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FEEDBACK LAWS FOR N-PERSON QUASI-VARIATIONAL DIFFERENTIAL GAMES

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Abstract. Some optimal feedback control laws for N-person non-zero sum quasi-variational differential games are obtained.

Let $H, \{U_j\}_{j=1}^N$ be real Hilbert spaces and let S be an upper semi-continuous set-valued mapping of H into the closed convex subsets of H. For each t in $[0,T]$, let $\varphi(t;.)$ be a lower semi-continuous mapping of $H \times H$ into R^+ with $\varphi(t;x,.)$ convex on H for every x in H. Let $\{M_j\}_{j=1}^N$ be bounded linear mappings of the space of controls $L^2(0,T;U_j)$ into $L^2(0,T;H)$.

Consider the differential inclusion

$$\begin{aligned} &y' + \partial_y\{\varphi(t;x,y) + I_{Sx}(y)\} \mid_{x=y} \ni M_j v_j + \sum_{k\neq j, k=1}^{N} M_k u_k \\ &y(t) \in Sy(t), \quad 0 \le \tau \le t \le T; \quad y(\tau) = \xi. \end{aligned} \tag{0.1}$$

The indicator function of the closed convex set Sx of H is $I_{Sx}(.)$. With some additional hypotheses on φ, the set $\mathcal{R}(\tau;\xi;v_j;\pi_j u)$ of all solutions of (0.1) is non-empty. We associate with the problem and with the j-player, $j = 1,\dots,N$ the cost functional

$$J_j(\tau;\xi;y;v_j,\pi_j u) = f_j(y(T)) + \int_\tau^T \big\{g_j(y(t)) + h_j(v_j(t))\big\}dt \tag{0.2}$$

with $\pi_j u = (u_1,\dots,u_{j-1},u_{j+1},\dots,u_N)$.

In this paper we shall introduce the notion of a j-equilibrium point of the system (0.1)–(0.2), generalizing that of open loop equilibrium and use it to study feedback laws for an N-person non-zero sum differential game.

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It seems only natural and important for the j-producer to minimize his production costs regardless of those of his competitors in the open economy.

In Section 2 of the paper, the existence of a j-equilibrium point $\{u_j^*, \pi_j u^*\}$ of (0.1)–(0.2) as well as that of a continuous value vector-function $V(\tau, \xi) = \{V_j(\tau; \xi)\}_{j=1}^N$ is established.

It is known that feedback control laws can be expressed in terms of the value function if the latter is continuously differentiable, an assumption which does not hold even for the simplest cases. We are led to the study of suboptimal control problems in Section 3. Various approximating schemes based on a decomposition of the Hamilton-Jacobi equation and on the Trotter product formula were introduced by Barbu [2–5], by Popa [8–10] and by others from the Romanian school. We shall follow Barbu's approach and consider an approximating problem on small time-intervals for which the associated value-function is locally Lipschitzian and hence has a generalized Clarke gradient.

In Section 4, feedback laws which are expressed in terms of the value-function of the suboptimal control problems are given. They are obtained by using an auxiliary optimization problem of Clarke and Vinter [6], connecting the maximum principle and dynamic programming. In Section 5, constrained non-cooperative N-person games are considered. Applications to parabolic quasi-variational inequalities and to a problem for a compactified economy with N-consumers and K-producers are as in [12, 13]. The results obtained in this paper seem new as the literature on feedback controls for N-person non-zero sum differential games is almost non-existent.

1. Notations and assumptions. Let $H, \{U_j\}_{j=1}^N$ and $\varphi(t;.)$ be as in the introduction with domain $D(\varphi(t;.))$ dense in $H \times H$. We shall make the following assumption.

Assumption I. Let $\varphi(t;.)$ be lower semi-continuous (l.s.c.) mappings of $H \times H$ into R^+ with $D(\varphi(t;x,.))$ dense in H for each $x \in H$. We assume that

(1) For each $\{t, x\} \in [0, T] \times H$, the mapping $\varphi(t; x, .)$ of H into R^+ is convex.
(2) There exists a positive constant c such that $\varphi(t; x, y) \geq c\|y\|^2$ for all $\{t, x, y\} \in [0, T] \times D(\varphi(t; .))$.
(3) If $x_n \to x$ in $L^2(0, T; H)$, then

$$\int_0^T \varphi(t; x, y)dt = \lim_{n\to\infty} \int_0^T \varphi(t; x_n, y)dt$$

for all $y \in L^2(0, T; H) \cap D(\varphi)$.

(4) For each positive constant C,

$$\mathcal{B}_C = \left\{ y : \|y'\|_{L^2(0,T;H)} + \sup_{x \in L^2(0,T;H)} \int_0^T \varphi(t;x,y)dt \leq C \right\}$$

is a compact subset of $L^2(0,T;H)$.

Let S be a set-valued mapping of H into the closed convex subsets of H. The mapping S is said to be upper semi-continuous (u.s.c.) at $x_0 \in D(S)$ if for any open set $\mathcal{M}$ of H with $Sx_0 \subset \mathcal{M}$, there exists a neighborhood $\mathcal{N}(x_0)$ of x_0 such that $Sx \subset \mathcal{M}$ for all x in $\mathcal{N}(x_0)$.

We say that S is u.s.c. on H if it is u.s.c. at each point of its domain.

We shall make the following assumption on S.

Assumption II. Let S be an u.s.c. set-valued mapping of H into the closed convex subsets of H. We assume that

(1) $0 \in \bigcap_{x \in D(S)} Sx$.
(2) S takes compact subsets of $D(S)$ into compact subsets of $D(S)$.
(3) For each $x \in H$ and each $t \in [0,T]$; $D(\varphi(t;x,.)) \subset D(S)$.

It follows from the Kakutani theorem that any mapping S verifying Assumption II has a fixed point.

To get strong solutions of evolutionary quasi-variational inequalities, we need a continuity hypothesis on φ and on the set $Sw(t)$ for w in $D(S)$, as done earlier by Yamada [14] when $Sw(t) = K(t)$ for all w in $D(S)$.

Assumption III. Let $w \in C(0,T;H)$ and for each t let $w(t) \in D(\varphi(t;.))$. Suppose that

$$\|w(t) - w(t_0)\|_H \leq c_1 \mid t - t_0 \mid^{\frac{1}{2}} .$$

Let $y_0 \in Sw(t_0) \cap D(\varphi(t_0;w(t_0),.))$ with $\|y_0\| \leq r$. Then we assume that there exists $y(t) \in Sw(t) \cap D(\varphi(t;w(t),.))$ such that

$$\|y(t) - y_0\|^2 \leq \mid k_r(t) - k_r(t_0) \mid^2 \left\{ K_r + \varphi(t_0; w(t_0, y_0)) \right\}$$

and

$$0 \leq \varphi(t;w(t),y(t)) \leq \varphi(t_0;w(t_0),y_0) + |l_r(t) - l_r(t_0)| \{K_r + \varphi(t_0;w(t_0),y_0)\}.$$

K_r is a positive constant ; k_r, l_r are two absolutely continuous functions on [0,T] with k_r', l_r' in $L^2(0,T)$.

For the cost functionals (0.2) we need the following assumption.

Assumption IV. We assume that $\{f_j, g_j, h_j\}_{j=1}^N$ are Lipschitz continuous convex functions of $H \times L^2(\tau,T;H) \times L^2(\tau,T;U_j)$ into R^3.

Throughout the paper

$$\mathcal{U} = \prod_{j=1}^{N} \mathcal{U}_j.$$

and $\mathcal{U}_j$ is a compact convex subset of $L^2(\tau, T; U_j)$.

2. Equilibrium point and the value function. In this section the existence of an equilibrium point of the system (0.1)–(0.2) is established. The value function associated with the problem is shown to be continuous. Let $x \in \mathcal{B}_C$, then it verifies all the assumptions on w of Assumption III. With φ as in Assumption I, the l.s.c. mapping $\phi(t; y) = \varphi(t; x, y) + I_{Sx}(y)$ verifies all the hypotheses of Yamada's theorem [14].

Applying a well-known fixed point theorem and taking into account the estimates obtained in [14], we have proved in [11] the following theorem.

Theorem 2.1. *Let $\varphi(t; .)$ be a l.s.c. mapping of $H \times H$ into R^+ with dense domain $D(\varphi)$ and satisfying Assumption I. Let S be an u.s.c. set-valued mapping of $D(S)$ into the closed convex subsets of H verifying Assumption II. Let $\{M_j\}_{j=1}^N$ be bounded linear mappings of $L^2(\tau, T; U_j)$ into $L^2(\tau, T; H)$ and suppose that Assumption III is verified. Then for any given $\{\xi, v_j, u_k\}$ in $D(\varphi(\tau; .)) \times \mathcal{U}_j \times \mathcal{U}_k$ with $\xi \in S\xi$, there exists a solution y of* (0.1)*. Moreover,*

$$\begin{aligned}\|y'\|^2_{L^2(t,T;H)} &+ \varphi(t; y, y) + \|A(.; y, y)\|^2_{L^2(t,T;H)} \\ &\leq C\big\{1 + \varphi(t; \xi, \xi) + \|v_j\|^2_{L^2(t,T;U_j)} + \sum_{k \neq j, k=1}^{N} \|u_k\|^2_{L^2(t,T;U_k)}\big\}\end{aligned}$$

for $t \in [\tau, T]$. C is independent of ξ, v_j and of u_k. $A(t; y, y)$ is an element of the set $\partial_y\{\varphi(t; x, y) + I_{Sx}(y)\}_{x=y}$.

Throughout the paper, we shall denote by $\mathcal{R}(\tau, \xi; v_j, \pi_j u)$, the set of all solutions of (0.1) with control $\{v_j, \pi_j u\}$ and

$$\pi_j u = (u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_N).$$

Let $\mathcal{X}_a = \big\{y : \ y \in C(\tau, T; H);\ 0 \leq \|y'\|_{L^2(\tau,T;H)} + \sup_{t \in [\tau,T]} \varphi(t; y, y) \leq a\big\}$ for some positive number a. Then $co(\mathcal{X}_a)$ is a closed convex subset of $C(\tau, T; H)$.

Lemma 2.1. *Suppose all the hypotheses of Theorem* 2.1 *are satisfied, then $\mathcal{R}(\tau; \xi;\, v_j, \pi_j u)$ is a compact subset of $L^2(\tau, T; H)$. Moreover, $\mathcal{R}(\tau; .; v_j, \pi_j u)$ maps $\mathcal{X}_a$ into $\mathcal{X}_{C(1+a)}$ for some positive constant C.*

Proof. It follows from Theorem 2.1 and from part 4 of Assumption I that $\mathcal{R}(\tau;\xi;v_j,\,\pi_j u_j)$ is a non-empty subset of $C(\tau,T;H)$ and

$$\|y'\|^2_{L^2(t,T;H)} + \varphi(t;y,y) + \|A(.;y,y)\|^2_{L^2(t,T;H)} \leq C\{1+\varphi(t;\xi,\xi) + \|v_j\|^2_{L^2(t,T;U_j)} + \sum_{j\neq k,k=1}^{N} \|u_k\|^2_{L^2(t,T;U_k)}$$

for all $y \in \mathcal{R}(\tau,\xi;v_j,\pi_j u_j)$

1) Let $\{y_n\}$ be in $\mathcal{R}(\tau;\xi;v_j,\pi_j u)$ and suppose that $\|y_n\|_{L^2(\tau,T;H)} \leq C$. We now show that there exists a subsequence $\{y_{n_k}\}$ such that: $y_{n_k} \to y$ in $L^2(\tau,T;H)$ with $y \in \mathcal{R}(\tau;\xi;v_j,\pi_j u) \cap C(\tau,T;H)$.

From Theorem 2.1 we have

$$\|y_n'\|_{L^2(t,T;H)} + \varphi(t;y_n,y_n) + \|A(.;y_n,y_n)\|_{L^2(t,T;H)} \leq C.$$

for $t \in [\tau,T]$. Thus there exists a subsequence $\{y_{n_k}\}$ such that

$$\{y_{n_k},\ y'_{n_k},\ A(.;y_{n_k},y_{n_k})\} \to \{y,y',\chi\}$$

in $(L^\infty(\tau,T;H))_{weak*} \times (L^2(\tau,T;H))^2_{weak}$ as $k\to\infty$ with $y \in C(\tau,T;H)$. Moreover it follows from part 4 of Assumption I that $y_{n_s} \to y$ in $L^2(\tau,T;H)$ and a.e. on (τ,T).

2) We have: $y_{n_k}(t) \in Sy_{n_k}(t);\ t\in[\tau,T]$. Since S is u.s.c. in H, its graph is closed and hence $y(t) \in Sy(t)$ for $t \in [\tau,T]$.

We now show that

$$Sy(t) = \bigcap_{k\geq k_0} Sy_{n_k}(t); \quad t\in[\tau,T] \tag{2.1}$$

It is clear that

$$\bigcap_{k\geq k_0} Sy_{n_k}(t) \subset Sy(t); \quad t\in[\tau,T].$$

Suppose that there exists $x(t) \in Sy(t)$ with $x(t)$ not in $\bigcap_{k\geq k_0} Sy_{n_k}(t)$. Then

$$d = \inf\{\|x(t)-\alpha\|_H :\ \forall\alpha \in \bigcap_{k\geq k_0} Sy_{n_k}(t)\}$$

is a positive number.

It follows from the definition of u.s.c. that there exists a neighborhood $\mathcal{N}(y)$ of $y(t)$ such that

$$Sz \subset \{w:\ \|w-\alpha(t)\| \leq \eta\}$$

for all z in $\mathcal{N}(y)$. Thus, $0 < d \leq \eta$.

Since η is arbitrary, we get a contradiction and therefore (2.1) is proved.

3) We have from the definition of sub-differential

$$\int_\tau^T \big\{\varphi(t; y_{n_k}, x) + I_{Sy_{n_k}}(x) - \varphi(t; y_{n_k}, y_{n_k}) - I_{Sy_{n_k}}(y_{n_k})\big\}dt \geq \int_\tau^T (A(t; y_{n_k}, y_{n_k}), x - y_{n_k})dt$$

for all $x \in C(\tau, T; H); x(t) \in D(\varphi(t; y; .))$ for $t \in [\tau, T]$ with $x(t) \in Sy_{n_k}(t)$.

We obtain by taking (2.1) into account

$$\int_\tau^T \{\varphi(t; y_{n_k}, x) - \varphi(t; y_{n_k}, y_{n_k}\}dt \geq \int_\tau^T (A(t; y_{n_k}, y_{n_k}), x - y_{n_k})dt \quad (2.2)$$

for all $x \in C(\tau, T; H); x(t) \in D(\varphi(t; y; .))$ for $t \in [\tau, T]$ with $x(t) \in Sy(t)$.

Let $k \to \infty$ in (2.2) and we get

$$\int_\tau^T \{\varphi(t; y, x) - \varphi(t; y, y)\}dt \geq \int_\tau^T (\chi, x - y)dt \quad (2.3)$$

with $x \in C(\tau, T; H); x(t) \in D(\varphi(t; y; .))$ for $t \in [\tau, T]$ and $x(t) \in Sy(t)$ on $[\tau, T]$. We may re-write (2.3) as

$$\int_\tau^T \{\varphi(t; y, x) + I_{Sy}(x) - \varphi(t; y, y) - I_{Sy}(y)\}dt \geq \int_\tau^T (\chi, x - y)dt$$

for all $x \in L^2(\tau, T; H)$. It follows from the definition of the sub-differential that

$$\chi \in \partial_y\{\varphi(t; x, y) + I_{Sx}(y)\} \mid_{x=y} .$$

The lemma is proved. □

The set-valued mapping $\mathcal{R}(\tau; \xi; v_j, \pi_j u)$ of $\mathcal{X}_a \times \mathcal{U}_j \times \prod_{k\neq j,k=1}^N \mathcal{U}_k$ into $C(\tau, T; H)$ has closed images which may not be convex. In order to use the approximate selection theorem, we need an u.s.c. set-valued mapping with closed convex image and thus, we have the following lemma.

Lemma 2.2. *Suppose all the hypotheses of Theorem* 2.1 *are satisfied. Then for any* $a > 0$

$$R(\tau; \xi; v_j, \pi_j u) = co(\mathcal{R}(\tau; \xi; v_j, \pi_j u)) \quad (2.4)$$
$$= \big\{\sum_{p\in P} \lambda_p y^p : y^p \in \mathcal{R}(\tau; \xi; v_j, \pi_j u), 0 \leq \lambda_p \leq 1; \sum_{p\in P} \lambda_p = 1\big\}$$

is an u.s.c. set-valued mapping of $co(\mathcal{X}_a)\times\mathcal{U}_j\times\prod_{k\neq j,k=1}^{N}\mathcal{U}_k$ *into* $L^2(\tau,T;H)$ *with closed convex images.*

Proof. Since $\mathcal{R}(\tau;\xi;v_j,\pi_j u)$ is a compact subset of $L^2(\tau,T;H)$, to show that $R(\tau;\xi;v_j,\pi_j u)$ is u.s.c., it suffices to prove that the graph of $R(\tau;\xi;v_j,\pi_j u)$ is closed. Let $\{y_n\}$ be in $R(\tau;\xi_n;v_j^n,\pi_j u^n)$ and suppose that

$$\{\xi_n,y_n,v_j^n,\pi_j u^n\}\to\{\xi,y,v_j,\pi_j u\}$$

in $H\times L^2(\tau,T;H)\times L^2(\tau,T;U_j)\times\prod_{k\neq j;k=1}^{N}L^2(\tau,T;U_k)$ with $\xi_n\in co(\mathcal{X}_a)$. Then $\xi\in co(\mathcal{X}_a)$ and we now show that: $y\in R(\tau;\xi;v_j,\pi_j u)$. By definition we have

$$y_n=\sum_{k=1}^{K}\lambda_n^k y_n^k;\quad y_n^k\in\mathcal{R}(\tau;\xi_n;v_j^n,\pi_j u^n)$$

with

$$\sum_{k=1}^{K}\lambda_n^k=1,\quad 0\le\lambda_n^k\le 1.$$

From the estimates of Theorem 2.1 we obtain

$$\|(y_n^k)'\|_{L^2(t,T;H)}+\varphi(t;y_n^k,y_n^k)+\|A(.;y_n^k,y_n^k)\|_{L^2(t,T;H)}\le C.$$

for $t\in[\tau,T]$. Thus there exists a subsequence $\{y_{n_s}^k\}$ such that

$$\{y_{n_s}^k,(y_{n_s}^k)',\ A(.;y_{n_s}^k,y_{n_s}^k)\}\to\{y^k,(y^k)',\chi_k\}$$

in $(L^\infty(\tau,T;H))_{weak*}\times(L^2(\tau,T;H))^2_{weak}$. From part 4 of Assumption I, we get: $y_{n_s}^k\to y^k$ in $L^2(\tau,T;H)$. It is clear that : $y^k(\tau)=\xi$.

A proof as in that of Lemma 2.1 gives

$$y^k(t)\in Sy^k(t);\quad \chi_k\in\partial_{y^k}\{\varphi(t;x,y^k)+I_{Sx}(y^k)\}_{x=y^k}.$$

Hence, $y^k\in\mathcal{R}(\tau;\xi;v_j,\pi_j u)$. It is now easy to see that

$$y_{n_s}\to y=\sum_{k=1}^{K}\lambda^k y^k$$

in the weak*-topology of $L^\infty(\tau,T;H)$ with $0\le\lambda^k\le 1$; $\sum_{k=1}^{K}\lambda^k=1$. Since the image of $\mathcal{R}$ and hence that of R are in a compact subset of $L^2(\tau,T;H)$ and since $\mathcal{R}$ is u.s.c., it follows that the graph of R is closed. Therefore R

is u.s.c. and it is clear that the images of R are closed convex subsets of $L^2(\tau,T;H)$. The lemma is proved.

Definition. A control $(u_j^*,\pi_j u_*)\in\mathcal{U}$ is said to be a j-equilibrium point of the system (0.1)–(0.2) if for any $y\in R(\tau;\xi;u_j^*;\pi_j u_*)$, we have

$$J_j(\tau;\xi;y;u_j^*,\pi_j u_*)\le J_j(\tau;\xi;x;v_j;\pi_j u_*)\le J_j(\tau;\xi;z;v_j;\pi_j w)$$

for all $v_j\in\mathcal{U}_j$, all $x\in R(\tau;\xi;v_j,\pi_j u_*)$, all $z\in R(\tau;\xi;v_j;\pi_j w)$ and all $\{v_j,w\}\in\mathcal{U}_j\times\mathcal{U}; j=1,\dots,N$.

It is clear that if $u_*=u^*$, then we have the open loop equilibrium point of the system (0.1)–(0.2).

For the study of feedback laws for N-person non-zero sum games, the notion of a j-equilibrium point of the system seems a much more appropriate one to use than that of open loop equilibrium. Indeed one of the important considerations of the j-producer is to minimize his production costs regardless of those of his competitors in the open economy.

We now have the following theorem on the existence of a j-equilibrium point of (0.1)–(0.2).

Theorem 2.2. *Suppose all the hypotheses of Theorem* 2.1 *are satisfied and suppose that Assumption IV is verified. Then there exists* u^*,u_* *in* $\mathcal{U}$ *and* $y_*\in R(\tau;\xi;u_j^*;\pi_j u_*)$ *such that*

$$J_j(\tau;\xi;y_*;u_j^*,\pi_j u_*)\le J_j(\tau;\xi;x;v_j,\pi_j u_*)\le J_j(\tau;\xi;z;v_j;\pi_j w)$$

for all $\{x,z\}\in R(\tau;\xi;v_j,\pi_j u_*)\times R(\tau;\xi;v_j;\pi_j w)$ *and all* $\{v_j,w\}\in\mathcal{U}_j\times\mathcal{U}$, *i.e., the system* (0.1)–(0.2) *has a* j *-equilibrium point for* $j=1,\dots,N$..

Proof. 1) Consider the optimization problem

$$d_j(\tau,\xi;\pi_j w)=\inf\big\{J_j(\tau;\xi;x;v_j,\pi_j w):\forall x\in R(\tau;\xi;v_j,\pi_j w),\forall v_j\in\mathcal{U}_j\big\}. \tag{2.5}$$

It is clear that the infimum exists and let $\{v_j^n,x_n\}$ be a minimizing sequence of (2.5). Then

$$d_j\le J_j(\tau;\xi;x_n;v_j^n,\pi_j w)\le d_j+n^{-1}. \tag{2.6}$$

Since $\mathcal{U}_j$ is a compact subset of $L^2(\tau,T;U_j)$, we have $v_j^{n_s}\to\tilde v_j$ in $L^2(\tau,T;U_j)$ with $\tilde v_j\in\mathcal{U}_j$.

The image of $R(\tau;\xi;.)$ is in a compact subset of $L^2(\tau,T;H)$ and therefore, $x_{n_s}\to x$ in $L^2(\tau,T;H)$. Since $R(\tau,\xi;.)$ is u.s.c., its graph is closed and hence, $x\in R(\tau;\xi;\tilde v_j,\pi_j w)$. It is now clear that

$$d_j(\tau;\xi;\pi_j w)=J_j(\tau;\xi;x;\tilde v_j,\pi_j w);\quad j=1,\dots,N \tag{2.7}$$

2) Consider the optimization problem

$$D_j(\tau,\xi) = \inf\big\{d_j(\tau,\xi;\pi_j w) : \forall w_k \in \mathcal{U}_k, k \neq j, k = 1,\dots,N\big\}. \tag{2.8}$$

Let $\{\tilde{v}_j^n, x_n, \pi_j w^n\}$ be a minimizing sequence of (2.8), i.e.,

$$D_j(\tau,\xi) \leq J_j(\tau,\xi;x_n;\tilde{v}_j^n,\pi_j w^n) \leq n^{-1} + D_j(\tau,\xi). \tag{2.9}$$

As in the first part, we get:

$$\{\tilde{v}_j^{n_s}, x_{n_s}, \pi_j w^{n_s}\} \to \{v_j^*, x_*, \pi_j w^*\}$$

in $L^2(\tau,T;U_j) \times L^2(\tau,T;H) \times \prod_{k\neq j,k=1}^{k=N} L^2(\tau,T;U_k)$ and

$$D_j(\tau,\xi) = J_j(\tau;\xi;x_*;v_j^*,\pi_j w^*) = d_j(\tau;\xi;\pi_j w^*) \tag{2.10}$$

with $x_* \in R(\tau;\xi;v_j^*,\pi_j w^*)$; $j = 1,\dots,N$.

3) Let $u_* = \{w_j^*\}_{j=1}^{j=N}$ and consider the problem

$$\inf\big\{J_j(\tau;\xi;y;u_j;\pi_j u_*) : y \in R(\tau;\xi;u_j,\pi_j u_*), \forall u_j \in \mathcal{U}_j\big\}; j = 1,\dots,N \tag{2.11}$$

Then the first part of the proof yields the existence of $u^* \in \mathcal{U}$ such that

$$J_j(\tau;\xi;y_*;u_j^*;\pi_j u_*) \leq J_j(\tau;\xi;y;v_j;\pi_j u_*)$$

for all $y \in R(\tau;\xi;v_j,\pi_j u_*)$, all $v_j \in \mathcal{U}_j$ and $j = 1,\dots,N$.

Taking into account the definition of $D_j(\tau,\xi)$, we get

$$J_j(\tau;\xi;y_*;u_j^*;\pi_j u_*) \leq D_j(\tau,\xi) \leq d_j(\tau;\xi;\pi_j w) \leq J_j(\tau;\xi;x;v_j;\pi_j w)$$

for all $x \in R(\tau;\xi;v_j;\pi_j w)$, all $v_j \in \mathcal{U}_j$, all $w \in \mathcal{U}$. The theorem is proved. □

We now define the value vector-function of the system (0.1)–(0.2) Let

$$V_j(\tau,\xi) = \inf\big\{J_j(\tau;\xi;y;u_j^*;\pi_j u_*) : y \in R(\tau;\xi;u_j^*,\pi_j u_*), \forall u_j^*, \forall u_*\big\} \tag{2.12}$$

with u^*, u_* as in Theorem 2.2 and $j = 1,..,N$.

Theorem 2.3. *Suppose all the hypotheses of Theorem* 2.1 *are satisfied and let* $\{V_j(\tau,\xi)\}_{j\in N}$ *be as in* (2.11)*. Then there exists* $\hat{u},\tilde{u}\in\mathcal{U}$ *and* $y^*\in R(\tau;\xi;\hat{u}_j,\pi_j\tilde{u})$ *such that*

$$V_j(\tau,\xi)=J_j(\tau;\xi;y^*;\hat{u}_j,\pi_j\tilde{u});\quad j=1,\dots,N.$$

Proof. In view of Theorem 2.2, the admissible set of the optimization problem for V_j is non-empty. Using a minimizing sequence we get the stated result. As the proof is similar to that of Theorem 2.2, we shall not reproduce it.

3. Suboptimal control problems. The value vector-function $V(\tau,\xi)$ of the differential inclusion (0.1) with cost functionals (0.2) can be shown to be continuous in τ and in ξ on $[0,T]\times D(\varphi)$. Applications of the maximum principle or of the Hamilton Jacobi equations are needed in order to obtain feedback control laws. Thus the existence of $\partial_\xi V_j(\tau,\xi)$ is crucial and we are led to the study of suboptimal control problems for which the value-function is at least locally Lipschitzian.

Let $Q=[T/\varepsilon];\tilde{q}=[\tau/\varepsilon]$ with

$$u_j^q=\varepsilon^{-1}\int_{q\varepsilon}^{(q+1)\varepsilon}u_j(t)dt;\quad j=1,\dots,N,\quad q=\tilde{q},\dots,Q.$$

For simplicity we shall assume that $\tilde{q}\varepsilon=\tau$; the other case can be treated in exactly the same fashion.

Let $\tilde{u}$ be as in Theorem 2.3 and consider the expression

$$\begin{aligned}V_j^\varepsilon(\tau,\xi)=\inf\Big\{&f_j(y_Q(Q\varepsilon))+\varepsilon\sum_{q=1}^{q=Q}\{g_j(y_q(q\varepsilon))+h_j(v_j^q)\}:\forall v_j\in\mathcal{U}_j,\\ &y_{q+1}(t)\in R(q,y_q(q\varepsilon);\ v_j^q,\pi_j\tilde{u}^q);t\in[q\varepsilon,(q+1)\varepsilon,\ y_{\tilde{q}}(\tau)=\xi\Big\};\end{aligned}\tag{3.1}$$

$j=1,\dots,N$, where R is as in Lemma 2.2 and where we write $R(q;.)$ for $R(q\varepsilon;.)$.

The first result of the section is the following theorem.

Theorem 3.1. *Suppose all the hypotheses of Theorem* 2.1 *are satisfied and suppose that Assumption IV is verified. Let* $\{V_j(\tau,\xi),V_j^\varepsilon(\tau,\xi)\}_{j=1}^{j=N}$ *be as in* (2.12) *and in* (3.1) *respectively. Then*

$$\lim_{\varepsilon\to0}V_j^\varepsilon(\tau,\xi)=V_j(\tau,\xi);\quad j=1,\dots,N$$

for any τ, ξ *in* $[0, T] \times D(\varphi(\tau; .))$ *with* $\xi \in S\xi$.

Let $\{\tilde{v}_j^q\}_{q=1}^{Q}$ *be an optimal control of* (3.1) *and let* $\hat{v}_j^\varepsilon(t) = \tilde{v}_j^q$ *for* $t \in [q\varepsilon, (q+1)\varepsilon]$, *then the* $L^2(\tau, T; U_j)$*-limit of* $\hat{v}_j^\epsilon$, *is an optimal control of* (2.12) *for* $j = 1, \dots, N$.

Proof. 1) Let $y_{q+1}^s(t)$ be a solution of the differential inclusion

$$(y_{q+1}^s)' + \partial\{\varphi(t; x, y_{q+1}^s) + I_{Sx}(y_{q+1}^s)\}_{x=y_{q+1}^s} \ni M_j v_j^q + \sum_{k\neq j, k=1}^{k=N} M_k \tilde{u}_k^q,$$

$$y_{q+1}^s(t) \in S y_{q+1}^s(t), \quad t \in [q\varepsilon, (q+1)\varepsilon]; \quad y_{q+1}^s(q\varepsilon) = y_q^s(q\varepsilon)$$

and let

$$y_{q+1}(t) = \sum_{s=1}^{K} \lambda_s y_{q+1}^s(t); \quad y_{q+1}^s \in \mathcal{R}(q; y_q^s(q\varepsilon); \quad v_j^q, \pi_j \tilde{u}_k^q)$$

with

$$0 \leq \lambda_s \leq 1; \quad \sum_{s=1}^{s=K} \lambda_s = 1.$$

We have : $y_{q+1}(t)$ in $R(q; y_q(q\varepsilon); v_j^q, \pi_j \tilde{u}_k^q)$. Let

$$\tilde{v}_j^\varepsilon(t) = v_j^q; \quad t \in [q\varepsilon, (q+1)\varepsilon]$$

and similarly for $\tilde{u}_k^\varepsilon$. Let y_s^ε be a $C(\tau, T; H)$-function with $y_s^\varepsilon(t) = y_{q+1}^s(t)$ for $t \in [q\varepsilon, q(1+\varepsilon)]$ and set

$$\tilde{y}^\varepsilon(t) = \sum_{s=1}^{K} \lambda_s y_s^\varepsilon(t).$$

Then

$$(y_s^\varepsilon)' + \partial\{\varphi(t; x, y_s^\varepsilon) + I_{Sx}(y_s^\varepsilon)\}_{x=y_s^\varepsilon} \ni M_j v_j^\varepsilon + \sum_{k\neq j, k=1}^{k=N} M_k \tilde{u}_k^\varepsilon$$

$$y_s^\varepsilon(t) \in S y_s^\varepsilon(t); \quad t \in [\tau, T], \quad y_s^\varepsilon(q\varepsilon) = y_q(q\varepsilon).$$

We get $\tilde{y}^\varepsilon(t)$ in $R(\tau; \xi; v_j^\varepsilon, \pi_j \tilde{u}^\varepsilon)$ with $y_s^\varepsilon \in \mathcal{R}(\tau; \xi; v^j; \pi_j \tilde{u})$. From the estimate of Theorem 2.1 we have $y_s^\varepsilon, \tilde{y}_\varepsilon$ in a compact subset of $L^2(\tau, T; H)$. Therefore, there exists a subsequence, denoted again by $\{y_s^\varepsilon, \tilde{y}^\varepsilon\}$ such that $y_s^\varepsilon \to y_s$ in $L^2(\tau, T; H)$. Since $\mathcal{R}(\tau, \xi; .)$ is u.s.c. and $\{\tilde{v}_j^\varepsilon, \pi_j \tilde{u}^\varepsilon\} \to \{\tilde{v}_j, \pi_j \tilde{u}_*\}$ in

$$L^2(\tau, T; U_j) \times \prod_{k \neq j, k=1}^{k=N} L^2(\tau, T; U_k),$$

we get $y_s \in \mathcal{R}(\tau; \xi; \tilde{v}_j, \pi_j \tilde{u})$. Thus, $\tilde{y} \in R(\tau; \xi; \tilde{v}_j, \pi_j \tilde{u})$.

2) Taking into account the definition of $\{\tilde{y}^\varepsilon, \tilde{u}^\varepsilon, \tilde{v}_q^\varepsilon\}$,we may re-write (3.1) as

$$\begin{aligned} V_j^\varepsilon(\tau, \xi) = \inf\big\{ & f_j(\tilde{y}^\varepsilon(T)) + \varepsilon \sum_{q=1}^{Q} \{g_j(\tilde{y}^\varepsilon(q\varepsilon)) + h_j(v_j^q))\} : \\ & \forall v_j \in \mathcal{U}_j; \tilde{y}^\varepsilon(t) \in R(\tau; \xi; v_j, \pi_j \tilde{u})\big\}; \quad j = 1, \dots, N \end{aligned} \tag{3.2}$$

Let $\{y_*^\varepsilon, v_j^*\}$ be an optimal solution of (3.2) for which the existence can be established easily by using a minimizing sequence. Then

$$V_j^\varepsilon(\tau, \xi) = f_j(y_*^\varepsilon(T)) + \varepsilon \sum_{q=1}^{Q} \{g_j(y_*^\varepsilon(q\varepsilon)) + h_j(v_j^*(q\varepsilon))\}$$

with $y_*^\varepsilon \in R(\tau; \xi; v_j^*; \pi_j \tilde{u}^\varepsilon)$.

As above $y_*^\varepsilon \to y$ in $L^2(\tau, T; H) \cap (L^\infty(\tau, T; H))_{weak*}$ with $y \in C(\tau, T; H)$. Thus,

$$f_j(y(T)) + \int_\tau^T \{g_j(y(t)) + h_j(v_j(t))\} dt \leq \liminf_{\varepsilon \to 0} V_j^\varepsilon(\tau, \xi). \tag{3.3}$$

Since $y \in R(\tau; \xi; v_j; \pi_j \tilde{u})$, it is in the admissible set of $V_j(\tau, \xi)$ and we have

$$V_j(\tau, \xi) \leq \liminf_{\varepsilon \to 0} V_j^\varepsilon(\tau, \xi).$$

3) Let $\eta > 0$, then from the definition of V_j we deduce that there exists $w_j \in \mathcal{U}_j$ such that

$$f_j(z(T)) + \int_\tau^T \{g_j(z(t)) + h_j(w_j(t))\} dt \leq \eta + V_j(\tau, \xi) \tag{3.4}$$

for all $z \in R(\tau; \xi; w_j, \pi_j \tilde{u})$. With the control $\{w_j, \pi_j \tilde{u}\}$, we proceed as in the first part by considering the inclusion (0.1) on sub-intervals $(q\varepsilon, (q+1)\varepsilon)$

and finally we get: $z_\varepsilon(t) \in R(\tau;\xi;w_j^\varepsilon,\pi_j\tilde{u}^\varepsilon)$. Again, $\{z_\varepsilon\}$ is in a compact subset of $L^2(\tau,T;H)$. Thus there exists a subsequence such that: $z_\varepsilon \to \tilde{z}$ in $L^2(\tau,T;H)$.

The mapping $R(\tau;\xi;.)$ is u.s.c. and so $\tilde{z} \in R(\tau;\xi;w_j,\pi_j\tilde{u}_*)$. We have

$$f_j(z_\varepsilon(T)) + \int_\tau^T \{g_j(z_\varepsilon(t)) + h_j(w_j^\varepsilon(t)\}\,dt \\ \to f_j(\tilde{z}(T)) + \int_\tau^T \{g_j(\tilde{z}(t)) + h_j(w_j(t))\}\,dt.$$

In view of (3.4), we get

$$\limsup_{\varepsilon\to 0} V_j^\varepsilon(\tau,\xi) \le \eta + V_j(\tau,\xi).$$

Hence,

$$\lim_{\varepsilon\to 0} V_j^\varepsilon = V_j(\tau,\xi)$$

for $j = 1,\dots,N$.

4)Let $\{\tilde{v}_j^q\}_{q=1}^Q$ be an optimal control of (3.1) and let $\hat{v}_j^\varepsilon(t)$ be as in the theorem. Then we know that: $\hat{v}_j^\varepsilon \to \tilde{v}_j$ in $L^2(\tau,T;U_j)$. With $\tilde{y}_\varepsilon$ as an optimal solution corresponding to the control $\{\hat{v}_j^\varepsilon,\pi_j\tilde{u}_*^\varepsilon\}$, we have

$$V_j^\varepsilon(\tau,\xi) = f_j(\tilde{y}_\varepsilon(T)) + \varepsilon\sum_{q=1}^Q \{g_j(\tilde{y}_\varepsilon(q\varepsilon)) + h_j(\hat{v}_j^\varepsilon(q\varepsilon))\}.$$

Thus,

$$\lim_{\varepsilon\to 0} V_j^\varepsilon(\tau,\xi) = f_j(\tilde{y}(T)) + \int_\tau^T \{g_j(\tilde{y}(t)) + h_j(\tilde{v}_j(t))\}dt = V_j(\tau,\xi).$$

Therefore, $\{\tilde{v}_j\}$ is an optimal control of (2.12) and the theorem is proved. □

We shall now introduce a suboptimal control problem for which the value function has a generalized gradient.

Let $R(q;\xi;v_j^q,\pi_j\tilde{u})$ be as in Lemma 2.2, then $R(r;.)$ is an u.s.c. set-valued mapping of $co(\mathcal{X}_a) \times \mathcal{U}_j \times \prod_{k\ne j,k=1}^N \mathcal{U}_k$ into the closed convex subsets of $L^2(q\varepsilon,(q+1)\varepsilon;H)$. It follows from the approximate selection theorem ([1], p. 84) that there exist single-valued mappings $R_n(q;.)$ of $co(\mathcal{X}_a) \times \mathcal{U}_j \times \prod_{k\ne j,k=1}^N \mathcal{U}_k$ into $L^2(q\varepsilon,(q+1)\varepsilon;H)$ such that

(i) $R_n(q;.)$ are locally Lipschitzian,
(ii) Range$(R_n(q;.)) \subset$ co(Range $R(q;.)$),
(iii) Graph$(R_n(q;.)) \subset$ Graph$(R(q;.)) + n^{-1}$ (unit ball about the graph of $R(q;.)$).

Let $\tilde{u}$ be as in Theorem 2.3 and let

$$V_j^{\varepsilon,n}(\tau,\xi) = \inf\{f_j(z_Q^n(Q\varepsilon)) + \varepsilon\{\sum_{q=1}^{Q} g_j(z_q^n(q\varepsilon)) + h_j(v_j^q)\} : \\ \forall v_j \in \mathcal{U}_j; z_{q+1}^n(t) = R_n(q; z_q^n(q\varepsilon); \varepsilon v_j^q, \varepsilon\pi_j\tilde{u}^q); z_{\tilde{q}}^n(\tau) = \xi\} \tag{3.5}$$

Lemma 3.1. *Suppose all the hypotheses of Theorem* 3.1 *are satisfied and let*

$$\{V_j^{\varepsilon,n}(\tau,\xi)\}_{j=1}^N$$

be as in (3.5), *then the generalized Clarke gradient* $\{\partial_\xi V_j^{\varepsilon,n}(\tau,\xi)\}_{j=1}^N$ *exists.*

Proof. We show that $V_j^{\varepsilon,n}(\tau,\xi)$ is locally Lipschitzian. Let ζ be in $co(\mathcal{X}_a)$ and such that $\zeta \in S\zeta$. The existence of an optimal control $\{\tilde{v}_{j,n}^q\}_{q=1}^Q$ of (3.5) and of an optimal solution $\{\tilde{z}_q^n\}_{q=1}^Q$ of (3.5) corresponding to the initial value ξ can be shown as in the proof of Theorem 2.2. Let

$$y_{q+1}^n(t) = R_n(q; y_q^n(q\varepsilon); \varepsilon\tilde{v}_{j,n}^q, \varepsilon\pi_j\tilde{u}); t \in [q\varepsilon, (q+1)\varepsilon]$$

with $y_{\tilde{q}}^n(\tau) = \zeta$. Then $\{y_q^n\}_{q=1}^Q$ is in the admissible set of the optimization problem for $V_j^{\varepsilon,n}(\tau,\zeta)$. We get

$$\begin{aligned} &V_j^{\varepsilon,n}(\tau,\zeta) - V_j^{\varepsilon,n}(\tau,\xi) \\ &\quad\le f_j(y_Q^n(Q\varepsilon)) - f_j(\tilde{z}_Q^n(Q\varepsilon)) + \varepsilon\sum_{q=1}^{Q}\{g_j(y_q^n(q\varepsilon)) - g_j(\tilde{z}_q^n(q\varepsilon))\} \\ &\quad\le C\sum_{q=1}^{Q}\|y_q^n(q\varepsilon) - \tilde{z}_q^n(q\varepsilon)\|. \end{aligned}$$

Since $R_n(q;.)$ are locally Lipschitzian, we obtain from the definition of $R_n(q;.)$

$$V_j^{\varepsilon,n}(\tau,\zeta) - V_j^{\varepsilon,n}(\tau,\xi) \le \sum_{q=1}^{Q} C_{q,\varepsilon}\|\zeta - \xi\|.$$

Similarly, by considering an optimal control for $V_j^{\varepsilon,n}(\tau,\zeta)$ instead of that of $V_j^{\varepsilon,n}(\tau,\xi)$, we have

$$V_j^{\varepsilon,n}(\tau,\xi) - V_j^{\varepsilon,n}(\tau,\zeta) \le \sum_{q=1}^{Q} c_{q,\varepsilon}\|\zeta - \xi\|.$$

Hence,

$$| V_j^{\varepsilon,n}(\tau,\xi) - V_j^{\varepsilon,n}(\tau,\zeta) | \leq \sum_{q=1}^{Q} c_q^{\varepsilon} \|\zeta - \xi\|.$$

Since $V_j^{\varepsilon,n}(\tau,\xi)$ is locally Lipschitzian, the generalized Clarke gradient $\partial_\xi V_j^{\varepsilon,n}(\tau,\xi)$ exists and the lemma is proved. □

The main result of the section is the following theorem.

Theorem 3.2. *Suppose all the hypotheses of Theorem* 3.1 *are satisfied. Let*

$$\{V_j^{\varepsilon}(\tau,\xi)\}_{j=1}^{N}$$

be as in (3.1) *and let* $V_j^{\varepsilon,n}(\tau,\xi)$ *be as in* (3.5)*, then*

$$\lim_{n\to\infty} V_j^{\varepsilon,n}(\tau,\xi) = V_j^{\varepsilon}(\tau,\xi); \quad j = 1,\ldots,N$$

for any τ,ξ *in* $[0,T] \times co(\mathcal{X}_a)$ *with* $\xi \in S\xi$*. Moreover, if* $\{\tilde{v}_j^{q,n}\}_{q=1}^{Q}$ *is an optimal control of* (3.5)*, then:* $\tilde{v}_j^{q,n} \to \tilde{v}_j^{q}$ *in* U_j *and* $\{\tilde{v}_j^{q}\}_{q=1}^{Q}$ *is an optimal control of* (3.1)*.*

Proof. Let $\{z_q^n(t)\}$ be as in (3.5) and suppose that $t \in [q\varepsilon,(q+1)\varepsilon]$. We have

$$z_{\tilde{q}+1}^n(t) = R_n(\tilde{q};\xi;\varepsilon\tilde{v}_j^{\tilde{q},n},\varepsilon\pi_j\tilde{u}^{\tilde{q}}).$$

We know from Lemmas 2.1–2.2 that $R(\tilde{q};.;v_j,\pi_j u)$ maps $co(\mathcal{X}_a)$ into $co(\mathcal{X}_{C(1+a)})$ and since the range of $R_n(\tilde{q};.)$ is in the convex hull of the range of $R(\tilde{q};.)$, the mapping $R_n(\tilde{q};.;v_j,\pi_j u)$ takes $co(\mathcal{X}_a)$ into $co(co(\mathcal{X}_{C(1+a)})) = co(\mathcal{X}_{C(1+a)})$. We have

$$\|(z_{\tilde{q}+1}^n)'\|_{L^2(\tau,\tau+\varepsilon;H)} + \sup_{(\tau,\tau+\varepsilon)} \varphi(t;z_{\tilde{q}+1}^n;z_{\tilde{q}+1}^n) \leq C\{1+\varphi(\tau;\xi,\xi)\}$$

for all $t \in [\tau,\tau+\varepsilon]$. C is independent of n. Moreover,

$$z_{\tilde{q}+1}^n = \sum_{k=1,p=1}^{k=K,p=P} \lambda_k\nu_p z_{\tilde{q}+1,k}^{n,p}; \quad 0 \leq \lambda_k,\nu_p \leq 1$$

and

$$\sum_{k=1}^{K} \lambda_k = 1 = \sum_{p=1}^{P} \nu_p$$

with $z^{n,p}_{\tilde{q}+1,k} \in \mathcal{X}_{C(1+a)}$ and $a = \varphi(\tau;\xi,\xi)$. By induction we have

$$\|(z^n_{q+1})'\|_{L^2(q\varepsilon,(q+1)\varepsilon;H)} + \sup_{(q\varepsilon,(q+1)\varepsilon)} \varphi(t; z^n_{q+1}; z^n_{q+1}) \leq C_q\{1 + \varphi(\tau;\xi,\xi)\}$$

for all $t \in [q\varepsilon,(q+1)\varepsilon]$ with C_q independent of n. Thus, by taking subsequences we obtain $\{z^n_q,(z^n_q)'\} \to \{z_q, z'_q\}$ in $\{L^\infty((q-1)\varepsilon, q\varepsilon; H)\}_{weak*} \cap L^2((q-1)\varepsilon, q\varepsilon; H) \times (L^2((q-1)\varepsilon, q\varepsilon; H))_{weak}$ for $q = 1,\ldots,Q$. It is clear that $z^n_q \in C((q-1)\varepsilon, q\varepsilon; H)$.

Since $z^n_q(t) \in Sz^n_q(t)$, it follows from the u.s.c. of S in H that $z_q(t) \in Sz_q(t)$; $t \in [(q-1)\varepsilon, q\varepsilon]$. Let $\chi^n_{q+1} \in R(q; z^n_q(q\varepsilon); \varepsilon v^q_{j,n}, \varepsilon\pi_j\tilde{u}^q)$, then from the definition of $R_n(q;.)$ we obtain

$$\|z^n_{q+1} - \chi^n_{q+1}\|_{L^2(q\varepsilon,(q+1)\varepsilon;H)} \leq Cn^{-1}.$$

Therefore, $\chi^n_{q+1} \to z_{q+1}$ in $L^2(q\varepsilon,(q+1)\varepsilon;H)$. Since $R(q;.)$ is u.s.c., its graph is closed and hence $z_{q+1}(t) \in R(q; z_q(q\varepsilon); \varepsilon v^q_j, \varepsilon\pi_j\tilde{u})$ for $t \in [q\varepsilon,(q+1)\varepsilon]$; $q = 1,\ldots,Q$.

2) We shall now use the above result to show that

$$V^\varepsilon_j(\tau,\xi) \leq \liminf_{n\to\infty} V^{\varepsilon,n}_j(\tau,\xi); \quad j = 1,\ldots,N$$

for any τ,ξ as in the theorem.

Let $\{\tilde{z}^n_q, \tilde{v}^{q,n}_j\}^Q_{q=1}$ be an optimal solution of (3.5); i.e.,

$$V^{\varepsilon,n}_j(\tau,\xi) = f_j(\tilde{z}^n_Q(Q\varepsilon)) + \varepsilon\sum_{q=1}^{Q}\{g_j(\tilde{z}^n_q(q\varepsilon)) + h_j(\tilde{v}^{q,n}_j)\}. \tag{3.6}$$

From the first part we get, by taking subsequences $\{\tilde{z}_{n_q}, \tilde{v}^{q,n}_j\} \to \{\tilde{z}_q, \tilde{v}^q_j\}$ in $(L^\infty((q-1)\varepsilon, q\varepsilon; H))_{weak*} \cap L^2((q-1)\varepsilon, q\varepsilon; H) \times U_j$ with

$$\tilde{z}_{q+1}(t) \in R(q; \tilde{z}_q(q\varepsilon); \varepsilon\tilde{v}^q_j, \varepsilon\pi_j\tilde{u}^q).$$

It follows from (3.6) that

$$V^\varepsilon_j(\tau,\xi) \leq \liminf_{n\to\infty} V^{\varepsilon,n}_j(\tau,\xi); \quad j = 1,\ldots,N.$$

3) Let $\eta > 0$ be an arbitrary number. A proof as in part 4 of Theorem 3.1 shows that

$$\limsup_{n\to\infty} V^{\varepsilon,n}_j(\tau,\xi) \leq V^\varepsilon_j(\tau,\xi).$$

Therefore,

$$V_j^{\varepsilon}(\tau,\xi) = \lim_{n\to\infty} V_j^{\varepsilon,n}(\tau,\xi); \quad j = 1,\dots,N.$$

It is now clear that $\{\tilde{v}_j^q\}_{q=1}^Q$ is an optimal control of (3.1) and the theorem is proved.

4. Feedback laws. In this section we shall establish some feedback control laws for (3.5). In [5] Clarke and Vinter used an auxiliary optimization problem to connect the maximum principle and dynamic programming. The idea has been applied by Popa in [8–9] in order to obtain feedback laws for a single control problem.

Let $V_j^{\varepsilon,n}, z_q^n, R_n(q;.)$ be as in (3.5) and let $k_{q,\delta}^j(.)$ be the convex continuous function on H defined by

$$k_{q,\delta}^j(\omega) = \sup\big\{(p,\omega) : y \in co\mathcal{X}_a; p \in \partial_y V_j^{\varepsilon,n}(q\varepsilon, y), \|y - z_q^n(q\varepsilon)\| \le \delta\big\} \quad (4.1)$$

for $q = 1,\dots,Q; j = 1,\dots,N$ with $a = 1 + \sup\varphi(q\varepsilon; z_q^n(q\varepsilon), z_q^n(q\varepsilon))$. Let

$$\mathcal{K}_q = \big\{z : z \in co(\mathcal{X}_a); \|\omega_{q+1}\| \le \delta, \|z - z_q^n(q\varepsilon)\| \le \delta; \|z + \omega_{q+1} - z_q^n(q\varepsilon)\| \le \delta\big\};$$

$q = 1,\dots,Q$.

Lemma 4.1. *Suppose all the hypotheses of Theorem* 3.1 *are satisfied and let* $\{z_q^n, R_n(q;.)\}$ *be as in* (3.5)*. Then for any given* ω_q *with* $\|\omega_q\| \le \delta$*, there exists a unique* $y_q^n \in C((q-1)\varepsilon, q\varepsilon; H)$ *such that*

$$(y_q^n(t) - R_n(q; y_{q-1}^n((q-1)\varepsilon) + \omega_q; \varepsilon v_j^q, \varepsilon\pi_j\tilde{u}^q), x - y_q^n(t)) \ge 0 \qquad (4.2)$$

for all $x \in \mathcal{K}_q$ *with* $y_{\tilde{q}}^n(\tau) = \xi$ *and* $y_q^n(t) \in \mathcal{K}_q$ *for almost all* $t \in [(q-1)\varepsilon, q\varepsilon]$.

Proof. The lemma is an immediate consequence of the existence theorem of the theory of variational inequalities as $\mathcal{K}_q$ is a closed convex subset of H. □

We now consider the following optimization problem

$$V_{j,\delta}^{\varepsilon,n}(\tau,\xi) = \inf\big\{f_j(y_Q^n(Q\varepsilon)) + \varepsilon\sum_{q=1}^Q\{g_j(y_q^n(q\varepsilon)) + h_j(v_j^q)\} + \sum_{q=1}^Q k_{q,\delta}^j(-\omega_q) : \forall\|\omega_q\| \le \delta; \forall v_j \in \mathcal{U}_j\big\} \qquad (4.3)$$

with $y_q^n(t)$ as in (4.2).

Lemma 4.2. *Suppose all the hypotheses of Theorem* 3.1 *are satisfied. Then there exists* $\{\omega_q^n, \tilde{v}_{j,n}^q, \tilde{y}_q^n\}_{q=1}^Q$ *such that for* $j = 1, \dots, N$,

$$V_{j,\delta}^{\varepsilon,n}(\tau,\xi) = f_j(\tilde{y}_Q^n(Q\varepsilon)) + \varepsilon \sum_{q=1}^{Q} \{g_j(\tilde{y}_q^n(q\varepsilon)) + h_j(\tilde{v}_{j,n}^q)\} + \sum_{q=1}^{Q} k_{q,\delta}^j(-\omega_q^n).$$

Proof. The existence of an optimal solution of (4.3) can be proved by using a minimizing sequence and by applying Lemma 2.1. The proof is similar to that of Theorem 2.2 and we shall not reproduce it. □

The following lemma is crucial for the proof of the main theorem of the section.

Lemma 4.3. *Suppose all the hypotheses of Theorem* 3.1 *are satisfied and let* $\{y_q^n\}$ *be as in Lemma* 4.1. *Then*

$$f_j(y_Q^n(Q\varepsilon)) + \sum_{q=1}^{Q} \{\varepsilon g_j(y_q^n(q\varepsilon)) + \varepsilon h_j(v_j^q) + k_{q,\delta}^j(-\omega_q)\} \geq V_j^{\varepsilon,n}(\tau, \xi + \omega_{\tilde{q}}). \quad (4.4)$$

Proof. 1) By definition, we have

$$V_j^{\varepsilon,n}(q\varepsilon, y_q^n(q\varepsilon)) = f_j(\tilde{z}_Q^n(Q\varepsilon)) + \varepsilon \sum_{s=q}^{Q} \{g_j(\tilde{z}_s^n(s\varepsilon)) + h_j(\tilde{v}_j^{s,n})\}$$

where $\{\tilde{z}_q^n, \tilde{v}_j^{q,n}\}_{q=1}^Q$ is an optimal solution of (3.5) with

$$\tilde{z}_s^n(t) = R_n(s-1; \tilde{z}_{s-1}((s-1)\varepsilon); \varepsilon \tilde{v}_j^{s,n}, \varepsilon \pi_j \tilde{u}^q)$$

if $q \leq s-1$ and $\tilde{z}_k^n(q\varepsilon) = y_q^n(q\varepsilon)$ if $k \leq q$. Similarly,

$$\begin{aligned} &V_j^{\varepsilon,n}((q-1)\varepsilon, y_{q-1}^n((q-1)\varepsilon)) + \omega_q) \\ &\quad = f_j(\tilde{x}_Q^n(Q\varepsilon)) + \varepsilon \sum_{s=q-1}^{Q} \{g_j(\tilde{x}_s^n(s\varepsilon)) + h_j(\tilde{v}_j^{s,n})\} \end{aligned}$$

with

$$\tilde{x}_s^n(t) = R_n(s-1; \tilde{x}_{s-1}^n((s-1)\varepsilon); \varepsilon \tilde{v}_j^{s,n}, \varepsilon \pi_j \tilde{u}^s) \quad (4.5)$$

if $q \leq s$ and

$$\tilde{x}_{q-1}^n((q-1)\varepsilon) = y_{q-1}^n((q-1)\varepsilon) + \omega_q.$$

It follows that

$$\tilde{x}_q^n(q\varepsilon) = R_n(s-1; y_{q-1}^n((q-1)\varepsilon) + \omega_q; \varepsilon\tilde{v}_j^{s,n}, \varepsilon\pi_j\tilde{u}^s).$$

Since the inequality (4.2) has a unique solution, we deduce that $\tilde{x}_q^n(q\varepsilon) = y_q^n(q\varepsilon)$. Therefore, (4.5) gives

$$\tilde{x}_{q+1}^n(t) = R_n(q; y_q^n(q\varepsilon); \varepsilon\tilde{v}_j^q, \varepsilon\pi_j\tilde{u}^q) = \tilde{z}_{q+1}^n(t).$$

Thus,

$$\tilde{x}_k^n(t) = \tilde{z}_k^n(t); \quad q+1 \le k \le Q.$$

Hence,

$$\begin{aligned} &\varepsilon\big\{g_j(y_{q-1}^n((q-1)\varepsilon)) + h_j(\tilde{v}_j^{q-1})\big\} + V_j^{\varepsilon,n}(q\varepsilon, y_q^n(q\varepsilon)) \\ &\quad = V_j^{\varepsilon,n}((q-1)\varepsilon, y_{q-1}^n((q-1)\varepsilon) + \omega_q). \end{aligned} \tag{4.6}$$

2) On the other hand, we have

$$\begin{aligned} &\sum_{q=1+\tilde{q}}^{Q} \big\{V_j^{\varepsilon,n}(q\varepsilon, y_q^n(q\varepsilon)) - V_j^{\varepsilon,n}((q-1)\varepsilon, y_{q-1}^n((q-1)\varepsilon) + \omega_q)\big\} \\ &\quad = \varepsilon\{g_j(y_Q^n(Q\varepsilon)) + h_j(\tilde{v}_j^Q)\} + f_j(y_Q^n(Q\varepsilon)) - V_j^{\varepsilon,n}(\tau, \xi + \omega_{\tilde{q}}) \\ &\quad + \sum_{q=1+\tilde{q}}^{Q} \big\{V_j^{\varepsilon,n}(q\varepsilon, y_q^n(q\varepsilon)) - V_j^{\varepsilon,n}(q\varepsilon, y_{q-1}^n((q-1)\varepsilon + \omega_q)\big\}. \end{aligned} \tag{4.7}$$

From (4.6)–(4.7), we get

$$\begin{aligned} &f_j(y_Q^n(Q\varepsilon)) - V_j^{\varepsilon,n}(\tau, \xi + \omega_{\tilde{q}}) + \varepsilon\sum_{s=\tilde{q}}^{Q}\{g_j(y_s^n(s\varepsilon) + h_j(\tilde{v}_j^s)\} \\ &\quad = \sum_{q=1+\tilde{q}}^{Q} \big\{-V_j^{\varepsilon,n}(q\varepsilon, y_q^n(q\varepsilon)) + V_j^{\varepsilon,n}(q\varepsilon, y_{q-1}^n((q-1)\varepsilon) + \omega_q)\big\}. \end{aligned} \tag{4.8}$$

3) Applying the mean-value theorem for generalized gradients, we obtain

$$\begin{aligned} &V_j^{\varepsilon,n}(q\varepsilon, y_q^n(q\varepsilon)) - V_j^{\varepsilon,n}(q\varepsilon, y_{q-1}^n((q-1)\varepsilon) + \omega_q) \\ &\quad = \big\{-(p,\omega) : p \in \partial_y V_j^{\varepsilon,n}(q\varepsilon, y); y \in \{y_q^n(q\varepsilon)\} + \delta B(y_q^n(q\varepsilon))\big\}, \end{aligned} \tag{4.9}$$

where B is the unit ball in H with center at the indicated point.

Setting $k^j_{\tilde{q}-1,\delta}(.) = k^j_{\tilde{q},\delta}(-\omega_{\tilde{q}}) = k^j_{Q+1,\delta}(.)$, we get from (4.8)–(4.9) and from the definition of $k^j_{q,\delta}$

$$\begin{aligned} &f_j(y^n_Q(Q\varepsilon)) + \sum_{q=\tilde{q}}^{Q} k^j_{q-1,\delta}(-\omega_{q-1})\varepsilon\{g_j(y^n_q(q\varepsilon)) + h_j(\tilde{v}^q_j)\} \\ &\quad \geq V^{\varepsilon,n}_j(\tau, \xi + \omega_{\tilde{q}}) + k^j_{\tilde{q},\delta}(-\omega_{\tilde{q}}) - k^j_{Q,\delta}(-\omega_Q), \end{aligned} \tag{4.10}$$

thus changing the index of summation in (4.10) and we get the lemma by noting that $k^j_{\tilde{q}-1,\delta}(.) = k^j_{\tilde{q},\delta}(-\omega_{\tilde{q}})$.

Lemma 4.4. *Suppose all the hypotheses of Theorem* 3.1 *are satisfied. Let* $\{z^n_q\}$ *be as in* (3.5)*, then :* $\{z^n_q, \omega_q = 0\}^Q_{q=\tilde{q}}$ *is an optimal solution of* (4.3)*.*

Proof. It is clear that $\{z^n_q, \omega_q = 0\}^Q_{q=\tilde{q}}$ is in the admissible set of the optimization problem (4.3). The stated assertion is now an immediate consequence of Lemma 4.3.

Lemma 4.5. *Suppose that* $\{M_j\}^N_{j=1}$ *are time-independent 1-1 bounded linear mappings of* U_j *onto* H *and that all the hypotheses of Theorem* 3.1 *are satisfied. Then there exists* $\{p^j_{q,\delta}\}^Q_{q=1}$ *in* H *such that*

(1) $M^*_j p^j_{q-1,\delta} \in \partial h_j(\tilde{v}^q_j); j = 1, \ldots, N; q = 1, \ldots, Q.$
(2) $-p^j_{q-1,\delta} \in \partial k^j_{q-1,\delta}(0).$
(3) $-p^j_{q,\delta} \in \overline{co} \bigcup \{\partial_y V^{\varepsilon,n}_j(q\varepsilon, y) : y \in co(\mathcal{X}_a); \|y - z^n_q(q\varepsilon)\|_{\mathcal{H}} \leq \delta\}$*, where* z^n_q *is as in (3.5) with* $a = 1 + \sup \varphi(q\varepsilon; z^n_q(q\varepsilon), z^n_q(a\varepsilon))$.
(4) $\|p^j_{q,\delta}\| \leq C$. C *is independent of* ε, δ.

Proof. 1) Consider the Hamiltonian of (4.3)

$$\begin{aligned} H_j(y_q, \omega_q, p_{q,\delta}) = &\sum_{q=1}^{Q} \big\{\varepsilon[g_j(y_q) + h_j(\tilde{v}^q_j)] + k^j_{q-1,\delta}(-\omega_{q-1})\big\} \\ &- \sum_{q=1}^{Q} (y_{q-1} + \omega_q + \varepsilon M_j \tilde{v}^q_j + \varepsilon \sum_{k \neq j, k=1}^{N} M_k \tilde{u}^q_k, p^j_{q-1,\delta}). \end{aligned}$$

It follows from Lemma 4.4 and the necessary optimality conditions of (4.3) that

(i) $M^*_j p^j_{q-1,\delta} \in \partial h_j(\tilde{v}^q_j)$;
(ii) $-p^j_{q-1,\delta} \in \partial k^j_{q-1,\delta}(-\omega_q) \mid_{\omega_q=0}$.

A detailed argument is as in Popa [10], pp. 995–996. See also [6], p. 1309.

2) We have from the definition of the sub-differential

$$k^j_{q-1,\delta}(w) - k^j_{q-1,\delta}(0) \geq (p^j_{q-1,\delta}, w)$$

for all $w \in H$. Thus,

$$(p^j_{q-1,\delta}, w) \leq k^j_{q-1,\delta}(w) \tag{4.11}$$

for all $w \in H$. We deduce from (4.11) and from the definition of $k^j_{q,\delta}$ that $-p^j_{q,\delta} \in Z^{\varepsilon,n}_j$ with

$$Z^{\varepsilon,n}_j = \overline{co} \bigcup \big\{ \partial_y V^{\varepsilon,n}_j(q\varepsilon, y) : y \in co(\mathcal{X}_a); \|y - z^n_q(q\varepsilon)\| \leq \delta \big\}.$$

Indeed, if it is not true, then $p^j_{q,\delta}$ can be separated from the closed convex set of the right hand side. Therefore, there exists $x \in H$ such that

$$-(p^j_{q,\delta}, x) > \sup\{(x, \alpha) : \forall \alpha \in Z^{\varepsilon,n}_j\}.$$

Hence $-(p^j_{q,\delta}, x) > k^j_{q,\delta}$ which is a contradiction with (4.11).

3) From the definition of the sub-differential, we have

$$(M^*_j p^j_{q-1,\delta}, w - \tilde{v}^q_j)_{U_j} \leq h_j(w) - h_j(\tilde{v}^q_j)$$

for all $w \in U_j$.Thus,

$$(p^j_{q-1,\delta}, M_j \omega) \leq h_j(\omega + \tilde{v}^q_j) - h_j(\tilde{v}^q_j)$$

for all $\omega \in U_j$. Since M_j is assumed to be a 1-1 mapping of U_j onto H, we get

$$(p^j_{q-1,\delta}, x) \leq h_j(M^{-1}_j x + \tilde{v}^q_j) - h_j(\tilde{v}^q_j)$$

for all $x \in H$. Therefore,

$$\|p^j_{q-1,\delta}\| \leq 2|h_j(\tilde{v}^q_j| + \sup\big\{|h_j(M^{-1}_j x + \tilde{v}^q_j)| + |h_j(-M^{-1}_j x + \tilde{v}^q_j)| : \|x\| \leq 1\big\}.$$

Since h_j is continuous and $\tilde{v}^q_j \in \mathcal{U}_j$, we get the stated result. □

The main result of the section is the following theorem.

Theorem 4.1. *Suppose all the hypotheses of Theorem* 3.1 *are satisfied and that* $\{M_j\}_{j=1}^N$ *are time-independent 1-1,linear bounded mappings of* U_j *onto* H. *Let* $\{V_j^{\varepsilon,n}(\tau,\xi)\}_{j=1}^N$ *and let* $\{z_q^n, \tilde{v}_j^{q,n}\}$ *be as in* (3.5). *Then*

$$\tilde{v}_j^{q,n} \in \partial h_j^*(-M_j^* \partial_z V_j^{\varepsilon,n}((q-1)\varepsilon, z_{q-1}^n((q-1)\varepsilon)))$$

for $q = 1+\tilde{q}, \dots, Q; j = 1, \dots, N$.

Proof. Let $\{p_{q,\delta}^j\}$ be as in Lemma 4.5. Then we have a subsequence such that $p_{q,\delta}^j \to p_q^j$ weakly in H. Since $M_j^* p_{q-1,\delta}^j \in \partial h_j(\tilde{v}_j^{q,n})$, and $\partial h_j(\tilde{v}_j^{q,n})$ is a closed convex subset of U_j, it follows that

$$M_j^* p_{q-1}^j \in \partial h_j(\tilde{v}_j^{q,n}); \quad j = 1, \dots, N, \quad q = 1+\tilde{q}, \dots, Q. \tag{4.12}$$

2) We have

$$-p_{q-1}^j \in \bigcap_{\delta>0} \overline{co}\big\{\partial_y V_j^{\varepsilon,n}((q-1)\varepsilon, y) : y \in co(\mathcal{X}_a); \|y - z_{q-1}^n((q-1)\varepsilon)\| \le \delta\big\}.$$

Now a proof by contradiction as in [6, p. 1310] gives

$$-p_{q-1}^j \in \partial_y V_j^{\varepsilon,n}((q-1)\varepsilon, y) \mid_{y=z_{q-1}^n((q-1)\varepsilon)} . \tag{4.13}$$

3) From (4.12) we obtain $\tilde{v}_j^{q,n} \in \partial h_j^*(-M_j^* p_{q-1}^j)$ where h_j^* is the conjugate of h_j. Taking (4.13) into account, we get the stated result.

5. Non-cooperative N-person games. In this section we shall consider an optimal control problem for a constrained non-cooperative N-person game. By connecting the problem with (0.1) via a perturbation, we can apply the results of the previous sections and thus obtain a feedback law for the j-player.

Lemma 5.1. *Let* $X = \{X_j\}_{j=1}^N, H = \{H_j\}_{j=1}^N$ *with* X_j *being a compact convex subset of the Hilbert space* H_j *and with* $0 \in X$. *Let* $S = \{S_j\}_{j=1}^N$ *with* S_j *being an u.s.c. set-valued mapping of* $\prod_{k\neq j,k=1}^N X_k$ *into the closed convex subsets of* X_j. *Let* $\{\tilde{f}_j\}_{j=1}^N$ *be continuously differentiable functions from* $[0,T] \times H$ *into* R^+ *and suppose that* $\{\tilde{f}_j(t;.,\pi_j x)\}_{j=1}^N$ *are convex functions from* H_j *to* R^+. *Let* $\{M_j\}_{j=1}^N$ *be bounded linear mappings of* $L^2(0,T;U_j)$ *into* $L^2(0,T;H)$ *and suppose that Assumption IV is verified.*

Then for any given $\{\xi, v_j, u_k\} \in X \times \mathcal{U}_j \times \mathcal{U}_k$, *there exists* $y \in C(\tau, T; X)$ *with* $y' \in L^2(\tau, T; H)$, *the solution of* (0.1) *with*

$$\varphi(t; x, z) = \sum_{j=1}^{N} \{ \tilde{f}_j(t; z_j, \pi_j x) - \tilde{f}_j(t; x_j, \pi_j x) \}. \tag{5.1}$$

Proof. It is clear that φ is a l.s.c. mapping of $H \times H$ into R. Moreover, for all $x, z \in X$, we have $\varphi(t; x, z) \geq -\gamma$. Let

$$\varphi_\eta(t; x, y) = \eta \|y\|_H^2 + \varphi(t; x, y) + \gamma + I_{Sx}(y)$$

for $x, y \in X$. It is now easy to check that φ_η satisfies Assumption I–III by noting that X is a compact subset of H and by applying the Ascoli-Arzela theorem. Thus it follows from Theorem 2.1 that there exists $y_\eta \in C(\tau, T; X)$ with $y'_\eta \in L^2(\tau, T; H)$ such that

$$y'_\eta + 2\eta y_\eta + \partial_y \{\varphi(t; x, y) + I_{Sx}(y)\} \mid_{x = y_\eta} \ni M_j v_j + \sum_{k \neq j, k=1}^{N} M_k u_k \tag{5.2}$$

$$y_\eta \in S y_\eta(t); \quad t \in [\tau, T]; \quad y_\eta(\tau) = \xi.$$

Moreover,

$$\|y'_\eta\|_{L^2(\tau, T; H)} \leq C.$$

Thus, with $y_\eta(t) \in X$ and X being a compact subset of H, the above estimate together with the Ascoli-Arzela theorem yields: $y_\eta \to y$ in $C(\tau, T; H)$ with $y \in X$ for all t and $y'_\eta \to y'$ weakly in $L^2(\tau, T; H)$. Now a proof as in that of Lemma 2.1 shows that y is a solution of (0.1).

We denote by $\mathcal{P}(\tau; \xi; v_j, \pi_j u)$, the set of all solutions of (0.1) with φ given by (5.1).

Definition. A control $(u_j^*, \pi_j u_*)$; $j = 1, ., N$ is said to be a j- equilibrium point of the system (0.1) with cost functionals (0.2) and with φ given by (5.1) if there exist $(u_j^*, \pi_j u_*) \in \mathcal{U}$ such that

$$J_j(\tau, \xi; y; u_j^*, \pi_j u_*) \leq J_j(\tau; \xi; x; v_j, \pi_j u_*) \leq J_j(\tau; \xi; z; v_j; \pi_j w) \tag{5.3}$$

for all $y \in \mathcal{P}(\tau; \xi; u_j^*, \pi_j u_*)$, $z \in \mathcal{P}(\tau; \xi; v_j; \pi_j w)$, $x \in \mathcal{P}(\tau; \xi; v_j, \pi_j u_*)$ and all $\{v, w\} \in \mathcal{U} \times \mathcal{U}$. If $u_* = u^*$, then we say that u^* is an open loop equilibrium point of the system.

Let $(u^*_{j,\eta}, \pi_j u_{*,\eta})$ be a j-equilibrium point of (5.2) with cost functionals (0.2) given by Theorem 2.2. Then an easy argument shows that $(u^*_{j,\eta}, \pi_j u_{*,\eta}) \to (u^*_j, \pi_j u_*)$ in $L^2(\tau, T; U)$ with $(u^*_j, \pi u_*)$ being a j-equilibrium point of (0.1)–(0.2).

We can now define the value function of the system (0.1)–(0.2) with φ as in (5.1) by $\tilde{V}(\tau, \xi) = \{\tilde{V}_j(\tau, \xi)\}_{j=1}^N$ and

$$\tilde{V}_j(\tau, \xi) = \inf\big\{ J_j(\tau; \xi; y; u^*_j, \pi_j u_*) : \forall y \in \mathcal{P}(\tau; \xi; u^*_j, \pi_j u_*); \forall u^*, \forall u_* \big\} \quad (5.4)$$

with u^*, u_* as in (5.3). It is clear that the admissible set of the problem (5.4) is non-empty and that $\tilde{V}_j(\tau, \xi)$ exists.

The main result of the section is the following theorem.

Theorem 5.1. *Suppose all the hypotheses of Lemma* 5.1 *are satisfied and let* $\{\tilde{V}_j(\tau, \xi)\}_{j=1}^N$ *be as in* (5.4). *Let* $\{V_{j,\eta}(\tau, \xi)\}_{j=1}^N$ *be the value function of* (5.2) *as in Theorem* 2.2. *Then*

$$\lim_{\eta \to 0} V_{j,\eta}(\tau, \xi) = \tilde{V}_j(\tau, \xi)$$

for any τ, ξ *in* $[0, T] \times X$ *with* $\xi \in S\xi$.

Proof. Let $\{\tilde{y}_\eta, \tilde{v}^\eta_j, \tilde{u}_\eta\}$ be an optimal solution for $V_{j,\eta}(\tau, \xi)$; i.e.,

$$V_{j,\eta}(\tau, \xi) = J_j(\tau; \xi; \tilde{y}_\eta; \tilde{v}^\eta_j, \pi_j \tilde{u}_\eta).$$

The existence of an optimal solution is given by Lemma 2.2 and we have

$$\tilde{y}_\eta \in X; \quad \|\tilde{y}'_\eta\|_{L^2(\tau, T; H)} \le C.$$

Since X is a compact subset of H, it follows from the above estimate and from the Ascoli-Arzela theorem that $\tilde{y}_\eta \to y$ in $C(\tau, T; H)$ with $y(t) \in X$ on $[\tau, T]$. Moreover, $\tilde{y}'_\eta \to y'$ weakly in $L^2(\tau, T; H)$.

On the other hand, $\{\tilde{v}^\eta_j, \tilde{u}_\eta\} \to \{\tilde{v}_j, \tilde{u}\}$ in $L^2(\tau, T; U_j) \times L^2(\tau, T; U)$. As in the proof of Lemma 2.1, $\{y, \tilde{v}_j, \tilde{u}\}$ is a solution of (0.1). Thus,

$$J_j(\tau; \xi; y; \tilde{v}_j, \pi_j \tilde{u}) \le \liminf_{\eta \to 0} V_{j,\eta}(\tau, \xi).$$

Hence,

$$\tilde{V}_{j,\eta}(\tau, \xi) \le \liminf_{\eta \to 0} V_{j,\eta}(\tau, \xi).$$

2) Let $\{\hat{v}_j, \hat{u}\}$ be an optimal control for $\tilde{V}_j(\tau, \xi)$ and let ν be an arbitrary positive number. Then from the definition of $\tilde{V}_j$, we deduce the existence of $v_j \in \mathcal{U}_j$ such that

$$J_j(\tau; \xi; x; v_j, \pi_j \hat{u}) \le \nu + \tilde{V}_j(\tau, \xi)$$

for all $x \in \mathcal{P}(\tau; \xi; v_j, \pi_j \hat{u})$.

Let y_η be a solution of (5.1) corresponding to the control $\{v_j, \pi_j \hat{u}\}$. Then

$$J_j(\tau; \xi; y_\eta; v_j, \pi_j \hat{u}) \to J_j(\tau; \xi; y; v_j, \pi_j \hat{u})$$

where $y \in \mathcal{P}(\tau; \xi; v_j, \pi_j \hat{u})$. But

$$V_{j,\eta}(\tau, \xi) \le J_j(\tau; \xi; y_\eta; v_j, \pi_j \hat{u}).$$

Therefore,

$$\limsup_{\eta \to 0} V_{j,\eta}(\tau, \xi) \le \nu + \tilde{V}_j(\tau, \xi).$$

The theorem is proved.

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