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STRONG CONVERGENCE OF BOUNDED SEQUENCES OF SOLUTIONS OF POROUS MEDIUM EQUATIONS

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Abstract. We show that smooth solutions of porous medium equations satisfy a simple L^2 gradient estimate on sets where the solution itself is small. Along with known continuity estimates for solutions and an estimate on second derivatives of smooth solutions, this estimate allows us to show that approximating smooth solutions of a porous medium equation converge strongly to the weak solution.

When studying degenerate parabolic equations, it is often useful to approximate the solutions by solutions of approximating, nondegenerate equations. Under reasonable assumptions the solutions of the approximating problems will converge weakly to the solution of the degenerate equation. Here we describe a situation in which the convergence is actually strong.

We consider the problem

$$\begin{aligned} \beta(u)_t &\ni \Delta u + g(x,t) \text{ in } \Omega \times (0,T), \\ u &= \varphi \text{ on } \partial\Omega \times (0,T) \cup \overline{\Omega} \times \{0\}, \end{aligned} \tag{0.1}$$

where β is a maximal monotone graph, Ω is a smooth domain in $\mathbb{R}^n$ and g and φ are given continuous functions. If $\partial\Omega$ and φ are sufficiently smooth (for example, C^2) and if β^{-1} is a continuous function, then (0.1) has a unique weak solution. Moreover, if (β_i) is a sequence of C^1, strictly increasing functions which converges β in the sense of maximal monotone graphs with β_i' strictly positive and if (g_i) is a sequence of C^1 functions which converges uniformly to g, then there is a unique classical solution u_i to the problem

$$\begin{aligned} \beta_i(u_i)_t &= \Delta u_i + g_i(x,t) \text{ in } \Omega \times (0,T), \\ u_i &= \varphi \text{ on } \partial\Omega \times (0,T) \cup \overline{\Omega} \times \{0\} \end{aligned} \tag{0.1i}$$

for each i. Under suitable additional conditions on β and φ, it is easy to check that (Du_i) converges to Du weakly in $L^2(\Omega\times(0,T))$. Typical such conditions

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are that $\beta(u) = |u|^{p-1}u$ for some positive constant p and $\varphi = 0$. Our goal here is to show that (Du_i) converges strongly to Du. The primary tool is a new gradient estimate for solutions of the approximating problems. This estimate seems to have some independent interest. We prove the estimate in Section 1 and then study the strong convergence in Section 2.

1. The gradient estimate. For the model case $\beta(u) = |u|^{p-1}u$, we know that it is possible to estimate the $C^{1,\alpha}$ norm (for any $\alpha \in (0,1)$) on the set $\{|u| > \varepsilon\}$ for any positive ε with the estimate depending on ε. In this section we obtain a gradient estimate on the set $\{|u| \le \varepsilon\}$.

We shall assume in this section that β is C^2 and that β' is uniformly bounded from above and below by positive constants; however, the size of these constants will not enter into our estimates. We also suppose that $g \in C^1$. These assumptions guarantee that u is a classical solution of (0.1).

In the special case that $\varphi = 0$, our estimate takes the following simple form.

Lemma 1.1. *Suppose that u is a classical solution of* (0.1) *with $\varphi = 0$ and set*

$$c_1 = |\Omega|(\sup|\beta(u)| + T\sup|g|). \tag{1.1}$$

Then for any constant $\alpha \in (0,1)$, there is a constant $C(\alpha)$ such that

$$\int_{\{|u|<\varepsilon\}} |Du|^2\ dX \le C\Big(\frac{\varepsilon}{\sup|u|}\Big)^\alpha\Big(\int_{\Omega\times(0,T)} |Du|^2\ dX + c_1\sup|u|\Big) \tag{1.2}$$

for any $\varepsilon \in (0, \sup|u|)$.

Proof. Define

$$f(s) = \begin{cases} s & \text{if } 0 \le s < \varepsilon \\ 2\varepsilon - s & \text{if } \varepsilon \le s < 2\varepsilon \\ 0 & \text{if } 2\varepsilon \le s \end{cases}$$

and extend f to all of $\mathbb{R}$ as an odd function. We then multiply the differential equation for u by $f(u)$ and define

$$F(s) = \int_0^s \beta'(\sigma)f(\sigma)\ d\sigma$$

to see that

$$\int_\Omega F(u(x,T))\,dx - \int_\Omega F(u(x,0))\,dx + \int_{\Omega\times(0,T)} f'(u)|Du|^2 dX = \int_{\Omega\times(0,T)} g(X)f(u)\,dX.$$

Now we note that $s \geq 0$ implies that

$$0 \leq F(s) \leq \varepsilon \int_0^s \beta'(\sigma)\ d\sigma = \varepsilon\beta(s),$$

and $s < 0$ implies that

$$0 \geq F(s) \geq -\varepsilon|\beta(s)|.$$

It follows that $|F(u)| \leq \varepsilon \sup|\beta(u)|$ and hence,

$$\int_{\Omega\times(0,T)} f'(u)|Du|^2\ dX \leq c_1\varepsilon.$$

On the other hand, the definition of f gives us

$$\begin{aligned}\int_{\Omega\times(0,T)} f'(u)|Du|^2\ dX &= \int_{\{|u|<\varepsilon\}} |Du|^2\ dX - \int_{\{\varepsilon<|u|<2\varepsilon\}} |Du|^2\ dX \\ &= 2\int_{\{|u|<\varepsilon\}} |Du|^2\ dX - \int_{\{|u|<2\varepsilon\}} |Du|^2\ dX.\end{aligned}$$

Therefore, ω defined by

$$\omega(\varepsilon) = \int_{\{|u|<\varepsilon\}} |Du|^2\ dX$$

satisfies the inequality

$$\omega(\varepsilon) \leq \frac{1}{2}\omega(2\varepsilon) + \frac{c_1}{2}\varepsilon,$$

and a standard iteration scheme then gives

$$\omega(\varepsilon) \leq C(\alpha)\big(\big(\frac{\varepsilon}{\sup|u|}\big)^\alpha \omega(\sup|u|) + c_1\varepsilon^\alpha(\sup|u|)^{1-\alpha}\big).$$

Since

$$\omega(\sup|u|) = \int_{\Omega\times(0,T)} |Du|^2\ dX,$$

this inequality easily implies (1.1). □

For nonzero boundary data, we shall use some fairly strong regularity assumptions. Specifically, we assume that

$$\partial\Omega \in C^{1,1}, \tag{1.3a}$$

$$\varphi \text{ is independent of time,} \tag{1.3b}$$

$$\varphi \in C^{1,1}(\overline{\Omega}). \tag{1.3c}$$

It then follows from the boundary gradient estimates of Kamynin and Khimchenko [3] (see also [5, Corollary 10.5]) that there is a constant K determined only by Ω, $\sup|g|$, and φ such that

$$|u-\varphi| \le Kd, \tag{1.4}$$

where d denotes distance to the boundary of Ω. Therefore,

$$Kd \le u-\varphi+2Kd \le 3Kd \tag{1.5a}$$

and

$$\{d<\frac{\varepsilon}{3K}\} \subset \{|u-\varphi+2Kd|<\varepsilon\} \subset \{d<\frac{\varepsilon}{K}\}. \tag{1.5b}$$

Since $|Dd|=1$ almost everywhere, it follows that

$$\int_{\{|u-\varphi+2Kd|<\varepsilon\}} |D(\varphi-2Kd)|^2\ dX \le c_2\varepsilon \tag{1.6}$$

for some positive constant c_2 determined only by K, T, $\sup|D\varphi|$, and Ω. With this information, we can derive an estimate for Du near $\partial\Omega\times(0,T)$.

Lemma 1.2. *Suppose that u is a classical solution of* (0.1) *and that Ω and φ satisfy* (1.3)*. Then there is a constant $C(\alpha)$ such that*

$$\int_{\{d<\varepsilon/3K\}} |Du|^2\ dX \le C\big(\frac{\varepsilon}{\sup|u|}\big)^{\alpha}\big(\int_{\Omega\times(0,T)} |Du|^2\ dX + (c_1+c_2)\sup|u|\big) \tag{1.7}$$

for $\varepsilon\in(0,\sup|u|)$

Proof. Now we use the test function $f(u-\varphi+2Kd)$ in place of $f(u)$ in the proof of Lemma 1.1 and we define

$$U(X)=\int_0^{u(X)} f(\sigma-\varphi(x)+2Kd(x))\beta'(\sigma)\ d\sigma, \bar{u}=u-\varphi+2Kd$$

to see that

$$\int_{\Omega\times(0,T)} f'(\bar{u})D\bar{u}\cdot Du\,dX = \int_\Omega U(x,0)\,dx - \int_\Omega U(x,T)\,dx + \int_{\Omega\times(0,T)} gf(\bar{u})\,dX.$$

As before, $|U|\le\varepsilon\sup|\beta(u)|$, so

$$\int_{\Omega\times(0,T)} f'(\bar{u})D\bar{u}\cdot Du\ dX \le c_1\varepsilon.$$

Writing I for the integral in this inequality, we have

$$I = 2\int_{\{|\bar{u}|<\varepsilon\}} |Du|^2\ dX - 2\int_{\{|\bar{u}|<\varepsilon\}} D(\varphi - 2Kd)\cdot Du\ dX$$
$$- \int_{\{|\bar{u}|<2\varepsilon\}} |Du|^2\ dX + \int_{\{|\bar{u}|<2\varepsilon\}} D(\varphi - 2Kd)\cdot Du\ dX.$$

Hence, for any $\delta > 0$, there is $C(\delta)$ such that

$$I \geq 2(1-\delta)\int_{\{|\bar{u}|<\varepsilon\}} |Du|^2\ dX - (1+\delta)\int_{\{|\bar{u}|<2\varepsilon\}} |Du|^2\ dX$$
$$- C(\delta)\int_{\{|\bar{u}|<\varepsilon\}} |D(\varphi - 2Kd)|^2\ dX,$$

and hence, if $\delta \leq 1/2$, we have

$$\int_{\{|\bar{u}|<\varepsilon\}} |Du|^2\ dX \leq \frac{1+\delta}{2(1-\delta)}\int_{\{|\bar{u}|<2\varepsilon\}} |Du|^2\ dX + (C(\delta)2c_2 + c_1)\varepsilon.$$

We now take δ so small that

$$\frac{1+\delta}{2(1-\delta)} \leq \big(\frac{1}{2}\big)^{(1+\alpha)/2}$$

and then iterate to conclude that

$$\int_{\{|\bar{u}|<\varepsilon\}} |Du|^2\ dX \leq C(\alpha)\big(\frac{\varepsilon}{\sup|u|}\big)^{\alpha}\big(\int_{\Omega\times(0,T)} |Du|^2\ dX + (c_1 + c_2)\sup|u|\big).$$

Combining this inequality with (1.5b) yields (1.7). □

Our final lemma gives an interior gradient estimate for u.

Lemma 1.3. *Suppose that u is a classical solution of* (0.1) *and that Ω satisfies* (1.3a). *Then for any $\alpha \in (0,1)$, there are positive $C(\alpha)$ and $c_3(\Omega, T)$ such that*

$$\int_{\{|u|<\varepsilon, d>\delta\}} |Du|^2\, dX \tag{1.8}$$
$$\leq C\big(\frac{\varepsilon}{\sup|u|}\big)^{\alpha}\big(\frac{\sup|u|}{\delta}\big)\big(\int_{\Omega\times(0,T)} |Du|^2\, dX + (c_1 + c_3)\sup|u|\big)$$

for any constants ε *and* δ *in the interval* $(0, \sup|u|)$.

Proof. Set $\zeta = \max\{1, d/\delta\}$ and use the test function $f(u)\zeta^2$ to infer that

$$\int_\Omega F(u)(x,T)\zeta^2 dx - \int_\Omega F(u)(0,T)\zeta^2 dx = \int_{\Omega\times(0,T)} |Du|^2 f'\zeta^2 + 2Du\cdot D\zeta f\zeta \, dX,$$

and hence,

$$\int_{\{|u|<\varepsilon\}} |Du|^2\zeta^2 \, dX - \int_{\{\varepsilon<|u|<2\varepsilon\}} |Du|^2\zeta^2 \, dX \\ \leq c_1\varepsilon + \eta\int_{\{|u|<2\varepsilon\}} |Du|^2\zeta^2 \, dX + \frac{\varepsilon^2}{\eta}\int_{\Omega\times(0,T)} |D\zeta|^2 \, dX$$

for any $\eta > 0$. Since $\partial\Omega$ is Lipschitz, there is a constant $c_3(\Omega, T)$ such that

$$\int_{\{d<\delta\}} |D\zeta|^2 \, dX \leq c_3/\delta,$$

and hence

$$\omega(\varepsilon) = \int_{\{|u|<\varepsilon\}} |Du|^2\zeta^2 \, dX$$

satisfies the inequality

$$2\omega(\varepsilon) - \omega(2\varepsilon) \leq c_1\varepsilon + 2\eta\omega(2\varepsilon) + \frac{c_3\varepsilon^2}{\delta\eta}.$$

Since $\delta \leq \sup|u|$ and $\varepsilon \leq \sup|u|$, it follows that

$$\omega(\varepsilon) \leq (\frac{1}{2} + \eta)\omega(2\varepsilon) + C(\eta)(c_1 + c_3)\frac{\sup|u|}{\delta}.$$

The proof is completed by taking η so small that

$$\tfrac{1}{2} + \eta \leq \left(\tfrac{1}{2}\right)^{(1+\alpha)/2}$$

and then iterating. □

The preceding two lemmata allow us to conclude a result on uniform integrability over sets on which $|u|$ is small, similar to Lemma 1.1.

Theorem 1.4. *Suppose u is a classical solution of* (0.1) *and that Ω and φ satisfy* (1.3). *Then for any $\eta \in (0,1/2)$, there is a positive constant C_0 determined only by K and η such that*

$$\int_{\{|u|<\varepsilon\}} |Du|^2 \, dX < C_0 \big(\frac{\varepsilon}{\sup|u|}\big)^\eta \big(\int_{\Omega\times(0,T)} |Du|^2 \, dX + (c_1+c_2+c_3)\sup|u|\big) \tag{1.9}$$

for all sufficiently small $\varepsilon > 0$.

Proof. First, choose $\alpha \in (0,1)$ so that $\eta = \alpha^2/(1+\alpha)$. Then set

$$K_1 = C(\alpha)\big(\int_{\Omega\times(0,T)} |Du|^2 \, dX + (c_1+c_2+c_3)\sup|u|\big),$$

and let $\delta \leq \sup|u|$ be a positive constant to be further determined. With $\overline{\delta} = \delta/\sup|u|$ and $\overline{\varepsilon} = \varepsilon/|\sup|u|$, Lemma 1.2 implies that

$$\int_{\{|u|<\varepsilon, d<\delta\}} |Du|^2 \, dX \leq K_1(3K\overline{\delta})^\alpha,$$

and Lemma 1.3 implies that

$$\int_{\{|u|<\varepsilon, d\geq\delta\}} |Du|^2 \, dX \leq K_1(\overline{\varepsilon})^\alpha/(3K\overline{\delta}).$$

Adding these two inequalities gives

$$\int_{\{|u|<\varepsilon\}} |Du|^2 \, dX \leq K_1 c(K)[\overline{\varepsilon}^\alpha/\overline{\delta} + \overline{\delta}^\alpha].$$

Now we take δ so that the two terms in square brackets are equal. In other words, $\overline{\delta}^{1+\alpha} = \overline{\varepsilon}^\alpha$, and hence,

$$\int_{\{|u|<\varepsilon\}} |Du|^2 \, dX \leq K_1 c(K)\overline{\varepsilon}^{\alpha-\alpha/(1+\alpha)}.$$

The proof is completed by noting that $\alpha - \alpha/(1+\alpha) = \eta$. $\square$

2. Strong convergence results. From Theorem 1.4, it is a simple matter to find sufficient conditions for our convergence results. As an immediate consequence, we have the following result when β is a power function.

Theorem 2.1. *Let Ω and φ satisfy* (1.3). *Suppose that* (u_i) *is a sequence of solutions to the problems* (0.1*i*) *with* $\beta_i(z) = (z^2 + i^{-2})^{p/2} \operatorname{sgn} z$ *for some positive constant p with (g_i) a sequence of continuous functions converging uniformly to some limit function g. Then there is a subsequence $(u_{i(m)})$ converging strongly in $L^2(0,T;W^{1,2}(\Omega))$ to the solution of* (0.1) *with* $\beta(z) = |z|^{p-1}z$.

Proof. First, we use the uniform continuity of the family (u_i) [6] to infer that there is a subsequence which converges uniformly to some limit function u. In addition, a simple calculation shows that we can extract a further subsequence $(u_{i(m)})$ so that $(Du_{i(m)})$ converges weakly to Du in $L^2(\Omega \times (0,T))$. To see that this convergence is actually strong, we first note that the constants C_0, c_1, c_2, and c_3 in Theorem 1.4 corresponding to u_i and β_i (and any fixed η) are independent of i. In addition, by taking $i(1)$ sufficiently large, we may assume that $\frac{1}{2}\sup|u| \le \sup|u_{i(m)}| \le 2\sup|u|$ for all m. In particular, choosing $\eta = 1/4$, it follows that there is a constant K_1, independent of i, such that

$$C_0(\sup|u_{i(m)}|)^{-1/4}\Big(\int_{\Omega\times(0,T)} |Du_{i(m)}|^2\, dX + (c_1+c_2+c_3)\sup|u_{i(m)}|\Big) \le K_1.$$

Now fix a positive δ and then let ε be so small that $K_1\varepsilon^{1/4} \le \delta/4$. Then there is a positive integer M such that $|u_{i(m)} - u| < \varepsilon/3$ for $m \ge M$. Hence, $\{|u| < 2\varepsilon/3\} \subset \{|u_{i(m)}| < \varepsilon\}$ and $\{|u| \ge 2\varepsilon/3\} \subset \{|u_{i(m)}| \ge \varepsilon/3\}$ for $m \ge M$. Then Theorem 1.4 gives

$$\int_{\{|u|<2\varepsilon/3\}} |Du_{i(m)}|^2\, dX \le \delta/4,$$

so the weak convergence of $Du_{i(m)}$ to Du gives

$$\int_{\{|u|<2\varepsilon/3\}} |Du|^2\, dX \le \delta/4$$

and hence,

$$\int_{\{|u|<2\varepsilon/3\}} |Du_{i(m)} - Du|^2\, dX \le \delta/2$$

for $m \ge M$. On the other hand, since $\beta'_{i(m)}(u_{i(m)})$ is uniformly continuous on $\Sigma(\delta) = \{|u| \ge 2\varepsilon/3\}$, the linear theory gives a uniform bound on $[Du_{i(m)}]_{\beta,\Sigma(\delta)}$. Hence, there is a positive integer N such that

$$\int_{\{|u|\ge 2\varepsilon/3\}} |Du_{i(m)} - Du|^2\, dX \le \delta/2$$

for $m \geq N$. Since δ is arbitrary, it follows that $(Du_{i(m)})$ converges strongly to Du in $L^2(\Omega \times (0,T))$. $\square$

Note that the only important properties of our approximation for β are that β' is nonzero, continuous, and finite except at $z = 0$ and that the sequence of approximating solutions (u_i) is equicontinuous. By taking full advantage of this information along with known estimates, it is possible to show the strong convergence when β is in a much more general class of functions.

To consider this more general class, we begin with an integral estimate for solutions of linear parabolic equations.

Lemma 2.2. *Let a be a uniformly positive, bounded function on $\Omega \times (0,T)$, let $h \in L^\infty(\Omega \times (0,T))$, and set $D = \Omega \times (0,T)$. Suppose that $u \in W_2^{2,1}$ is a strong solution of*

$$u_t = a\Delta u + h \text{ in } D \tag{2.1}$$

with $u(\cdot,0) = u_0$ on Ω for some $u_0 \in H_0^1(\Omega)$. Let
(a) If the restriction of u to $\partial\Omega \times (0,T)$ is independent of t, then

$$\int_D a(\Delta u)^2 \, dX \leq \int_D \frac{1}{a} h^2 \, dX + \int_\Omega |Du_0|^2 \, dx. \tag{2.2a}$$

(b) If $\eta \in C(\overline{D})$ with η_t and $D\eta$ in $L^\infty(D)$ and if $\eta = 0$ on $\partial\Omega \times (0,T)$, then

$$\begin{aligned}\int_D \eta^2 a(\Delta u)^2 \, dX \leq 10\Big(\int_D \frac{1}{a}\eta^2 h^2 dX + \int_D a(Du \cdot D\eta)^2 dX\Big) \\ + 2\Big(\int_\Omega \eta^2 |Du_0|^2 dx + \int_D \eta\eta_t |Du|^2 dX\Big).\end{aligned} \tag{2.2b}$$

Proof. Following the argument in Lemma 10 of [1], we multiply the differential equation by Δu and integrate by parts to see that

$$\begin{aligned}\int_D a(\Delta u)^2 + h\Delta u \, dX &= \int_D \Delta u u_t \, dX \\ &= -\int_D Du \cdot Du_t \, dX = -\frac{1}{2}\int_D \big(|Du|^2\big)_t \, dX \\ &= \frac{1}{2}\int_\Omega |Du_0|^2 \, dx - \frac{1}{2}\int_{\Omega \times \{T\}} |Du|^2 \, dx\end{aligned}$$

provided $Du_t \in L^2(D)$. A simple approximation argument shows that

$$\int_D a(\Delta u)^2 + h\Delta u \, dX = \frac{1}{2}\int_\Omega |Du_0|^2 \, dx - \frac{1}{2}\int_{\Omega \times \{T\}} |Du|^2 \, dx$$

under our hypotheses and (2.2a) follows immediately.

For part (b), we multiply the differential equation by $\eta^2\Delta u$ and find that

$$\int_D \eta^2 \Delta u u_t \, dX = \int_D a(\Delta u)^2 \eta^2 \; dX + \int_D \eta^2 h(\Delta u) \; dX.$$

On the other hand, integration by parts yields

$$\int_D \eta^2 \Delta u u_t dX = -2 \int_D \eta u_t Du \cdot D\eta \, dX + \int_D \eta\eta_t |Du|^2 dX - \tfrac{1}{2} \int_D \left(\eta^2 |Du|^2\right)_t dX.$$

Since

$$\int_D \left(\eta^2 |Du|^2\right)_t \; dX = \int_{\Omega\times\{T\}} \eta^2 |Du|^2 \; dx - \int_\Omega \eta^2 |Du_0|^2 \; dx,$$

we infer (2.2b) by combining these equalities and using Cauchy's inequality.

We are now ready to prove our general result on strong convergence.

Theorem 2.3. *Let Ω and φ satisfy (1.3), let β be a maximal monotone graph on $\mathbb{R}$ which is bounded on bounded subsets, and let (β_i) be a sequence of C^2, strictly increasing functions converging to β in the sense of maximal monotone graphs. Let (g_i) be a family of uniformly bounded functions which converge a.e to g, and let u_i be the solution of (0.1i). Suppose also that a uniform modulus of continuity is known for the family (u_i), and that there is a constant $\eta \in (0, 1/2)$ such that for any $\zeta > 0$, there are positive constants $\lambda(\zeta)$ and $\Lambda(\zeta)$ along with a finite family of intervals $(I_j^{(\zeta)})_{j=1}^J$ such that*

$$\sum_{j=1}^J l(I_j^{(\zeta)})^\eta < \zeta, \tag{2.3a}$$

and

$$\lambda(\zeta) \le \beta_i'(s) \le \Lambda(\zeta) \tag{2.3b}$$

for all $s \in \mathbb{R} \setminus \cup_{j=1}^J I_j^{(\zeta)}$ and all i. Suppose finally that β^{-1} is continuous. Then $u_i \to u$ uniformly for the solution u of (0.1), and $Du_i \to Du$ strongly in $L^2(\Omega \times (0,T))$.

Proof. As noted in Section 1, the sequence (u_i) is uniformly bounded and hypothesis (i) implies that this sequence is equicontinuous, and $(Du_{i(m)})$ is uniformly bounded in L^2. Hence there is a subsequence $(u_{i(m)})$ which converges uniformly to a limit function u with $(Du_{i(m)})$ converging weakly in L^2 to Du; hence u solves (0.1). Since the solution of (0.1) is unique (see [4, Lemma 1.3]), $u_i \to u$ uniformly.

Now let $\delta > 0$ be arbitrary and take M_1 so large that $i \geq M_1$ implies that $\sup |u_i| \geq \frac{1}{2} \sup |u|$. Then, we note that Theorem 1.4 can be applied to $u_i - r$ for any constant r. Thus there is a constant C_1 (independent of δ, r, and i) such that

$$\int_{\{|u_i - r| < \varepsilon\}} |Du_i|^2 \, dX \leq C_1 \varepsilon^\eta \tag{2.4}$$

for all sufficiently small $\varepsilon > 0$. Now let (I_j) be the finite collection of intervals corresponding to $\zeta = \delta/8C_1$, and write r_j for the center of I_j. Replacing r by r_j and ε by $2l(I_j)$ in (2.4), we see that

$$\int_{\{|u_i - r_j| < 2l(I_j)\}} |Du_i|^2 \, dX \leq 2^\eta C_1 l(I_j)^\eta.$$

Now choose M_2 so that $i \geq M_2$ implies $|u_i - u| < \frac{1}{2} \min_j l(I_j)$. As before, we have

$$\int_{\{|u - r_j| < (3/2) l(I_j)\}} |Du_i|^2 \, dX \leq C_1 [2l(I_j)]^\eta,$$

and hence, if $(3/2)I_j$ denotes the interval centered at r_j with length $\frac{3}{2} l(I_j)$, then

$$\int_{\{u \in \cup (3/2) I_j\}} |Du_i|^2 \, dX \leq 4^\eta C_1 \sum l(I_j)^\eta < \delta/4. \tag{2.5a}$$

In addition, we have

$$\int_{\{d < c(\delta)\}} |Du_i|^2 \, dX \leq \delta/4 \tag{2.5b}$$

for some positive constant $c(\delta)$. Now we write E for the subsets of $\{d \geq c(\delta)\}$ on which $u \notin \cup (3/2) I_j$ and E' for the subset of $\Omega \times (0,T)$ on which $u \notin \cup I_j$. Since $\overline{E}$ is compact, we can cover E by finitely many cylinders $Q_1, \ldots, Q_K$, such that $2Q_k \subset E'$ for each k. (Here if Q is the cylinder

$$Q = \{X : |x - x_0| < r, \ t_0 - r^2 < t < t_0\},$$

then

$$2Q = \{X : |x - x_0| < 2r, \ t_0 - 4r^2 < t < t_0\} \cap \Omega \times (0,T).)$$

From Lemma 2.2 along with the L^2 Schauder theory for elliptic equations and a suitable cutoff function η, we infer a uniform bound on $\|D^2 u_i\|_{2,Q_k}$. It follows from the Rellich compactness theorem (or by interpolation between the L^2 bound on D^2u_i and the L^2 convergence of u_i) that (Du_i) is Cauchy in $L^2(Q_k)$ and hence in $L^2(E)$. Therefore, we can find M_3 such that $i,k \geq M_3$ implies

$$\int_E |Du_i - Du_k|^2 \, dX < \delta/4.$$

Combining this estimate with (2.5a,b), we see that

$$\int_{\Omega\times(0,T)} |Du_i - Du_k|^2 < \delta.$$

provided $i,k \geq \max\{M_1, M_2, M_3\}$. Hence, (Du_i) is Cauchy in L^2, so $Du_i \to Du$ strongly in L^2. □

Note that, if we remove the hypothesis that β^{-1} is continuous, then we only conclude that some subsequence $(u_{i(k)})$ of (u_i) converges uniformly to u with $Du_{i(k)} \to Du$ in L^2.

We illustrate this theorem with some examples. The important element of these examples is the uniform modulus of continuity for u_i, and our examples are based on structure conditions which imply this equicontinuity.

Our first example comes from [6]. We define

$$S(r) = \begin{cases} 1 & \text{if } r > 0 \\ [-1,1] & \text{if } r = 0 \\ -1 & \text{if } r < 0, \end{cases}$$

and we suppose that $\beta = \nu S + b$ for some nonnegative constant ν and some locally absolutely continuous function b with $b(0) = 0$. We also suppose that b' is bounded away from zero and infinity uniformly on compact subsets of $\mathbb{R} \setminus \{0\}$. (In other words, if K is a compact subset of $\mathbb{R} \setminus \{0\}$, there is a positive constant $p(K)$ such that $1/p(K) < b' < p(K)$ a.e on K.) We now define S_i by

$$S_i(r) = \begin{cases} 1 & \text{if } r > 1/i \\ ir & \text{if } |r| \leq 1/i \\ -1 & \text{if } r < -1/i, \end{cases}$$

and then take β_i to be a mollification of $\nu S_i + b + 1/i$ with mollification parameter less than $1/i$. In other words, for some nonnegative function $\psi \in C^2(\mathbb{R})$ with support in the interval $[-1,1]$ and $\int_{\mathbb{R}} \psi(x)\, dx = 1$, we set

$$\beta_i(r) = \int [\nu S_i(r + s/i) + b(r + s/i) + 1/i]\psi(s)\, ds.$$

A uniform modulus of continuity for (u_i) then follows from Theorem 1.1 of [6]. The intervals $I_j^{(\zeta)}$ only need to contain the number 0, so the uniform upper and lower bounds on β_i' follow from the corresponding ones for β'.

As a second example, we suppose that β' is bounded from below by a positive constant and that $g \equiv 0$. The condition on β' means that there is a positive constant γ such that

$$\beta(s) - \beta(t) \geq \gamma[s - t] \tag{2.6}$$

whenever $s > t$. A uniform modulus of continuity for u follows from [2], and the modulus of continuity depends only on $\sup|u|$, $\sup|\beta(u)|$ and γ. Now we suppose that β' is unbounded on a sufficiently small set; specifically, we suppose that there is a constant $\eta \in (0, \frac{1}{2})$ such that, for any $\zeta > 0$, there is a finite collection of open intervals $(I_j^{(\zeta)})$ such that $|\beta'|$ is uniformly bounded off of the union of these intervals and

$$\sum l(I_j^{(\zeta)})^\eta < \zeta.$$

Taking β_i to be a mollification of β gives the desired result since condition (2.6) is preserved under mollification. (The condition $g \equiv 0$ appears in [2], but the proof there should give a modulus of continuity as long as $g \in L^\infty$.)

Note that in both examples, the condition involving the intervals $I_j^{(\zeta)}$ is similar to assuming that the set on which β' is infinite has Hausdorff dimension less than 1/2. It would be of interest to see if our condition can be relaxed to this set having Hausdorff dimension less than 1/2. The question of whether this hypothesis is needed is still open.

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