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# SOME REMARKS ON THE METHOD OF MOVING PLANES

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**Abstract.** We propose a variational approach to the method of moving planes which easily applies to quasilinear equations of type (1-1) with $f$ locally Lipschitz continuous. To do this we use a characterization of Lipschitz continuous functions which allows us to get symmetry results without writing an equation for the difference between the solution and its reflection.

**1. Introduction, notations and statement of the results.** In a famous paper Gidas, Ni and Nirenberg [4] investigated, using the technique of "moving planes", properties of symmetry and monotonicity of classical solutions to elliptic problems. More recently, the method has been substantially simplified by Berestycki and Nirenberg [1] with the aid of a form of the maximum principle in small domains due to Varadhan and based on the Alexandrov-Bakelman-Pucci's inequality. This new approach allows us to get symmetry results for classical or strong solutions to fully nonlinear elliptic equations in general domains (i.e., without supposing any smoothness of the boundary). The same tool is applied by the authors to improve also the so called sliding method to get monotonicity results in general domains.

The aim of this note is to propose a variational approach to the method of moving planes which is elementary and does not rely on the Alexandrov-Bakelman-Pucci's inequality. However we also exploit the idea of Berestycki and Nirenberg that to conclude the procedure it is enough to know how to deal with domains of small measure.

The method applies to weak solutions of nonlinear elliptic equations in divergence form in general domains and is based on comparison principles for nonlinear operators. This allows us to compare the solution $u$ and its reflection $u_\lambda$ (see notations that follow) without being forced to write an equation for the difference and hence applies nicely to nonlinear operators such as (1-1). This is mainly due to the fact that we split the nonlinearity $f$

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as the sum of a linear term and a monotone function. The same technique also works for the sliding method to get monotonicity results.

Since the aim of this paper is to give an idea of what ingredients are needed in our approach, we limit ourselves to the case of the solutions to the following problem

$$\begin{cases} -\mathrm{div}\, A(x, \nabla u) = f(x,u)\ , & u > 0 \quad \text{in} \quad \Omega \\ & u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1-1}$$

where we make the following assumptions on $A = (A_1, \dots, A_N)$ and $f$.

For $j = 1 \dots n$, $A_j$ is a Caratheodory function of $(x, \eta) \in \Omega \times \mathbb{R}^N$, i.e., measurable in $x$ and continuous in $\eta$, with $A_j(x, .) \in C^1(\mathbb{R}^N \setminus \{0\})$ and

$$A_j(x, 0) = 0 \tag{1-2}$$

$$|A_j(x, \eta)| \le c_1(1 + |\eta|) \tag{1-3}$$

$$\sum_{i,j=1}^{N} |\frac{\partial A_j}{\partial \eta_i}(x, \eta)| \le c_2 \tag{1-4}$$

$$\sum_{i,j=1}^{N} \frac{\partial A_j}{\partial \eta_i}(x, \eta)\xi_i\xi_j \ge \alpha|\xi|^2 \tag{1-5}$$

for $x \in \Omega, \eta \in \mathbb{R}^N \setminus \{0\}$ and suitable positive constants $c_1, c_2, \alpha$.

$f \in C(\overline{\Omega} \times \mathbb{R}^N)$ satisfies

$$\begin{cases} \forall M > 0\, \exists \Lambda_i = \Lambda_i(M) \ge 0, \quad i = 1, 2 \text{ such that for } x \text{ fixed in } \Omega : \\ \qquad g_1(x, s) = f(x, s) + \Lambda_1 s \quad \text{ is nondecreasing in } s \in [0, M] \\ \qquad g_2(x, s) = f(x, s) - \Lambda_2 s \quad \text{ is nonincreasing in } s \in [0, M]. \end{cases} \tag{1-6}$$

Note that the condition (1-6) is equivalent to require that $f$ is locally Lipschitz continuous in $s$ uniformly in $x$. In order to state the result we introduce some more notations.

Let $n$ be a fixed direction in $\mathbb{R}^N$ and for $\lambda \in \mathbb{R}$ let $T_\lambda$ be the hyperplane

$$T_\lambda = \{x \in \mathbb{R}^N : x \cdot n = \lambda\}, \tag{1-7}$$

where the dot stands for scalar product in $\mathbb{R}^N$. Let us indicate with $R_\lambda : \mathbb{R}^N \to \mathbb{R}^N$ the reflection through $T_\lambda$:

$$R_\lambda(x) = x + 2(\lambda - x \cdot n)n. \tag{1-8}$$

If $v$ is a $C^1$ (or $H^1$) function $\mathbb{R}^N \to \mathbb{R}$, we define

$$v_\lambda(x) = v(x_\lambda), \text{ where } x_\lambda = R_\lambda(x) \tag{1-9}$$

and we note that $v_\lambda \in C^1$ (or $H^1$) with $\nabla v_\lambda(x) = R_0(\nabla v(x_\lambda))$. In our result, $\Omega$ will be a bounded domain which is convex in the $n-$direction and symmetric with respect to $T_0$; for such a domain we put

$$\Omega_\lambda = \{x \in \Omega : x \cdot n < \lambda\}, \quad \Omega'_\lambda = R_\lambda(\Omega_\lambda) \tag{1-10}$$

and note that if $-b = inf_{x\in\Omega} x \cdot n < \lambda \leq 0$ then $\Omega_\lambda \neq \emptyset$, $\Omega'_\lambda \subseteq \Omega$ and $\Omega \setminus T_0 = \Omega_0 \cup \Omega'_0$. The symmetry and monotonicity properties we assume on $A$ and $f$ are the following:

$$A(x,\eta) = A(x - (x \cdot n)n, \eta) \qquad \text{if } x \in \Omega, \eta \in \mathbb{R}^N \tag{1-11}$$

$$A(x, R_0(\eta)) = R_0(A(x,\eta)) \qquad \text{if } x \in \Omega, \eta \in \mathbb{R}^N \tag{1-12}$$

$$f(x,s) = f(R_0(x), s) \qquad \text{if } x \in \Omega_0, s \in \mathbb{R} \tag{1-13}$$

$$f(x,s) \leq f(x_\lambda, s) \qquad \text{if } \lambda < 0, x \in \Omega_\lambda, s \in \mathbb{R} \tag{1-14}$$

If, for instance $n = (1, 0, \ldots, 0)$, then (1-11)–(1-14) mean that $A(x,\eta)$ is independent of $x_1$ with $A_1$ odd in $\eta_1$ and $A_j$ even in $\eta_1$ for $j > 1$, while $f(x,s)$ is even in $x_1$ and nondecreasing in $x_1$ for $x_1 < 0$.

The result we get is the following:

**Theorem 1-1.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ which is convex in a direction $n$ and symmetric with respect to the hyperplane $T_0 = \{x \in \mathbb{R}^N : x \cdot n = 0\}$. Let us suppose that $A$ and $f$ satisfy the hypotheses* (1-2)–(1-6) *as well as* (1-11)–(1-14). *If $u \in H^1_0(\Omega) \cap C(\overline{\Omega})$ is a weak solution to the problem* (1-1) (*with $u = 0$ pointwise on $\partial\Omega$*), *then $u$ is symmetric with respect to $T_0$; i.e., $u(x) = u(R_0(x))$ if $x \in \Omega_0$, and $u(x) < u(x_\lambda)$ if $x \in \Omega_\lambda, \lambda < 0$.*

An immediate consequence of Theorem 1-1 is the following

**Corollary 1-1.** *Let $\Omega$ be a ball and $u \in H^1_0(\Omega) \cap C(\overline{\Omega})$ be a weak solution to the problem* (1-1), *where $A$ and $f$ satisfy* (1-2)–(1-6) *and have the form $A(x,\eta) = a(|\eta|)\eta$, $f(x,s) = f_1(|x|, s)$ with $f_1(r,s)$ nonincreasing in $r$. Then $u$ is radial and radial decreasing.*

The method works also for operators with different growth conditions, provided the (weak) solutions are more regular. For instance, we can prove the following result about the operator approximating the $p$-Laplacian operator which extends to weak solutions of a proposition in [6]:

**Theorem 1-2.** *Let $u \in C^1(\overline{\Omega})$ be a weak solution to the problem*

$$\begin{cases} -div\left((\epsilon + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u\right) = f(x,u) & , u > 0 \quad in \quad \Omega \\ & u = 0 \quad on \quad \partial\Omega, \end{cases} \tag{1-15}$$

*where $\epsilon > 0$, $1 < p < \infty$. If $\Omega, f$ satisfy the assumptions in Theorem* 1-1*, then the conclusions of the Theorem hold.*

In a forthcoming paper [3] we apply the technique developed here to study the symmetry and monotonicity of positive weak solutions of some problems which involve more general quasilinear operators in divergence form that can also be degenerate. This class of operators includes the $p$- Laplacian, $1 < p < \infty$.

The paper is organized as follows: in Section 2 we state and prove the comparison theorems needed in the proofs of Theorems 1-1 and 1-2, that follow in Section 3.

**2. Preliminaries.** Throughout this section, $\Omega$ will be a bounded open set in $\mathbb{R}^N$, $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ a function verifying conditions (1-2)–(1-5), and $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ a continuous function.

If $u \in H^1(\Omega) \cap L^\infty(\Omega)$ and $\Lambda \in \mathbb{R}$, we say that $u$ satisfies (in a weak sense) the inequality

$$-div\, A(x, \nabla u) + \Lambda u \le g(x,u) \qquad \text{in} \quad \Omega$$

if for each $\varphi \in C_c^\infty(\Omega), \varphi \ge 0$, we have

$$\int_\Omega [A(x,\nabla u)\cdot\nabla\varphi + \Lambda u\varphi]\, dx \le \int_\Omega g(x,u)\varphi\, dx. \tag{2-1}$$

Since $u \in L^\infty(\Omega)$, $g$ is continuous and $A$ satisfies (1-3), the inequality (2-1) is still true if $\varphi \in H_0^1(\Omega), \varphi \ge 0$.

If $u, v \in H^1(\Omega)$, we say that $u \le v$ on $\partial\Omega$ (in the weak sense) if $(u-v)^+ \in H_0^1(\Omega)$. Recall that if $u, v \in H^1(\Omega) \cap C(\overline{\Omega})$ and $u \le v$ pointwise on $\partial\Omega$, then $u \le v$ on $\partial\Omega$ in the weak sense (see [2], page 172).

Let us recall the following form of the Poincaré inequality (see [5], page 164):

$$\int_\Omega |\phi|^2\, dx \le \left(\frac{|\Omega|}{\omega_N}\right)^{\frac{2}{N}} \int_\Omega |\nabla\phi|^2\, dx \qquad \forall \phi \in H_0^1(\Omega) \tag{2-2}$$

where $|.|$ stands for N-dimensional Lebesgue measure and $\omega_N = |B(1,0)|$. In [4] (2-2) is proved for $\Omega$ a bounded domain but if $\Omega$ is a bounded open set it has at most countably many components so that (2-2) is still true.

**Theorem 2-1** (Weak comparison principle)**.** *Let $u, v \in H^1(\Omega) \cap L^\infty(\Omega)$ and let us suppose that for $|s| \le max\{\|u\|_\infty, \|v\|_\infty\}$ $g(x,s)$ is nonincreasing in $s$ and that for $\Lambda \in \mathbb{R}$, $\Omega'$ open $\subseteq \Omega$ :*

$$u \le v \qquad on\ \partial\Omega' \tag{2-3}$$

$$\begin{cases} -divA(x,\nabla u) + \Lambda u \le g(x,u) & in\ \Omega' \\ -divA(x,\nabla v) + \Lambda v \ge g(x,v) & in\ \Omega' \end{cases} \tag{2-4}$$

(1) *If $\Lambda \ge 0$, then $u \le v$ in $\Omega'$ (whatever $\Omega' \subseteq \Omega$)*

(2) *If $\Lambda < 0$ and $|\Lambda| \left(\frac{|\Omega'|}{\omega_N}\right)^{\frac{2}{N}} < \alpha$ ($\alpha$ being defined in* 1-5*), then $u \le v$ in $\Omega'$.*

**Proof.** Since $A_j(x,.) \in C(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\})$ and (1-2) and (1-4) hold, we have that

$$\sum_{j=1}^N [A_j(x,\eta) - A_j(x,\zeta)][\eta_j - \zeta_j] = \int_0^1 \sum_{i,j=1}^N \frac{\partial A_j}{\partial \eta_i}[\zeta + t(\eta-\zeta)][\eta_i - \zeta_i][\eta_j - \zeta_j]\, dt$$

so that by (1-5),

$$\sum_{j=1}^N [A_j(x,\eta) - A_j(x,\zeta)][\eta_j - \zeta_j] \ge \alpha|\eta - \zeta|^2. \tag{2-5}$$

By hypothesis $(u-v)^+ \in H_0^1(\Omega')$, so it can be used as a test function in (2-1). Since $g(x,u) \le g(x,v)$, if $u \ge v$ we get, with $[u \ge v] = \{x \in \Omega' : u(x) \ge v(x)\}$:

$$\begin{aligned} \int_{[u\ge v]} \Big[\sum_{j=1}^N A_j(x,\nabla u)(\frac{\partial(u-v)}{\partial x_j}) + \Lambda u(u-v)\Big]\, dx \\ \le \int_{[u\ge v]} g(x,u)(u-v)\, dx \le \int_{[u\ge v]} g(x,v)(u-v)\, dx \\ \le \int_{[u\ge v]} \Big[\sum_{j=1}^N A_j(x,\nabla v)\frac{\partial(u-v)}{\partial x_j} + \Lambda v(u-v)\Big]\, dx. \end{aligned}$$

If $\Lambda \ge 0$, we get from (2-5)

$$\alpha \int_{[u\ge v]} |\nabla(u-v)|^2 \le \Lambda \int_{[u\ge v]} (v-u)(u-v)\, dx \le 0;$$

i.e., $\alpha\|(u-v)^+\|^2_{H^1_0(\Omega')} \le 0$, so that $(u-v)^+ = 0$ and $u \le v$ in $\Omega'$.

If $\Lambda < 0$, we get from (2-5) and (2-2)

$$\alpha \int_{\Omega'} |\nabla(u-v)^+|^2 dx \le |\Lambda| \int_{\Omega'} \left[(u-v)^+\right]^2 dx$$
$$\le |\Lambda| \left(\frac{|\Omega'|}{\omega_N}\right)^{\frac{2}{N}} \int_{\Omega'} |\nabla(u-v)^+|^2 dx$$

from which we easily conclude. □

**Theorem 2-2** (Strong comparison principle)**.** *Suppose* $u, v \in H^1(\Omega) \cap C(\Omega)$ *satisfy for* $\Lambda \ge 0$

$$u \le v, \qquad -divA(x, \nabla u) + \Lambda u \le -divA(x, \nabla v) + \Lambda v \qquad in\ \Omega.$$

*If* $u(x_0) = v(x_0)$ *for* $x_0 \in \Omega$, *then* $u \equiv v$ *in the component of* $\Omega$ *containing* $x_0$.

**Remark.** The conclusion is true also if $\Lambda < 0$ since being $u \le v$, we have

$$-divA(x, \nabla u) \le -divA(x, \nabla v) + \Lambda(v-u) \le -divA(x, \nabla v)$$

in this case.

**Proof.** If $\varphi \in H^1_0(\Omega), \varphi \ge 0$, by hypothesis $(u-v) \le 0$ in $\Omega$ and

$$0 \ge \int_\Omega \sum_{j=1}^N \left[A_j(x, \nabla u) - A_j(x, \nabla v)\right] \frac{\partial \varphi}{\partial x_j}\, dx + \Lambda \int_\Omega (u-v)\varphi\, dx$$
$$= \int_\Omega \sum_{i,j=1}^N \Big[\int_0^1 \frac{\partial A_j}{\partial \eta_i}\left(x, \nabla v + t(\nabla u - \nabla v)\right) dt\Big] \frac{\partial(u-v)}{\partial x_i} \frac{\partial \varphi}{\partial x_j}\, dx$$
$$+ \Lambda \int_\Omega (u-v)\varphi\, dx.$$

So if we put

$$a_{ij}(x) = \int_0^1 \frac{\partial A_j}{\partial \eta_i}\left(x, \nabla v(x) + t(\nabla u(x) - \nabla v(x))\right) dt,$$

then $a_{ij} \in L^\infty(\Omega)$ by (1-4), $\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \ge \alpha|\xi|^2$ by (1-5), and

$$-\sum_{i,j=1}^N \frac{\partial}{\partial x_j}\big[a_{ij}\frac{\partial(u-v)}{\partial x_i}\big] + \Lambda(u-v) \le 0 \quad \text{ in } \Omega$$

in the weak sense, with $u - v \leq 0$ in $\Omega$.

If $u(x_0) = v(x_0)$, with $x_0 \in \Omega$, then $\sup_B(u-v) = \sup_\Omega(u-v) = 0$ for every ball $B$ around $x_0$ since $u$ and $v$ are continuous in $\Omega$. The conclusion of the Theorem then follows from Theorem 8.19 in [5] (Strong maximum principle for weak solutions of elliptic differential inequalities). □

### 3. Proof of Theorems 1-1 and 1-2.

**Proof of Theorem 1-1.** Let us fix $\Lambda_1(M), \Lambda_2(M)$ so that (1-6) holds with $M = \|u\|_\infty$. With the notations of Section 1, if $-b < \lambda < 0$, then $u_\lambda$ satisfy, in the weak sense, the equation

$$-divA(x, \nabla u_\lambda) = f(x_\lambda, u_\lambda) \qquad \text{in } \Omega_\lambda. \tag{3-1}$$

In fact, by (1-11), (1-12) if $x \in \Omega_\lambda$, we have that

$$A(x, \nabla u_\lambda(x)) = A(x_\lambda, \nabla u_\lambda(x)) = A(x_\lambda, R_0[\nabla u(x_\lambda)]) = R_0[A(x_\lambda, \nabla u(x_\lambda)].$$

If $\varphi \in C_c^\infty(\Omega_\lambda)$ and $\psi(y) = \varphi(y_\lambda)$ for $y \in \Omega'_\lambda$, then $\psi \in C_c^\infty(\Omega'_\lambda) \subseteq C_c^\infty(\Omega)$, $\varphi(x) = \psi(x_\lambda) = \psi_\lambda(x)$ if $x \in \Omega_\lambda$ and we get

$$\int_{\Omega_\lambda} A(x, \nabla u_\lambda(x))\,.\nabla\varphi(x)\,dx = \int_{\Omega_\lambda} R_0\,A(x_\lambda, \nabla u(x_\lambda))\,.R_0\,\nabla\psi(x_\lambda)\,dx$$

$$\int_{\Omega_\lambda} A(x_\lambda, \nabla u(x_\lambda))\,.\nabla\psi(x_\lambda)dx = \int_{\Omega'_\lambda} A(y, \nabla u(y))\,.\nabla\psi(y)\,dy$$

$$= \int_{\Omega'_\lambda} f(y, u(y))\psi(y)\,dy = \int_{\Omega_\lambda} f(x_\lambda, u_\lambda(x))\varphi(x)\,dx$$

so that (3-1) holds. Since (1-14) and (1-6) holds we have that in $\Omega$:

$$\begin{cases} -div\,A(x, \nabla u_\lambda) - \Lambda_2 u_\lambda \geq f(x, u_\lambda) - \Lambda_2 u_\lambda = g_2(x, u_\lambda) \\ \quad -div\,A(x, \nabla u) - \Lambda_2 u = f(x, u) - \Lambda_2 u = g_2(x, u) \end{cases}$$

with $g_2(x, .)$ nonincreasing in the range of values of $u, u_\lambda$. If $\lambda > -b$, $\lambda$ is close to $-b$, then $|\Omega_\lambda|$ is small; more precisely, there exists $\underline{\lambda}$ such that if $-b < \lambda \leq \underline{\lambda}$ then $|\Omega_\lambda| < \delta$ where $\delta = \omega_N \left(\frac{\alpha}{\Lambda_2}\right)^{\frac{N}{2}}$. Since $u_\lambda \geq u$ on $\partial\Omega_\lambda$ ($u_\lambda = u$ on $\partial\Omega_\lambda \cap T_\lambda$, $u_\lambda \geq 0 = u$ on $\partial\Omega_\lambda \cap \partial\Omega$) we get $u_\lambda \geq u$ in $\Omega_\lambda$ by Theorem 2-1 (2).

We note that if $\lambda < 0$ then for every component $C_\lambda$ of $\Omega_\lambda$ there exists $x \in \overline{C}_\lambda \cap \partial\Omega$ such that $x_\lambda \in \Omega$ (because $\Omega$ is convex in the $n-$direction) so that $u(x) = 0 < u_\lambda(x)$ and $u_\lambda \not\equiv u$ in any component of $\Omega_\lambda$. Moreover, if $u_\lambda \geq u$ in $\Omega_\lambda$, then

$$\begin{aligned} -divA(x, \nabla u) + \Lambda_1 u &= f(x,u) + \Lambda_1 u = g_1(x,u) \leq g_1(x, u_\lambda) \\ &= f(x, u_\lambda) + \Lambda_1 u_\lambda \leq f(x_\lambda, u_\lambda) + \Lambda_1 u_\lambda = -divA(x, \nabla u_\lambda) + \Lambda_1 u_\lambda \end{aligned}$$

because $g_1(x,.)$ is nondecreasing. So for $\lambda$ close to $-b$ by Theorem 2-2 we have $u_\lambda > u$ in $\Omega_\lambda$.

Let $\lambda_0 = \sup\{\mu \in (-b, 0) : u_\lambda > u \text{ in } \Omega_\lambda \quad \forall \lambda \in (-b, \mu)\}$.

By continuity $u_{\lambda_0} \geq u$ in $\Omega_{\lambda_0}$ and if we show that $\lambda_0 = 0$ the theorem will be proved because the symmetry hypotheses on $\Omega$, $A$ and $f$ allow us to make the same reasoning in the symmetric cap $\Omega_0'$.

Suppose $\lambda_0 < 0$. Then by continuity, $u_{\lambda_0} \geq u$ in $\Omega_{\lambda_0}$ and since $\lambda_0 < 0$, as before, by Theorem 2-2, we get $u_{\lambda_0} > u$ in $\Omega_{\lambda_0}$. Since $u_{\lambda_0} - u$ has a positive minimum in a compact $K \subseteq \Omega_{\lambda_0}$ and $\min_K(u_\lambda - u)$ depends continuously on $\lambda$ as well as the measure $|\Omega_\lambda \setminus K|$ we can find a compact $K \subseteq \Omega_{\lambda_0}$ and $\lambda_0'$ with $\lambda_0 < \lambda_0' < 0$ such that for $\lambda_0 < \lambda < \lambda_0'$ we have

$$u_\lambda > u \ \text{ in } K \quad , \qquad |\Omega_\lambda \setminus K| < \delta,$$

where $\delta$ is as before. In this way if $\Omega_\lambda^K = \Omega_\lambda \setminus K$ we have that for such $\lambda$ $u_\lambda \geq u$ on $\partial\Omega_\lambda^K$ (as before on $\partial\Omega_\lambda^K \cap \partial\Omega_\lambda$, by construction on $\partial\Omega_\lambda^K \cap \partial K \subseteq K$). Since $|\Omega_\lambda^K|$ is small, exactly the same argument as before gives us $u_\lambda \geq u$ in $\Omega_\lambda^K$ (by Theorem 2-1) so that $u_\lambda \geq u$ in $\Omega_\lambda$ and finally $u_\lambda > u$ in $\Omega_\lambda$ because $\lambda < 0$ (by Theorem 2-2). This contradicts the definition of $\lambda_0$ and ends the proof. $\square$

**Proof of Theorem 1-2.** Here the operator is $A(\eta) = (\epsilon + |\eta|^2)^{\frac{p-2}{2}} \eta$ and

$$\frac{\partial A_j}{\partial \eta_i}(\eta) = \delta_{ij}(\epsilon + |\eta|^2)^{\frac{p-2}{2}} + (p-2)(\epsilon + |\eta|^2)^{\frac{p-4}{2}} \eta_i \eta_j$$

$$\sum_{i,j=1}^{N} \frac{\partial A_j}{\partial \eta_i}(\eta) \xi_i \xi_j = (\epsilon + |\eta|^2)^{\frac{p-2}{2}} |\xi|^2 + (p-2)(\epsilon + |\eta|^2)^{\frac{p-4}{2}} (\eta.\xi)^2.$$

Hence,

$$|\frac{\partial A_j}{\partial \eta_i}(\eta)| \leq \begin{cases} (p-1)(\epsilon + |\eta|^2)^{\frac{p-2}{2}} & (p \geq 2) \\ (3-p)(\epsilon + |\eta|^2)^{\frac{p-2}{2}} & (1 < p < 2) \end{cases} \tag{3-2}$$

$$\sum_{i,j}^{N} \frac{\partial A_j}{\partial \eta_i}(\eta) \xi_i \xi_j \geq \begin{cases} (\epsilon + |\eta|^2)^{\frac{p-2}{2}} |\xi|^2 & (p \geq 2) \\ (p-1)(\epsilon + |\eta|^2)^{\frac{p-2}{2}} |\xi|^2 & (1 < p < 2). \end{cases} \tag{3-3}$$

From (3-2) and (3-3) we deduce that

$$|\frac{\partial A_j}{\partial \eta_i}(\eta)| \leq c \tag{3-4}$$

$$\sum_{i,j=1}^{N} \frac{\partial A_j}{\partial \eta_i}(\eta)\xi_i\xi_j \geq \alpha|\xi|^2 \tag{3-5}$$

if $\eta$ is a convex combination of $\nabla u$, $\nabla v$ for any two solutions $u$ and $v$ of (1-15) which belongs to $C^1(\overline{\Omega})$ This is enough to get the weak and strong comparison principles for $C^1(\overline{\Omega})$ solutions of (1-15) in the same way as in the previous section. Then proceeding as in the case of Theorem 1-1 we get the assertion. $\square$

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