\Omega)} \|u\|^{1-j/m}_{L^r(\Omega)} + C\|u\|_{L^s(\Omega)}, \tag{2.14}$$

with $C > 0$ independent of u.

We now present some specialized versions of these inequalities, which we will prove in Section 7. In the sequel, we set $Q = \Omega \times (a,b)$, where (a,b) is an interval of $\mathbb{R}$, and let ∂_t, ∂ denote differentiation respectively along (a,b) and a generic direction of $\mathbb{R}^n$; $C > 0$ will denote several positive quantities independent of the function u.

Proposition 2.2. *Let $u \in W^{m,p}(Q) \cap L^r(Q)$, and p, r, m, j, q, s satisfy* (2.13). *Then*

$$\|\partial_t^j u\|_{L^q(Q)} \le C\|\partial_t^m u\|^{j/m}_{L^p(Q)} \|u\|^{1-j/m}_{L^r(Q)} + C\|u\|_{L^s(Q)}. \tag{2.15}$$

Proposition 2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a ball, assume that* (2.13) *holds, and that $u \in W^{m,p}(\Omega) \cap L^r(\Omega)$. Then for any k, $1 \le k \le n$,*

$$\|\partial_k^j u\|_{L^q(\Omega)} \le C\|\partial_k^m u\|^{j/m}_{L^p(\Omega)} \|u\|^{1-j/m}_{L^r(\Omega)} + C\|u\|_{L^s(\Omega)}. \tag{2.16}$$

Conversely, if δ denotes differentiation along any direction of $\mathbb{R}^n$ except the k-th, for all $\beta \in \mathbb{N}^n$ with $|\beta| = j$ and $\beta_k = 0$, (*with slight abuse of notation*)

$$\|\delta^\beta u\|_{L^q(\Omega)} \le C \sum_{\substack{|\alpha|=m \\ \alpha_k=0}} \|\delta^\alpha u\|^{j/m}_{L^p(\Omega)} \|u\|^{1-j/m}_{L^r(\Omega)} + C\|u\|_{L^s(\Omega)}. \tag{2.17}$$

Proposition 2.4. *Let* (2.13) *hold and assume that for $\ell \in \mathbb{N}$, $\partial_t^\ell u \in W^{m,p}(Q) \cap L^r(Q)$. Then, if $\beta \in \mathbb{N}^n$ with $|\beta| = j$,*

$$\|\partial^\beta \partial_t^\ell u\|_{L^q(Q)} \le C \sum_{|\alpha|=m} \|\partial^\alpha \partial_t^\ell u\|^{j/m}_{L^p(Q)} \|\partial_t^\ell u\|^{1-j/m}_{L^r(Q)} + C\|\partial_t^\ell u\|_{L^s(Q)}. \tag{2.18}$$

We shall also need the following extension of (2.15) to Sobolev spaces of noninteger order like $H^s(0,T;L^2(\Omega))$, with $s \in \mathbb{R}^+$ (see Lions-Magenes, [11] for the definition and principal properties of these spaces). We claim

Proposition 2.5. *Let $(a,b) \subseteq \mathbb{R}$ be an interval, $s \in \mathbb{R}^+$ and, for integer j such that $0 \leq j \leq [s]$, let $q = \frac{2s}{j}$ if $j = 0$, and $q = \infty$ if $j = 0$. Let $u \in H^s(a,b;L^2(\Omega)) \cap L^\infty(Q)$, then $\partial_t^j u \in L^q(Q)$, and*

$$\|\partial_t^j u\|_{L^q(Q)} \leq C\|u\|^{j/s}_{H^s(a,b;L^2(\Omega))}\|u\|^{1-j/s}_{L^\infty(Q)}, \tag{2.19}$$

with $C > 0$ independent of u.

This result is proven in Section 7.3; we recall from Lions-Magenes, [11], Theorem 4.1, that (in particular)

$$\|u\|_{H^{1/2}(0,T;L^2)} \leq C\|u\|^{1/2}_{L^2(0,T;H^1)}\big(\|u_t\|^{1/2}_{L^2(0,T;H^{-1})} + \|u\|^{1/2}_{L^2(0,T;L^2)}\big) \tag{2.20}$$

for all $u \in X_1(T)$, with C independent of u. This interpolation inequality is a generalization of the inequalities described by

Proposition 2.6. *Let $\Omega = \mathbb{R}^n$ or Ω be a bounded domain with smooth boundary, and I be an interval. If $\theta \in (0,1)$, $l, l_1, l_2 \in \mathbb{N}$, $k, k_1, k_2 \in \mathbb{N}$, and $(l,k) = \theta(l_1,k_1) + (1-\theta)(l_2,k_2)$, then*

$$\begin{aligned} &\|\partial_t^l u\|_{L^2(I,H^k(\Omega))} \\ &\leq C\|\partial_t^{l_1} u\|^{\theta}_{L^2(I,H^{k_1}(\Omega))} \cdot \big(\|\partial_t^{l_2} u\|^{1-\theta}_{L^2(I,H^{k_2}(\Omega))} + \|\partial_t^{l_1} u\|^{1-\theta}_{L^2(I,H^{k_2}(\Omega))}\big), \end{aligned} \tag{2.21}$$

with C independent of u.

This Proposition will be proved in Section 7.4; now, we proceed with a result concerning composite functions.

Proposition 2.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open domain satisfying the cone property, and assume that, for $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, $u \in W^{m,p}(\Omega) \cap L^\infty(\Omega)$ has support in Ω. If Φ is a C^m scalar function, then $\Phi(u) \in W^{m,p}(\Omega) \cap L^\infty(\Omega)$, and for all $r = 0, \ldots m$, for all multi-indices α with $|\alpha| = r$, there exists a constant $C_r > 0$, depending on $|u|_\infty$ and $\|\Phi\|_{C^r(B)}$, $B \doteq [-|u|_\infty, |u|_\infty]$, such that*

$$\|D^\alpha \Phi(u)\|_{L^p(\Omega)} \leq C_r\big(1 + \sum_{|\alpha|=r} \|D^\alpha u\|_{L^p(\Omega)}\big). \tag{2.22}$$

This result follows in a standard way from classical calculus inequalities; for a proof, see e.g. [14], Proposition 2.4, or Hajlasz, [3], in the context of Sobolev spaces defined on manifolds. As a specialized version which privileges derivatives, for $Q = \Omega \times (a,b)$ as before, we have

Proposition 2.8. *Given $m \in \mathbb{N}$ and $\mu \in [1, +\infty]$, set*

$$W_t^{m,\mu}(Q) = \{u \in L^\infty(Q) \mid \partial_t^j u \in L^\mu(Q), 0 \leq j \leq m\}.$$

If Φ is a C^m scalar function and $u \in W_t^{m,\mu}(Q)$, then $\Phi(u) \in W_t^{m,\mu}(Q)$ and for all $r = 1, \ldots, m$ there is $C_r > 0$ as in Proposition 2.7, such that

$$\|\partial_t^r \Phi(u)\|_{L^\mu(Q)} \leq C_r(\|\partial_t^r u\|_{L^\mu(Q)} + 1). \tag{2.23}$$

We conclude by recalling some elliptic estimates:

Proposition 2.9. *Let L be a second order strongly elliptic operator with coefficients in $C^1(\bar{\Omega})$. If $u \in H_*^2$ and $Lu \in H^1$, then in fact $u \in H_*^3$, and*

$$\|u\|_3 \leq C\big\{\|Lu\|_1 + \|u\|\big\}, \tag{2.24}$$

with C independent of u. Assume instead that the coefficients of L are in H^r, with $r > \frac{n}{2}$: if $u \in H_*^2$ and, for $m = 0, \ldots r$, $Lu \in H^m$, then in fact $u \in H_*^{m+2}$, and

$$\|u\|_{m+2} \leq C\big\{\|Lu\|_m + \|u\|\big\}, \tag{2.25}$$

with C depending only on the norm of the coefficients of L in H^r.

The standard elliptic estimates can be found, for instance, in Gilbarg-Trudinger, [1]; the second part of the Proposition is proved in [16].

3. Proof of Theorem 4.

3.1. We now assume that (2.11) holds for $2 \leq m \leq s$, and proceed to prove that, in fact, $u \in X_{m+2}(T)$: setting again $k = [\frac{m}{2}]$, we claim

Proposition 3.1. *Assume that Theorem 4 holds for $2 \leq m \leq s$, then it also holds for $m+1$; that is, there exist C_{m+1}, D_{m+1} and $E_{m+1} \in \Gamma$ such that*

$$\int_0^{T'} \|u\|_{m+2}^2 dt \leq C_{m+1}, \tag{3.1}$$

$$\int_0^{T'} \|\partial_t^{k+1} u\|_{m-2k}^2 dt \leq D_{m+1}, \tag{3.2}$$

and, if $m = 2k+1$,

$$\int_0^{T'} \|\partial_t^{k+2} u\|_{-1}^2 dt \leq E_{m+1}. \tag{3.3}$$

We shall prove Proposition 3.1 in three steps, described by the following Propositions:

Proposition 3.2. *For all $\eta > 0$ there exists $M_\eta \in \Gamma$ such that*

$$\int_0^{T'} \|\partial_t^{k+1} u\|_{m-2k}^2 dt \le M_\eta + \eta \|u\|_{X_{m+2}(T')}^2. \tag{3.4}$$

Proposition 3.3. *For all $\eta > 0$ there exists $M_\eta \in \Gamma$ such that*

$$\int_0^{T'} \|u\|_{m+2}^2 dt \le M_\eta + \eta \|u\|_{X_{m+2}(T')}^2. \tag{3.5}$$

Proposition 3.4. *Let $m = 2k + 1$: for all $\eta > 0$ there exists $M_\eta \in \Gamma$ such that*

$$\int_0^{T'} \|\partial_t^{k+2} u\|_{-1}^2 dt \le M_\eta + \eta \|u\|_{X_{m+2}(T')}^2. \tag{3.6}$$

We shall prove these estimates in the next sections; assuming them true, we can immediately deduce Proposition 3.1, recalling (2.1).

3.2. As a preliminary step to the proof of Propositions 3.2, 3.3 and 3.4, for $0 \le r \le k + 1$ we differentiate equation (1.1) r times with respect to time, so that $\partial_t^r u$ solves the equation

$$\begin{aligned} &(\partial_t^r u)_t - a(\nabla u)\partial^2(\partial_t^r u) = \partial_t^r f + G_r, \\ &(\partial_t^r u)(0) = u_r, \quad (\partial_t^r u)|_{\partial\Omega} = 0, \end{aligned} \tag{3.7}$$

where we have set $G_0 = 0$ and, for $r \ge 1$,

$$G_r \doteq \sum_{l=0}^{r-1} \binom{r}{l} \partial_t^{r-l} a(\nabla u) \partial_t^l \partial^2 u. \tag{3.8}$$

We recall that $\partial_t^r f \in L^2(0, T; H^{s-2r}) \subset L^2(0, T; H^{-1})$ and $u_r \in H^{s+1-2r} \subset L^2$ for $0 \le r \le k + 1$ (since $s \ge m = 2k + 1$); however, if $s = m = 2k + 1$ and $r = k + 1$, the equation in (3.7) doesn't necessarily make sense, since in this case we only know that $\partial_t^r u \in L^2(0, T'; H^1)$ (although we still have $u_{k+1} \in L^2$). Thus, if $m = s$ our procedure is formal, and to justify it we should introduce a suitable regularization, resorting for instance to Ikawa's mollifiers of [4]. For simplicity, we abbreviate $a(\nabla u) \equiv a$ and note that, by Proposition 2.7, Theorem 3 and (2.11), we can assume that

$$\alpha \in X_m(T)\,, \qquad a(\cdot, t) \in C^1(\overline{\Omega}) \quad \text{for} \quad 0 \le t \le T. \tag{3.9}$$

4. Proof of Proposition 3.2.

4.1. If $m = 2k+1$, we consider (3.7) for $r = k+1$, and multiply in L^2 by $2\partial_t^{k+1}u$, we obtain

$$\frac{d}{dt}\|\partial_t^{k+1}u\|^2 + 2(a\partial\partial_t^{k+1}u, \partial\partial_t^{k+1}u) = 2(\partial_t^{k+1}f + G_{k+1} - \partial a\partial\partial_t^{k+1}u, \partial_t^{k+1}u). \tag{4.1}$$

We have

$$2(\partial_t^{k+1}f, \partial_t^{k+1}u) \le \|\partial_t^{k+1}f\|_{-1}\|\partial_t^{k+1}u\|_1;$$

thus, recalling (2.5) and (3.9), and that $\partial_t^{k+1}u(0) = u_{k+1} \in H^{s+1-2(k+1)} = H^{s-m} \subset L^2$, we obtain from (4.1) that for suitable $M \in \Gamma$

$$\begin{aligned}
&\frac{d}{dt}\|\partial_t^{k+1}u\|^2 + \nu\|\nabla\partial_t^{k+1}u\|^2 \\
&\quad \le M\|\partial_t^{k+1}f\|_{-1}^2 + M\|\partial_t^{k+1}u\|^2 + 2(G_{k+1}, \partial_t^{k+1}u), \\
&\|\partial_t^{k+1}u(t)\|^2 + \nu\int_0^t \|\nabla\partial_t^{k+1}u\|^2 dt \\
&\quad \le M + M\int_0^t \|\partial_t^{k+1}u\|^2 dt + 2\int_0^t \|G_{k+1}\,\partial_t^{k+1}u\|_{L^1(\Omega)} dt.
\end{aligned} \tag{4.2}$$

To estimate the last term of (4.2), recalling (3.8) it is sufficient to estimate, for $0 \le \ell \le k$, each term

$$I_\ell = \int_0^t \|\partial_t^{k+1-\ell}a(\nabla u)\,\partial_t^\ell\partial^2 u\,\partial_t^{k+1}u\|_{L^1(\Omega)} dt,$$

which we do resorting to the generalized Gagliardo-Nirenberg inequality of Proposition 2.5. Defining p, q and $r \ge 1$ by $\frac{1}{p} = \frac{k+1-\ell}{2(k+1)}$, $\frac{1}{q} = \frac{\ell}{2k+1}$ and $\frac{1}{r} = \frac{k}{2k+1}$, we have that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, and therefore, recalling that we denote by $|\,.\,|_p$ the norm in $L^p(Q)$,

$$I_\ell \le M|\partial_t^{k+1-\ell}a(\nabla u)|_p|\partial_t^\ell\partial^2 u|_q|\partial_t^{k+1}u|_r; \tag{4.3}$$

by Propositions 2.8 and 2.5 we have

$$\begin{aligned}
|\partial_t^{k+1-\ell}a(\nabla u)|_p &\le C(1 + |\partial_t^{k+1-\ell}\nabla u|_p) \le C(1 + |\partial_t^{k+1}\nabla u|_2^{2/p}|\nabla u|_\infty^{1-2/p}), \\
|\partial_t^\ell\partial^2 u|_q &\le C\|\partial^2 u\|_{H^{k+1/2}(0,T';L^2(\Omega))}^{2/q}|\partial^2 u|_\infty^{1-2/q}, \\
|\partial_t^{k+1}u|_r &\le C\|u_t\|_{H^{k+1/2}(0,T';L^2(\Omega))}^{2/r}|u_t|_\infty^{1-2/r} \\
&\le C\|u\|_{H^{k+3/2}(0,T';L^2(\Omega))}^{2/r}|u_t|_\infty^{1-2/r}.
\end{aligned}$$

Recalling then (2.12) and (2.20), we deduce that, for $\eta > 0$,

$$
\begin{aligned}
I_\ell &\le C\big(1 + |\partial_t^{k+1}\nabla u|_2^{2/p}\big)\|\partial^2 u\|^{2/q}_{H^{k+1/2}(0,T';L^2(\Omega))}\|u\|^{2/r}_{H^{k+3/2}(0,T';L^2(\Omega))} \\
&\le M + \eta|\partial_t^{k+1}\nabla u|_2^2 + \eta\|\partial^2 u\|^2_{H^{k+1/2}(0,T';L^2(\Omega))} + \eta\|u\|^2_{H^{k+3/2}(0,T';L^2(\Omega))} \\
&\le M + 2\eta\int_0^t \|\partial_t^{k+1}\nabla u\|^2 dt + \eta\int_0^t \|\partial_t^k\partial^2 u\|_1^2 dt \\
&\quad + \eta\int_0^t \|\partial_t^{k+1}\partial^2 u\|_{-1}^2 dt + \eta\int_0^t \|\partial_t^{k+2} u\|_{-1}^2 dt \\
&\le M + 2\eta\int_0^t \|\partial_t^{k+1}\nabla u\|^2 dt + 3\eta\|u\|^2_{X_{m+2}(T')}.
\end{aligned}
$$

Inserting this into (4.2) and renaming η, by Gronwall's inequality we obtain (3.4) when $m = 2k+1$.

4.2. If $m = 2k$, we consider (3.7) for $r = k$; now, we have $\partial_t^k f \in L^2(0,T;H^{s-2k}) \subset L^2(0,T;L^2)$ and $u_k \in H_*^{s+1-2k} \subset H^1$, and (3.7) makes sense in L^2 at least for almost all $t \in [0,T']$. Multiplying (3.7) in L^2 by $2\partial_t^{k+1}u$, and recalling (3.9), we have, for $\eta > 0$:

$$
\begin{aligned}
&2\|\partial_t^{k+1}u\|^2 + \frac{d}{dt}(a\partial\partial_t^k u, \partial\partial_t^k u) \\
&\quad = 2(\partial_t^k f + G_k - \partial a\partial\partial_t^k u, \partial_t^{k+1}u) + (a_t\partial_t^k\partial u, \partial_t^k\partial u) \\
&\quad \le M\big(\|\partial_t^k f\|^2 + \|G_k\|^2 + \|\nabla\partial_t^k u\|^2\big) + \|\partial_t^{k+1}u\|^2 + \eta\|a_t\,\nabla\partial_t^k u\|^2;
\end{aligned}
\tag{4.4}
$$

integrating this and recalling (2.5), we have

$$
\begin{aligned}
&\int_0^t \|\partial_t^{k+1}u\|^2 dt + \nu\|\nabla\partial_t^k u(t)\|^2 \le \mu\|\nabla\partial_t^k u(0)\|^2 \\
&\qquad + M\int_0^t \big(\|\partial_t^k f\|^2 + \|G_k\|^2 + \|\nabla\partial_t^k u\|^2\big)dt + \eta\int_0^t \|a_t\,\nabla\partial_t^k u\|^2 dt;
\end{aligned}
$$

therefore, since $\nabla\partial_t^k u(0) = \nabla u_k \in H^{s+1-2k-1} = H^{s-m} \subset L^2$ and $\partial_t^k f \in L^2(0,T;L^2)$,

$$
\begin{aligned}
\int_0^t \|\partial_t^{k+1}u\|^2 dt + \nu\|\nabla\partial_t^k u(t)\|^2 &\le M + M\int_0^t \|G_k\|^2 dt \\
&\quad + M\int_0^t \|\nabla\partial_t^k u\|^2 dt + \eta\int_0^t \|a_t\,\nabla\partial_t^k u\|^2 dt,
\end{aligned}
\tag{4.5}
$$

with $M \in \Gamma$. To estimate the last term of (4.5), we define α, β and $\gamma \geq 2$ by $\frac{1}{\alpha} = \frac{1}{2m}$, $\frac{1}{\beta} = \frac{m-1}{2m} = \frac{1}{2} - \frac{1}{\alpha}$, $\frac{1}{\gamma} = \frac{k-1}{2k} = \frac{1}{2} - \frac{1}{m}$: noting that $\frac{1}{\beta} = \frac{1}{2}(\frac{1}{2} - \frac{1}{\gamma}) + \frac{1}{\gamma}$, by Propositions 2.8 and 2.1 we have, for suitable (different) M's $\in \Gamma$:

$$\begin{aligned}
J_1 &\doteq \int_0^t \|a_t \nabla \partial_t^k u\|^2 dt \leq M \int_0^t \|\nabla u_t\|^2_{L^\alpha(\Omega)} \|\nabla \partial_t^k u\|^2_{L^\beta(\Omega)} dt \\
&\leq M \int_0^t \big(\|D^m u_t\|^{2/\alpha}_{L^2(\Omega)} |u_t|^{1-2/\alpha}_\infty + |u_t|_\infty\big)^2 \\
&\quad \cdot \big(\|D^2 \partial_t^k u\|^{1/2}_{L^2(\Omega)} \|\partial_t^k u\|^{1/2}_{L^\gamma(\Omega)} + \|\partial_t^k u\|_{L^\gamma(\Omega)}\big)^2 dt \\
&\leq M \int_0^t \big(\|D^m u_t\|^{4/\alpha}_{L^2(\Omega)} + 1\big)\big(\|D^2 \partial_t^k u\|_{L^2(\Omega)} \|\partial_t^k u\|_{L^2(\Omega)} + \|\partial_t^k u\|^2_{L^\gamma(\Omega)}\big) dt \\
&= M \Big\{ \int_0^t \|D^m u_t\|^{4/\alpha}_{L^2(\Omega)} \|D^2 \partial_t^k u\|_{L^2(\Omega)} \|\partial_t^k u\|^2_{L^\gamma(\Omega)} dt \\
&\quad + \int_0^t \|D^m u_t\|^{4/\alpha}_{L^2(\Omega)} \|\partial_t^k u\|^2_{L^\gamma(\Omega)} dt + \int_0^t \|D^2 \partial_t^k u\|_{L^2(\Omega)} \|\partial_t^k u\|_{L^\gamma(\Omega)} dt \\
&\quad + \int_0^t \|\partial_t^k u\|^2_{L^\gamma(\Omega)} dt \Big\} \equiv M(J_{11} + J_{12} + J_{13} + J_{14})
\end{aligned}$$

($M \in \Gamma$ because $|u_t|_\infty \in \Gamma$). Using the fact that $\frac{1}{\gamma} = \frac{1}{2} - \frac{2}{\alpha}$, and then Proposition 2.2 to estimate

$$|\partial_t^k u|_\gamma \leq C\big(|\partial_t^{k+1} u|_2^{2/\gamma} |u_t|_\infty^{1-2/\gamma} + |u_t|_\infty\big) \leq M\big(1 + |\partial_t^{k+1} u|_2^{2/\gamma}\big)$$

(recall that $|\,.\,|_p$ denotes the norm in $L^p(Q)$), by repeated applications of Hölder's inequality we obtain the estimate

$$\begin{aligned}
J_{11} &\leq M |D^m u_t|_2^{4/\alpha} |D^2 \partial_t^k u|_2 (1 + |\partial_t^{k+1} u|_2^{2/\gamma}) \\
&\leq M \|u\|^{1+4/\alpha}_{X_{m+2}(T')} (1 + |\partial_t^{k+1} u|_2^{2/\gamma}) \leq M(1 + \|u\|^2_{X_{m+2}(T')} + |\partial_t^{k+1} u|_2^2).
\end{aligned}$$

Acting similarly for the other terms of J_1, we conclude that

$$J_1 \leq M\big(1 + \|u\|^2_{X_{m+2}(T')} + |\partial_t^{k+1} u|_2^2\big);$$

inserting this into (4.5), we obtain that

$$\begin{aligned}
\int_0^t \|\partial_t^{k+1} u\|^2 dt + \|\nabla \partial_t^k u(t)\|^2 &\leq M + \eta \|u\|^2_{X_{m+2}(T')} \\
&\quad + M \int_0^t \|G_k\|^2 dt + M \int_0^t \|\nabla \partial_t^k u\|^2 dt.
\end{aligned} \tag{4.6}$$

To estimate $|G_k|_2$, recalling (3.8) it is sufficient to estimate, for $0 \le \ell \le k-1$, each term

$$I_\ell = \int_0^t \|\partial_t^{k-\ell} a(\nabla u)\, \partial_t^\ell \partial^2 u\|^2 dt :$$

defining p, $q \ge 2$ by $\frac{1}{p} = \frac{k-\ell}{2k}$, $\frac{1}{q} = \frac{\ell}{2k} = \frac{1}{2} - \frac{1}{p}$, recalling Propositions 2.1, 2.2 and (2.12) we have that, for $\eta > 0$ and (different) $M \in \Gamma$,

$$\begin{aligned} I_\ell &\le M|\partial_t^{k-\ell} a(\nabla u)|_p^2 |\partial_t^\ell \partial^2 u|_q^2 \\ &\le M\big(|\partial_t^k \nabla u|_2^{2/p} |\nabla u|_\infty^{1-2/p} + 1\big)^2 \big(|\partial_t^k \partial^2 u|_2^{2/q} |\partial^2 u|_\infty^{1-2/q} + 1\big)^2 \\ &\le M\big(1 + |\partial_t^k \nabla u|_2^2\big) + \eta |\partial_t^k \partial^2 u|_2^2 \\ &\le M\Big(1 + \int_0^t \|\partial_t^k \nabla u\|^2 dt\Big) + \eta \|u\|_{X_{m+2}(T')}^2. \end{aligned}$$

Inserting these estimates into (4.6) and renaming η, by Gronwall's inequality we obtain (3.4) when $m = 2k$ as well: this concludes the proof of Proposition 3.2.

5. Proof of Proposition 3.3.

5.1. In [14], the estimate of $\int_0^{T'} \|u\|_{m+2}^2 dt$, together with that of all terms $\int_0^{T'} \|\partial_t^r u\|_{m+2-2r}^2 dt$, $0 \le r \le k = [\frac{m}{2}]$, was achieved by applying the elliptic estimates (2.25) to equations (3.7), so as to have

$$\int_\tau^t \|\partial_t^r u\|_{m-2r+2}^2 \le C\Big\{1 + \int_0^t \|G_r\|_{m-2r}^2 + \int_0^t \|\partial_t^{r+1} u\|_{m-2r}^2 + \int_0^t \|\partial_t^r f\|_{m-2r}^2\Big\}, \tag{5.1}$$

and then by estimating $\int_0^t \|G_r\|_{m-2r}^2$. Here, we cannot pursue this strategy directly, since Proposition 2.9 ensures that (5.1) will hold on any interval $[0, \bar{t}]$ where the coefficients $a(\cdot, t) \in H^\ell(\Omega)$, with $\ell > \frac{n}{2}$; thus, while it is true that the m-induction assumption (2.11) implies that, since $a(\cdot, t) \in H^s(\Omega)$ for $0 \le t \le T'$ and $s > \frac{n}{2}$, (5.1) will hold in $[0, T']$, the constant C will in general depend explicitly on T' if $m - 1 \le \frac{n}{2}$ (this explains the limitation $n \le 3$ of [14], for in that case $m-1 \ge 2 > \frac{3}{2} \ge \frac{n}{2}$). Nevertheless, we are going to show that a refined version of the elliptic estimates (2.25), based on the extended Gagliardo-Nirenberg inequalities (2.15)…(2.23), actually leads to (5.1) under the m-induction assumptions (2.11), (3.9), at least when $r = 0$; analogous computations (see [15]) show that (5.1) also holds for $0 < r \le k$.

5.2. We consider the equation in (1.1) as linear in u, with coefficients $\tilde{a}(x,t) \doteq a(\nabla u(x,t))$ satisfying the regularity assumptions (3.9); for simplicity, we will continue to write a instead of $\tilde{a}$. As is customary in this process, we resort to localization and flattening of the boundary, and estimate the tangential and normal derivatives of u separately. More precisely, we consider a partition of unity $\mathcal{Z} = \{\zeta_0, \dots, \zeta_N\}$ subordinate to a finite covering of $\overline{\Omega}$, and a generic function $\zeta \in \mathcal{Z}$ whose support intersects $\partial\Omega$: after flattening this intersection by means of a suitable change of local coordinates, we may assume that in the new local coordinates the function $v = \zeta u$ satisfies the equation

$$v_t - a\,\partial^2 v = \zeta f - 2a\,\partial\zeta\,\partial u - a(\partial^2\zeta)u \tag{5.2}$$

in a set $Q' = \Omega' \times (0, T')$, with $\overline{\Omega'} \subset \mathbb{R}^{n-1} \times \mathbb{R}^+$ compact (recall that ζ is independent of t). In the sequel, we denote by ∂_n the derivative normal to the flattened portion of the boundary $\partial\Omega'$, i.e. in the direction of $\mathbb{R}^+$, and by δ any tangential derivative in $\mathbb{R}^{n-1}$ (i.e. $\delta \in \{\partial_1, \dots, \partial_{n-1}\}$). We also denote by ∂ either δ or ∂_n and, finally, we set $\nabla = \{\delta, \partial_n\}$; thus, v satisfies the boundary conditions

$$\delta^\beta v(y_1, \dots, y_{n-1}, 0, t) = 0, \quad 0 \le \beta \le \alpha - 1. \tag{5.3}$$

We first estimate the tangential derivatives of v by

Proposition 5.1. *Assume* (2.11)*, and* (3.9) *hold. For all* $\eta > 0$*, there exists* $M \in \Gamma$ *such that for all* $\zeta \in \mathcal{Z}$ *and all multi-indices* α*, with* $1 \le |\alpha| \le m+1$*,* $\forall t \in [0, T']$

$$\begin{aligned} \|\delta^\alpha(\zeta u)(t)\|^2 + \int_0^t \|\nabla\delta^\alpha(\zeta u)\|^2 dt \le\ & M + \eta\|u\|^2_{X_{m+2}(T')} \\ & + \eta\int_0^t \|\nabla\delta^\alpha\zeta u\|^2 dt + M\int_0^t \|\delta^{\alpha-1}(\zeta u)\|^2\, dt. \end{aligned} \tag{5.4}$$

Before proving (5.4), we remark that this inequality makes sense, since $\zeta \in C_0^\infty$, and $\delta^\alpha u \in C([0,T'];\ H^{s+1-|\alpha|}) \cap L^2(0,T'; H^{s+2-|\alpha|}) \subseteq C([0,T']; L^2) \cap L^2(0,T; H^1)$, because $s \ge m \ge |\alpha| - 1$; also, by Proposition 2.6 we have

$$\begin{aligned} \int_0^t \|\delta^{\alpha-1}(\zeta u_t)\|^2 dt &\le M\|u_t\|^2_{L^2(0,T';H^m)} \\ &\le M\|u\|^{2-2/(k+1)}_{L^2(0,T';H^{m+2})}\|\partial_t^{k+1}u\|^{2/(k+1)}_{L^2(0,T';H^{m-2k})} + \|u\|^{2/(k+1)}_{L^2(0,T';H^{m-2k})} \end{aligned}$$

($k = [\frac{m}{2}]$), so that, for $\eta > 0$ and $C_\eta \in \Gamma$, recalling (2.12),

$$\int_0^t \|\delta^{\alpha-1}(\zeta u_t)\|^2 dt \le \eta\int_0^t \|u\|^2_{m+2} dt + C_\eta\Big(1 + \int_0^t \|\partial_t^{k+1}u\|^2_{m-2k} dt\Big), \tag{5.5}$$

and therefore, recalling (3.4) and renaming the constants, we obtain from (5.4), by means of Gronwall's inequality, that

$$\|\delta^\alpha(\zeta u)(t)\|^2 + \int_0^t \|\nabla\delta^\alpha(\zeta u)\|^2 dt \le M + \eta\|u\|^2_{X_{m+2}(T')}. \tag{5.6}$$

Proof. To prove (5.4), we take α tangential derivatives of (5.2) and multiply in $L^2(Q')$ by $2\delta^\alpha v$: by the boundary condition (5.3) we obtain

$$\begin{aligned}\frac{d}{dt}\|\delta^\alpha v\|^2 + 2(a\partial\delta^\alpha v\,,\,\partial\delta^\alpha v) &= 2(\delta^\alpha(\zeta f) - \delta^\alpha(a\Phi) - \partial a\partial\delta^\alpha v + C^\alpha\,,\,\delta^\alpha v)\\ &= -2(\delta^{\alpha-1}(\zeta f) - \delta^{\alpha-1}(a\Phi)\,,\,\delta^{\alpha+1}v) + 2(\partial a\,\partial\delta^\alpha v + C^\alpha\,,\,\delta^\alpha v),\end{aligned} \tag{5.7}$$

where we have set

$$\Phi \doteq \partial\zeta\,\partial u + (\partial^2\zeta)u \tag{5.8}$$

and introduced the commutator

$$C^\alpha \doteq \delta^\alpha(a\partial^2 v) - a\delta^\alpha\partial^2 v. \tag{5.9}$$

Given then $\eta > 0$ we have from (5.7)

$$\begin{aligned}&\frac{d}{dt}\|\delta^\alpha v\|^2 + 2\nu\|\nabla\delta^\alpha v\|^2\\ &\quad\le M\,\|\zeta f\|_m^2 + M\|\delta^{\alpha-1}(a\Phi)\|^2 + \eta\|\nabla\delta^\alpha v\|^2 + \eta\|C^\alpha\|^2 + M\|\delta^\alpha v\|^2,\end{aligned}$$

so that, for $\eta \le \frac{\nu}{2}$ and $M \in \Gamma$,

$$\begin{aligned}\|\delta^\alpha v(t)\|^2 + \tfrac{3}{2}\nu\int_0^t \|\nabla\delta^\alpha v\|^2 dt &\le \|\delta^\alpha v(0)\|^2 + M\int_0^T \|\zeta f\|_m^2 dt\\ &+ M\int_0^t \|\delta^{\alpha-1}(a\Phi)\|^2 dt + \eta\int_0^t \|C^\alpha\|^2 dt + M\int_0^t \|\delta^\alpha v\|^2 dt.\end{aligned} \tag{5.10}$$

Since $\delta^\alpha v(0) \in H^{s+1-|\alpha|} \subseteq H^{s-m} \subseteq L^2$; since also $\zeta f \in L^2(0,T;H^m)$ and, by the m-induction assumption, $\delta^\alpha v \in L^2(0,T;H^{m+1-|\alpha|}) \subseteq L^2(0,T;L^2)$, from (5.10), we deduce that

$$\|\delta^\alpha v(t)\|^2 + \tfrac{3}{2}\nu\int_0^t \|\nabla\delta^\alpha v\|^2 dt \le M + M\int_0^t \|\delta^{\alpha-1}(a\Phi)\|^2 dt + \eta\int_0^t \|C^\alpha\|^2 dt. \tag{5.11}$$

In the next sections we shall prove the estimates

$$\int_0^t \|\delta^{\alpha-1}(a\Phi)\|^2 dt \leq M, \tag{5.12}$$

$$\int_0^t \|C^\alpha\|^2 dt \tag{5.13}$$
$$\leq M\big\{1 + \int_0^t \|\nabla\delta^\alpha(\zeta u)\|^2 dt + \int_0^t \|\delta^{\alpha-1}(\zeta u_t)\|^2 dtin + \|u\|^2_{X_{m+2}(T')}\big\};$$

assuming these to hold, it is then easy to obtain (5.4) from (5.11). We remark that these same estimates, with δ replaced by ∂, would also yield the desired additional regularity of u in the interior of Ω (that is, when the support of ζ does not intersect $\partial\Omega$).

5.3. To obtain (5.12), recalling (5.8), it is sufficient to estimate the more difficult term $|\delta^{\alpha-1}(a\,\partial\zeta\,\nabla u)|_2$ (here and in the sequel we abuse notation, writing e.g. α instead of $|\alpha|$, etc). By Leibniz' formula, we estimate for $\ell = 0, \dots, \alpha - 1$ each term $|\delta^\ell(a\,\partial\zeta)\delta^{\alpha-1-\ell}\nabla u|_2$ as follows: defining p, $q \geq 2$ by $\frac{1}{p} = \frac{\ell}{2m}$ and $\frac{1}{q} = \frac{\alpha-1-\ell}{2m}$, we have $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$, and

$$|\delta^\ell(a\partial\zeta)|_p \leq C|\partial^m(a\partial\zeta)|_2^{\ell/m}|a\partial\zeta|_\infty^{1-\ell/m} + C|a\partial\zeta|_\infty \leq M,$$
$$|\delta^{\alpha-1-\ell}\nabla u|_q \leq C|\partial^{m+1}u|_2^{2/q}|\nabla u|_\infty^{1-2/q} + C|\nabla u|_\infty \leq M,$$

with $M \in \Gamma$ because of (3.9) and the m-induction assumption (2.11); hence, (5.12) follows.

5.4. To estimate $|C^\alpha|_2$, we shall proceed in two steps, showing that

$$\int_0^t \|C^\alpha\|^2 dt \tag{5.14}$$
$$\leq M\big\{1 + \int_0^t \|\nabla\delta^\alpha(\zeta u)\|^2 dt + \int_0^t \|\delta^{\alpha-1}\partial_n^2(\zeta u)\|^2 dt + \|u\|^2_{X_{m+2}(T')}\big\},$$

$$\int_0^t \|\delta^{\alpha-1}\partial_n^2(\zeta u)\|^2 dt \leq M\big\{1 + \int_0^t \|\nabla\delta^\alpha(\zeta u)\|^2 dt + \int_0^t \|\delta^{\alpha-1}(\zeta u_t)\|^2 dt\big\}, \tag{5.15}$$

again for $0 \leq \alpha - 1 \leq m$.

5.4.1. We prove (5.14): from (5.9) we decompose

$$C^\alpha = \sum_{\beta=0}^{\alpha-1} \binom{\alpha}{\beta} \delta^{\alpha-\beta} a\,\delta^\beta \partial^2 v; \tag{5.16}$$

if $\alpha \le m$, defining p and $q \ge 2$ by $\frac{1}{p} = \frac{\alpha-\beta}{2m}$, $\frac{1}{q} = \frac{\beta}{2m} \le \frac{1}{2} - \frac{1}{p}$, we estimate

$$|\delta^{\alpha-\beta} a(\nabla u)|_p \le C|\partial^m a(\nabla u)|_2^{2/p} |a(\nabla u)|_\infty^{1-2/p} + C|a(\nabla u)|_\infty \le M,$$
$$|\delta^\beta \partial^2 v|_q \le C|\partial^{m+2} v|_2^{2/q} |\partial^2 v|_\infty^{1-2/q} + C|\partial^2 v|_\infty;$$

therefore, by (3.9), (2.12) and the m-induction assumption (2.11), we have that

$$|C^\alpha|_2 \le M\Big(1 + \int_0^t \|\partial^{m+2}(\zeta u)\|^2 dt\Big) \le M\big(1 + \|u\|^2_{X_{m+2}(T')}\big). \tag{5.17}$$

If $\alpha = m+1$, we set $\frac{1}{p} = \frac{\alpha-\beta-1}{2(\alpha-1)}$, $\quad \frac{1}{q} = \frac{\beta}{2(\alpha-1)} \le \frac{1}{2} - \frac{1}{p}$, so that

$$\begin{aligned}
&|\delta^{\alpha-\beta} a\, \delta^\beta \partial^2(\zeta u)|_2 \le C|\delta^{\alpha-\beta} a|_p\, |\delta^\beta \partial^2(\zeta u)|_q \\
&\le C(|\delta^\alpha a|_2^{2/p} |\delta a|_\infty^{1-2/p} + |\delta a|_\infty)(|\delta^{\alpha-1}\partial^2(\zeta u)|_2^{2/q} |\partial^2(\zeta u)|_\infty^{1-2/q} + |\partial^2(\zeta u)|_\infty) \\
&\le M(|\delta^\alpha a|_2^{2/p} + 1)(|\delta^{\alpha-1}\partial^2(\zeta u)|_2^{2/q} + 1);
\end{aligned}$$

therefore,

$$\begin{aligned}
&\int_0^t \|\delta^{\alpha-\beta} a\, \delta^\beta \partial^2(\zeta u)\|^2 dt \\
&\qquad \le M\Big\{1 + \int_0^t \|\delta^\alpha \nabla(\zeta u)\|^2 dt + \int_0^t \|\delta^{\alpha-1}\partial_n^2(\zeta u)\|^2 dt + \|u\|_{X_{m+2}(T')}\Big\},
\end{aligned}$$

which is (5.14).

5.4.2. We now prove (5.15). From equation (5.2) we deduce that

$$\partial_n^2 v = \frac{-1}{a_{nn}} \Big\{\zeta f - a\Phi - v_t + \sum_{ij \ne nn} a_{ij}\partial_i\partial_j\, v\Big\} \equiv \varphi(\zeta f - v_t) + \psi(\Phi + \nabla\delta v), \tag{5.18}$$

where $\varphi = \varphi(\nabla u)$ and $\psi = \psi(\nabla u)$ satisfy

$$\varphi, \psi \in X_m(T), \quad \varphi(\cdot,t), \psi(\cdot,t) \in C^1(\overline{\Omega}) \quad \text{for} \quad 0 \le t \le T, \tag{5.19}$$

(the claim that $\psi \in X_m(T)$ can easily be proven by means of the standard Gagliardo-Nirenberg inequalities; note also that $a_{nn}(\nabla u) \in C(\overline{Q})$ by (2.9), and that (2.2) implies $a_{nn}(\nabla u) \ge \nu > 0$, so that(5.18) makes sense).

Since $f \in L^2(0,T;H^s)$, we easily deduce that $\varphi\zeta f \in L^2(0,T;H^{\alpha-1})$; to estimate $\delta^{\alpha-1}(\varphi v_t)$, it is sufficient to estimate $J_2 \doteq |\delta^{\alpha-1-j}\varphi\, \delta^j(\zeta u_t)|_2$ for

$0 \le j \le \alpha - 1 \le m$. If $\alpha = 1$, we set $p = q = \infty$; otherwise, we define p and $q \ge 2$ by $\frac{1}{p} = \frac{\alpha-1-j}{2m}$ and $\frac{1}{q} = \frac{j}{2(\alpha-1)}$, and check that $\frac{1}{p} + \frac{1}{q} \le \frac{1}{2}$ because $j \le \alpha - 1 \le m$; thus, since

$$|\delta^{\alpha-1-j}\varphi|_p \le C|\partial^m\varphi|_2^{2/p}|\varphi|_\infty^{1-2/p} + C|\varphi|_\infty,$$
$$|\delta^j(\zeta u_t)|_q \le C|\delta^{\alpha-1}(\zeta u_t)|_2^{2/q}|\zeta u_t|_\infty^{1-2/q} + C|\zeta u_t|_\infty,$$

recalling (2.12) and the m-induction assumption (2.11) we deduce that

$$\int_0^t \|\delta^{\alpha-1}(\varphi\zeta u_t)\|^2 dt \le M\Big(1 + \int_0^t \|\delta^{\alpha-1}(\zeta u_t)\|^2 dt\Big), \tag{5.20}$$

in accord with the right side of (5.15).

To conclude the proof of (5.15), recalling (5.18) we need to estimate $\delta^{\alpha-1}(\psi(\Phi + \nabla\delta v))$; however, we can proceed exactly in the same way as above, replacing ζu_t with $\nabla\delta(\zeta u)$, and obtain the estimate

$$\int_0^t \|\delta^{\alpha-1}(\psi\nabla\delta(\zeta u)\|^2 dt \le M\Big(1 + \int_0^t \|\delta^\alpha\nabla(\zeta u)\|^2 dt\Big). \tag{5.21}$$

Together with (5.20), (5.21) allows us to conclude the proof of(5.15) and, therefore, of (5.13); consequently, (5.4) holds.

5.5. Having thus estimated the tangential derivatives of ζu, we estimate its normal derivatives by means of

Proposition 5.2. *Assume* (2.11) *and* (3.9) *hold. For all* $\eta > 0$, *there exists* $M \in \Gamma$ *such that for all* $\zeta \in \mathcal{Z}$, $\ell = 0, \dots, m$ *and* $t \in [0, T']$,

$$\int_0^t \|\delta^{m-\ell}\,\partial_n^{\ell+2}(\zeta u)\|^2 dt \le M + \eta\|u\|^2_{X_{m+2}(T')}. \tag{5.22}$$

Again, we note that inequality (5.22) makes sense for $u \in X_{s+1}(T')$, since then $\delta^{m-\ell}\,\partial_n^{\ell+2}(\zeta u) \in L^2(0,T;H^{s-m}) \subseteq L^2(0,T;L^2)$.

Proof. We proceed by induction on ℓ. For $\ell = 0$, (5.22) is a consequence of (5.15), (5.4), (5.6) with $\alpha = m+1$; thus, we assume that (5.22) holds for $\ell - 1$; i.e., that

$$\int_0^t \|\delta^{m-r}\partial_n^{r+2}(\zeta u)\|^2 dt \le M + \eta\|u\|^2_{X_{m+2}(T')} \tag{5.23}$$

for $0 \le r \le \ell - 1$, and proceed to prove (5.22) for ℓ. From (5.18) we have

$$\delta^{m-\ell}\,\partial_n^{\ell+2}\,v = \delta^{m-\ell}\,\partial_n^{\ell}\{\varphi(\zeta f - v_t) + \psi(\Phi + \nabla\delta v)\}; \tag{5.24}$$

since $f \in L^2(0,T;H^s)$, it is not difficult to prove that $\delta^{m-\ell}\partial_n^{\ell}(\varphi\zeta f) \in L^2(0,T,L^2)$. We can estimate $\delta^{m-\ell}\partial_n^{\ell}(\psi\Phi)$ exactly as in Section 5.4, where we did not need to distinguish between tangential and normal derivatives. Thus, we have

$$\int_0^t \|\delta^{m-\ell}\,\partial_n^{\ell}\big(\varphi\zeta f + \psi\Phi\big)\|^2 \le M, \tag{5.25}$$

with $M \in \Gamma$. To estimate the term $|\delta^{m-\ell}\partial_n^{\ell}(\varphi v_t)|_2$ of (5.24), we decompose

$$\delta^{m-\ell}\,\partial_n^{\ell}(\varphi\zeta u_t) = \sum_{j=0}^{m-\ell}\sum_{i=0}^{\ell}\binom{m-\ell}{j}\binom{\ell}{i}\delta^{m-\ell-j}\partial_n^{\ell-i}(\varphi\zeta)\cdot\delta^j\partial_n^i u_t; \tag{5.26}$$

setting then $\frac{1}{p} = \frac{m-i-j}{2m}$, $\frac{1}{q} = \frac{i+j}{2m} = \frac{1}{2} - \frac{1}{p}$, we estimate

$$|\delta^{m-\ell-j}\partial_n^{\ell-i}(\varphi\zeta)|_p \le C|\partial^m(\varphi\zeta)|_2^{2/p}|\varphi\zeta|_\infty^{1-2/p} + C|\varphi\zeta|_\infty,$$
$$|\delta^j\partial_n^i(\varphi\zeta)|_q \le C|\partial^m u_t|_2^{2/q}|u_t|_\infty^{1-2/q} + C|u_t|_\infty,$$

so that, by (2.12) and the m-induction assumption (2.11) we obtain that

$$J_3^2 \doteq \int_0^t \|\delta^{m-\ell-j}\partial_n^{\ell-i}(\varphi\zeta)\,\delta^j\partial_n^i u_t\|^2 dt \le M\Big(1 + \int_0^t \|u_t\|_m^2 dt\Big).$$

Acting then as in (5.5) and recalling (3.4), renaming the constants we obtain that

$$J_3 \le M + \eta\|u\|_{X_{m+2}(T')}, \tag{5.27}$$

with $M \in \Gamma$. Finally, we estimate the last term of (5.24) in the same way: decomposing it as in (5.26) and keeping the same values of p and q, we are led to the estimates

$$\begin{aligned} J_4 &\doteq |\delta^{m-\ell-j}\,\partial_n^{\ell-i}\psi\cdot\delta^j\partial_n^i\nabla\delta(\zeta u)|_2 \le M|\delta^{m-\ell-j}\,\partial_n^{\ell-i}\psi|_p\,|\delta^j\partial_n^i\nabla\delta(\zeta u)|_q \\ &\le M\big(|\partial^m\psi(\nabla u)|_2^{2/p}|\psi(\nabla u)|_\infty^{1-2/p} + |\psi(\nabla u)|_\infty\big) \\ &\times\big(|\partial^m\nabla\delta(\zeta u)|_2^{2/q}|\nabla\delta(\zeta u)|_\infty^{1-2/q} + |\nabla\delta(\zeta u)|_\infty\big). \end{aligned} \tag{5.28}$$

If $i + j \le m - 1$, so that $\frac{2}{q} < 1$, we obtain that, for $\eta > 0$ and $M \in \Gamma$,

$$J_4^2 \le M + \eta\int_0^t \|\partial^{m+2}(\zeta u)\|^2 dt \le M + C\eta\|u\|_{X_{m+2}(T')}^2, \tag{5.29}$$

with C independent of η; if $i+j=m$, from the inequalities $0 \le j = m-i \le m-\ell$ and $0 \le i \le \ell$ we see that $i=\ell$ and $j=m-\ell$, so that

$$J_4 = |\psi\, \delta^{m-\ell} \partial_n^\ell \nabla \delta(\zeta u)|_2 \le M\big(|\delta^{m-\ell+2}\partial_n^\ell(\zeta u)|_2 + |\delta^{m-\ell+1}\partial_n^{\ell+1}(\zeta u)|_2\big),$$

by the ℓ-induction assumption (5.23) with $r=\ell-2$ and $r=\ell-1$, we conclude that (5.29) still holds. Hence, (5.22) follows from (5.25), (5.23) and (5.29), and the proof of Proposition 5.2 is complete.

5.6. We can now conclude the proof of Proposition 3.3: since we are assuming $u \in L^2(0,T;H^{m+1})$, to estimate u in $L^2(0,T;H^{m+2})$ it is sufficient to estimate its derivatives of order $m+2$, which was precisely the purpose of Propositions 5.1 and 5.2. Thus, by (5.6) and (5.22) we have that, for each $\zeta \in \mathcal{Z}$,

$$\sum_{i=0}^{m+2} |\delta^{m+2-i}\, \partial_n^i(\zeta u)|_2 \le |\delta^{m+2}(\zeta u)|_2 \\ + |\delta^{m+1}\nabla(\zeta u)|_2 + \sum_{\ell=0}^{m} |\delta^{m-\ell}\partial_n^{\ell+2}(\zeta u)|_2 \le M + \eta \|u\|^2_{X_{m+2}(T')},$$

with $M \in \Gamma$; thus, writing $u = \sum_{j=0}^N \zeta_j u$, we deduce that

$$\int_0^{T'} \|\partial^{m+2} u\|^2 dt \le M + \eta \|u\|^2_{X_{m+2}(T')},$$

which allows us to prove (3.5).

6. Proof of Proposition 3.4.

6.1. We rewrite(3.7) for $r=k+1$ as

$$\partial_t^{k+2} u = \partial_t^{k+1} f \, + \sum_{l=0}^{k+1} \binom{k+1}{\ell} A_\ell, \tag{6.1}$$

where $A_\ell = \partial_t^{k+1-\ell} a(\nabla u)\partial_t^\ell \partial^2 u$. If $\ell = k+1$, we integrate by parts: given any $\phi \in L^2(0,T';H_0^1)$, we write

$$\int_0^{T'} (A_{k+1},\phi)\, dt = -\int_0^{T'} (a(\nabla u)\partial_t^{k+1}\partial u, \partial\phi)\, dt \\ - \int_0^{T'} (a'(\nabla u)\cdot \nabla\partial u \partial_t^{k+1}\partial u, \phi)\, dt;$$

since $|a(\nabla u)|_\infty$, $|a'(\nabla u)\cdot\nabla\partial u|_\infty \in \Gamma$, recalling (3.4) we have that

$$\|A_{k+1}\|^2_{L^2(0,T';H^{-1})} \le M\int_0^{T'} \|\nabla\partial_t^{k+1}u\|^2 dt \le M + \eta\|u\|^2_{X_{m+2}(T')}. \quad (6.2)$$

If $\ell = 0$, we have $A_0 = \partial_t^{k+1}a(\nabla u)\partial^2 u$, so that by Proposition 2.8 we obtain

$$|A_0|_2 \le M|\partial_t^{k+1}a(\nabla u)|_2 \le M(1+|\partial_t^{k+1}\nabla u|_2) \le M + \eta\|u\|_{X_{m+2}(T')}; \quad (6.3)$$

thus, we can assume that $1 \le \ell \le k$. Defining p, q, α and $\beta \ge 1$ by

$$\frac{1}{\alpha} = \frac{\ell-1}{m}, \quad \frac{1}{p} = \frac{1}{m+2-2\ell}\Big(\frac12 - \frac{1}{\alpha}\Big) + \frac{1}{\alpha} = \frac{2\ell-1}{2m},$$
$$\frac{1}{\beta} = \frac{k-\ell}{m}, \quad \frac{1}{q} = \frac{2}{2\ell+1}\Big(\frac12 - \frac{1}{\beta}\Big) + \frac{1}{\beta} = \frac{m-2\ell+1}{2m},$$

we have that $\frac1p + \frac1q = \frac12$ and therefore, by Proposition 2.8,

$$|A_\ell|_2^2 \le C|\partial_t^\ell a(\nabla u)|_p|\partial_t^{k+1-\ell}\partial^2 u|_q \le M(1+|\partial_t^\ell\nabla u|_p)|\partial_t^{k+1-\ell}\partial^2 u|_q. \quad (6.4)$$

Setting $\theta_p = \frac{1}{m+2-\ell}$, we easily verify that $\frac{2}{p\theta_p} \ge 1$ (because $m \ge 3$), and $\frac{2p(1-\theta_p)}{2-\theta_p} = \alpha$, so that, by the standard Gagliardo-Nirenberg inequalities (2.14) and by Hölder's inequality, we have

$$\begin{aligned}|\partial_t^\ell\nabla u|_p^p &\le M\int_0^t \|\partial_t^\ell u\|^{p\theta_p}_{m+2-2\ell}\|\partial_t^\ell u\|^{p(1-\theta_p)}_{L^\alpha(\Omega)}dt \\ &\le M\|\partial_t^\ell\|^{p\theta_p}_{L^2(0,T';H^{m+2-2\ell})}\|\partial_t^\ell u\|^{p(1-\theta_p)}_{L^\alpha(Q)},\end{aligned}$$

with $M \in \Gamma$; therefore, by Proposition 2.5,

$$\begin{aligned}|\partial_t^\ell\nabla u|_p &\le M\|u\|^{\theta_p}_{X_{m+2}(T')}|\partial_t^\ell u|_\alpha^{1-\theta_p} \\ &\le M\|u\|^{\theta_p}_{X_{m+2}(T')}\big(\|u\|^{2/\alpha}_{H^{k+3/2}(0,T';L^2(\Omega))}|u_t|_\infty^{1-2/\alpha}\big)^{1-\theta_p} \\ &\le M\|u\|^{\theta_p}_{X_{m+2}(T')}\|u\|^{2(1-\theta_p)/\alpha}_{H^{k+3/2}(0,T';L^2(\Omega))},\end{aligned} \quad (6.5)$$

with $M \in \Gamma$. We estimate the last term of (6.4) analogously: setting $\theta_q = \frac{2}{2\ell+1}$, we verify that $\frac{2}{q\theta_q} \ge 1$, and estimate

$$\begin{aligned}|\partial_t^{k+1-\ell}\partial^2 u|_q^q &\le M\int_0^t \|\partial_t^{k+1-\ell}u\|^{q\theta_q}_{2\ell+1}\|\partial_t^{k+1-\ell}u\|^{q(1-\theta_q)}_{L^\beta(\Omega)}dt \\ &\le M\|\partial_t^{k+1-\ell}u\|^{q\theta_q}_{L^2(0,T';H^{2\ell+1})}\|\partial_t^{k+1-\ell}u\|^{q(1-\theta_q)}_{L^\beta(Q)},\end{aligned}$$

so that, as before,

$$\begin{aligned} |\partial_t^{k+1-\ell}\partial^2 u|_q &\le M\|u\|_{X_{m+2}(T')}^{\theta_q}|\partial_t^{k+1-\ell}u|_\beta^{1-\theta_q} \\ &\le M\|u\|_{X_{m+2}(T')}^{\theta_q}\left(\|u\|_{H^{k+3/2}(0,T';L^2(\Omega))}^{2/\beta}|u_t|_\infty^{1-2/\beta}\right)^{1-\theta_q} \\ &\le M\|u\|_{X_{m+2}(T')}^{\theta_q}\|u\|_{H^{k+3/2}(0,T';L^2(\Omega))}^{2(1-\theta_q)/\beta}, \end{aligned} \tag{6.6}$$

with $M \in \Gamma$. From (6.5) and (6.6) we have then

$$|\partial_t^\ell \nabla u|_p |\partial_t^{k+1-\ell}\partial^2 u|_q \le M\|u\|_{X_{m+2}(T')}^{\theta_p+\theta_q}\|u\|_{H^{k+3/2}(0,T';L^2(\Omega))}^{2\theta}, \tag{6.7}$$

with $\theta = \frac{1}{\alpha}(1-\theta_p)+\frac{1}{\beta}(1-\theta_q)$. We compute that $(\theta_p+\theta_q)+2\theta = 1$, because

$$\frac{1}{2} = \frac{1}{p}+\frac{1}{q} = \frac{1}{2}\theta_p + \frac{1}{\alpha}(1-\theta_p) + \frac{1}{2}\theta_q + \frac{1}{\beta}(1-\theta_q);$$

moreover, $\theta_p+\theta_q > 0$, and we easily verify that $\theta_p+\theta_q < 1$, unless $m = 3$. Thus, if $m > 3$, recalling also (2.20) we deduce from (6.7) that, for ϵ_1, $\epsilon_2 > 0$,

$$\begin{aligned} &|\partial_t^\ell \nabla u|_p |\partial_t^{k+1-\ell}\partial^2 u|_q \\ &\le \epsilon_1\|u\|_{X_{m+2}(T')} + C_{\epsilon_1}\|\partial_t^{k+1}\nabla u\|_{L^2(0,T';L^2)}^{1/2}\|\partial_t^{k+2}u\|_{L^2(0,T';H^{-1})}^{1/2} \\ &\le \epsilon_1\|u\|_{X_{m+2}(T')} + C_{\epsilon_1\epsilon_2}\|\partial_t^{k+1}\nabla u\|_{L^2(0,T';L^2)} + \epsilon_2\|\partial_t^{k+2}u\|_{L^2(0,T';H^{-1})}. \end{aligned}$$

Putting this together with (6.4), (6.3) and (6.2) and renaming the constants, we finally obtain that, if $m > 3$,

$$\int_0^t \|\partial_t^{k+2}u\|_{-1}^2 dt \le M + \eta\|u\|_{X_{m+2}(T')}^2 + \epsilon\int_0^t \|\partial_t^{k+2}u\|_{-1}^2 dt,$$

from which (3.5) follows. If $m = 3$, then $\ell = k = 1$, $A_1 = a'(\nabla u)\nabla u_t \partial^2 u_t$, and we modify the above procedure as follows:

$$\begin{aligned} \int_0^t \|A_1\|^2 dt &\le M\int_0^t \|\nabla u_t\|_{L^6(\Omega)}^2\|\partial^2 u_t\|_{L^3(\Omega)}^2 dt \\ &\le M\int_0^t \|D^3 u_t\|_{L^2(\Omega)}^{2/3}\|u_t\|_{L^\infty(\Omega)}^{4/3}\|D^3 u_t\|_{L^2(\Omega)}^{4/3}\|u_t\|_{L^\infty(\Omega)}^{2/3} dt \\ &\le M\int_0^t \|u_t\|_3^2 dt, \end{aligned}$$

with $M \in \Gamma$; applying then Proposition 2.6, we obtain that

$$\int_0^t \|A_1\|^2 dt \le \|u\|^{1/2}_{L^2(0,T';H^5)} \big(\|u_{tt}\|^{1/2}_{L^2(0,T';H^1)} + \|u\|^{1/2}_{L^2(0,T';H^1)} \big) \le \eta \|u\|^2_{X_{m+2}(T')} + M,$$

because of (3.4) with $k = 1$ and $m = 3$. This allows us to conclude the proof of Proposition 3.4 (and, therefore, of Proposition 3.1 and Theorem 4).

7. Appendix: the specialized Gagliardo-Nirenberg inequalities.

In this section we prove Propositions 2.2, 2.3, 2.4, 2.8, 2.5 and 2.6.

7.1. Proof of Propositions 2.2 and 2.3.

a) Assume first that $s < \infty$: for each $x \in \Omega$ we apply the standard Gagliardo - Nirenberg inequality (2.14) to $u(x, \cdot)$ to obtain

$$\int_a^b |\partial_t^j u(x,t)|^q dt \le C \Big\{ \int_a^b |\partial_t^m u(x,t)|^p dt \Big\}^{\frac{q}{p} : \frac{j}{m}} \Big\{ \int_a^b |u(x,t)|^r dt \Big\}^{\frac{q}{r}(1-\frac{j}{m})} + C \Big\{ \int_a^b |u(x,t)|^s dt \Big\}^{q/s}; \tag{7.1}$$

we write (2.13) as

$$1 = \frac{j}{m} : \frac{q}{p} + \frac{q}{r}\left(1 - \frac{j}{m}\right) \equiv \frac{1}{\lambda} + \frac{1}{\mu}, \tag{7.2}$$

and integrate (7.1) in Ω, applying the Cauchy-Schwartz inequality to the functions

$$\varphi(x) \doteq \Big\{ \int_a^b |\partial_t^m u(x,t)|^p dt \Big\}^{\frac{1}{\lambda}}, \quad \psi(x) \doteq \Big\{ \int_a^b |u(x,t)|^r dt \Big\}^{\frac{1}{\mu}}.$$

Since, as we easily check, $q \le s$, we obtain (for different C's, and $\theta = 1 - \frac{q}{s} \ge 0$):

$$\begin{aligned}
\int_\Omega \int_a^b |\partial_t^j u(x,t)|^q dt\, dx &\le C \int_\Omega \varphi(x)\, \psi(x)\, dx + C \int_\Omega \Big\{ \int_a^b |u(x,t)|^s dt \Big\}^{\frac{q}{s}} dx \\
&\le C \Big\{ \int_\Omega |\varphi(x)|^\lambda dx \Big\}^{\frac{1}{\lambda}} \Big\{ \int_\Omega |\psi(x)|^\mu dx \Big\}^{\frac{1}{\mu}} \\
&\quad + C(\mathrm{meas}(\Omega))^\theta \Big\{ \int_\Omega \int_a^b |u(x,t)|^s dt dx \Big\}^{\frac{q}{s}} \\
&\le C \Big\{ \int_\Omega \int_a^b |\partial_t^m u(x,t)|^p dt\, dx \Big\}^{\frac{1}{\lambda}} \Big\{ \int_\Omega \int_a^b |u(x,t)|^r dt\, dx \Big\}^{\frac{1}{\mu}} \\
&\quad + C \Big\{ \int_\Omega \int_a^b |u(x,t)|^s dt\, dx \Big\}^{\frac{q}{s}},
\end{aligned}$$

from which (2.15) follows by Fubini's theorem and (7.2). If $s=\infty$, the proof is analogous; note that if $q=\infty$, (2.13) implies that $j=0$ and $r=\infty$, in which case (2.15) is obvious.

b) (2.16) of Proposition 2.3 is proved in the same way, considering Ω as embedded in a cylinder $Q_k=\Omega_k\times(a,b)$, with $\Omega_k=\{x\in\Omega\mid x_k=0\}$. Finally, (2.17) is proven as (2.15), but applying first (2.14) to the function $h(\hat{x})=u(\hat{x},x_k)$ in Ω_k, and then integrating in (a,b) with respect to x_k.

7.2. Proof of Propositions 2.4 and 2.8.

a) With abuse of notation, by (2.14) in Ω we have, if $s<\infty$:

$$
\begin{aligned}
\|\partial^\beta\partial_t^\ell u\|^q_{L^q(Q)} &= \int_a^b \|\partial^\beta\partial_t^\ell u\|^q_{L^q(\Omega)}dt \\
&\le C\int_a^b \|\partial^\alpha\partial_t^\ell u\|^{q\frac{j}{m}}_{L^p(\Omega)}\,\|\partial_t^\ell u\|^{q(1-\frac{j}{m})}_{L^r(\Omega)}dt + C\int_a^b\|\partial_t^\ell u\|^q_{L^s(\Omega)}dt \\
&\le C\Big\{\int_a^b \|\partial^\alpha\partial_t^\ell u\|^{q\frac{j}{m}\lambda}_{L^p(\Omega)}dt\Big\}^{\frac{1}{\lambda}}\Big\{\int_a^b\|\partial_t^\ell u\|^{q(1-\frac{j}{m})\mu}_{L^r(\Omega)}\,dt\Big\}^{\frac{1}{\mu}} \\
&\quad + C\Big(\int_a^b\|\partial_t^\ell u\|^s_{L^s(\Omega)}dt\Big)^{q/s},
\end{aligned}
$$

for $\frac{1}{\lambda}+\frac{1}{\mu}=1$. Choosing $\lambda=\frac{mp}{qj}$ (≥ 1), recalling (2.13) we check that $q(1-\frac{j}{m})\mu=r$, so that

$$
\|\partial^\beta\partial_t^\ell u\|^q_{L^q(Q)} \le C\|\partial^\alpha\partial_t^\ell u\|^{p/\lambda}_{L^p(Q)}\,\|\partial_t^\ell u\|^{r/\mu}_{L^r(Q)} + C\|\partial_t^\ell u\|^q_{L^s(Q)},
$$

from which (2.18) follows. The case $s=\infty$ is analogous.

b) As in the proof of Proposition 2.2, fix $x\in\Omega$: by Proposition 2.7 applied to $u(x,\cdot)$ on (a,b), there exists a constant $c_r(x)>0$, depending on the norm of Φ in $C^m\big(B(x)\big)$, $B(x)=[-r(x),r(x)]$, $r(x)=|u(x,\cdot)|_{L^\infty(a,b)}$, such that

$$
\int_a^b|\partial_t^r\Phi\big(u(x,t)\big)|^\mu\,dt \le c_r(x)\Big\{\int_a^b|\partial_t^r u(x,t)|^\mu\,dt+1\Big\}. \tag{7.3}
$$

Since $|u(x,\cdot)|_{L^\infty(a,b)}\le|u|_{L^\infty(Q)}$, we have $B(x)\subseteq B$, and therefore there exists C_r such that $c_r(x)\le C_r$ $\forall x\in\Omega$. Consequently, we can integrate (7.3) in Ω, to obtain

$$
\int_\Omega\int_a^b|\partial_t^r\Phi\big(u(x,t)\big)|^\mu dt\,dx \le C_r\Big\{\int_\Omega\int_a^b|\partial_t^r u(x,t)|^\mu dt\,dx+1\Big\},
$$

from which (2.23) follows.

7.3. Proof of Proposition 2.5. If $j = 0$ the result is obvious; if s is an integer, the result follows from Proposition 2.2; thus, we shall assume that $j > 0$ and s is not an integer.

(a) We start with the one-dimensional case: assuming that $u \in H^s(a, b) \cap L^\infty(a, b)$, we claim that

$$\|\partial_t^j u\|_{L^q(a,b)} \le C\|u\|_{H^s(a,b)}^{j/s}\,\|u\|_{L^\infty(a,b)}^{1-j/s}, \tag{7.4}$$

with j and q as above. By considering a proper extension of u, it is sufficient to establish (2.19) when $(a, b) = \mathbb{R}$; to this end, following Triebel, [20], we resort to a series of results on interpolation theory between the Sobolev spaces $W^{m,p}$, the Bessel potential spaces H_p^s, the Besov-Triebel spaces $F_{p,q}^s$ and the inhomogeneous space *bmo* of functions with bounded mean oscillations. As in Triebel, [19], we define $\theta \in (0, 1)$, $p \ge 1$ and $q > 1$ by $(1 - \theta)s = j$, $\frac{1}{q} = \frac{1-\theta}{2}$, and $\frac{1}{p} = 1 - \frac{1}{q}$: then, applying subsequently formulas (11) of page 37, (2) of page 51, (2) of page 178 and (15) of page 93 of [19], we obtain that

$$[H^s, bmo]_\theta = [H_2^s, bmo]_\theta = [F_{2,2}^s, bmo]_\theta = [(F_{2,2}^{-s})', h_1']_\theta; \tag{7.5}$$

since $F_{2,2}^{-s} = (H_2^s)'$ is reflexive, by duality we obtain that

$$[H^s, bmo]_\theta = [F_{2,2}^{-s}, h_1]_\theta'\,; \tag{7.6}$$

applying then formulas (6) of page 92, (3) of page 69 (note that the value $p_1 = 1$ is admissible), (2) of page 178, (2) of page 51 and (19) of page 38 of[19] again, we proceed to obtain from (7.6) that

$$[H^s, bmo]_\theta = [F_{2,2}^{-s}, F_{1,2}^0]_\theta' = (F_{p,2}^{-(1-\theta)s})' = F_{q,2}^j = H_q^j = W^{j,q},$$

the last equality holding because $j \in \mathbb{N}$. Consequently, for all $U \in H^s(\mathbb{R}) \cap bmo(\mathbb{R})$

$$\|\partial_t^j U\|_{L^q(\mathbb{R})} \le C\|U\|_{H^s(\mathbb{R})}^{j/s}\,\|U\|_{bmo(\mathbb{R})}^{1-j/s}; \tag{7.7}$$

since $L^\infty(\mathbb{R}) \hookrightarrow bmo(\mathbb{R})$, we deduce from (7.7) that, if $U \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$,

$$\|\partial_t^j U\|_{L^q(\mathbb{R})} \le C_1\|U\|_{H^s(\mathbb{R})}^{j/s}\,\|U\|_{L^\infty(\mathbb{R})}^{1-j/s}, \tag{7.8}$$

from which (2.19) follows, considering any proper extension U of u.

(b) Let now $u \in H^s\big(a,b;L^2(\Omega)\big)\cap L^\infty(Q)$, and apply (2.19) to the function $u(x,\cdot)$, where $x \in \Omega$ is fixed. We obtain

$$\int_a^b |\partial_t^j u(x,t)|^q dt \le C\|u(x,\cdot)\|^2_{H^s(a,b)}\,\|u(x,\cdot)\|^{q-2}_{L^\infty(a,b)},$$

from which

$$I = \int_\Omega\int_a^b |\partial_t^j u(x,t)|^q dt\,dx \le C\|u\|^{q-2}_{L^\infty}(Q)\int_\Omega \|u(x,\cdot)\|^2_{H^s(a,b)}dx. \tag{7.9}$$

We now have that

$$\int_\Omega \|u(x,\cdot)\|^2_{H^s(a,b)}dx \le \|u\|^2_{H^s(a,b;L^2(\Omega))}. \tag{7.10}$$

Indeed, if $U \in H^s\big(\mathbb{R};L^2(\Omega)\big)$ is any extension of u outside (a,b), by Fubini's theorem we have

$$\begin{aligned}&\int_\Omega \|u(x,\cdot)\|^2_{H^s(a,b)}\,dx \le \int_\Omega \|U(x,\cdot)\|^2_{H^s(\mathbb{R})}\,dx\\ &= \int_\Omega\int_\mathbb{R}\big(1+|\tau|^2\big)^s|\hat U(x,\tau)|^2\,d\tau\,dx = \int_\mathbb{R}(1+|\tau|^2)^s\int_\Omega|\hat U(x,\tau)|^2\,dx\,d\tau\\ &= \int_\mathbb{R}\big(1+|\tau|^2\big)^s\,\|\hat U(\cdot,\tau)\|^2_{L^2(\Omega)}\,d\tau = \|U\|^2_{H^s(\mathbb{R};L^2(\Omega))},\end{aligned}$$

so that (7.10) follows by taking the infimum over all extensions U of u. Estimate (2.19) follows then from (7.9) and (7.10).

7.4. Proof of Proposition 2.6. We can assume for simplicity that $l_1 = 0$; the general case would follow then by replacing $\tilde u = \partial_t^{l_1}u$, $\tilde l_2 = l_2 - l_1$, $\tilde l = l - l_1$. Then if $l_1 = l_2 = 0$, the proof follows from Proposition 2.2; if $l_2 > 0$, by Theorems 2.3 and 12.4 of [11] we obtain that

$$\|\partial_t^l u\|_{L^2(I,H^k(\Omega))} \le C(\|u\|_{L^2(I,H^{k_1}(\Omega))} + \|\partial_t^{l_2}u\|_{L^2(I,H^{k_2}(\Omega))}), \tag{7.11}$$

with C independent of u. Assume first that $I = \mathbb{R}$: given then $\delta > 0$, we set $u_\delta(t,x) = u(\delta t, x)$ and apply (7.11) to u_δ, so as to obtain

$$\|\partial_t^l u\|_{L^2(\mathbb{R},H^k(\Omega))} \le C(\delta^{-l}\|u\|_{L^2(\mathbb{R},H^{k_1}(\Omega))} + \delta^{l_2-l}\|\partial_t^{l_2}u\|_{L^2(\mathbb{R},H^{k_2}(\Omega))}); \tag{7.12}$$

this is an inequality of the type

$$A \le \delta^{-l}B + \delta^{l_2-l}C,$$

so that, choosing $\delta = (B/C)^{1/l_2}$ we obtain

$$\|\partial_t^l u\|_{L^2(I,H^k(\Omega))} \le C\|\partial_t^{l_1} u\|^{\theta}_{L^2(I,H^{k_1}(\Omega))}\|\partial_t^{l_2} u\|^{1-\theta}_{L^2(I,H^{k_2}(\Omega))}. \tag{7.13}$$

If I is an arbitrary interval, by [11], Theorem 2.2 (and its proof), we can extend u to $\hat{u}$ defined on $\mathbb{R}$ in such a way that

$$\|\hat{u}\|_{L^2(\mathbb{R},H^{k_1}(\Omega))} \le C\|u\|_{L^2(I,H^{k_1}(\Omega))}, \tag{7.14}$$

$$\|\partial_t^{l_2}\hat{u}\|_{L^2(\mathbb{R},H^{k_2}(\Omega))} \le C(\|u\|_{L^2(I,H^{k_2}(\Omega))} + \|\partial_t^{l_2} u\|_{L^2(I,H^{k_2}(\Omega))}); \tag{7.15}$$

to verify these, we resort to extensions of u to half lines and a decomposition $1 = \alpha(t) + \beta(t)$, $t \in \mathbb{R}$, with α and β identically equal to 1 on such half lines, as in the proof of Theorem 2.1 of [11]. Then, (2.21) follows easily from (7.13), (7.14) and (7.15).

Acknowledgments. This paper was written while the second author was visiting the TH Darmstadt with a grant of the Alexander von Humboldt Foundation. Their support, as well as the kind hospitality and efficiency of the AG6 of the Mathematics Department, is gratefully appreciated. We are also grateful for partial support from the S. Banach Center of the Polish Academy of Sciences, and to Dr. A. Kalamajska of the University of Warsaw and the Referee for their important suggestions.

REFERENCES

[1] D. Gilbarg, N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order," 2nd ed., Springer-Verlag, Berlin, 1983.

[2] Guangchang Dong, *Nonlinear partial differential equations of second order,* Transl. AMS, 95, Am. Math. Soc., Providence, RI, 1991.

[3] P. Hajlasz, PhD Thesis, Warsaw University, 1994.

[4] M. Ikawa, *Mixed problems for hyperbolic equations of second order,* J. Math. Soc. Japan, 20(4) (1968), 580–608.

[5] T. Kato, "Abstract Differential Equations and Nonlinear Mixed Problems," Fermian Lectures, Sc. Norm. Sup., Pisa, 1986.

[6] N.V. Krylov, "Nonlinear Elliptic and Parabolic Equations of Second Order," D. Reidel, Dordrecht (NL), 1987.

[7] O.A. Ladyzenskaya, V.A. Solonnikov, and N.N. Ural'tseva, *Linear and quasilinear equations of parabolic type,* Transl. AMS, vol. 23, Am. Math. Soc. Providence, RI, 1968.

[8] O.A. Ladyzenskaya and N.N. Ural'tseva, *Estimates of* $\max|u_x|$ *for the solution of quasilinear elliptic and parabolic equations of the general form, and existence theorems,* Zap. Nauch. Sem. Len. Otd. Mat. Inst., 138 (1984), 90–107. (transl.)

[9] O.A. Ladyzenskaya and N.N. Ural'tseva, *Estimates of the Hölder constant for functions satisfying a uniformly elliptic or a uniformly parabolic quasilinear inequality with unbounded coefficients,* Zap. Nauch. Sem. Len. Otd. Mat. Inst., 147 (1985), 72–94. (transl.)

[10] O.A. Ladyzenskaya and N.N. Ural'tseva, *On the solvability of the first boundary value problem for quasilinear elliptic and parabolic equations in the presence of singularities,* Sov. Math. Dokl., 31(2) (1985), 296–300.

[11] J.L. Lions and E. Magenes, "Problèmes aux Limites Non Homogènes et Applications," vol. I, Dunod, Paris, 1968.

[12] A. Matsumura, *Global existence and asymptotics of the solutions of second order quasilinear hyperbolic equations with first order dissipation term,* Publ. Res. Inst. Math. Sci., Kyoto Univ., 13 (1977), 349–379.

[13] K. McLeod and A. Milani, *A global existence result for quasilinear parabolic equations,* Comm. PDEs, 15(4) (1990), 557–594.

[14] K. McLeod and A. Milani, *On some global well-posedness and asymptotic results for quasilinear parabolic equations,* NoDEA, 3 (1996), 79–114.

[15] K. McLeod and A. Milani, *Global Sobolev solutions of quasilinear parabolic equations,* preprint version of this paper, available at: ftp://www.uwm.edu/pub/Math/ajmilani/preprintpar.dvi.

[16] A. Milani, *A regularity result for strongly elliptic systems,* Boll. U.M.I., 6(2-B) (1983), 641–651.

[17] L. Nirenberg, *On elliptic partial differential equations,* Ann. Sc. Norm. Pisa, 13 (1959), 115–162.

[18] L. Nirenberg, *An extended interpolation inequality,* Ann. Sc. Norm. Pisa, 20 (1966), 733–737.

[19] H. Triebel, "Theory of Function Spaces," Birkhäuser Verlag, Basel, 1983.

[20] H. Triebel, *Personal communication.*