Differential and Integral Equations Volume 11, Number 3, May 1998, pp. 425–438.

A REMARK ON THE CONTINUOUS DEPENDENCE ON ϕ OF SOLUTIONS TO $U_T - \Delta\phi(U) = 0$

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1. Introduction. In this note we examine the singular Cauchy problem

$$\begin{cases} u_t - \Delta \ln u = 0 & \text{in } \mathbb{R}^N \times (0,T) \\ u(x,0) = u_0(x) \geq 0. \end{cases} \tag{1}$$

The partial differential equation in (1) arises in the dynamics of thin liquid films (see [8] and references therein), the expansion of an electron cloud (see [16]), and the Ricci flow on $\mathbb{R}^2$ (see [19]). It also arises formally as the limit as $m \to 0$ of the porous medium equation.

Through a rescaling in time the porous medium equation can be written

$$u_t - \frac{1}{m}\Delta u^m = 0 \quad \text{for all } 0 < m < 1$$

or equivalently

$$u_t - \Delta\Big(\frac{u^m - 1}{m}\Big) = 0.$$

Of course, for $u > 0$

$$\frac{u^m - 1}{m} \to \ln u \quad \text{as} \quad m \to 0.$$

Thus (1) can be viewed, at least formally, as the limit as $m \to 0$ of

$$\begin{cases} u_t - \frac{1}{m}\Delta u^m = 0 & \text{in } \mathbb{R}^N \times (0,T) \\ u(x,0) = u_0(x) \geq 0. \end{cases} \tag{2}$$

The partial differential equation in (2) arises in, for example, thermoconduction (see [15], Chapter 5), plasma physics (see [3]), and modeling the diffusion of impurities in silicon (see [12]).

Received for publication August 1997.
AMS Subject Classifications: 35K55, 35Q35.

This note is devoted to understanding the limit of solutions to (2) as $m \to 0$ with the primary focus being $N = 2$ which is the physical and geometrical dimension for (1).

The properties of (1) and (2) are drastically different. For any $0 \le u_0 \in L^1_{loc}(\mathbb{R}^N)$ (2) has a unique solution without any specification of the behavior at infinity (see [11])[1]. In particular, if $m \ge \dfrac{(N-2)_+}{N}$ and $u_0 \in L^1(\mathbb{R}^N)$, then the solutions of (2) are mass preserving in the sense that

$$\int_{\mathbb{R}^N} u(x,t)\,dx = \int_{\mathbb{R}^N} u_0\,dx \quad \text{for all } t > 0. \tag{3}$$

While (2) is extremely well-posed, (1) is extremely ill-posed. For $N \ge 3$ and $u_0 \in L^1(\mathbb{R}^N)$, (1) has no solution (see [6, 8, 18]). For $N = 2$, (1) is ill-posed for lack of uniqueness. Indeed, for any $0 \le u_0 \in L^1(\mathbb{R}^2)$, (1) has a continuum of solutions. More precisely, for $0 \le u_0 \in L^1(\mathbb{R}^2)$ and for each $s \ge 1$, (1) has a solution, u^s, in $\mathbb{R}^2 \times (0, \|u_0\|_1/4\pi s)$ which satisfies

$$\int_{\mathbb{R}^2} u^s(x,t)\,dx = \int_{\mathbb{R}^2} u_0\,dx - 4\pi s t \quad \text{for } 0 < t < \frac{\|u_0\|_1}{4\pi s}.$$

The solution, u^1, is the maximal solution (i.e., the pointwise largest solution corresponding to the given initial data). In particular, any solution to (1) with $u_0 \in L^1(\mathbb{R}^2)$ loses mass at a rate of at least 4π (for these results see [9,7]).

In the fundamental work of [2], Bénilan and Crandall showed under very mild restrictions on ϕ that if $\lim_{n\to\infty} \phi_n = \phi$, then solutions of

$$\begin{cases} u_t - \Delta\phi_n(u) = 0 & \text{in } \mathbb{R}^N \times (0,T) \\ u(x,0) = u_0(x) \in L^1(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N) & \end{cases}$$

converge in $C([0,\infty); L^1(\mathbb{R}^N))$ to solutions of

$$\begin{cases} u_t - \Delta\phi(u) = 0 & \text{in } \mathbb{R}^N \times (0,T) \\ u(x,0) = u_0(x) \in L^1(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N). & \end{cases}$$

Unfortunately, one of the key assumptions, namely that $0 \in \phi(0)$, fails in our situation. The case in question offers a concrete demonstration of the sharpness of the results of [2].

[1]The restriction that $u_0 \ge 0$ can be omitted if u^m is replaced by $|u|^{m-1}\,u$. We restrict our attention to nonnegative initial data to make sense of the partial differential equation in (1).

For $N \geq 3$, we show that if $u_0 \in L^1(\mathbb{R}^N)$, then for any $t_0 > 0$

$$\lim_{m\to 0} \big(\sup_{t>t_0} \int_{\mathbb{R}^N} |u_m| \, (x,t)\, dx \big) = 0,$$

where u_m is the unique solution of (2) with initial data u_0. Since (1) has no solutions for $N \geq 3$ and $u_0 \in L^1(\mathbb{R}^N)$ this result is consistent with the results of [2]. For $N = 2$ and $u_0 \in L^1(\mathbb{R}^2)$ the solutions of (2) satisfy (3) whereas the solutions to (1) lose mass at least at the rate of 4π. Thus the solutions of (2) cannot converge in $C([0,\infty), L^1(\mathbb{R}^2))$ to a solution of (1). We do, however, show that the solutions of (2) converge in $C((0,T), L^1_{loc}(\mathbb{R}^2))$ to the maximal solution of (1) where $T = \|u_0\|_1/4\pi$.

As in [2], we approach this limit through semi-groups; i.e., we consider

$$\lambda u - \Delta \ln u = f \tag{4}$$

and

$$\lambda u - \frac{1}{m}\Delta(u\,|u|^{m-1}) = f. \tag{5}$$

Note that here we allow f to have varying sign. We only allow nonnegative solutions of (4), but we do allow solutions of (5) to have varying sign. The reason for this will become clear.

Equation (5) has a unique solution for any $f \in L^1_{loc}(\mathbb{R}^N)$ (see [4]). Also, the maximum principle is valid for solutions of (5); i.e., if u and v are solutions of (5) with data f and g respectively with $f \leq g$, then $u \leq v$. Furthermore, if $f, g \in L^1(\mathbb{R}^N)$, then

$$\lambda \int_{\mathbb{R}^N} |u - v| \; dx \leq \int_{\mathbb{R}^N} |f - g| \; dx. \tag{6}$$

If, in addition, $m \geq \frac{(N-2)_+}{N}$, then the solutions of (5) are mass preserving; i.e.,

$$\lambda \int_{\mathbb{R}^N} u \, dx = \int_{\mathbb{R}^N} f \, dx. \tag{7}$$

In contrast, (4) is not as well-posed. For example, if $N \geq 3$ and $f \in L^1(\mathbb{R}^N)$, then (4) has no solution (see [6, 10, 18]). Furthermore, if $N = 2$, $f \in L^1(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2} f \, dx \leq 4\pi,$$

then (4) has no solution (see [10]). But under mild restrictions on the negative part of f if for some $s \geq 1$

$$\int_{\mathbb{R}^2} f \, dx > 4\pi s,$$

then (4) has a unique solution (see [10]) satisfying

$$\lambda \int_{\mathbb{R}^2} u^s \, dx = \int_{\mathbb{R}^2} f \, dx - 4\pi s. \tag{8}$$

In particular, for $N = 2$ there are no solutions of (4) which are mass preserving in the sense of (7). We stress that for $f \in L^1(\mathbb{R}^2)$ either (4) has no solution or it has a continuum of solutions. In light of (7) and (8) for $f \in L^1(\mathbb{R}^2)$ it is impossible for the solutions of (5) to converge in $L^1(\mathbb{R}^2)$ to a solution of (4).

For $m \ll 1$, $N \geq 3$ and $f \in L^1(\mathbb{R}^N)$ the solutions of (5) are not mass preserving. In fact, for $f \in L^1(\mathbb{R}^N)$ the solutions of (5) converge to 0 in $L^1(\mathbb{R}^N)$ as $m \to 0$ (see Example 2.1).

2. Some examples.

Example 2.1. For $N \geq 3$ set

$$v_m(x) = \Big(\frac{Cm}{(1+ |x|^2)^{\frac{N-2}{2}}} \Big)^{\frac{1}{m}} \text{ for } C, m > 0.$$

Then

$$\begin{aligned} -\frac{1}{m} \Delta v^m &= C \operatorname{div}\Big(\frac{(N-2)x}{(1+ |x|^2)^{\frac{N}{2}}} \Big) \\ &= \frac{CN(N-2)}{(1+ |x|^2)^{\frac{N}{2}}} - \frac{CN(N-2)\,|x|^2}{(1+ |x|^2)^{\frac{N}{2}+1}} = \frac{CN(N-2)}{(1+ |x|^2)^{\frac{N}{2}+1}}. \end{aligned}$$

Thus v_m is a solution of (5) with datum g_m where g_m is given by

$$g_m(x) = \lambda v_m(x) + \frac{CN(N-2)}{(1+ |x|^2)^{\frac{N}{2}+1}}.$$

Notice that for $0 < m < \dfrac{(N-2)_+}{N}$, $v_m, g_m \in L^1(\mathbb{R}^N)$, v_m converges to 0 in $L^1(\mathbb{R}^N)$ as $m \to 0$, but

$$g_m(x) \geq \frac{CN(N-2)}{(1+ |x|^2)^{\frac{N}{2}+1}} \quad \text{for all } m \in \Big(0, \frac{N-2}{N}\Big).$$

Of course, $-v_m$ is a solution of (5) with datum $-g_m$. Thus, if for some $C > 0$

$$|f(x)| \leq \frac{CN(N-2)}{(1+ |x|^2)^{\frac{N}{2}+1}},$$

the solutions, u_m, of (5) with datum f converge to 0 in $L^1(\mathbb{R}^N)$ as $m \to 0$. Since C can be chosen arbitrarily large, a standard density argument using (6) shows that for any $f \in L^1(\mathbb{R}^N)$ the solutions of (5) converge to 0 in $L^1(\mathbb{R}^N)$ as $m \to 0$.

Example 2.2. For $N = 2$, $M > 0$, and $s > 1$, set

$$w_m(x) = \frac{M}{(1+ |x|^2)^s}$$

and

$$\phi_m(x) = \lambda w_m(x) + \frac{4sM^m(sm+1)}{(1+ |x|^2)^{2+sm}} - \frac{4s^2M^m m}{(1+ |x|^2)^{1+sm}}.$$

Then w_m is a solution of (5) with datum ϕ_m. Notice that

$$\lambda \int_{\mathbb{R}^2} w_m \, dx = \int_{\mathbb{R}^2} \phi_m \, dx.$$

Of course, $w_m = w$ is independent of m and

$$\phi_m(x) \to \phi(x) = \lambda w(x) + \frac{4s}{(1+ |x|^2)^2} \text{ in } L^1_{loc}(\mathbb{R}^2) \text{ (but not in } L^1(\mathbb{R}^2)),$$

where w is a solution of (4) with datum ϕ satisfying

$$\lambda \int_{\mathbb{R}^2} w \, dx = \int_{\mathbb{R}^2} \phi \, dx - 4\pi s.$$

This example is central to the results for $N = 2$.

3. The results for $N = 2$.

3.1. The elliptic equation.

Theorem 3.1. *Let $f \in L^1(\mathbb{R}^2) \bigcap L^\infty(\mathbb{R}^2)$. Suppose that there is an $s > 1$ such that*

$$\int_{\mathbb{R}^2} f \, dx > 4\pi s.$$

Set

$$f_m(x) = f(x) - \frac{4s^2 m}{(1+ |x|^2)^{1+sm}} \quad \text{for } m > 0. \tag{9}$$

Let u_m be the solution of (5) with datum f_m. Then $u_m \to u^s$ in $L^1(\mathbb{R}^2)$ where u^s is the unique solution of (4), with datum f, satisfying

$$\lambda \int_{\mathbb{R}^2} u^s \, dx = \int_{\mathbb{R}^2} f \, dx - 4\pi s. \tag{10}$$

Furthermore, if v_m *is the unique solution of* (5) *with datum* f *then* $v_m \to u^1$ *in* $L^1_{loc}(\mathbb{R}^2)$ *where* u^1 *is the unique solution of* (4) *with datum* f *satisfying* (10) *with* $s = 1$*.*

Remark 3.2. With the choice, (9), of f_m a certain amount of mass $(-4\pi s)$ is pushed to infinity. As a result, the forcing term actually gains mass in the limit $m \to 0$. To a certain extent, this explains the loss of mass in passing from $m > 0$ to $m = 0$ for fixed f.

We delay the proof of this result until Section 4. The proof hinges on the following pair of lemmas.

Lemma 3.3. *Let* $f \in L^1(\mathbb{R}^2) \bigcap L^\infty(\mathbb{R}^2)$*. Let* u *be a solution to* (5) *with datum* f*; then,*

$$|u|^{m-1}\, u(x) = |u|^{m-1}\, u(0) - m \int_{\mathbb{R}^2} K(x,y)(\lambda u - f)\, dy, \tag{11}$$

where

$$K(x,y) = \frac{1}{2\pi}(\ln |y| - \ln |x-y|).$$

Remark 3.4. For $|y| \geq \max\{2\,|x|, 1\}$

$$|K(x,y)| \leq \frac{\ln 2}{2\pi}.$$

If $f \in L^1(\mathbb{R}^2) \bigcap L^\infty(\mathbb{R}^2)$, then from the maximum principle and (6) the solution u of (5) satisfies

$$\lambda \|u\|_{\infty,\mathbb{R}^2} \leq \|f\|_{\infty,\mathbb{R}^2} \text{ and } \lambda \|u\|_{1,\mathbb{R}^2} \leq \|f\|_{1,\mathbb{R}^2}.$$

As a result the integral in (11) is convergent, and

$$\left| \int_{\mathbb{R}^2} K(x,y)(\lambda u - f)\, dy \right| \leq \gamma(|x|, \|f\|_1, \|f\|_\infty).$$

We stress that γ is independent of m. Suppose that we have a sequence of solutions $\{u_m\}_{m>0}$ to (5) with data $\{f_m\}_{m>0}$ satisfying

$$\sup_{m \in (0,1)} (\|f_m\|_1 + \|f_m\|_\infty) \leq C \quad \text{for some } C > 0.$$

Note that the f_m given in (9) satisfy this estimate. Then for $|x| < R$ ($R > 0$ fixed)

$$|u_m|^{m-1}\, u_m(x) \geq |u_m|^{m-1}\, u_m(0) - m\gamma(C,R).$$

If $u_m(0) \geq \eta > 0$ for all $m > 0$, then for m sufficiently small

$$u_m(x) \geq \big(\eta^m - m\gamma(C,R)\big)^{\frac{1}{m}} \geq \frac{\eta}{2} e^{-\gamma(C,R)} \quad \text{for all } |x| < R.$$

Thus once a suitable sequence of solutions is bounded away from 0 at one point, it is bounded away from 0 uniformly on compact sets (for m sufficiently small depending on the compact set).

We delay the proof of Lemma 3.3 until Section 5.

Lemma 3.5. *Let u be a solution of* (4) *with datum f where $f \in C_0^\infty(\mathbb{R}^2)$. If for some $s > 1$ and $M > 0$*

$$u(x) \leq \frac{M}{(1+|x|^2)^s},$$

then

$$\lambda \int_{\mathbb{R}^2} u(x)\,dx \leq \int_{\mathbb{R}^2} f\,dx - 4\pi s.$$

Remark 3.6. For a complete proof of Lemma 3.5 see [10]. We sketch the proof in Section 5.

3.2. The parabolic equation.

Theorem 3.7. *Suppose that $0 \leq u_0 \in L^1(\mathbb{R}^N)$. Let u_m be the unique solution of* (2) *with initial data u_0. If $N \geq 3$, then for any $t_0 > 0$*

$$\lim_{m\to 0} \Big(\sup_{t_0<t<\infty} \int_{\mathbb{R}^N} u_m\,dx \Big) = 0.$$

For $N = 2$, u_m converges in $C((0,T); L^1_{loc}(\mathbb{R}^2))$ where $T = \|u_0\|_1/4\pi$ to u^1, where u^1 is the maximal solution of (1) *with initial data u_0.*

Proof of Theorem 3.7 from Theorem 3.1. Take the case $N = 2$. Let $0 \leq u_0 \in L^1(\mathbb{R}^2)$. For $m > 0$ let u_m be the solution of (2) with this initial data. Let u be the maximal solution of (1) with this initial data. For $h > 0$, $m \geq 0$ and $k = 0, 1, 2, \ldots$, define $u_{m,h,k}(\cdot)$ via

$$u_{m,h,0}(x,0) = u_0(x) \quad \text{for } m \geq 0.$$

Then inductively for $k = 1, 2, \ldots$ let $u_{m,h,k}$ for $m > 0$ be the unique solution of

$$\frac{1}{h}u - \frac{1}{m}\Delta u^m = \frac{1}{h}u_{m,h,k-1},$$

and let $u_{0,h,k}$ be the maximal solution of

$$\frac{1}{h}u - \Delta \ln u = \frac{1}{h}u_{0,h,k-1}.$$

For $(k-1)h \le t \le kh$ and $m \ge 0$ set

$$u_{m,h}(\cdot,t) = \Big(\frac{kh-t}{h}\Big)u_{m,h,k-1}(\cdot) + \Big(\frac{t+h-kh}{h}\Big)u_{m,h,k}(\cdot).$$

For $m > 0$ $u_{m,h}$ is an approximate solution to (2) (see [5] and [1]) in the sense that

$$\lim_{h\to 0}\Big(\sup_{t_1<t<t_2}\int_{\mathbb{R}^2} |u_m - u_{m,h}|\,(x,t)\,dx\Big) = 0 \quad \text{for all } 0<t_1<t_2<\infty,$$

and $u_{0,h}$ is an approximate solution to (1) (see [9]) in the sense that

$$\lim_{h\to 0}\Big(\sup_{t_0<t<T}\int_{\mathbb{R}^2} |u_{0,h} - u|\ dx\Big) = 0 \quad \text{for all } 0<t_0<T,$$

where $T = \|u_0\|_1/4\pi$. Let $R > 0$ and $\epsilon > 0$. Then for $0 < h \ll 1$

$$\begin{aligned}\sup_{t_0<t<T}\int_{B_R} |u_m - u|\,(x,t)\,dx &\le \sup_{t_0<t<T}\int_{\mathbb{R}^2} |u_m - u_{m,h}|\,(x,t)\,dx \\ &\quad + \sup_{t_0<t<T}\int_{\mathbb{R}^2} |u_{0,h} - u|\,(x,t)\,dx + \sup_{t_0<t<T}\int_{B_R} |u_{0,h} - u_{m,h}|\,(x,t)\,dx \\ &\le 2\epsilon + \sup_{t_0<t<T}\int_{B_R} |u_{0,h} - u_{m,h}|\,(x,t)\,dx.\end{aligned}$$

To show that

$$\limsup_{m\to 0}\Big(\sup_{t_0<t<T}\int_{B_R} |u_{0,h} - u_{m,h}|\,(x,t)\,dx\Big) = 0$$

we need only show that

$$\lim_{m\to 0}\int_{B_R} |u_{0,h,k} - u_{m,h,k}|\ dx = 0 \tag{12}$$

for each $k = 1, 2, \ldots$. We prove this by induction on k. For $k = 1$, $u_{0,h,1}$ and $u_{m,h,1}$ are solutions of (4) and (5) respectively with the same forcing term, u_0/h. Thus (12) for $k = 1$ follows from Theorem 3.1. Assume that (12) holds for $k - 1 \geq 1$. Then

$$\int_{B_R} |u_{0,h,k} - u_{m,h,k}|\ dx \leq \int_{B_R} |u_{0,h,k} - u^*_m|\ dx + \int_{B_R} |u^*_m - u_{m,h,k}|\ dx, \tag{13}$$

where u^*_m is the unique solution of

$$\frac{1}{h}u - \frac{1}{m}\Delta u^m = \frac{1}{h}u_{0,h,k-1}.$$

Since $u_{0,h,k}$ and u^*_m have the same forcing term,

$$\lim_{m\to 0}\int_{B_R} |u_{0,h,k} - u^*_m|\ dx = 0 \tag{14}$$

again by Theorem (3.1). Finally, both $u_{m,h,k}$ and u^*_m are solutions of (5) with $\lambda = 1/h$ and forcing terms $u_{m,h,k-1}$ and $u_{0,h,k-1}$ respectively. Thus by (6)

$$\int_{B_R} |u_{m,h,k} - u^*_m|\ dx \leq \int_{B_R} |u_{m,h,k-1} - u_{0,h,k-1}|\ dx. \tag{15}$$

Of course, by the induction hypothesis

$$\lim_{m\to 0}\int_{B_R} |u_{m,h,k-1} - u_{0,h,k-1}|\ dx = 0.$$

This together with (13), (14) and (15) proves (12) for all $k = 1, 2, 3, \ldots$. Thus

$$\limsup_{m\to 0}\Big(\sup_{t_0<t<T}\int_{B_R} |u^m - u|\,(x,t)\,dx\Big) \leq 2\epsilon.$$

This completes the proof for $N = 2$. The proof for $N \geq 3$ is virtually identical.

4. The proof of Theorem 3.1 from Lemmas 3.3 and 3.5. Suppose that for some $s > 1$

$$\int_{\mathbb{R}^2} f\,dx > 4\pi s.$$

For the moment assume also that $f \in C_0^\infty(\mathbb{R}^2)$. Then for some $M \gg 1$

$$f(x) \leq \frac{\lambda M}{(1+ |x|^2)^s}. \tag{16}$$

For this M and s fixed, set

$$f_{m,M}(x) = f(x) - \frac{4s^2 M^m m}{(1+ |x|^2)^{1+sm}}.$$

Also, for this M and s let w_m denote the solution of (5) with datum ϕ_m given in Example (2.2). Then by (16) $f_{M,m} \leq \phi_m$. Let $u_{M,m}$ be the solution of (5) with datum $f_{M,m}$. By the maximum principle $u_{M,m} \leq w_m$. Thus by (6) for $R \gg 1$ fixed

$$\begin{aligned}\int_{B_R} \lambda(w_m - u_{M,m})\,dx =& \lambda \int_{B_R} \left|w_m - u_{M,m}\right| dx \leq \lambda \int_{\mathbb{R}^2} \left|w_m - u_{M,m}\right| dx \\ &\leq \int_{\mathbb{R}^2} \left|\phi_m - w_{M,m}\right| dx = \int_{\mathbb{R}^2} (\phi_m - w_{M,m})\,dx.\end{aligned}$$

Thus

$$\lambda \int_{B_R} u_{M,m}\,dx \geq \int_{\mathbb{R}^2} f(x)\,dx - M^m 4\pi s - \int_{\mathbb{R}^2/B_R} \frac{\lambda M}{(1+ |x|^2)^s}\,dx. \tag{17}$$

For R chosen sufficiently large, $u_{M,m}$ cannot approach 0 uniformly in B_R as $M \to 0$. By Lemma 3.3 and Remark 3.4, this means that $u_{M,m}$ is locally uniformly bounded away from 0 in $\mathbb{R}^2$. Hence by the standard regularity results for uniformly elliptic equations (see [14, 17]) and the Ascoli–Arzela Theorem on a subsequence $u_{M,m} \to u$ in $C^\infty(\mathbb{R}^2)$ where u is a solution of (4) with datum f. Since $u_{M,m} \leq w_m$ for all $m > 0$,

$$u(x) \leq \frac{M}{(1+ |x|^2)^s}. \tag{18}$$

This, (17), and Lemma 3.5 imply that

$$\lambda \int_{\mathbb{R}^2} u\,dx = \int_{\mathbb{R}^2} f\,dx - 4\pi s. \tag{19}$$

Since $u = u^s$ is the only solution of (4) with datum f satisfying (19), any subsequence of $\{u_{M,m}\}_{m>0}$ necessarily has a subsequence converging to u^s. Thus the entire sequence converges to u^s. Also from (18)

$$\lim_{m\to 0} (u_{M,m})_+ = u \text{ in } L^1(\mathbb{R}^2). \tag{20}$$

In addition,

$$\lim_{m\to 0} \int_{\mathbb{R}^2} u_{M,m}\,dx = \int_{\mathbb{R}^2} u\,dx.$$

This and (20) imply that $u_{M,m} \to u$ in $L^1(\mathbb{R}^2)$ as $m \to 0$.

To see that the solution u_m of (5) with datum f_m given in (9) converges to u in $L^1(\mathbb{R}^2)$ as m approaches 0 note that

$$\lim_{m\to 0}\int_{\mathbb{R}^2} \big|f_{M,m} - f_m\big|\, dx = 0,$$

and use (6). To prove Theorem 3.1 for $f \notin C_0^\infty(\mathbb{R}^2)$, use a density argument from (6).

Let v_m be the unique solution of (5) with data f. Since $f \geq f_m$ for all $m > 0$, $v_m \geq u_m$ for all $m > 0$ and any $s > 1$. For some $s > 1$ fixed the solutions u_m are locally bounded away from zero. Thus the v_m are locally bounded away from zero. Arguing as above $v_m \to v$ in $C^\infty(\mathbb{R}^2)$ where v is a solution of (4) with datum f. Since $v_m \geq u_m$ for any $s > 1$

$$\lambda \int_{\mathbb{R}^2} v\, dx \geq \int_{\mathbb{R}^2} f\, dx - 4\pi.$$

But (see [10]) any solution of (4) satisfies

$$\lambda \int_{\mathbb{R}^2} v\, dx \leq \int_{\mathbb{R}^2} f\, dx - 4\pi$$

with equality if and only if v is the maximal solution with datum f. Thus $v = u^1$ is the maximal solution of (1) with datum f.

5. The proofs of Lemmas 3.3 and 3.5.

5.1. The proof of Lemma 3.3. Let u be a solution of (5) with datum f where $f \in L^1(\mathbb{R}^2)\bigcap L^\infty(\mathbb{R}^2)$. Then by (6) $\lambda\|u\|_1 \leq \|f\|_1$, and by the maximum principle $\lambda\|u\|_\infty \leq \|f\|_\infty$.

For any $R > 0$ and any $x \in B_R$

$$\begin{aligned}\Big(\frac{|u|^{m-1}\ u(x) - 1}{m}\Big) =& \frac{1}{2\pi R}\int_{\partial B_R}\Big(\frac{R^2 - |x|^2}{|x-y|^2}\Big)\Big(\frac{|u|^{m-1}\ u(y) - 1}{m}\Big)\, d\sigma(y) \\ &+ \int_{B_R} G(x,y)(f - \lambda u)\, dy,\end{aligned} \tag{21}$$

where $G(x, y)$ is the Green's function for the Laplacian in two dimensions. Thus

$$\begin{aligned}\Big(\frac{|u|^{m-1}\ u(x) - 1}{m}\Big) =& \Big(\frac{|u|^{m-1}\ u(0) - 1}{m}\Big) + \int_{B_R}[G(x,y) - G(0,y)](f - \lambda u)\, dy \\ &+ \frac{1}{2\pi R}\int_{\partial B_R}\Big(\frac{R^2 - |x|^2}{|x-y|^2} - 1\Big)\Big(\frac{|u|^{m-1}\ u(y) - 1}{m}\Big)\, d\sigma(y) \\ =& \frac{|u|^{m-1}\ u(0) - 1}{m} + I_1 + I_2.\end{aligned} \tag{22}$$

Fix $x \in \mathbb{R}^2$, and chose $R > 2\,|x|$. First, with $M = \|f\|_{\infty,\mathbb{R}^2}$

$$
\begin{aligned}
|I_2| &= \left| \frac{1}{2\pi R} \int_{\partial B_R} \Big(\frac{R^2 - |x|^2 - |y-x|^2}{|y-x|^2}\Big)\Big(\frac{|u|^{m-1}\,u(y) - 1}{m}\Big)\, d\sigma(y) \right| \\
&\le \frac{M^m}{2m\pi R} \int_{\partial B_R} \frac{|x \cdot y - 2\,|x|^2|}{R^2 - |x|^2}\, d\sigma(y) \le \frac{C(M,m,|x|)}{R} \to 0 \quad \text{as } R \to \infty.
\end{aligned}
$$

Next, in two dimensions the Green's function for the Laplacian is constructed as

$$
G(x,y) = \frac{1}{2\pi}\big(\phi(x,y) - \ln|x-y|\,\big),
$$

where ϕ satisfies

$$
\begin{cases} \Delta_y \phi(x,y) = 0 & \text{in } B_R \\ \phi(x,y) = \ln|x-y| & \text{for } y \in \partial B_R. \end{cases}
$$

Notice that for $|y| = R$

$$
\ln(R - |x|) \le \phi(x,y) \le \ln(R + |x|). \tag{23}
$$

Since $\phi(x,\cdot)$ is harmonic it satisfies (23) in all of B_R. As a result

$$
\ln\Big(\frac{R - |x|}{R}\Big) \le \phi(x,y) - \phi(0,y) \le \ln\Big(\frac{R + |x|}{R}\Big) \quad \text{for all } y \in B_R.
$$

Thus for x fixed

$$
\lim_{R\to\infty} |\phi(x,y) - \phi(0,y)| = 0
$$

uniformly in y. Thus (11) follows from (22) by letting $R \to \infty$.

5.2. A sketch of the proof of Lemma 3.5. Let u be a solution of (4) with datum f where $f \in C_0^\infty(\mathbb{R}^2)$. Furthermore assume that

$$
u(x) \le \frac{M}{(1 + |x|^2)^s}.
$$

Suppose also that

$$
\lambda \int_{\mathbb{R}^2} u\,dx = \int_{\mathbb{R}^2} f\,dx - 4\pi s + 4\pi\delta
$$

for some $\delta > 0$. Chose $R_0 > 0$ such that if $R > R_0$ then

$$
\lambda \int_{B_R} u\,dx \ge \int_{B_R} f\,dx - 4\pi s + 2\pi\delta.
$$

Let $R > R_0$. Again by the Green's function representation

$$\begin{aligned}\ln u(0) &= \frac{1}{2\pi R}\int_{\partial B_R} \ln u(y)\, d\sigma(y) + \frac{1}{2\pi}\int_{B_R} (\ln R - \ln|y|)(f - \lambda u)\, dy \\ &\le C(M,f) - \ln R\big(2s - \frac{1}{2\pi}\int_{B_R} (f - \lambda u)\, dy\big) \\ &\le C(M,f) - \ln R\big(2s - \frac{4\pi s - 2\pi\delta}{2\pi}\big) \\ &= C(M,f) - \delta \ln R \to -\infty \text{ as } R \to \infty.\end{aligned}$$

Thus $u(0) = 0$. Of course, we could make this same argument for any point in the plane. This leads us to the contradiction that $u \equiv 0$.

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