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ON THE COUNT AND THE CLASSIFICATION OF PERIODIC SOLUTIONS TO FORCED PENDULUM–TYPE EQUATIONS

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Abstract. We provide a method for the count and the variational characterization of periodic solutions to forced pendulum-type equation. This is achieved by reducing the problem to the study of a real function of one variable. This paper generalizes some results obtained by G. Tarantello.

0. Introduction. Let $V : \mathbf{R} \to \mathbf{R}$ be a smooth, S-periodic function and let $h : \mathbf{R} \to \mathbf{R}$ be a continuous and T-periodic with zero mean value (i.e., $\int_0^T h dt = 0$).

The aim of this paper is to evaluate the exact number of solutions to

$$\ddot{u}(t) + V'(u(t)) = h(t) + \lambda, \quad u \ T\text{–periodic} \qquad (P_{h,\lambda})$$

where $\lambda \in \mathbf{R}$, under suitable assumptions on the period T and the forcing term h; at the same time we will provide some information on the variational nature of these solutions. Since $u + kS$ $(k \in \mathbf{Z})$ is a solution to $(P_{h,\lambda})$ wherever u is, of course the count will be intended up to S-translations.

It is well known that problem $(P_{h,\lambda})$ is solvable for λ in a closed, bounded interval $I_h = [d_h, D_h]$ containing zero ([7], [10], [11], [5]), and it has been shown recently that the nature of the problem is strongly affected by the possible degeneracy of I_h ([12], [13], and also [6] for a particular case). Here we look for the *exact number of solutions* to $(P_{h,\lambda})$ when the parameter λ ranges over I_h.

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For the pendulum equation

$$\ddot{u}(t) + A\sin(u) = h(t) + \lambda, \tag{0.1}$$

where $V(u) = -A\cos(u)$, the answer has been presented in [15], a paper where results are given also in the presence of friction terms in the equation. There, by taking advantage of the explicit form of the nonlinearity, the author proves that if

$$A < (\frac{2\pi}{T})^2 =: \omega^2, \quad \min\{\frac{T}{12}(1-\frac{A}{\omega^2})^{-1}||h||_1, \frac{\sqrt{T}}{2\sqrt{3}\omega}(1-\frac{A}{\omega^2})^{-1}||h||_2\} \leq \frac{\pi}{4},$$

then I_h is non degenerate, namely $d_h < 0 < D_h$, and the problem $(P_{h,\lambda})$ has

1. *exactly one solution* if $\lambda = d_h$ or $\lambda = D_h$;
2. *exactly two solutions* if $d_h < \lambda < D_h$.

The key argument in [15] is the fact that for h small, the situation is not too different from the case $h \equiv 0$, where the smallness condition on T guarantees that all the solutions are constant, so that they can be computed explicitly.

The meaning of the result in [15] is that under convenient assumptions on T and h, the number of solutions to (0.1) behaves exactly like the number of pre-images of $A\sin(x)$ (restricted to $[0, 2\pi[$). This remark which appears in [15], is the main motivation of the present work, its purpose being to achieve a similar statement for a more general class of periodic nonlinearities.

Of course, in replacing $A\sin(u)$ with a generic periodic function, even the unperturbed situation ($h \equiv 0$) becomes more complex to describe, and we need first of all to re-express the statement in a more convenient way.

It seems to us that the simplest way to reach our aim is to make use of a tool introduced in [3], [4] and developed in [13] based on the variational nature of problem $(P_{h,\lambda})$ and which enjoys various stability properties with respect to perturbations. This tool is the *reduction function* $\varphi_h : \mathbf{R} \to \mathbf{R}$ associated to problem $(P_{h,0})$ (see Section 2 for its definition). The link between our problem and the reduction function is given by the remarkable property that when

$$\max V'' < (\frac{2\pi}{T})^2, \tag{$H1$}$$

the solutions of $(P_{h,\lambda})$ are in 1-1 correspondence with the solutions (in ξ) of the equation

$$\varphi_h'(\xi) = -T\lambda$$

(see Proposition 2.1). Moreover, the situation for $h \equiv 0$ is explicitly known (and computable), since, as it will be proved in this case,

$$\varphi_0'(\xi) = -TV'(\xi).$$

We can therefore view the number of solutions to $(P_{h,\lambda})$ as the number of pre-images of φ_h', a function that is very likely to be similar, when h is small, to $\varphi_0' = -TV'$, which are data of the problem.

Having established this link, our main result (Theorem 3.1) states that "up to re–parametrizations" in the variable λ

$$\#\{\xi : \varphi_h'(\xi) = -T\lambda\} = \#\{\xi : \varphi_0'(\xi) = -T\lambda\} \qquad \forall\lambda.$$

Of course we need some smallness conditions on h, as does [15], and some suitable nondegeneracy hypotheses on the nonlinearity V', which are automatically satisfied for the pendulum equation and that are, anyhow, necessary even to obtain a perturbative result (see Lemma 1.5).

We wish to point out that our result, as well as those in [15], is not of a perturbative nature. Indeed much effort in this paper is devoted to obtain explicit estimates, in terms of V and T, of the set of forcing terms h to which our results apply.

Until now we have used the derivative φ_h' of the reduction function, instead of φ_h itself. Indeed, it wouldn't even be necessary to define the function φ_h in order to obtain the previous result; this is exactly the approach adopted in literature, namely the well known Lyapunov-Schmidt reduction method. The function φ_h, however, is a more general tool than its derivative (which, for instance, does not exist everywhere when $(H1)$ does not hold) and, above all, has much more direct and clear geometrical meaning in the variational context (see Section 2, [13], [14]). In Section 3 we will take advantage of this geometrical meaning to produce a very simple variational characterization of the solutions to $(P_{h,0})$, still in the spirit of [15].

Notations. We denote by H the Hilbert space $H^1_T(\mathbf{R};\mathbf{R})$ of T-periodic and $H^1_{loc}(\mathbf{R};\mathbf{R})$. $||u|| = ||u||_2 + ||\dot{u}||_2$ is its norm, where we use $||u||_p$ to denote the $L^P(0,T)$ norm. F or every u in $L^1(0,T)$ we write $\overline{u}$ for the mean value $\frac{1}{T}\int_0^T u(t)dt$. C^k_T will denote the space of T-periodic functions which are continuous with their k derivatives.

The space H splits orthogonally as $H = \mathbf{R} \oplus H_0$, where $\mathbf{R}$ is the space of constant functions and H_0 is the subspace of functions with zero mean value. Similarly, for $\xi \in \mathbf{R}$ we write H_ξ for the hyperplane of functions with mean value ξ. We write a generic element $u \in H$ as $u = \xi + w$ or $u = \xi + \tilde{u}$, where $\xi \in \mathbf{R}$ and w and $\tilde{u}$ are in H_0. The norm of $w \in H_0$ will be denoted by $||w||$, and we tacitly use the fact that it is equivalent to $||\dot{w}||_2$. Likewise, the norm of $u \in H$ is equivalent to $||\dot{u}||_2 + |\overline{u}|$. For notational convenience, sometimes we will write ω instead of $2\pi/T$.

1. Critical equivalence and stability. First of all we need to define a notion of equivalence of functions for which, roughly speaking, the numbers of the pre-images is equal up to parametrizations. Precisely:

Definition 1.1. Two Morse functions $f,\ g \in C^2_S(\mathbf{R};\mathbf{R})$ are said to be *critically equivalent* when

$$\#f^{-1}(\lambda) = \#g^{-1}(\eta(\lambda)) \qquad \forall\lambda$$

for a suitable increasing and continuous map $\eta : \mathbf{R} \to \mathbf{R}$.

Here the count is intended up to S-translations. Of course the well known concept of differential equivalence (see [8]) applies but, on one hand, it is too strong for our purposes and, on the other hand, it is certainly not easy to perform any explicit estimate concerning the domain of differential stability of a stable map.

First of all, note that bounding itself to Morse functions is the most natural way to guarantee that integer valued function

$$f^{\#}(\lambda) := \#f^{-1}(\lambda)$$

is well defined, which is necessary in order to show a counting procedure, as we want to do. However, there is also a stability reason for working with Morse functions, since it is well known (see [8]) that it is a persistent property under small perturbations. We prove it here because, for later use, we need both explicit estimates and to locate critical points in some precise way.

From now on in this section, let $f \in C^2_S(\mathbf{R};\mathbf{R})$ be a fixed Morse function, and define

$$\beta_f := \min\{|f'(x)| : f''(x) = 0\}.$$

Clearly, f is a Morse function if and only if $\beta_f > 0$. Now, if for $\alpha > 0$, we define

$$A(\alpha) := \{x \in \mathbf{R} : |f'(x)| < \alpha\}$$

then the following lemma holds.

Lemma 1.2. *If $\alpha \le \beta_f$, then each connected component A_j of $A(\alpha)$ contains exactly one zero of f' and no zeros of f''.*

Proof. Because of the definition of β_f, it is clear that f'' cannot vanish in $A(\alpha)$. Let now I be a connected component of $A(\alpha)$, and suppose that f' does not vanish on I. Then f' has constant sign on I, and hence it assumes the same value on the endpoints of the interval I. By the Rolle Theorem, it should exist $\bar{x} \in I$ such that $f''(\bar{x}) = 0$, that is a contradiction. On the other hand, we cannot have more than one critical point of f in the interval

I, since otherwise, once more due to the Rolle Theorem, f'' should vanish on it. □

Note that the critical points to which the previous lemma refers are non degenerate local minima or maxima only, since f is a Morse function. Hence the sign of f'' on each connected component of $A(\alpha)$ is completely determined by the nature of the unique critical point contained in it. Later on we will show that the same happens to all functions g which are near to f in some suitable sense.

To this aim define

$$\alpha_f := \min\{|f'(x)| + |f''(x)|\}$$

and note that f is a Morse function if and only if $\alpha_f > 0$. Clearly, $\alpha_f \leq \beta_f$ so that the previous lemma holds in correspondence of $\alpha = \alpha_f$. In fact, choosing $\alpha = \alpha_f/2$, the behavior of f is shared by the functions g near to it.

Lemma 1.3. *If $g \in C^2_S(\mathbf{R};\mathbf{R})$ satisfies*

$$||f' - g'||_\infty < \frac{\alpha_f}{2} \quad \text{and} \quad ||f'' - g''||_\infty < \frac{\alpha_f}{2},$$

then g is a Morse function, and each connected component of $A(\alpha_f/2)$ contains exactly one zero of g' and no zeros of g''. Moreover, the sign of g'' on each component is equal to the sign of f''.

In particular, we can establish a 1-1 correspondence between the critical points of g and the ones of f. Moreover, this correspondence preserves the nature of the critical points.

Proof. Every critical point of g is contained in $A(\alpha_f/2)$. Indeed if $g'(y) = 0$; then

$$|f'(y)| = |f'(y) - g'(y)| \leq \|f' - g'\|_\infty < \frac{\alpha_f}{2}.$$

Moreover, g'' cannot vanish in $A(\alpha_f/2)$. Indeed, if $g''(y) = 0$ with $y \in A(\alpha_f/2)$, then

$$\frac{\alpha_f}{2} > |f''(y) - g''(y)| = |f''(y)| \geq \alpha_f - |f'(y)| \geq \frac{\alpha_f}{2}.$$

These remarks prove that g is a Morse function which has at most one critical point in each I, connected component of $A(\alpha_f/2)$; we will now show that it has exactly one critical point in I. To prove it, assume by contradiction that I does not contain critical points of g. Then g' has constant sign on I, say positive (the negative case can be handled in a similar way). Now, due to Lemma 1.2, f'' has constant sign on I, and hence we can choose $x \in \partial I$ is such that $f'(x) = -\frac{\alpha_f}{2}$. Then we should have

$$\frac{\alpha_f}{2} > |f'(x^-) - g'(x^-)| = \frac{\alpha_f}{2} + g'(x^-) \geq \frac{\alpha_f}{2}.$$

Finally, let us show that, on I, g'' has the same sign as f''. To prove it, let $x, y \in I$ the unique critical points of f and g, respectively. We will show that if x is a maximum point of f, then y is a maximum point of g. Since $f''(x) < 0$ and f'' cannot change sign on I, we have that $f''(y) < 0$. Now, if y were a minimum point of g, we should have

$$\frac{\alpha_f}{2} > g''(y) - f''(y) \geq -f''(y) \geq \alpha_f - |f'(y)| \geq \frac{\alpha_f}{2}.$$

The same holds for minimum points. □

Of course, we are interested in localizing the critical points of a Morse function because the integer $f^{\#}(\lambda)$ may change only when the parameter λ crosses a critical value of f. Since the critical points of f are finite, so are its critical values, say

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n$$

and it is easy to show that, if $n_j = f^{\#}(\lambda)$ for $\lambda \in (\lambda_j, \lambda_{j+1})$, then

$$n_j = n_{j-1} + 2(m_j - M_j) \qquad \text{and} \qquad f^{\#}(\lambda_j) = n_{j-1} + m_j - M_j,$$

where m_j, M_j are the number of local minima and maxima respectively at the level λ_j (no inflection points are allowed since f is a Morse function).

Going back to Definition 1.1, to say that f and g are critically equivalent means that g has the same number of critical levels as f, say $\mu_1 < \mu_2 < \cdots < \mu_n$ and that

$$\begin{aligned} f^{\#}(\lambda_j) =& g^{\#}(\mu_j) \qquad \forall j \\ f^{\#}(\lambda) =& g^{\#}(\mu) \qquad \forall \lambda \in (\lambda_{j-1}, \lambda_j) \;\text{ and }\; \forall \mu \in (\mu_{j-1}, \mu_j). \end{aligned}$$

In other words, the counting procedure performed on the function g is qualitatively equivalent to that one performed on f. To obtain a result like that in [15], we need a bit more; this must be true for all g which are near enough to f.

Definition 1.4. A Morse function $f \in C^2_S(\mathbf{R};\mathbf{R})$ is said to be *critically stable* when there exists a neighborhood $U \ni f$ in $C^2_S(\mathbf{R};\mathbf{R})$ such that all $g \in U$ are critically equivalent to f.

To be critically stable of course involves strong restrictions on a map. Indeed, by an easy modification of the arguments in [8] (see Proposition 2.2 there), it is easy to prove the following lemma.

Lemma 1.5. *If a Morse function $f \in C^2_S(\mathbf{R};\mathbf{R})$ is critically stable, then its critical values are distinct, namely*

$$f'(x) = f'(y) = 0 \quad \textit{and} \quad x \neq y \pmod S \qquad \textit{implies} \qquad f(x) \neq f(y).$$

Remark 1.6. Note that for a function to have distinct critical values it turns out to be also an implicit restriction on the period S; it has to be the minimal period of the (nonconstant) function f. Nevertheless, this is not really a problem for us, since the functions g we are interested in are S-periodic too.

The converse of the previous lemma is also true by the arguments in [8]. However, we prefer to give here a different proof in order to provide explicit estimates for the neighborhood of stability.

Let us now define

$$d_f = \min_{j \neq k} |\lambda_j - \lambda_k|$$

to be the minimal separation between these levels, which we will assume to be distinct from now on. Our main result in this section can be finally stated as follows.

Proposition 1.7. *If $f \in C^2_S(\mathbf{R};\mathbf{R})$ is a Morse function with distinct critical values, and $g \in C^2_S(\mathbf{R};\mathbf{R})$ satisfies*

$$||f - g||_\infty < \frac{d_f}{2}, \quad ||f' - g'||_\infty < \frac{\alpha_f}{2}, \quad ||f'' - g''||_\infty < \frac{\alpha_f}{2},$$

then g is a Morse function with distinct critical values. Moreover, g is critically equivalent to f.

Proof. Let I be a connected component of $A(\alpha_f/2)$, and let $x, y \in I$ be the unique critical points in I of f and g respectively. For definiteness, suppose that x and y are maximum points. Note that $g(y) \geq g(x)$ and $f(x) \geq f(y)$. If $f(x) \geq g(y)$ we have

$$|f(x) - g(y)| = f(x) - g(y) \leq f(x) - g(x) < \frac{d_f}{2}.$$

On the contrary, if $f(x) < g(y)$, we have

$$|f(x) - g(y)| = g(y) - f(x) \leq g(y) - f(y) < \frac{d_f}{2}.$$

This shows that g has distinct critical values, and their number equals that of the critical values of f. $\square$

In the last section we will make use of these explicit estimates to find an upper bound for the forcing term in Theorem 0.1.

2. Stability estimates on the reduction function. The *reduction function* associated with the problem $(P_{h,0})$ is defined to be the following real function of one real variable

$$\varphi_h(\xi) = \min_{\overline{u}=\xi} J_h(u),$$

where $J_h : H \to \mathbf{R}$ is the usual action functional on the Sobolev space of H^1, T-periodic functions, namely

$$J_h(u) = \int_0^T [\frac{|\dot{u}|^2}{2} - V(u) + hu]dt.$$

The function φ_h is easily seen to be well defined, and its fine regularity properties have been studied in [13] and [14]. Because of the growth assumption

$$C_2 := \max V'' < (\frac{2\pi}{T})^2, \tag{$H2$}$$

here the situation is simpler, and the properties of the reduction can be easily proved. Let us state the main ones.

The condition $(H2)$ clearly is a convexity assumption for the action functional J_h on the hyperplanes $H_\xi = \{u : \overline{u} = \xi\}$. This implies that the action functional J_h has exactly one critical point when restricted to H_ξ (in fact a minimum point), which we call u_ξ (for notation convenience we don't make explicit its dependence on the forcing term h).

By means of the Implicit Function Theorem, it can be easily proved that

$$\mathbf{R} \ni \xi \mapsto u_\xi \in H$$

is in fact a smooth map (see [13] and also [15]) and, since by definition

$$\varphi_h(\xi) = J_h(u_\xi)$$

it follows that φ_h is a smooth function too. Moreover, it is easy to prove that u_ξ solves the equation (see once more [13] and [15])

$$\ddot{u}_\xi + V'(u_\xi) = h - \frac{1}{T}\varphi_h'(\xi). \tag{2.1}$$

The relevance of the reduction function φ_h in connection to the problem $(P_{h,0})$ is clear by noting the following facts. First of all, $J'_h(u) \equiv 0$ implies that $u \equiv u_\xi$ for some ξ, and since

$$\varphi'_h(\xi) = \frac{\partial J_h}{\partial \xi}(u_\xi) = -\int_0^T V'(u_\xi)dt,$$

clearly u_ξ is a classical solution of $(P_{h,0})$ if and only if ξ is a critical point of φ_h. Of course, this is nothing else than the Lyapunov-Schmidt approach in a variational guise. In fact, in this section we don't make full use of the reduction function φ_h, being interested in the zeroes of its derivative only. As we will see in the next section, this will be no longer the case when we'll argue about the variational nature of the solutions to $(P_{h,0})$.

Concerning the problem $(P_{h,\lambda})$ instead of $(P_{h,0})$, the following proposition can be easily proved.

Proposition 2.1. *The function u is a classical solution of the problem $(P_{h,\lambda})$ if and only if $u = u_\xi$ for some ξ and*

$$\varphi'_h(\xi) = -T\lambda.$$

Proof. It is enough to note that the functional and the reduction function associated with $(P_{h,\lambda})$ are respectively

$$J_{h,\lambda}(u) = J_h(u) + T\lambda\bar{u} \quad \text{and} \quad \varphi_{h,\lambda}(\xi) = \varphi_h(\xi) + T\lambda\xi.$$

In other words, the previous proposition guarantees that to count the solution of the problem $(P_{h,\lambda})$ is exactly the same as to count the solutions of a much simpler equation.

Moreover, the situation becomes extremely simple when $h \equiv 0$.

Proposition 2.2. $\varphi_0(\xi) = -TV(\xi)$

Proof. Let u_ξ be the solution of $\ddot{u}_\xi + V'(u_\xi) = -\frac{1}{T}\varphi'_0(\xi)$. If we multiply the equation by $\ddot{u}_\xi$ and integrate, we obtain

$$\int_0^T |\ddot{u}_\xi|^2 dt = \int_0^T V''(u_\xi)\ |\dot{u}_\xi|^2 dt \le C_2 \int_0^T |\dot{u}_\xi|^2 \le \frac{C_2}{\omega^2}||\ddot{u}_\xi||_2^2.$$

Then $\ddot{u}_\xi = 0$ and, since u_ξ is T-periodic, $u_\xi = \text{constant} = \xi$. $\square$

Finally, we are ready to prove that, when the forcing term h is small, then φ_h is close to φ_0, in some suitable sense.

Proposition 2.3. *If for every k we define $C_k = \max V^{(k)}$ then*

$$||\varphi_h' - \varphi_0'||_\infty \leq \frac{C_2 T^{1/2}}{\omega^2 - C_2}||h||_2,$$

$$||\varphi_h'' - \varphi_0''||_\infty \leq \frac{C_3 T^{1/2}\omega^2}{(\omega^2 - C_2)^2}||h||_2,$$

$$||\varphi_h''' - \varphi_0'''||_\infty \leq \Big[\frac{C_4 T^{1/2}}{\omega^2 - C_2} + \frac{3C_3 T^{3/2}\omega}{(\omega^2 - C_2)^2}\big(1 + \frac{C_2 T\omega}{\omega^2 - C_2}\big)^2\Big]||h||_2.$$

Note that, since the potential V is not constant, necessarily, we have that $C_k > 0$ for every k, so that the previous estimates make sense. To prove them, we need a preliminary lemma.

Lemma 2.4. *For all ξ, we have*

$$||u_\xi - \xi||_2 \leq \frac{\omega^2}{\omega^2 - C_2^2}||h||_2,$$

$$||\frac{\partial \dot{u}_\xi}{\partial \xi}||_2 \leq \frac{C_3\omega}{(\omega^2 - C_2)^2}||h||_2,$$

$$||\frac{\partial u_\xi}{\partial \xi}||_\infty \leq 1 + \frac{C_2 T\omega}{\omega^2 - C_2}.$$

Proof. Let us first remark that $J_h'(u_\xi) \cdot v = \varphi_h'(\xi)\bar{v}, \quad \forall v \in H$. Then choosing $v = u_\xi - \xi$, we have $J_h'(u_\xi) \cdot (u_\xi - \xi) = 0$; that is,

$$\int_0^T |\dot{u}_\xi|^2 = \int_0^T V'(u_\xi)(u_\xi - \xi)dt - \int_0^T h(u_\xi - \xi)dt.$$

We can also write

$$\int_0^T |\dot{u}_\xi|^2 = \int_0^T [V'(u_\xi) - V'(\xi)](u_\xi - \xi)dt - \int_0^T h[u_\xi - \xi]dt,$$

so that

$$||\dot{u}_\xi||_2^2 \leq C_2 \; ||u_\xi - \xi||_2^2 + ||h||_2 \; ||u_\xi - \xi||_2.$$

Now, by Wirtinger inequality,

$$||u_\xi - \xi||_2^2 \leq \frac{1}{\omega^2}(C_2 \; ||u_\xi - \xi||_2^2 + ||h||_2 \; ||u_\xi - \xi||_2)$$

and

$$||u_\xi - \xi||_2 \leq \frac{\omega^2}{\omega^2 - C_2}||h||_2.$$

For the second estimate, we differentiate $J_h(u_\xi)$ twice in ξ to obtain

$$J_h''(u_\xi)(\frac{\partial u_\xi}{\partial \xi}, v) = \varphi_h''(\xi)\bar{v}, \quad \forall v \in H.$$

By choosing $v = \frac{\partial u_\xi}{\partial \xi} - 1$, since $\frac{\partial u_\xi}{\partial \xi} - 1$ has zero mean value, we have

$$J_h''(u_\xi)(\frac{\partial u_\xi}{\partial \xi}, \frac{\partial u_\xi}{\partial \xi} - 1) = 0,$$

then it follows that

$$\begin{aligned}\int_0^T \big|\frac{\partial \dot{u}_\xi}{\partial \xi}\big|^2 dt &= \int_0^T \big[V''(u_\xi)\frac{\partial u_\xi}{\partial \xi} - V''(\xi)\big]\big(\frac{\partial u_\xi}{\partial \xi} - 1\big)dt \\ &\leq \int_0^T \big|V''(u_\xi)\frac{\partial u_\xi}{\partial \xi} - V''(u_\xi)\big|\big|\frac{\partial u_\xi}{\partial \xi} - 1\big|dt \\ &+ \int_0^T \big|V''(u_\xi) - V''(\xi)\big|\big|\frac{\partial u_\xi}{\partial \xi} - 1\big|dt \\ &\leq C_2\big\|\frac{\partial u_\xi}{\partial \xi} - 1\big\|_2^2 + C_3\big\|u_\xi - \xi\big\|_2 \ \big\|\frac{\partial u_\xi}{\partial \xi} - 1\big\|_2.\end{aligned}$$

From Wirtinger Inequality and the previous estimate, we have

$$\big\|\frac{\partial \dot{u}_\xi}{\partial \xi}\big\|_2^2 \leq \frac{C_2}{\omega^2}\big\|\frac{\partial \dot{u}_\xi}{\partial \xi}\big\|_2^2 + \frac{C_3}{\omega}\frac{1}{\omega^2 - C_2}\big\|\frac{\partial \dot{u}_\xi}{\partial \xi}\big\|_2 \ \big\|h\big\|_2,$$

and, therefore,

$$\big\|\frac{\partial \dot{u}_\xi}{\partial \xi}\big\|_2 \leq \frac{C_3\omega}{(\omega^2 - C_2)^2}\big\|h\big\|_2.$$

For the third estimate we first note that

$$\big|\frac{\partial u_\xi}{\partial \xi}(t)\big| = \lim_{\eta \to 0}\big|\frac{u_{\xi+\eta}(t) - u_\xi(t)}{\eta}\big|.$$

To estimate the difference $u_{\xi+\eta} - u_\xi$, let us remark that since $J'(u_\xi) \cdot w = 0$, $\forall w \in H_0$, then

$$[J(u_{\xi+\eta}) - J'(u_\xi)](u_{\xi+\eta} - u_\xi - \eta) = 0.$$

Therefore, we have

$$\begin{aligned}\int_0^T \big|\dot{u}_{\xi+\eta} - \dot{u}_\xi\big|^2 dt &= \int_0^T \big[V'\big(u_{\xi+\eta}\big) - V'\big(u_\xi\big)\big]\big(u_{\xi+\eta} - u_\xi - \eta\big)dt \\ &\leq C_2\int_0^T \big|u_{\xi+\eta} - u_\xi\big|\big|u_{\xi+\eta} - u_\xi - \eta\big|dt \\ &\leq C_2\int_0^T \big[\big|u_{\xi+\eta} - u_\xi - \eta\big|^2 + |\eta|\big|u_{\xi+\eta} - u_\xi - \eta\big|\big]dt.\end{aligned}$$

Using Wirtinger Inequality, we obtain

$$\|u_{\xi+\eta} - u_\xi\|_2^2 \leq \frac{C_2}{\omega^2}\|\dot u_{\xi+\eta} - \dot u_\xi\|_2^2 + C_2\frac{\sqrt{T}}{\omega^2}|\eta|\|\dot u_{\xi+\eta} - \dot u_\xi\|_2;$$

that is,

$$\|\dot u_{\xi+\eta} - \dot u_\xi\|_2 \leq C_2\sqrt{T}|\eta|\frac{\omega}{\omega^2 - C_2}.$$

Let $\bar t \in [0,T]$ such that $u_{\xi+\eta}(\bar t) - u_\xi(\bar t) = \eta$; there results

$$u_{\xi+\eta}(t) - u_\xi(t) - \eta = \int_{\bar t}^t \big[\dot u_{\xi+\eta}(s) - \dot u_\xi(s)\big]dt.$$

Therefore, we have

$$\begin{aligned}
|u_{\xi+\eta}(t) - u_\xi(t)| &\leq |\eta| + |u_{\xi+\eta}(t) - u_\xi(t) - \eta| \\
&\leq |\eta| + \int_0^T |\dot u_{\xi+\eta} - \dot u_\xi|dt \\
&\leq |\eta| + \sqrt{T}\|\dot u_{\xi+\eta} - \dot u_\xi\|_2 \\
&\leq |\eta| + TC_2|\eta|\frac{\omega}{\omega^2 - C_2}
\end{aligned}$$

and

$$\Big\|\frac{\partial u_\xi}{\partial \xi}\Big\|_\infty \leq 1 + \frac{TC_2\omega}{\omega^2 - C_2}.$$

Proof of Proposition 2.3. Concerning the first derivatives, since

$$\varphi_h'(\xi) = -\int_0^T V'(u_\xi)dt \quad \text{and} \quad \varphi_0'(\xi) = -TV'(\xi),$$

we have

$$\begin{aligned}
|\varphi_h'(\xi) - \varphi_0'(\xi)| &\leq \int_0^T |V'(u_\xi) - V'(\xi)|dt \leq C_2\int_0^T |u_\xi - \xi|dt \\
&\leq C_2\sqrt{T}\|u_\xi - \xi\|_2 \leq \frac{C_2\sqrt{T}}{\omega^2 - C_2}\|h\|_2.
\end{aligned}$$

Concerning the second derivatives, since

$$\varphi_h'(\xi) = -\int_0^T V'(u_\xi)dt \quad \text{and} \quad \varphi''(\xi) = -\int_0^T V''(u_\xi)\frac{\partial u_\xi}{\partial \xi}dt,$$

we have

$$\begin{aligned}
\big|\varphi_h''(\xi)-\varphi_0''(\xi)\big| &= \Big|\int_0^T \big[V''(\xi)-V''(u_\xi)\frac{\partial u_\xi}{\partial \xi}\big]dt\Big| \\
&\le \int_0^T \big|V''(\xi)-V''(u_\xi)\big|dt+\int_0^T \big|V''(u_\xi)\big|\big|1-\frac{\partial u_\xi}{\partial \xi}\big|dt \\
&\le C_3\int_0^T \big|u_\xi-\xi\big|dt+C_2\int_0^T \big|1-\frac{\partial u_\xi}{\partial \xi}\big|dt \\
&\le C_3\sqrt{T}\big\|u_\xi-\xi\big\|_2+C_2\sqrt{T}\big\|1-\frac{\partial u_\xi}{\partial \xi}\big\|_2 \\
&\le \frac{C_3\sqrt{T}}{1-\omega^2}\big\|h\big\|_2+\frac{C_2C_3\sqrt{T}}{\big(\omega^2-C_2\big)^2}\big\|h\big\|_2 \\
&= \frac{C_3\sqrt{T}\omega^2}{\big(\omega^2-C_2\big)^2}\big\|h\big\|_2.
\end{aligned}$$

In order to obtain φ_h''', let us remark that

$$\varphi_h'''(\xi)\bar{v}=J'''(u_\xi)(\frac{\partial u_\xi}{\partial \xi},\frac{\partial u_\xi}{\partial \xi},v)+J''(u_\xi)(\frac{\partial^2 u_\xi}{\partial \xi^2},v).$$

Let us choose $v=\frac{\partial u_\xi}{\partial \xi}$; since $\frac{\partial^2 u_\xi}{\partial \xi^2}$ has zero mean value, the previous equation becomes

$$\varphi_h'''(\xi)=J'''(u_\xi)(\frac{\partial u_\xi}{\partial \xi},\frac{\partial u_\xi}{\partial \xi},\frac{\partial u_\xi}{\partial \xi});$$

that is,

$$\varphi_h'''(\xi)=-\int_0^T V'''(u_\xi)\big(\frac{\partial u_\xi}{\partial \xi}\big)^3dt.$$

Hence, we have

$$\begin{aligned}
\big|\varphi_h'''(\xi)-\varphi_0'''(\xi)\big| &= \Big|-\int_0^T \big[V'''(u_\xi)\big(\frac{\partial u_\xi}{\partial \xi}\big)^3-V'''(\xi)\big]dt\Big| \\
&\le \int_0^T \big|V'''(u_\xi)-V'''(\xi)\big|dt+\int_0^T \big|V'''(u_\xi)\big|\big|\big(\frac{\partial u_\xi}{\partial \xi}\big)^3-1\big|dt \\
&\le C_4\int_0^T \big|u_\xi-\xi\big|dt+C_3\int_0^T \big|\big(\frac{\partial u_\xi}{\partial \xi}\big)^3-1\big|dt \\
&\le C_4\sqrt{T}\big\|u_\xi-\xi\big\|_2+C_3\int_0^T \big|\big(\frac{\partial u_\xi}{\partial \xi}\big)^3-1\big|dt.
\end{aligned}$$

Finally, we will estimate the last integral in the previous formula. Since $\overline{\frac{\partial u_\xi}{\partial \xi}} = 1$, there exists $\bar{t}$ such that $\frac{\partial u_\xi}{\partial \xi}(\bar{t}) = 1$; then

$$\begin{aligned}\int_0^T \big|\big(\frac{\partial u_\xi}{\partial \xi}\big)^3 - 1\big|dt &= \int_0^T \big|\int_{\bar{t}}^t 3\big(\frac{\partial u_\xi}{\partial \xi}\big)^2 \frac{\partial \dot{u}_\xi}{\partial \xi} ds\big|dt \\ &\leq 3T\int_0^T \big|\frac{\partial u_\xi}{\partial \xi}\big|^2\big|\frac{\partial \dot{u}_\xi}{\partial \xi}\big|dt \\ &\leq 3T^{\frac{3}{2}}\big\|\frac{\partial u_\xi}{\partial \xi}\big\|_\infty^2\big\|\frac{\partial \dot{u}_\xi}{\partial \xi}\big\|_2.\end{aligned}$$

Now, from Lemma 2.4, we conclude that

$$\big|\varphi'''(\xi) - \varphi_0'''(\xi)\big| \leq \big[\frac{C_4\sqrt{T}}{\omega^2 - C_2} + \frac{3C_3T^{\frac{3}{2}}\omega}{(\omega^2 - C_2)^2}\big(1 + \frac{C_2T\omega}{\omega^2 - C_2}\big)^2\big]\big\|h\big\|_2.$$

3. Conclusions. Collecting Proposition 1.7 and Proposition 2.3, we can conclude the following:

Theorem 3.1. *Assume that*

$$\max V'' < (\frac{2\pi}{T})^2$$

and that V' is a Morse function with distinct critical values. Then there exists a constant K such that if the forcing term h satisfies

$$||h||_2 < K,$$

then φ_h' is critically equivalent to φ_0'.

The constant K can be explicitly determined

$$\begin{aligned}K = \min\big\{&\frac{\omega^2 - C_2}{C_2T^{1/2}}\frac{d_{\varphi_0'}}{2}, \frac{(\omega^2 - C_2)^2}{C_3T^{1/2}\omega^2}\frac{\alpha_{\varphi_0'}}{2}, \\ &\frac{\alpha_{\varphi_0'}}{2}\big[\frac{C_4T^{1/2}}{\omega^2 - C_2} + \frac{3C_3T^{3/2}\omega}{(\omega^2 - C_2)^2}\big(1 + \frac{C_2T\omega}{\omega^2 - C_2}\big)^2\big]^{-1}\big\}\end{aligned}$$

and it is easy to see that the comparison between our constant and that one found in [15] for the pendulum case, $V(u) = -A\cos(u)$, strictly depends on the choices of the period T and of the amplitude A; our estimates, however, apply to a wider class of nonlinearities.

We would like to conclude with some remarks about the utility of the reduction function φ_h in classifying the critical points of the action functional J_h, associated with the problem $(P_{h,0})$.

Let us first recall that a critical point u_0 for the functional J_h is called an *inflection point* ([1], [2], [15]) when

(a) $\dim \ker J_h''(u_0) = 1$;
(b) if $v_0 \in \ker J_h''(u_0)$ and $v_0 \neq 0$ then $J_h'''(u_0) \cdot v_0^3 \neq 0$.

Whereas u_0 is called of mountainpass type ([9]) if, for every sufficiently small neighborhood U of u, the set $U \cap \{u \in H : J_h(u) < J_h(u_0)\}$ is neither empty nor path-connected.

Theorem 3.2. *Assume*

$$\max V'' < (\frac{2\pi}{T})^2$$

and let $u_0 = u_{\xi_0}$ be a critical point of the action functional J_h. Then u_0 is an isolated critical point of J_h if and only if ξ_0 is an isolated critical point of φ_h. Moreover, in this case, u_0 is a local minimum point, a point of mountainpass type or an inflection point for J_h if and only if ξ_0 is a local minimum point, a local maximum point or a nondegenerate inflection point (namely $\varphi_h'(\xi_0) = \varphi_h''(\xi_0) = 0$ and $\varphi_h'''(\xi_0) \neq 0$) for the reduction function φ_h, respectively.

Proof. We know that all the critical points of J_h are contained in the set $\{u_\xi \in H : \xi \in \mathbf{R}\}$, that $\xi \mapsto u_\xi$ is a smooth map, and that u_ξ is a critical point of J_h if and only if ξ is a critical point of φ_h.

Then the statement about the isolatedness of critical points is trivial. Take now $\xi_0 \in \mathbf{R}$ for which $\varphi_h'(\xi_0) = 0$ and $\varphi_h'(\xi) \neq 0$ for all $0 < |\xi - \xi_0| < \delta$ for a suitable small δ (namely an isolated critical point for φ_h), and let $u_0 = u_{\xi_0}$ be the correspondent isolated critical point of J_h. We can distinguish between three different situations, according to the sign of φ_h' close to to ξ_0:

(a) $\varphi_h'(\xi)$ is negative to the left and positive to the right of ξ_0;
(b) $\varphi_h'(\xi)$ is positive to the left and negative to the right of ξ_0;
(c) $\varphi_h'(\xi)$ has the same sign in a neighborhood of ξ_0.

It is trivial to show that, if (a) holds then u_0 is a local minimum for J_h, whereas if (b) holds then u_0 is a point of mountainpass type. On the other hand, if (c) holds then u_0 cannot be a local minimum, since $\varphi_h(\xi) < \varphi_h(\xi_0)$ in a one sided neighborhood of ξ_0, nor a point of mountainpass-type, since locally in u_0 the sublevel $J_h(u) < J_h(u_0)$ is path-connected. Then the conclusions about local minima and points of mountain-pass type follows easily.

Concerning the last statement, let us begin by noting that since

$$J_h''(u_0) \cdot (\frac{\partial u_\xi}{\partial \xi}\Big|_{\xi=\xi_0}, v) = \varphi_h''(\xi_0)\overline{v}$$

holds for all $v \in H$, then

$$\varphi_h''(\xi_0) = 0 \qquad \text{if and only if} \qquad \frac{\partial u_\xi}{\partial \xi}\Big|_{\xi=\xi_0} \in \ker J_h''(u_0)$$

On the other hand, note that for all $w \in H_0$, we have

$$\begin{aligned} 0 = J_h''(u_0) \cdot w^2 &= \int_0^T |\dot{w}|^2 dt - \int_0^T V''(u_0) w^2 dt \\ &\geq [(\frac{2\pi}{T})^2 - \max V'']||w||_2. \end{aligned}$$

In particular, if it happens that $w \in \ker J_h''(u_0)$, then clearly $w = 0$.

We can use it to prove that $\frac{\partial u_\xi}{\partial \xi}\big|_{\xi=\xi_0}$ is in fact the only candidate to be en element of the kernel of $J_h''(u_0)$. Assume indeed that $v_0 \in \ker J_h''(u_0)$ and $\overline{v}_0$ (which is not restrictive if $v_0 \neq 0$). Then for all $w \in H_0$ we have

$$\begin{aligned} 0 &= J_h''(u_0) \cdot (\frac{\partial u_\xi}{\partial \xi}\Big|_{\xi=\xi_0}, w) - J_h''(u_0) \cdot (v_0, w) \\ &= J_h''(u_0) \cdot (\frac{\partial u_\xi}{\partial \xi}\Big|_{\xi=\xi_0} - v_0, w) \end{aligned}$$

which shows that in fact $v_0 = \frac{\partial u_\xi}{\partial \xi}\big|_{\xi=\xi_0}$. Summing up, $J_h''(u_0)$ has a non trivial kernel if and only $\varphi_h(\xi_0) = 0$. In this case, moreover, the kernel is one–dimensional, and it is spanned by $\frac{\partial u_\xi}{\partial \xi}\big|_{\xi=\xi_0}$.

To conclude, it is enough to note that

$$\varphi_h'''(\xi_0) = J_h'''(u_0) \cdot (\frac{\partial u_\xi}{\partial \xi}\Big|_{\xi=\xi_0})^3.$$

Let us note that the remarkable simplicity of the previous statement is essentially due to the clear geometrical meaning of the reduction function itself: this is a main advantage on using it instead of its derivative (namely the well known Liapunov-Schmidt reduction).

Remark 3.3. It is easy to see that the characterization for the minimum points holds without assumption on the isolatedness hypothesis, whereas this is no longer the case for points of mountainpass type.

Of course, all the critical points of φ_h (and hence of J_h) are isolated if, for instance, the potential V is an analytic function, which implies the analyticity of J_h and φ_h.

A more precise result can be obtained for small forcing terms.

Theorem 3.4. *Assume that*

$$\max V'' < (\frac{2\pi}{T})^2$$

and that V is a Morse function with distinct critical values. Then there exists a constant K_0 such that if the forcing term h satisfies

$$||h||_2 < K_0$$

then each critical point of J_h is a local minimum or a point of mountainpass type.

Proof. Take

$$K_0 = \min\big\{\frac{(\omega^2 - C_2)^2}{C_3 T^{1/2}\omega^2}\frac{\alpha_{\varphi'_0}}{2}, \big[\frac{C_4 T^{1/2}}{\omega^2 - C_2} + \frac{3C_3 T^{3/2}\omega}{(\omega^2 - C_2)^2}\big(1 + \frac{C_2 T\omega}{\omega^2 - C_2}\big)^2\big]^{-1}\frac{\alpha_{\varphi'_0}}{2}\big\}$$

and use Proposition 2.3 and Lemma 1.3 to prove that φ_h is a Morse function. Then use Theorem 3.2 to conclude.

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