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A FREE BOUNDARY PROBLEM FOR QUASILINEAR ELLIPTIC EQUATIONS IN EXTERIOR DOMAINS*

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Abstract. We prove the existence of weak solutions that belong to some local Sobolev space of a quasilinear elliptic problem in an exterior domain. The free-boundary problem considered here arises from the discontinuous behavior of the nonlinearity involved. The method of upper and lower solutions, extremality results for quasilinear elliptic problems in bounded domains, gradient estimates and abstract fixed point principles in partially ordered sets are the main tools used in the proof of our main result. An application to a superlinear discontinuous elliptic problem with the p-Laplacian is given.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. This paper deals with quasilinear elliptic equations in the exterior domain $E := \mathbb{R}^N \setminus \overline{\Omega}$ in the form

$$Au = f(x, u, u, \nabla u), \quad x \in E, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where the operator A given by

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(\cdot, u, \nabla u) \tag{1.2}$$

is an operator of Leray–Lions type. The free-boundary problem is expressed by the nonlinearity $f : E \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ that may be discontinuous not only in the space variable but also with respect to the unknown function

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u. Since we do not impose any additional decay behaviour of the solutions of (1.1) as $|x| \to \infty$, the underlying solution space will be a suitable local Sobolev space.

Our main goal is to show that by assuming an ordered pair of appropriately defined upper and lower solutions there exist solutions of the free-boundary problem (FBP) (1.1) enclosed by the given upper and lower solution. The method of proof developed in this paper yields not only the existence of solutions but the existence of *extremal* ones.

For bounded domains and under smooth-enough data extremality results of this kind for quasilinear elliptic boundary-value problems with a general Leray–Lions operator of the form (1.2) have been obtained by Puel in [12] and by a different method by the first author in [3] as well as for differential inclusions in [4], and in special cases of the operator A and the right-hand side in [6, 9].

The upper and lower solution method has been successfully applied recently by Pao ([11]—cf. also the references therein) to semilinear elliptic problems in unbounded domains when the operator A is linear and the data of the problems are smooth enough. However, due to the lack of regularity of the data of problem (1.1), especially the discontinuous behavior of the right-hand side function f, and due to the fact that the operator A is quasilinear, the extension of his method to our problem is by no means straightforward and requires different tools.

2. Notations and hypotheses. We denote by $\mathcal{W} := W^{1,p}_{loc}(E)$ the local Sobolev space, i.e., the space of all functions $u\colon E \mapsto \mathbb{R}$ which belong to the Sobolev space $W^{1,p}(\Gamma)$ for every domain Γ with compact closure in E. The space $\mathcal{W}$ is a locally convex space with its topology induced by the family of seminorms $\{p_l \mid l = 1, 2, \dots\}$ given by $p_l(u) = \|u\|_{W^{1,p}(E_l)}$ where $E_l \subset \mathbb{R}^N$ is defined by $E_l := E \cap B_l$ and $(B_l)_{l=1}^{\infty}$ is a sequence of increasing open balls in $\mathbb{R}^N$ such that $\overline{\Omega} \subset B_1$, and $B_l \to \mathbb{R}^N$ as $l \to \infty$. A sequence (u_n) is convergent to u with respect to this topology if and only if $p_l(u_n - u) \to 0$ as $n \to \infty$, for all $l = 1, 2, \dots$. By $\mathcal{D} := C_0^\infty(E)$ we denote the space of all infinitely differentiable functions with compact support in E. The local Lebesgue space $\mathcal{L}^p := L^p_{loc}(E)$ is equipped with the natural partial ordering $\le$ introduced by means of the order cone $\mathcal{L}^p_+ = L^p_{loc,+}(E)$, which is the set of all nonnegative functions of $\mathcal{L}^p$. If $u, v \in \mathcal{L}^p$ satisfy $u \le v$, then $[u, v] = \{z \in \mathcal{L}^p : u \le z \le v\}$ denotes the order interval.

On the coefficients a_i, $i = 1, \cdots, N$, of the differential operator A we impose the following conditions of Leray–Lions type.

(A1) Each $a_i : E \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies Carathéodory conditions; i.e., $a_i(x, t, \xi)$ is measurable in $x \in E$ for all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous in (t,ξ) for almost every (a.e.) $x \in E$. There exist a constant $c_0 > 0$ and a function $k_0 \in \mathcal{L}^q, 1/p + 1/q = 1$, such that

$$|a_i(x,t,\xi)| \le k_0(x) + c_0(|t|^{p-1} + |\xi|^{p-1}),$$

for a.e. $x \in E$ and for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

(A2)
$$\sum_{i=1}^{N}(a_i(x,t,\xi) - a_i(x,t,\xi'))(\xi_i - \xi_i') \ge \mu|\xi - \xi'|^p$$

for a.e. $x \in E$, for all $t \in \mathbb{R}$, and for all $\xi, \xi' \in \mathbb{R}^N$ with μ being some positive constant.

(A3) $|a_i(x,t,\xi) - a_i(x,t',\xi)| \le [k_1(x) + |t|^{p-1} + |t'|^{p-1} + |\xi|^{p-1}]\omega(|t-t'|), i = 1, \cdots, N$, for some function $k_1 \in \mathcal{L}^q$, for a.e. $x \in E$, for all $t, t' \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$, where $\omega : [0,\infty) \mapsto [0,\infty)$ is the *modulus of continuity* satisfying

$$\int_{0^+} \frac{dr}{\omega^q(r)} = +\infty.$$

Remark 2.1. Hypothesis (A3) is satisfied for example in the case that $\omega(|t - t'|) = c|t - t'|^{1/q}$ with some positive constant c; i.e., the coefficients $a_i(x,t,\xi)$ satisfy a Hölder condition with respect to t.

Let a denote the semilinear form associated with the operator A by

$$a(u,\varphi) = \int_E \sum_{i=1}^{N} a_i(x,u,\nabla u)\frac{\partial \varphi}{\partial x_i}\,dx,$$

which is well defined for all $u \in \mathcal{W}$ and $\varphi \in \mathcal{D}$, and let F denote the Nemytskij operator associated with f by $Fu(x) = f(x,u(x),u(x),\nabla u(x))$. Further, we introduce the function space $\mathcal{W}_0$ which is defined by

$$\mathcal{W}_0 := \{u \in \mathcal{W} : u = 0 \quad \text{on } \partial\Omega\}$$

Definition 2.1. A function $u \in \mathcal{W}_0$ is called a *solution* of problem (1.1) if

$$a(u,\varphi) = \int_E (Fu)\varphi\,dx \quad \text{for all } \varphi \in \mathcal{D}.$$

Definition 2.2. A function $\bar{u} \in \mathcal{W}$ is called an *upper solution* to problem (1.1) if $F\bar{u} \in \mathcal{L}^q$, $\bar{u} \ge 0$ on $\partial\Omega$ and

$$a(\bar{u},\varphi) \ge \int_E (F\bar{u})\varphi\,dx, \text{ for all } \varphi \in \mathcal{D}_+ := \{\varphi \in \mathcal{D} : \varphi \ge 0\}.$$

A lower solution is defined similarly, by reversing the inequality sign in Definition 2.2.

Let $\bar{u}$ and $\underline{u}$ be upper and lower solutions of (1.1), respectively, satisfying $\underline{u} \le \bar{u}$. We impose the following conditions on the nonlinearity f:

(H1) There exist a function $k_2 \in \mathcal{L}^q_+$ and a constant $c_1 \ge 0$ such that

$$|f(x,s,t,\xi)| \le k_2(x) + c_1|\xi|^{p-1} \tag{2.2}$$

for a.e. $x \in E$, for all $\xi \in \mathbb{R}^N$ and for all $s, t \in [\underline{u}(x), \bar{u}(x)]$.

(H2) The function $(t,\xi) \mapsto f(x,s,t,\xi)$ is continuous for a.e. $x \in E$ and for all $s \in \mathbb{R}$, and the function $s \mapsto f(x,s,t,\xi)$ is nondecreasing for a.e. $x \in E$ and for each $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

(H3) The function $(x,s) \mapsto f(x,s,t,\xi)$ is a standard function in $E \times \mathbb{R}$ in the sense of Shragin (cf. e.g. [7, 13]) for each $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

Finally, a solution $u^* \in \mathcal{W}_0$ of (1.1) is called the *greatest solution* of (1.1) within the order interval $[\underline{u}, \bar{u}]$ if any solution $u \in [\underline{u}, \bar{u}]$ of (1.1) satisfies $u \le u^*$. Accordingly, the *smallest solution* u_* of (1.1) in $[\underline{u}, \bar{u}]$ satisfies $u_* \le u$ for any solution $u \in [\underline{u}, \bar{u}]$. If both of these solutions exist, we call them the *extremal solutions* of (1.1) in $[\underline{u}, \bar{u}]$.

As the main result of the present paper we shall prove that if the hypotheses (A1)–(A3) and (H1)–(H3) are satisfied, then the FBP (1.1) has the extremal solutions u_*, u^* in $[\underline{u}, \bar{u}]$.

3. Preliminaries. Before proving our main result we shall first prove some preliminary results concerning a Carathéodory type BVP associated with the FBP (1.1). Throughout this section we shall assume that the hypotheses (A1)–(A3) and (H1)–(H3) are satisfied. The preliminaries given here are designed for proving the existence of the greatest solution of (1.1) within the sector formed by the upper and lower solutions. The corresponding preliminary results which are needed to prove the smallest solution can be obtained in a similar way.

Let $\bar{u}, \underline{u} \in \mathcal{W}$ be the given upper and lower solutions of the FBP (1.1). We assign to any fixed upper solution $v \in [\underline{u}, \bar{u}]$ of (1.1) the following Carathéodory-type BVP:

$$Au = f(x, v(x), u, \nabla u) \text{ in } E, \quad u = 0 \text{ on } \partial\Omega. \tag{3.1}$$

Lemma 3.1. *If $v \in [\underline{u}, \bar{u}]$ is any upper solution of* (1.1)*, then the BVP* (3.1) *has the greatest and smallest solution within the order interval $[\underline{u}, v]$.*

Proof. Let $v \in [\underline{u}, \bar{u}]$ be an upper solution of (1.1). By means of the monotonicity assumption (H2) it is easy to see that $\underline{u}$ is a lower solution of (3.1) and by definition v is also an upper solution of the BVP (3.1). First let us note that the hypotheses (H2) and (H3) ensure that the function $(x,t,\xi) \mapsto f(x,v(x),t,\xi)$ satisfies the Carathéodory conditions, and in view of condition (H1) we have

$$|f(x,v(x),t,\xi)| \le k_2(x) + c_1|\xi|^{p-1} \tag{3.2}$$

for a.e. $x \in E$, for all $\xi \in \mathbb{R}^N$ and for all $t \in [\underline{u}(x), \bar{u}(x)]$, and that (3.2) holds for any $v \in [\underline{u}, \bar{u}]$. Let us denote the Nemytskij operator associated with the right-hand side of (3.1) by F_v, i.e., $F_v u(x) = f(x, v(x), u(x), \nabla u(x))$ which implies $F_u u = Fu$.

The proof will be done in several steps and is designed to show the existence of the greatest solution v^* of (3.1) within $[\underline{u}, v]$ only, since the proof for the smallest solution v_* in $[\underline{u}, v]$ can be done quite similarly.

a) Let $\{B_l\}$ be the sequence of open balls, $l = 1, 2, \dots$, in $\mathbb{R}^N$ such that $\bigcup_{l=1}^{\infty} B_l = \mathbb{R}^N$. We construct a sequence of functions $(U_l)_{l=0}^{\infty}$ in the following way: $U_0 := v$, and for $l = 1, 2, \dots$ we define

$$U_l = \begin{cases} u_l(x) & \text{when } \ x \in E_l \\ v(x) & \text{when } \ x \in \mathbb{R}^N \setminus B_l, \end{cases}$$

where $u_l \colon E_l \mapsto \mathbb{R}$ denotes the greatest solution with respect to the order interval $[\underline{u}|_{E_l}, v|_{E_l}]$ of the following Dirichlet problem in the bounded domain E_l having the boundary $\partial E_l = \partial\Omega \cup \partial B_l$:

$$Au_l = F_v u_l \ \text{ in } E_l, \quad u_l = v \text{ on } \partial B_l \text{ and } u_l = 0 \text{ on } \partial\Omega, \tag{3.3}$$

and where $u|_{E_l}$ denotes the restriction of the function u to E_l. Since $\underline{u}|_{E_l}$ and $v|_{E_l}$ are lower and upper solution, respectively, to the boundary value problem (3.3) on the bounded domain E_l the existence of such an extremal solution u_l has been proved in [3]. From the definition of U_l and in view of the fact that $u_{l+1}|_{E_l}$ is a lower solution for the problem defining the solution u_l it follows that for $l = 1, 2, \dots$ the functions $U_l : E \mapsto \mathbb{R}$ satisfy the inequalities

$$\underline{u} \le \cdots \le U_{l+1} \le U_l \le \cdots \le U_1 \le U_0 = v. \tag{3.4}$$

As a consequence of (3.4) the a.e. pointwise limit U^* of (U_l) exists; i.e.,

$$\lim_{l\to\infty} U_l(x) = U^*(x) \quad \text{ for a.e. } \ x \in E.$$

b) Here we show that $U^* \in \mathcal{W}_0$. To this end let $\Gamma \in E$ be any domain with compact closure. Then there exists a ball B_l such that $\Gamma \subset E_l$, and each U_k, $k \ge l+1$, satisfies

$$a(U_k, \varphi) = \int_{E_{l+1}} F_v(U_k)\varphi \, dx \tag{3.5}$$

for all $\varphi \in W_0^{1,p}(E_{l+1})$ extended by zero outside of B_{l+1}. Since $\underline{u} \le U_k \le v$, we get

$$\|U_k\|_{L^p(E_l)} \le C_l, \quad \text{for all} \quad k = 1, 2, \dots, \tag{3.6}$$

where C_l stands for some generic positive constant whose value may change within the same proof but which depends only on l. Our aim is to derive a uniform $L^p(E_l)$-bound for the gradients of U_k for $k \ge l+1$. This, together with (3.6) would imply that $\|U_k\|_{W^{1,p}(E_l)} \le C_l$ for all k, which due to the existence of the pointwise limit U^* of the monotone sequence $\{U_k\}$ shows that $U_k \rightharpoonup U^*$ in $W^{1,p}(E_l)$ and $U_k \to U^*$ in $L^p(E_l)$ for any l.

For this purpose we take in (3.5) the following special test function:

$$\phi = U_k \ \theta^p, \tag{3.7}$$

where $\theta = \hat{\theta}|_E : E \to \mathbb{R}$ is the restriction to the exterior domain E of the finite function $\hat{\theta} \in C_0^\infty(\mathbb{R}^N)$ satisfying

i) $0 \le \hat{\theta}(x) \le 1$ for all $x \in \mathbb{R}^N$,
ii) $\hat{\theta}(x) = 1$ for $x \in \overline{B}_l$,
iii) $\hat{\theta}(x) = 0$ for $x \in \mathbb{R}^N \setminus B_{l+1}$.

For the existence of such a function $\hat{\theta}$ we refer to [8, Theorem 1.2.2.]. Substituting ϕ given by (3.7) in (3.5) and taking the hypotheses (A1) and (A2) into account we obtain by means of Young's inequality for the left-hand side of (3.5) the following estimate:

$$\begin{aligned} a(U_k, \phi) &= \int_{E_{l+1}} \sum_{i=1}^N a_i(x, U_k, \nabla U_k)\big(\frac{\partial U_k}{\partial x_i}\theta^p + pU_k\theta^{p-1}\frac{\partial \theta}{\partial x_i}\big)\,dx \\ &\ge \int_{E_{l+1}} \big(\mu\theta^p|\nabla U_k|^p - \sum_{i=1}^N (c(\varepsilon_1)|a_i(x, U_k, 0)|^q + \varepsilon_1|\frac{\partial U_k}{\partial x_i}|^p)\theta^p\big)\,dx \\ &\quad - \int_{E_{l+1}} \sum_{i=1}^N p|a_i(x, U_k, \nabla U_k)\theta^{p-1}||U_k\frac{\partial \theta}{\partial x_i}|\,dx \\ &\ge \int_{E_{l+1}} \big(\mu\theta^p|\nabla U_k|^p - \varepsilon_1\theta^p|\nabla U_k|^p - c(\varepsilon_1)\theta^p(|k_0|^q + |U_k|^p)\big)\,dx \\ &\quad - \int_{E_{l+1}} \big(c(\varepsilon_2)|U_k|^p|\nabla\theta|^p + \varepsilon_2\theta^p(|k_0|^q + |U_k|^p + |\nabla U_k|^p)\big)\,dx, \end{aligned} \tag{3.8}$$

where ε_1 and ε_2 may be any positive real numbers.

From hypothesis (H1) and Young's inequality we get for any $\varepsilon_3 > 0$

$$\begin{aligned}\int_{E_{l+1}} |F_v(U_k)U_k\theta^p|\,dx &\le \int_{E_{l+1}} \Big(|k_2| + c_1|\nabla U_k|^{p-1}\Big)|U_k|\theta^p\,dx \\ &\le \int_{E_{l+1}} \Big(\varepsilon_3(|k_2|^q\theta^p + |\nabla U_k|^p\theta^p) + c(\varepsilon_3)|U_k|^p\theta^p\Big)\,dx.\end{aligned} \tag{3.9}$$

Selecting ε_1, ε_2 and ε_3 sufficiently small and taking into account that (U_k) possesses a uniform $L^p(E_{l+1})$-bound from (3.8) and (3.9) we get for all $k \ge l+1$ a uniform bound of the form

$$\int_{E_{l+1}} \theta^p|\nabla U_k|^p\,dx \le C_l,$$

which yields by the definition of θ for all $k \ge l+1$ the uniform gradient estimate

$$\int_{E_l} |\nabla U_k|^p\,dx \le \int_{E_{l+1}} \theta^p|\nabla U_k|^p\,dx \le C_l,$$

and hence there is a uniform bound

$$\|U_k\|_{W^{1,p}(E_l)} \le C_l, \qquad \text{for all } k = 1, 2, \ldots, \tag{3.10}$$

which implies the weak compactness of the sequence $(U_k)_{k=1}^{\infty}$ in $W^{1,p}(E_l)$. This, together with the compact imbedding of $W^{1,p}(E_l)$ in $L^p(E_l)$ and the existence of a unique pointwise limit U^*, proves that

$$U_k \rightharpoonup U^* \text{ in } W^{1,p}(E_l) \text{ and } U_k \to U^* \text{ in } L^p(E_l) \text{ as } k \to \infty; \tag{3.11}$$

i.e., $U^* \in W^{1,p}(E_l)$, and thus $U^* \in W^{1,p}(\Gamma)$. Since Γ was an arbitrarily chosen domain with compact closure in E, it follows that $U^* \in \mathcal{W}$. Furthermore, the restrictions $U_k|_{E_l}$ belong to the closed convex set $\{u \in W^{1,p}(E_l) : u = 0 \text{ on } \partial\Omega\}$ which is weakly closed in $W^{1,p}(E_l)$. Hence it follows that the weak limit $U^* \in \mathcal{W}_0$.

c) Next we show that $U^* \in \mathcal{W}_0$ is a solution of (3.1); that is,

$$a(U^*, \varphi) = \int_E F_v(U^*)\varphi\,dx \tag{3.12}$$

holds for any $\varphi \in \mathcal{D}$. The proof for this is done provided we can show that the sequence (U_k) is convergent to U^* with respect to the topology of the locally convex space $\mathcal{W}$, since then for any test function $\varphi \in \mathcal{D}$ whose support is contained in some set E_l one can pass to the limit as $k \to \infty$ in

$$a(U_k, \varphi) = \int_E F_v(U_k)\varphi \, dx. \tag{3.13}$$

Therefore, in view of (3.11) it remains to show that the sequence of the gradients (∇U_k) is strongly convergent in $L^p(E_l)$ to ∇U^* for any $l = 1, 2, \ldots$.

Let an arbitrary l be given and let $\hat{\theta} \in C_0^\infty(\mathbb{R}^N)$ have the properties i), ii), and iii) as given above and suppose $\theta = \hat{\theta}|_E$. First we show

$$\lim_{k\to\infty} \int_{E_{l+1}} \sum_{i=1}^N a_i(x, U_k, \nabla U_k) \frac{\partial(U_k - U^*)}{\partial x_i} \theta \, dx = 0. \tag{3.14}$$

Each U_k, $k \geq l+1$, satisfies

$$a(U_k, \varphi) = \int_{E_{l+1}} F_v(U_k)\varphi \, dx \tag{3.15}$$

for all $\varphi \in W_0^{1,p}(E_{l+1})$ extended by zero outside of B_{l+1}. We take in (3.15) as special test function $\varphi = (U_k - U^*)\theta$ which gives

$$\int_{E_{l+1}} \sum_{i=1}^N a_i(x, U_k, \nabla U_k) \frac{\partial((U_k - U^*)\ \theta)}{\partial x_i} \, dx = \int_{E_{l+1}} F_v(U_k)(U_k - U^*)\theta \, dx$$

and thus

$$\begin{aligned} &\int_{E_{l+1}} \sum_{i=1}^N a_i(x, U_k, \nabla U_k) \frac{\partial(U_k - U^*)}{\partial x_i} \theta \, dx \qquad (3.16) \\ &= \int_{E_{l+1}} -\sum_{i=1}^N a_i(x, U_k, \nabla U_k)(U_k - U^*) \frac{\partial \theta}{\partial x_i} \, dx + \int_{E_{l+1}} F_v(U_k)(U_k - U^*)\theta \, dx. \end{aligned}$$

The uniform $W^{1,p}(E_l)$-bound of (U_k) for each l implies by (A1) that $(a_i(\cdot, U_k, \nabla U_k))$ is bounded in $L^q(E_{l+1})$. This together with the strong convergence of (U_k) in $L^p(E_{l+1})$ and the boundedness of $(F_v(U_k))$ in $L^q(E_{l+1})$ implies that the right-hand side of (3.16) tends to zero as $k \to \infty$ which shows (3.14).

Furthermore, the convergence properties (3.11) imply

$$\lim_{k\to\infty} \int_{E_{l+1}} \sum_{i=1}^N a_i(x, U_k, \nabla U^*) \frac{\partial(U_k - U^*)}{\partial x_i} \theta \, dx = 0. \tag{3.17}$$

By (A2) and the definition of θ it follows that

$$\begin{aligned}0 &\le \mu \int_{E_l} |\nabla(U_k - U^*)|^p \, dx \le \mu \int_{E_{l+1}} |\nabla(U_k - U^*)|^p \theta \, dx \\ &\le \int_{E_{l+1}} \sum_{i=1}^{N} (a_i(x, U_k, \nabla U_k) - a_i(x, U_k, \nabla U^*)) \frac{\partial(U_k - U^*)}{\partial x_i} \theta \, dx,\end{aligned} \tag{3.18}$$

which yields by (3.14) and (3.17) that the right-hand side of (3.18) tends to zero as $k \to \infty$, and thus the $L^p(E_l)$-convergence of the gradients (∇U_k) to ∇U^* follows for any l. This and the convergence properties (3.11) prove that U^* is a solution of (3.1).

d) To complete the proof of Lemma 3.1 we show that $v^* := U^*$ is the greatest solution of (3.1) within the order interval $[\underline{u}, v]$. For this purpose let $\tilde{u} \in [\underline{u}, v]$ be any solution of (3.1). Then one can show that the constructed sequence (U_k) satisfies (3.4) when $\underline{u}$ is replaced by $\tilde{u}$. This proves that $\tilde{u} \le U^*$, and hence it follows that $v^* := U^*$ is the greatest solution of (3.1) within $[\underline{u}, v]$.

Remark 3.1. It should be noted that the strong monotonicity (p-ellipticity) condition (A2) which is related with the modulus-of-continuity condition (A3) depending on the dual real q is needed to ensure the existence of extremal solutions of the Dirichlet problem (3.3) in bounded domains E_l (cf. e.g. [3]), which is a crucial point in the proof of Lemma 3.1. The more relaxed assumptions of strict monotonicity, i.e.,

$$\sum_{i=1}^{N} (a_i(x, t, \xi) - a_i(x, t, \xi'))(\xi_i - \xi'_i) > 0, \quad \text{if } \xi \ne \xi', \tag{M}$$

along with a coercive condition, like

$$\sum_{i=1}^{N} a_i(x, t, \xi)\xi_i \ge \mu |\xi|^p - k(x), \tag{C}$$

would be sufficient for existence as well as enclosure results for the bounded domain problem (3.3). However, to prove the existence of *extremal* solutions of (3.3) the stronger conditions (A2) and (A3) are needed. In the special case that the coefficients $a_i(x, t, \xi)$ do not depend on the variable t standing for the solution u conditions (A2) and (A3) can be replaced by the weaker ones (M) and (C).

Let $\underline{u}$ and $\bar{u}$ be the given lower and upper solutions, respectively, of the FBP (1.1). Now we define a partially ordered set $P \subset \mathcal{W}$ by

$$P = \{v \in \mathcal{W} : v \in [\underline{u}, \bar{u}] \text{ and } v \text{ is an upper solution of the FBP (1.1)}\},$$

and an operator $G : P \mapsto \mathcal{W}_0$ that assigns to each $v \in P$ the greatest solution $Gv := v^*$ of the BVP (3.1) with respect to the interval $[\underline{u}, v]$. By Lemma 3.1 the operator G is well defined, and we always have $Gv \leq v$.

From the proof of Lemma 3.1 one easily derives the following result concerning the BVP (3.1) which can be rewritten in the form

$$Au = F_v u \quad \text{in } E, \quad u = 0 \quad \text{on } \partial\Omega. \tag{3.19}$$

Corollary 3.1. *Let $\underline{v}$ and $\bar{v}$ be any lower and upper solutions, respectively, of the BVP* (3.19) *satisfying $\underline{v} \leq \bar{v}$. Then there exist the extremal solutions of* (3.19) *within the order interval $[\underline{v}, \bar{v}]$.*

By means of Lemma 3.1 and Corollary 3.1 we are able to prove the following monotonicity result for the operator G.

Lemma 3.2. *The operator $G \; : P \to P$ is nondecreasing.*

Proof. By the definition of the operator G it follows that $G\colon P \to [\underline{u}, \bar{u}]$. To prove that $Gv \in P$ for any $v \in P$ we have to show that Gv is an upper solution of (1.1). This, however, readily follows by the monotonicity condition (H2), and due to $Gv \leq v$ which shows

$$A(Gv) = F_v(Gv) \geq F_{Gv}(Gv) = F(Gv) \quad \text{in } E, \quad Gv = 0 \quad \text{on } \partial\Omega.$$

To prove the monotonicity, let $v_1, v_2 \in P$, and assume that $v_1 \leq v_2$. Then we have

$$Gv_1 \in [\underline{u}, v_1], \quad A(Gv_1) = F_{v_1}(Gv_1) \quad \text{in } E, \quad Gv_1 = 0 \quad \text{on } \partial\Omega$$

and

$$Gv_2 \in [\underline{u}, v_2], \quad A(Gv_2) = F_{v_2}(Gv_2) \quad \text{in } E, \quad Gv_2 = 0 \quad \text{on } \partial\Omega. \tag{3.20}$$

Since $v_1 \leq v_2$ it follows that $Gv_1 \leq v_2$. Further we have

$$A(Gv_1) = F_{v_1}(Gv_1) \leq F_{v_2}(Gv_1) \quad \text{in } E, \quad Gv_1 = 0 \quad \text{on } \partial\Omega$$

which shows that Gv_1 is a lower solution and by definition v_2 is an upper solution for equation (3.20). By means of Corollary 3.1 this implies the existence of solutions of (3.20) within the interval $[Gv_1, v_2]$. But Gv_2 is the greatest solution of (3.20) in a bigger interval $[\underline{u}, v_2] \supset [Gv_1, v_2]$ so that $Gv_1 \leq Gv_2$. $\square$

The following lemma extends a result proved in [7, Lemma 4.1.2] to the local Sobolev space $\mathcal{W}$.

Lemma 3.3. *Let $\mathcal{C} \subset P$ be an inversely well-ordered chain which is bounded in $\mathcal{W}$. Then there exists a nonincreasing sequence of functions w_n in $\mathcal{W}$ whose restrictions to E_l converge weakly in $W^{1,p}(E_l)$ and strongly in $L^p(E_l)$ to* $\inf \mathcal{C} =: w \in \mathcal{W}$ *for each $l = 1, 2, \ldots$.*

Proof. Replacing $\mathcal{C}$ by $\mathcal{V} := \max \mathcal{C} - \mathcal{C}$, it is equivalent to show that if $\mathcal{V}$ is a bounded and well-ordered chain in $\mathcal{W}$, and if $u(x) \geq 0$ for a.e. $x \in E$ whenever $u \in \mathcal{V}$, there exists a nondecreasing sequence (v_n) which converges weakly in $W^{1,p}(E_l)$ and strongly in $L^p(E_l)$ to $\sup \mathcal{V} \in \mathcal{W}$ for each $l = 1, 2, \ldots$. To prove this, note first that for each $l = 1, 2, \ldots$ the space $W^{1,p}(E_l)$ is continuously embedded in $L^p(E_l)$, so that

$$c_l = \sup\{\|u|_{E_l}\|_{L^p(E_l)} : u \in \mathcal{V}\}$$

is finite. Choose a nondecreasing sequence $(u_n^l)_{n=1}^\infty$ from $\mathcal{V}$ so that

$$c_l = \lim_{n \to \infty} \|u_n^l|_{E_l}\|_{L^p(E_l)}.$$

Since $\mathcal{V}_l := \{u|_{E_l} : u \in \mathcal{V}\} \subset L_+^p(E_l)$, and since $L_+^p(E_l)$ is a fully regular order cone of $L^p(E_l)$, see e.g. [7], then

$$u^l = \lim_{n \to \infty} u_n^l|_{E_l} \tag{3.21}$$

exists in $L_+^p(E_l)$. Further, since $\|\cdot\|_{L^p(E_l)}$ is strictly monotone in $L_+^p(E_l)$, it is easy to see that $u^l = \sup \mathcal{V}_l$ in $L^p(E_l)$. Because $W^{1,p}(E_l)$ is reflexive, (u_n^l) converges to u^l weakly in $W^{1,p}(E_l)$. In particular, $u^l = \sup \mathcal{V}_l$ belongs to $W^{1,p}(E_l)$. Define a sequence $(v_n)_{n=1}^\infty$ by

$$v_n = \max_{1 \leq l \leq n} u_n^l, \quad n = 1, 2, \ldots. \tag{3.22}$$

The sequence $(v_n)_{n=1}^\infty$ is nondecreasing and

$$0 \leq u_n^l|_{E_l} \leq v_n|_{E_l} \leq u^l \quad \text{when} \quad 1 \leq l \leq n. \tag{3.23}$$

This, (3.21) and (3.22) imply that for each $l = 1, 2, \ldots$ the sequence $(v_n|_{E_l})_{n=1}^\infty$ converges weakly in $W^{1,p}(E_l)$ and strongly in $L^p(E_l)$ to $u^l = \sup \mathcal{V}_l \in W^{1,p}(E_l)$. Consequently, if $l \leq \hat{l}$, then $u^l = u^{\hat{l}}|_{E_l}$ (a.e.), so that the sequence (v_n) converges to $v = \bigcup_l u^l = \sup \mathcal{V} \in \mathcal{W}$ strongly in $L^p(E_l)$ and weakly in $W^{1,p}(E_l)$ for each $l = 1, 2, \ldots$. Hence it follows that $w_n := \max \mathcal{C} - v_n$ satisfies the assertions of the lemma.

4. Main result. The proof of our main result is based on the following abstract fixed-point result proved in [7, Proposition 1.2.1].

Lemma 4.1. *We are given a nondecreasing mapping $G : P \mapsto P$ of a partially ordered set P to itself and let $\bar{u} \in P$. Then there exists a unique inversely well-ordered chain $\mathcal{C}$ in P, called an i.w.o. chain of G-iterations of $\bar{u}$, satisfying*

$$\bar{u} = \max \mathcal{C} \quad \textit{and} \quad \bar{u} > u \in \mathcal{C} \quad \textit{iff} \quad u = \inf G\{z \in \mathcal{C} : z > u\}.$$

If $u^* = \inf G[\mathcal{C}]$ exists and $G\bar{u} \leq \bar{u}$, then u^* is the greatest fixed point of G in $(\bar{u}] := \{z \in P : z \leq \bar{u}\}$.

The main result of this paper is given by the following theorem.

Theorem 4.1. *Let $\bar{u}$ and $\underline{u}$ be upper and lower solutions to the FBP* (1.1)*, respectively, satisfying $\underline{u} \leq \bar{u}$. Then, under the hypotheses* (A1)–(A3) *and* (H1)–(H3) *there exist the extremal solutions of* (1.1) *within the order interval $[\underline{u}, \bar{u}]$.*

Proof. The proof is designed to show the existence of the greatest solution of the FBP (1.1) within $[\underline{u}, \bar{u}]$. The existence of the smallest solution can be proved analogously.

Let P and $G : \ P \mapsto P$ be the partially ordered set and the operator, respectively, as introduced in Section 3; i.e.,

$$P = \{v \in \mathcal{W} : v \in [\underline{u}, \bar{u}] \text{ and } v \text{ is an upper solution of the FBP (1.1)}\},$$

and the operator G assigns to each $v \in P$ the greatest solution $Gv := v^*$ of the BVP (3.1), respectively (3.19), given by

$$Au = F_v u \quad \text{in } E, \quad u = 0 \quad \text{on } \partial\Omega$$

with respect to the interval $[\underline{u}, v]$. By Lemma 3.1 the operator G is well defined, and we have $Gv \leq v$. In view of Lemma 3.2 $G \colon P \mapsto P$ is nondecreasing. Let $\mathcal{C} \subset P$ be the i.w.o. chain of G-iterations of the upper solution $\bar{u} \in P$. Because G is nondecreasing, $G[\mathcal{C}]$ is also an inversely well-ordered chain in P. The proof of Lemma 3.1, which is essentially based on an appropriate gradient estimate, can be applied to show that $G[\mathcal{C}]$ is bounded with respect to the topology of the locally convex space $\mathcal{W}$. Hence, by Lemma 3.3 there exists a nonincreasing sequence $(w_n)_{n=0}^{\infty}$ in $G[\mathcal{C}]$ that converges to $u^* = \inf G[\mathcal{C}] \in \mathcal{W}$ weakly in $W^{1,p}(E_l)$ and strongly in $L^p(E_l)$ for each $l = 1, 2, \ldots$. By definition, $w_n = Gz_n$, where $\underline{u} \leq w_n \leq z_n$, $z_n \in \mathcal{C} \subset P$ and

$$Aw_n = F_{z_n} w_n \quad \text{in } E, \quad w_n = 0 \quad \text{on } \partial\Omega. \tag{4.1}$$

The monotonicity of the Nemytskij operator F_v with respect to v yields

$$Aw_n \geq F_{w_n} w_n \geq F_{u^*} w_n \quad \text{in } E, \quad w_n = 0 \quad \text{on } \partial\Omega. \tag{4.2}$$

In addition to the convergence properties of the sequence (w_n) one can show in just the same way as in the proof of Lemma 3.1 that also the gradients ∇w_n converge strongly to ∇u^* in $L^p(E_l)$ for each $l = 1, 2, \dots$. Hence, we may pass to the limit in the inequality (4.2) as $n \to \infty$ which yields for $u^* = \inf G[\mathcal{C}]$ the inequality

$$Au^* \geq F_{u^*} u^* \quad \text{in } E, \quad u^* = 0 \quad \text{on } \partial\Omega. \tag{4.3}$$

This proves that u^* is an upper solution of the FBP (1.1); i.e., $u^* = \inf G[\mathcal{C}] \in P$. Furthermore, by definition of the operator G we have $G\bar{u} \leq \bar{u}$. Thus our partially ordered set P and the operator G along with the given upper solution $\bar{u}$ meet exactly the situation of Lemma 4.1 which implies that $u^* = \inf G[\mathcal{C}]$ is the greatest fixed point of G in $(\bar{u}] = \{z \in P : z \leq \bar{u}\}$. Since any fixed point of G is a solution of (1.1) and vice versa, it follows that u^* is the greatest solution of the FBP (1.1) in $(\bar{u}] = \{z \in P : z \leq \bar{u}\} \subset [\underline{u}, \bar{u}]$. Because any solution of (1.1) within $[\underline{u}, \bar{u}]$ is in particular an upper solution of (1.1) and thus belongs to $(\bar{u}]$ it follows that u^* must be also the greatest solution of (1.1) within the order interval $[\underline{u}, \bar{u}]$. This completes the proof of Theorem 4.1.

Remark 4.1. Theorem 4.1 remains true also in case that $\Omega = \emptyset$, i.e., for the FBP in all of $\mathbb{R}^N$

$$Au = Fu \quad \text{in } \mathbb{R}^N.$$

5. Example. Let $-A$ be the p-Laplacian, i.e.,

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i}),$$

and let $p > 2$. Then $a_i(x, t, \xi) = |\xi|^{p-2}\xi_i$, and the conditions (A1)–(A3) are satisfied.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain whose boundary is of class $C^{2,\beta}$. We are interested in the existence of nonnegative and nontrivial solutions of the following FBP:

$$Au = -u^\alpha + mu + r(x)h(u - s(x)) \text{ in } E = \mathbb{R}^N \setminus \overline{\Omega},\ u = 0 \text{ on } \partial\Omega, \tag{5.1}$$

with $\alpha > 1$, $m > 0$, $r \colon E \mapsto \mathbb{R}_+$, $r \in L^\infty(E)$ and $s \colon E \mapsto \mathbb{R}_+$ measurable, and where $h \colon \mathbb{R} \mapsto \mathbb{R}$ is the Heaviside step function. Obviously $u \equiv 0$ is a solution of (5.1). To prove the existence of nonnegative nontrivial solutions of (5.1) we construct an ordered pair of nonnegative lower and upper solutions and apply Theorem 4.1.

First, there exists always a constant $\varrho > 0$ sufficiently large such that the inequality

$$-\varrho^\alpha + m\varrho + \|r\|_{L^\infty(E)} \le 0 \tag{5.2}$$

holds. Then an upper solution of (5.1) is given by $\bar{u} = \varrho$. Take any fixed l; then, a nontrivial nonnegative lower solution $\underline{u}$ can be constructed in the following way:

$$\underline{u} = \begin{cases} \varepsilon\varphi & \text{when } \ x \in E_l \\ 0 & \text{when } \ x \in \mathbb{R}^N \setminus B_l, \end{cases} \tag{5.3}$$

where φ is the eigenfunction that corresponds to the first eigenvalue λ of the operator A satisfying

$$A\varphi = \lambda|\varphi|^{p-2}\varphi \quad \text{in } E_l, \quad \varphi = 0 \text{ on } \partial E_l = \partial\Omega \cup \partial B_l. \tag{5.4}$$

By results of Anane in [1] and Lindqvist in [10] the first eigenvalue $\lambda > 0$ is simple and the first eigenfunction φ is nonnegative and belongs to $C^{1,\gamma}$ for some $\gamma \in (0,1)$. Thus it can always be assumed that $0 \le \varphi \le 1$. By choosing $\varepsilon > 0$ appropriately we shall show that $\underline{u}$ given by (5.3) is a lower solution in the sense of the definition given in Section 2.

To this end consider first $\underline{u}$ in E_l, i.e., $\underline{u} = \varepsilon\varphi$, which yields

$$A\underline{u} - [-\underline{u}^\alpha + m\underline{u} + r(x)h(\underline{u} - s(x))] \le \ \lambda(\varepsilon\varphi)^{p-1} + (\varepsilon\varphi)^\alpha - m(\varepsilon\varphi) \le 0$$

for $\varepsilon > 0$ sufficiently small. Since $\underline{u} = 0$ on $\partial\Omega$, it follows that $\underline{u}$ is a lower solution of (5.1) in the domain E_l. Due to the fact that the outer normal derivative of the first eigenfunction φ satisfies

$$\frac{\partial\varphi}{\partial\nu} < 0 \quad \text{on } \partial E_l$$

(cf. [1]) one can prove that the function $\varepsilon\varphi$ extended by zero in $\mathbb{R}^N \setminus B_l$ is a lower solution in all of E. Finally ε can be chosen small enough such that also $\varepsilon\varphi \le \varrho$ is satisfied which shows that $\underline{u}$ given by (5.3) and $\bar{u} = \varrho$ are lower and upper solutions, respectively, of (5.1) satisfying $\underline{u} \le \bar{u}$. The hypotheses (H1)–(H3) can readily be verified, and thus Theorem 4.1 can be applied which proves the existence of nontrivial and nonnegative solutions of the FBP (5.1).

Remark 5.1. Semilinear elliptic equations with discontinuous nonlinearities in unbounded domains have been investigated e.g. by the authors in [5] and by different methods by Badiale in [2]. In [2] variational methods due to K.C. Chang are used to get existence of solutions in the sense of multivalued mappings by a constraint-minimization problem which is related to the original equation. However, this approach requires elliptic operators of potential type and yields solutions of an associated differential inclusion, i.e., of a relaxed problem, only. Furthermore, in order to apply variational methods the solutions need to be of finite energy which implies a certain decay behavior as $|x| \to \infty$. The method used by the authors is based on upper and lower solutions and an abstract fixed-point theorem in partially ordered sets which does not require potential-type elliptic operators or any kind of growth conditions on the solutions. Moreover, the solutions satisfy the equation itself rather than an inclusion.

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