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REMARKS ON A NONHOMOGENEOUS MODEL OF MAGNETOHYDRODYNAMICS

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Abstract. This paper is devoted to a model of magnetohydrodynamics described by a parabolic system of partial differential equations coupling the nonhomogeneous incompressible Navier–Stokes equations and Maxwell's equations. In the case of two-dimensional flows, we prove global regularity results under the assumption that the fluids' viscosities are close enough to their average. On the other hand, a more detailed description of the interface and of the regularity of the third component of the magnetic field is given when the fluids have the same viscosity.

1. Introduction. The motivation of this work is a model of multiphase conductive flows arising in the study of production of aluminium in electrolytic cells. Roughly speaking, a cell is a container filled with two immiscible conductive fluids: an electrolyte in the upper part and liquid aluminium in the lower part. These fluids are subject to gravity and strong electromagnetic forces generating instabilities at the interface. Magnetohydrodynamics has been extensively studied during the last two decades, primarily from a numerical point of view. Amongst other references, we refer the reader for instance to the textbook by Moreau ([14]) for an introduction, to LaCamera, Ziegler, and Kozarek ([12]) for an overview of the numerical simulations, and more specifically to Descloux, Frosio, and Flück ([6]) for the stationary free-boundary problems, to Bermùdez, Muñiz, and Quintela ([3]) for the modelling of the solidification process, to Descloux, Flück, and Romerio ([5]) and references therein for an example of a simulation of a time-dependent flow. On the other hand, some theoretical studies were also devoted to the MHD problem in this context; see the bibliography

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(mostly dealing with the case of *one* fluid) in Gerbeau, and Le Bris ([11]). Less numerous works address the difficult two-fluid problems. We mention for instance the recent work by Solonnikov ([19]) who requires the presence of surface tension to obtain existence results for the stationary problem. However, in the context of electrolytic cells, physical as well as numerical experiments show that capillary effects should be neglected. In that case, P.L. Lions ([13]) and Gerbeau and Le Bris ([11]) proved global existence of weak solutions for a system of MHD equations for which we want to obtain here global regularity properties. Let us mention that long-time existence results for the corresponding time-dependent problem with surface tension is still open, unless initial data are close to an equilibrium (see Solonnikov [18], [19] and Tanaka [20]).

Let Ω be a compact open subset of $\mathbb{R}^3$ and consider M immiscible incompressible fluids characterized by their densities $\{\rho_i\}_{1\leq i\leq M}$, $\rho_i > 0$, their viscosities $\{\mu_i\}_{1\leq i\leq M}$, $\mu_i = \mu(\rho_i) > 0$, and their electrical conductivities $\{\sigma_i\}_{1\leq i\leq M}$, $\sigma_i = \sigma(\rho_i) > 0$. The i^{th} fluid is supposed to fill a time-dependent subdomain $\Omega_i(t)$ ($i \in \mathbb{N}_M$) of Ω. The global density $\rho(t,.)$ at time $t \geq 0$ is defined as the piecewise-constant function which takes the values ρ_i on $\Omega_i(t)$. The nonhomogeneous MHD model we consider in the sequel can be written as follows:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho)d) + \nabla(p + \frac{B^2}{2}) = \rho f + \operatorname{div}(B \otimes B), \\ \partial_t B + \operatorname{curl}(\frac{\operatorname{curl} B}{\sigma(\rho)}) = \operatorname{curl}(u \times B), \\ \operatorname{div}\ u = 0, \quad \operatorname{div}\ B = 0, \quad d_{i,j} = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \end{cases} \tag{1}$$

with initial data

$$\begin{cases} \rho_{|t=0} = \rho_0, \text{ defined by a given initial partition } \{\Omega_i(0)\}_{1\leq i\leq M} \text{ of } \Omega, \\ \rho u_{|t=0} = m_0,\ B_{|t=0} = B_0, \end{cases} \tag{2}$$

and suitable boundary conditions on $\partial\Omega$ that we shall described in the next section. The physical system is described in terms of the density ρ, the velocity field u, the pressure p and the magnetic field B. The source term f denotes volumic external forces, for instance gravity. Global existence of weak solutions was first proven by P.-L. Lions ([13]) for the nonhomogeneous incompressible Navier–Stokes equations without magnetic field. He used in [13] compactness results on linear transport equations based on the method of renormalized solutions introduced by R.J. Di Perna and P.-L. Lions in [10]. J.-F. Gerbeau and C. Le Bris ([11]) recently proved a global existence theorem for the above MHD system. They obtained weak solutions (ρ, u, p, B)

such that for all $T > 0$, $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, $B \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, and $\rho \in L^\infty((0,T)\times\Omega) \cap C([0,T];L^q(\Omega))$ for all $q \in [1,\infty)$. In this article, we want to deal with the case $\Omega = \Omega_0 \times \mathbb{R}$ or $\Omega = \Omega_0 \times \mathbb{S}^1$, where Ω_0 is a bounded open domain of $\mathbb{R}^2 = \mathbb{R}e_1 \oplus \mathbb{R}e_2$, assuming that the physical system is invariant under translations in a given direction e_3. Here, (e_1, e_2, e_3) denotes an orthonormal basis of $\mathbb{R}^3$. Adapting arguments we gave in a previous work ([9]), we prove in Sections 3 and 4 that weak solutions of the corresponding two-dimensional system satisfy additional regularity properties as soon as the viscosities satisfy the following condition: for all i such that $1 \le i \le M$, one has

$$|\mu_i - \bar{\mu}| < \eta_0 \bar{\mu}, \text{ where } \bar{\mu} = \frac{1}{M}\sum_{i=1}^{M} \mu_i, \tag{3}$$

for some small enough η_0 which will be determined later. Let us emphasize that the only assumption on σ is

$$\sigma \in C^0(\mathbb{R}_+) \text{ and } 0 < \underline{\sigma} \le \sigma(.) \le \bar{\sigma} < +\infty, \tag{4}$$

for some given positive $\underline{\sigma}$ and $\bar{\sigma}$.

In Section 5, we deal with the case of constant viscosity μ. Using the results of Antontsev, Kazhikov, and Monakhov ([2]) on the regularity of the velocity and pressure field, we obtain regularity properties for the interface. A refined smoothness property on the third component of the magnetic field is also given whenever σ is piecewise constant and satisfies a condition similar to (3) for some small-enough κ_0:

$$\Big|\frac{1}{\sigma_i} - \frac{1}{\overline{\overline{\sigma}}}\Big| < \frac{\kappa_0}{\overline{\overline{\sigma}}}, \quad \text{where } \frac{1}{\overline{\overline{\sigma}}} = \frac{1}{M}\sum_{i=1}^{M}\frac{1}{\sigma_i}. \tag{5}$$

2. Description of the model. The physical model we consider here is based upon the nonhomogeneous Navier–Stokes equations coupled with Maxwell's equations. For a detailed derivation of the model, we refer to [11] and [14]. First of all, Maxwell Ampere's equations are expressed by

$$\operatorname{curl} B = \mu_0 j + \varepsilon_0 \mu_0 \partial_t E, \tag{6}$$

where B denotes the magnetic field, j the current density field, and E the electric field. Maxwell Faraday's equation can be written as

$$\partial_t B + \operatorname{curl} E = 0, \tag{7}$$

and Ohm's law as

$$j = \sigma(\rho)(E + u \times B), \tag{8}$$

denoting $\sigma(\rho_i) = \sigma_i$ the conditivity of the i^{th} fluid. Here, σ is a continuous function of ρ such that for some $\underline{\sigma} > 0$ and $\bar{\sigma} > \underline{\sigma}$,

$$0 < \underline{\sigma} \leq \sigma(.) \leq \bar{\sigma} < +\infty. \tag{9}$$

In the sequel, we shall assume as J.-F. Gerbeau and C. Le Bris ([11]) that the fluids in presence are conductive enough, so that the displacement currents $\varepsilon_0 \partial_t E$ can be neglected. For the sake of brevity, we take $\mu_0 = 1$ and rewrite (6) as follows:

$$\operatorname{curl} B = j. \tag{10}$$

From the additional magnetic equation div $B = 0$, we deduce that

$$\partial_t B + \operatorname{curl}\Big(\frac{\operatorname{curl} B}{\sigma(\rho)}\Big) = \operatorname{curl}(u \times B) = (B.\nabla)u - (u.\nabla)B. \tag{11}$$

Let us now recall the fluid's equations. The equation of mass conservation first reads as

$$\partial_t \rho + \operatorname{div}\ (\rho u) = 0. \tag{12}$$

Next, the equations of conservation of momentum including volumic external forces $f \in L^2((0,T) \times \Omega)^3$, for instance gravity, and Lorentz forces $j \times B$ are expressed by

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}\ (\rho u \otimes u) - \operatorname{div}\ (2\mu(\rho)d) + \nabla p - \rho f &= j \times B \\ &= -\nabla\Big(\frac{B^2}{2}\Big) + \operatorname{div}\ (B \otimes B), \end{aligned} \tag{13}$$

d denoting the strain tensor associated to u. The viscosity μ is supposed to be a continuous function of ρ such that $\mu_i = \mu(\rho_i)$ and

$$0 < \underline{\mu} \leq \mu(.) \leq \bar{\mu} < +\infty. \tag{14}$$

Finally, we require the incompressibility condition div $u = 0$. Let us now assume that the initial data lie in energy spaces

$$\rho_{|t=0} = \rho_0 \in L^\infty(\Omega), \rho u_{|t=0} = m_0 \in L^2(\Omega)^3, \text{ and } B_{|t=0} = B_0 \in L^2(\Omega)^3 \tag{15}$$

and set up the following natural boundary conditions:

$$u = 0, \quad B.n = 0, \text{ and } E \times n = 0 \text{ in } \mathcal{D}'((0,T) \times \partial\Omega). \tag{16}$$

Thus, we obtain the MHD model (1, 2, 16) for which J.-F. Gerbeau and C. Le Bris proved in [11] global existence of weak solutions:

Theorem 1. ([11]) *There exists a global weak solution* (ρ, u, p, B) *of* (1,2,16) *such that for all time* $T > 0$, ρ, $\sigma(\rho)$, *and* $\mu(\rho)$ *belong to* $L^\infty((0,T)\times\Omega)\cap C([0,T];L^q(\Omega))$ *for all* $q\in[1,\infty)$, $u\in L^\infty(0,T;L^2(\Omega)^3)\cap L^2(0,T;H_0^1(\Omega)^3)$ *and* $B\in L^\infty(0,T;L^2(\Omega)^3)\cap L^2(0,T;H^1(\Omega)^3)$. *Moreover, for all* $(\alpha,\beta)\in\mathbb{R}^2$ *such that* $0\le\alpha\le\beta$,

$$\textit{meas}\ \{\ x\in\Omega,\ \textit{such that}\ \alpha\le\rho(t,x)\le\beta\}\ \textit{is independent of}\ t\ge 0. \tag{17}$$

P. Secchi recently proved well posedness results for small time for an analogous nonhomogeneous MHD model corresponding to immiscible fluids in which the effects of viscosity and electrical resistivity can be neglected. The system of partial differential equations considered by P. Secchi in [15], [16], and [17] is obtained from (1) by dropping the diffusive terms $\mathrm{div}\,(2\mu(\rho)d)$ and $\mathrm{curl}\,((\mathrm{curl}\,B)/\sigma(\rho))$. Hereafter, we are interested in the model described by (1, 2, 16) under some symmetry properties. Let (e_1,e_2,e_3) be an orthonormal basis of $\mathbb{R}^3$ and assume that $\Omega=\{x=x_1e_1+x_2e_2+x_3e_3,\ \text{ such that }\ x_1e_1+x_2e_2\in\Omega_0\ \text{ and }\ x_3\in\mathbb{R}\}=\Omega_0\times\mathbb{R}$ or similarly $\Omega=\Omega_0\times\mathbb{S}^1$ for some bounded open domain $\Omega_0\subset\mathbb{R}^2$. We suppose in addition that the physical system is invariant under translations in the e_3 direction. In other words, the variables (ρ,u,p,B,E) and the initial data (ρ_0,m_0,B_0) are independent of x_3. For any $g\in\mathbb{R}^3$, we shall denote $g^0=(g|e_1)e_1+(g|e_2)e_2$ and $g_3=(g|e_3)$, and use the differential operators ∇_0, $\nabla_0^\perp$ and curl_0 defined by

$$\nabla_0=\Big\|{}^{\partial_1}_{\partial_2},\ \nabla_0^\perp=\Big\|{}^{-\partial_2}_{\partial_1},\ \text{and}\ \mathrm{curl}_0\,g^0=\nabla_0^\perp.g^0. \tag{18}$$

Thus, (16) can be rewritten as

$$\mathrm{curl}_0\ B^0=0,\quad B^0.n=0,\ \text{and}\ \frac{\partial B_3}{\partial n}=0\ \text{in}\ \mathcal{D}'((0,T)\times\partial\Omega_0), \tag{19}$$

and (1) becomes

$$\begin{cases}\partial_t\rho+\mathrm{div}_0\ (\rho u^0)=0,\ \mathrm{div}_0\ u^0=0\ \text{and}\ \mathrm{div}_0\ B^0=0,\\ \partial_t(\rho u^0)+\mathrm{div}_0\ (\rho u^0\otimes u^0-2\mu(\rho)d^0-B^0\otimes B^0)\\ \qquad\qquad+\nabla_0\big(p+\frac{B^{0\,2}+B_3^2}{2}\big)=\rho f^0,\\ \partial_tB^0+(u^0.\nabla_0)B^0-\nabla_0^\perp\big(\frac{\mathrm{curl}_0\ B^0}{\sigma(\rho)}\big)=(B^0.\nabla_0)u^0,\end{cases} \tag{20}$$

$$\begin{cases}\partial_t(\rho u_3)+\mathrm{div}_0\ (\rho u^0u_3)-\mathrm{div}_0\ (\mu(\rho)\nabla_0u_3)=\rho f_3+(B^0.\nabla_0)B_3,\\ \partial_tB_3+(u^0.\nabla_0)B_3-\mathrm{div}_0\ \big(\frac{\nabla_0B_3}{\sigma(\rho)}\big)=(B^0.\nabla_0)u_3.\end{cases} \tag{21}$$

Let us observe that global existence of weak solutions $(\rho, u^0, u_3, p, B^0, B_3)$ corresponding to initial data $(\rho_0, u_0^0, u_{3,0}, B_0^0, B_{3,0})$ can easily be proven using the same method as J.-F. Gerbeau and C. Le Bris ([11]). For the sake of simplicity, we will restrict ourselves in the following sections to periodic boundary conditions in (x_1, x_2) and explain later how to adapt the proofs to the case of a bounded domain Ω_0 in $\mathbb{R}^2$ with (16) as boundary conditions (see Remark 4). Such a weak solution has the regularity stated in Theorem 1; namely, ρ, $\sigma(\rho)$ and $\mu(\rho)$ belong to $L^\infty((0,T)\times\mathbb{T}^2)\cap C([0,T];L^q(\mathbb{T}^2))$ for all $q < +\infty$, and u and B belong to $L^\infty(0,T;L^2(\mathbb{T}^2)^2)\cap L^2(0,T;H^1(\mathbb{T}^2)^2)$.

3. Regularity theorem. In the sequel, we replace u^0, B^0, ∇_0 and curl_0 by u, B, ∇ and curl, and keep u_3 and B_3 in order to simplify the notation. We consider initial data satisfying

$$\begin{cases} \rho_{|t=0} = \rho_0 \in L^\infty(\mathbb{T}^2), & \\ u_{|t=0} = u_0 \in H^1(\mathbb{T}^2)^2, & u_{3\ |t=0} = u_{3,0} \in L^2(\mathbb{T}^2), \\ B_{|t=0} = B_0 \in H^1(\mathbb{T}^2)^2, & B_{3\ |t=0} = B_{3,0} \in L^2(\mathbb{T}^2), \end{cases} \tag{22}$$

and external forces $f \in L^2((0,T)\times\mathbb{T}^2)^3$ for all $T > 0$. This section is devoted to the following two-dimensional MHD system in $(0,T)\times\mathbb{T}^2$:

$$\begin{cases} \partial_t\rho + \text{div}\ (\rho u) = 0, \quad \text{div}\ u = 0 \text{ and div } B = 0, \\ \partial_t(\rho u) + \text{div}\ (\rho u\otimes u) - \text{div}\ (2\mu(\rho)d) + \nabla\big(p + \frac{B^2+B_3^2}{2}\big) \\ \qquad\qquad = \rho f + \text{div}\ (B\otimes B), \\ \partial_t B + (u.\nabla)B - \nabla^\perp\big(\frac{\text{curl}\, B}{\sigma(\rho)}\big) = (B.\nabla)u, \end{cases} \tag{23}$$

$$\begin{cases} \partial_t(\rho u_3) + \text{div}\ (\rho u u_3) - \text{div}\ (\mu(\rho)\nabla u_3) = \rho f_3 + (B.\nabla)B_3, \\ \partial_t B_3 + (u.\nabla)B_3 - \text{div}\big(\frac{\nabla B_3}{\sigma(\rho)}\big) = (B.\nabla)u_3. \end{cases} \tag{24}$$

Under the hypothesis (3) on the viscosities which was soon formulated in a previous work ([9]) for nonconductive fluids, we want here to prove

Theorem 2. *There exists $\eta_0 > 0$ such that if the viscosities $\{\mu_i\}_{1\le i\le M}$ satisfy*

$$|\mu_i - \bar\mu| < \eta_0\bar\mu \text{ where } \bar\mu = \frac{1}{M}\sum_{i=1}^M \mu_i, \tag{25}$$

then, for all $T > 0$,

$$\begin{aligned} &\partial_t u,\ \partial_t B,\ \text{and } \nabla\big(\frac{\text{curl}\, B}{\sigma(\rho)}\big) \in L^2((0,T)\times\mathbb{T}^2), \\ &\nabla u,\ \nabla B \in L^\infty(0,T;L^2(\mathbb{T}^2)) \\ &\nabla(p - R_iR_j(2\mu d_{ij})) \text{ and } \nabla((P\otimes Q)(2\mu d))_{kl} \in L^2((0,T)\times\mathbb{T}^2). \end{aligned} \tag{26}$$

Moreover, the pressure field p may be renormalized in such a way that

$$p \text{ and } \nabla u \in L^2(0,T;L^q(\mathbb{T}^2)) \text{ for all } q < q^*, \tag{27}$$

where $q^* > 4$ *is defined by*

$$\frac{1}{q^*} = \frac{1}{4\eta_0} \inf_{\ c>0} \ \big|\frac{\mu(\rho_0)}{c} - 1\big|_{L^\infty(\mathbb{T}^2)}. \tag{28}$$

As usual, R_i denotes the Riesz transform $\partial_i \Delta^{-1/2}$ on $\mathbb{T}^2$, where Δ^{-1} is the inverse Laplacian on $\mathbb{T}^2$ with zero mean value. P is defined as the projection on the space of divergence-free vector fields and $Q = I - P = R(R.)$ the projection on the space of gradients. As an easy consequence of Theorem 2, we have

Corollary 1. *For all* $T > 0$*, there exists* $\alpha \in L^2(0,T)$ *such that*

$$\exp\big\{\big(\frac{|\nabla B|}{\alpha}\big)^{\frac{2}{3}}\big\} \in L^\infty(0,T;L^1(\mathbb{T}^2)), \tag{29}$$

for all $(x,y) \in \mathbb{T}^2 \times \mathbb{T}^2$*,*

$$|B(t,x) - B(t,y)| \leq C\alpha(t)|x-y|\big\{ -\log\big(\frac{1}{2} \wedge |x-y|\big)\big\}^{\frac{3}{2}}. \tag{30}$$

Remark 1. In the case when the fluids have the same constant viscosity $\mu > 0$, Kazhikhov and Antontsev proved in [2] for the nonhomogeneous $2D$ Navier–Stokes flow that the velocity field satisfies under the same assumptions as in Theorem 2

$$u \in L^2(0,T;H^2(\mathbb{T}^2)^2) \cap L^\infty(0,T;H^1(\mathbb{T}^2)^2), \partial_t u \text{ and } \nabla p \in L^2((0,T)\times\mathbb{T}^2)^2, \tag{31}$$

$$u_3 \in L^2(0,T;H^2(\mathbb{T}^2)) \cap L^\infty(0,T;H^1(\mathbb{T}^2)), \partial_t u_3 \in L^2((0,T)\times\mathbb{T}^2). \tag{32}$$

It turns out that a similar result can be proven for the above two-dimensional MHD system. More precisely, u and p satisfy (31). Furthermore, in the more restrictive case when the two fluids have the same electrical conductivity $\sigma > 0$, the arguments of [2] can be adapted in order to show that B satisfies for all $T > 0$

$$B \in L^\infty(0,T;H^1(\mathbb{T}^2)^2) \cap L^2(0,T;H^2(\mathbb{T}^2)^2), \text{ and } \partial_t B \in L^2((0,T)\times\mathbb{T}^2)^2, \tag{33}$$

$$B_3 \in L^\infty(0,T;H^1(\mathbb{T}^2)) \cap L^2(0,T;H^2(\mathbb{T}^2)), \text{ and } \partial_t B_3 \in L^2((0,T)\times\mathbb{T}^2). \tag{34}$$

However, the assumption of a constant σ does not hold in real situations for which one of the two fluids is much more conductive than the other (consider for instance liquid aluminium compared with a solution containing dissolved alumina). In Section 5, we attempt to track more physical cases dealing with nonconstant σ. However, we have to assume that the conductivities are not too far from each other, which might not be the case in the applications (see Remark 9).

Proof of Theorem 2. Multiplying the equation of momentum conservation by $\partial_t u$, we obtain as in [9] for all $t \in [0,T]$

$$\begin{aligned} \int_0^t |\sqrt{\rho}\partial_t u|^2_{L^2(\mathbb{T}^2)}\, ds + \int_{\mathbb{T}^2} |\nabla u(t,x)|^2\, dx \le C \int_{\mathbb{T}^2} |\nabla u_0(x)|^2\, dx \\ + C\Big|\int_0^t \int_{\mathbb{T}^2} \big(\rho f + B.\nabla B - \rho u.\nabla u\big).\partial_t u\, dx\, ds\Big| \\ + \Big|\int_0^t \int_{\mathbb{T}^2} (u.\nabla)u.\mathrm{div}\,(2\mu(\rho)d)\, dx\, ds\Big| + \int_0^t |\nabla u|_{L^2(\mathbb{T}^2)} |\nabla u|^2_{L^4(\mathbb{T}^2)}\, ds. \end{aligned} \tag{35}$$

Using the fact that

$$\int_0^t \int_{\mathbb{T}^2} \partial_t \mu(\rho)\, d : d\, dx\, ds = \int_0^t \big(\mu(\rho)\partial_k u_i \partial_i u_j d_{kj} + (u.\nabla)u.\mathrm{div}\,(2\mu(\rho)d)\big)\, dx\, ds.$$

We deduce that

$$\begin{aligned} \int_0^t (|\sqrt{\rho}u|^2_{L^2(\mathbb{T}^2)} + |\sqrt{\rho}u.\nabla u|^2_{L^2(\mathbb{T}^2)})\, ds + |\nabla u(t,.)|^2_{L^2(\mathbb{T}^2)} \le C \\ + C\int_0^t (|B.\nabla B|^2_{L^2(\mathbb{T}^2)} + |\sqrt{\rho} f|^2_{L^2(\mathbb{T}^2)} + |\sqrt{\rho}u.\nabla u|^2_{L^2(\mathbb{T}^2)})\, ds \\ + \Big|\int_0^t \int_{\mathbb{T}^2} p\partial_i u_j \partial_j u_i\, dx\, ds\Big|. \end{aligned} \tag{36}$$

Let us now recall Gagliardo–Nirenberg's inequality valid for all $q \in [2,\infty)$

$$\begin{aligned} &\text{for all } f \in H^1(\mathbb{T}^2) \text{ such that } \int_{\mathbb{T}^2} f\ dx = 0, \\ &|f|_{L^q(\mathbb{T}^2)} \le C\sqrt{q}|f|^{\frac{2}{q}}_{L^2(\mathbb{T}^2)} |\nabla f|^{1-\frac{2}{q}}_{L^2(\mathbb{T}^2)}. \end{aligned} \tag{37}$$

Following [9], we write

$$-c\Delta u = \mathrm{div}\,(2(\mu(\rho) - c)\, d) - \mathrm{div}\,(2\mu(\rho)\, d); \tag{38}$$

hence, applying the projection operator P, we obtain for all $q \in [2, \infty)$

$$\begin{aligned} |\nabla u|_{L^q(\mathbb{T}^2)} \le 2C_0\ q\ |d|_{L^q(\mathbb{T}^2)}\ \inf_{c>0}\Big|\frac{\mu(\rho_0)}{c} - 1\Big|_{L^\infty(\mathbb{T}^2)} \\ +\ C\sqrt{q}|(P\otimes Q)(2\mu(\rho)d)|^{\frac{2}{q}}_{L^2(\mathbb{T}^2)}|P\mathrm{div}\,(2\mu(\rho)d)|^{1-\frac{2}{q}}_{L^2(\mathbb{T}^2)}, \end{aligned} \tag{39}$$

where C_0 is the smallest positive constant C such that for all $q \in [2, \infty)$ and for all M in $C^\infty(\mathbb{T}^2; \mathcal{M}_{2,2}(\mathbb{R}^2))$,

$$|(P\otimes Q)M|_{L^q(\mathbb{T}^2)} \le Cq|M|_{L^q(\mathbb{T}^2)}. \tag{40}$$

Choosing $\eta_0 = 1/(8C_0)$ in (3), we obtain for all $q \in [4, q^*)$ using the equation of conservation of momentum

$$|\nabla u|_{L^q(\mathbb{T}^2)} \le C\frac{q^*}{q^*-q}\sqrt{q}|d|^{\frac{2}{q}}_{L^2(\mathbb{T}^2)}|P(\rho\partial_t u+\rho u.\nabla u-\rho f-B.\nabla B)|^{1-\frac{2}{q}}_{L^2(\mathbb{T}^2)}, \tag{41}$$

q^* being defined by

$$\frac{1}{q^*} = 2C_0\inf_{c>0}\Big|\frac{\mu(\rho_0)}{c} - 1\Big|_{L^\infty(\mathbb{T}^2)}. \tag{42}$$

On the other hand, similar computations yield

$$|p - R.R.(2\mu(\rho)d)|_{BMO(\mathbb{T}^2)} \le |\nabla(p - R.R.(2\mu(\rho)d))|_{L^2(\mathbb{T}^2)} \tag{43}$$

$$\le |\rho\partial_t u + \rho u.\nabla u - \rho f - B.\nabla B|_{L^2(\mathbb{T}^2)}. \tag{44}$$

Using Gagliardo–Nirenberg's inequality with $q = 4$, we obtain

$$\begin{aligned} &|(u.\nabla)u|^2_{L^2(\mathbb{T}^2)} \le |u|^2_{L^4(\mathbb{T}^2)}|\nabla u|^2_{L^4(\mathbb{T}^2)} \\ &\le C|u|_{L^2(\mathbb{T}^2)}|\nabla u|_{L^2(\mathbb{T}^2)}|\nabla u|^2_{L^4(\mathbb{T}^2)}, \le C|\nabla u|_{L^2(\mathbb{T}^2)}|\nabla u|^2_{L^4(\mathbb{T}^2)}, \end{aligned} \tag{45}$$

observing that u belongs to $L^\infty(0, T; L^2(\mathbb{T}^2))$, since ρ is bounded from below by $\alpha = \inf_i \rho_i$. Denoting by $\mathcal{H}^1(\mathbb{T}^2)$ the usual Hardy space in $\mathbb{T}^2$ (see [4]), we deduce that

$$\begin{aligned} &\int_0^t |u.\nabla u|^2_{L^2(\mathbb{T}^2)}\,ds + \Big|\int_0^t\int_{\mathbb{T}^2} p\partial_i u_j\partial_j u_i\,dx\,ds\Big| \\ &\le C\int_0^t |p - R.R.(2\mu(\rho)d)|_{BMO(\mathbb{T}^2)}|\partial_i u_j\partial_j u_i|_{\mathcal{H}^1(\mathbb{T}^2)}\,ds \\ &+ C\int_0^t |\nabla u|_{L^2(\mathbb{T}^2)}|\nabla u|^2_{L^4(\mathbb{T}^2)}\,ds \qquad (46) \\ &\le C\int_0^t |\nabla u|^2_{L^2(\mathbb{T}^2)}|\rho\partial_t u + \rho u.\nabla u - \rho f - B.\nabla B|_{L^2(\mathbb{T}^2)}\,ds, \qquad (47) \end{aligned}$$

taking $q = 4$ in (41). Next, we treat similarly the vector field B and its derivatives by multiplying the corresponding equation in (24) by $\partial_t B$ and integrating by parts:

$$\int_0^t \int_{\mathbb{T}^2} |\partial_t B|^2 \, dx \, ds + \int_{\mathbb{T}^2} \frac{1}{\sigma(\rho)} |\operatorname{curl} B(t,x)|^2 \, dx \\ \leq C \int_{\mathbb{T}^2} |\operatorname{curl} B_0(x)|^2 \, dx + C \int_0^t \big(|B.\nabla u|^2_{L^2(\mathbb{T}^2)} + |u.\nabla B|^2_{L^2(\mathbb{T}^2)} \big) \, ds \\ + \Big| \int_0^t \int_{\mathbb{T}^2} u^\perp \operatorname{curl} B \nabla^\perp \Big(\frac{\operatorname{curl} B}{\sigma(\rho)} \Big) \, dx \, ds \Big|. \tag{48}$$

As a matter of fact, we have

$$-\frac{1}{2} \int_{\mathbb{T}^2} |\operatorname{curl} B|^2 \partial_t \Big(\frac{1}{\sigma(\rho)} \Big) \, dx = \frac{1}{2} \int_{\mathbb{T}^2} \Big| \frac{\operatorname{curl} B}{\sigma(\rho)} \Big|^2 \partial_t (\sigma(\rho)) \, dx \tag{49}$$

$$= \int_{\mathbb{T}^2} (\operatorname{curl} B)(u.\nabla) \Big(\frac{\operatorname{curl} B}{\sigma(\rho)} \Big) \, dx. \tag{50}$$

Hence we obtain

$$\int_0^t |\partial_t B|^2_{L^2(\mathbb{T}^2)} \, ds + |\operatorname{curl} B(t,.)|^2_{L^2(\mathbb{T}^2)} \\ \leq C + C \int_0^t \big(|\nabla B|_{L^2(\mathbb{T}^2)} |\nabla u|^2_{L^4(\mathbb{T}^2)} + |\nabla u|_{L^2(\mathbb{T}^2)} |\nabla B|^2_{L^4(\mathbb{T}^2)} \big) \, ds. \tag{51}$$

Moreover, considering once more the equation on B, we obtain the following estimate:

$$\Big| \nabla \Big(\frac{\operatorname{curl} B}{\sigma(\rho)} \Big) \Big|_{L^2(\mathbb{T}^2)} \leq C \big(|\partial_t B|_{L^2(\mathbb{T}^2)} + |u.\nabla B|_{L^2(\mathbb{T}^2)} + |B.\nabla u|_{L^2(\mathbb{T}^2)} \big). \tag{52}$$

Using the fact that $\operatorname{div} B = 0$, Gagliardo–Nirenberg's inequality leads to

$$|\nabla B|^2_{L^4(\mathbb{T}^2)} \leq C \Big| \frac{\operatorname{curl} B}{\sigma(\rho)} \Big|^2_{L^4(\mathbb{T}^2)} \tag{53}$$

$$\leq C \Big| \frac{\operatorname{curl} B}{\sigma(\rho)} \Big|_{L^2(\mathbb{T}^2)} \Big| \nabla \Big(\frac{\operatorname{curl} B}{\sigma(\rho)} \Big) \Big|_{L^2(\mathbb{T}^2)}. \tag{54}$$

We deduce that

$$\int_0^t \big(|\partial_t B|^2_{L^2(\mathbb{T}^2)} + |B.\nabla B|^2_{L^2(\mathbb{T}^2)} + \big| |u| |\nabla B| \big|^2_{L^2(\mathbb{T}^2)} + |B.\nabla u|^2_{L^2(\mathbb{T}^2)} \big) \, ds \\ + |\nabla B(t,.)|^2_{L^2(\mathbb{T}^2)} \leq C + C \int_0^t |\partial_t B + u.\nabla B - B.\nabla u|^2_{L^2(\mathbb{T}^2)} \\ \times \big(|\nabla u|_{L^2(\mathbb{T}^2)} |\nabla B|_{L^2(\mathbb{T}^2)} + |\nabla B|^2_{L^2(\mathbb{T}^2)} \big) \, ds. \tag{55}$$

Finally, inserting (47) into (55), we conclude that

$$\int_0^t \big(|\partial_t B|^2_{L^2(\mathbb{T}^2)} + |\partial_t u|^2_{L^2(\mathbb{T}^2)} + \big|\nabla\big(\frac{\operatorname{curl} B}{\sigma(\rho)}\big)\big|^2_{L^2(\mathbb{T}^2)}\big)\, ds + |\nabla u(t,.)|^2_{L^2(\mathbb{T}^2)}$$
$$+ |\nabla B(t,.)|^2_{L^2(\mathbb{T}^2)} \le C + C\int_0^t \big(|\nabla u|^4_{L^2(\mathbb{T}^2)} + |\nabla B|^4_{L^2(\mathbb{T}^2)}\big)\, ds. \quad (56)$$

Hence, Gronwall's lemma completes the proof, since

$$\nabla u \text{ and } \nabla B \in L^2(0,\infty; L^2(\mathbb{T}^2)). \quad (57)$$

In the proof of the above theorem, we only obtain a priori bounds assuming that the unknown functions ρ, u, p, B are C^∞. However, the above arguments can be made rigorous by mollifying the velocity u in the convection terms and passing to the limit, checking that the corresponding bounds are uniform.

4. Consequences on the integrability of ∇B. This section is devoted to the proof of Corollary 1. Let us first recall Trudinger's theorem:

There exists $\delta_0 > 0$ such that for all functions $f \in H^1(\mathbb{T}^2)$ satisfying

$$\int_{\mathbb{T}^2} f\, dx = 0, \text{ we have } \int_{\mathbb{T}^2} \exp\big(\frac{\delta_0 |f|^2}{|\nabla f|^2_{L^2(\mathbb{T}^2)}}\big)\, dx \le \frac{1}{\delta_0}. \quad (58)$$

For fixed $T > 0$, the integrability of $(\operatorname{curl} B)/\sigma(\rho)$ yields the existence of $\beta \in L^2(0,T)$ such that for all $p \in \mathbb{N}$

$$|\operatorname{curl} B(t,.)|_{L^{2p}(\mathbb{T}^2)} \le \beta(t)(p!)^{\frac{1}{2p}}, \quad (59)$$
$$\le C\beta(t)\sqrt{p}. \quad (60)$$

Using the magnetic equation div $B = 0$, a classical theorem about homogeneous Fourier multipliers on L^p provides

$$|\nabla B(t,.)|_{L^{2p}(\mathbb{T}^2)} \le Cp|\operatorname{curl} B(t,.)|_{L^{2p}(\mathbb{T}^2)}; \quad (61)$$

hence, we have

$$|\nabla B(t,.)|_{L^{2p}(\mathbb{T}^2)} \le C\beta(t)p^{\frac{3}{2}}. \quad (62)$$

It is now easy to extend the above estimate for p noninteger in $[2/3,\infty)$, so that we obtain for all $q \in \mathbb{N}$

$$|\nabla B(t,.)|^{\frac{2}{3}q}_{L^{\frac{2}{3}q}(\mathbb{T}^2)} \le C^q \beta(t)^{\frac{2}{3}q} q^q. \quad (63)$$

Choosing $\alpha(t) = 4C\beta(t)$, we deduce from (63) that for all $q \geq 1$

$$\int_{\mathbb{T}^2} \big(\frac{|\nabla B(t,x)|}{\alpha(t)}\big)^{\frac{2}{3}q}\, dx \leq \frac{q^q}{4^q}, \tag{64}$$

$$\sum_{q=0}^{\infty} \frac{1}{q!} \int_{\mathbb{T}^2} \big(\frac{|\nabla B(t,x)|}{\alpha(t)}\big)^{\frac{2}{3}q}\, dx \leq C \sum_{q=0}^{\infty} \frac{e}{4}^{q} < \infty, \tag{65}$$

so that (29) is proven. In order to prove (30), we observe that $W^{1,p}(\mathbb{T}^2) \hookrightarrow C^{1-\frac{2}{p}}(\mathbb{T}^2)$ for all $p > 2$. Hence, for all $(x,y) \in \mathbb{T}^2 \times \mathbb{T}^2$, we have

$$|B(t,x) - B(t,y)| \leq C|\nabla B(t,.)|_{L^p(\mathbb{T}^2)}|x-y|^{1-\frac{2}{p}} \leq C\beta(t)p^{\frac{3}{2}}|x-y|^{1-\frac{2}{p}}. \tag{66}$$

Thus, it suffices to choose

$$p = \big[-\log\big(\tfrac{1}{2} \wedge |x-y|\big)\big], \tag{67}$$

to deduce from (66) that

$$|B(t,x) - B(t,y)| \leq C\beta(t)|x-y|\big\{ -\log\big(\tfrac{1}{2} \wedge |x-y|\big)\big\}^{\frac{3}{2}}. \tag{68}$$

Remark 2. Let us observe that (68) is not enough to provide global existence of integral curves of B. However, since $\nabla B \in L^2((0,T) \times \mathbb{T}^2)$ and div $B = 0$, there exists a globally defined generalized flow X of B in the sense of Di Perna–Lions (see [10]).

Remark 3. Let us mention that the regularity of u_3 and B_3 can be improved if $\sigma(\rho)$ satisfies a condition similar to (3). Let C_1 be the smallest constant C such that for all $h \in C^\infty(\mathbb{T}^2; \mathbb{R}^2)$ and for all $q \in [2, \infty)$

$$|R.h|_{L^q(\mathbb{T}^2)} \leq Cq|h|_{L^q(\mathbb{T}^2)}. \tag{69}$$

Let us assume in addition to (4) and (22) that

$$\inf_{\,c>0} \big|\frac{1}{\sigma(\rho)c} - 1\big|_{L^\infty(\mathbb{T}^2)} < \frac{1}{8C_1} \tag{70}$$

and that the initial data satisfy

$$u_{3,0} \text{ and } B_{3,0} \in H^1(\mathbb{T}^2). \tag{71}$$

Then, arguing as in [9], we have for all $T > 0$

$$\nabla u_3 \text{ and } \nabla B_3 \in L^\infty(0,T; L^2(\mathbb{T}^2)), \tag{72}$$

$$\nabla u_3 \in L^2(0,T; L^q(\mathbb{T}^2)) \text{ and } \nabla B_3 \in L^2(0,T; L^r(\mathbb{T}^2)) \quad \text{for all } q < q^* \text{ and } r < r^*, \tag{73}$$

where $q^* > 4$ is defined as in Theorem 2 and $r^* > 4$ satisfies

$$\frac{1}{r^*} = 2C_1 \inf_{c>0} \Big|\frac{1}{c\sigma(\rho)} - 1\Big|_{L^\infty(\mathbb{T}^2)}. \tag{74}$$

Remark 4. In the case of a bounded domain Ω_0, the proof of Theorem (2) can be adapted taking care of boundary conditions. As a matter of fact, the proof is mainly based upon integration by parts after multiplication of the equation on the momentum by $\partial_t u$ and the equation on the magnetic field by $\partial_t B$. Since $\partial_t u$ and $\partial_t B$ satisfy homogeneous Dirichlet boundary conditions on $\partial\Omega_0$, they can formally be taken as test functions as in the periodic case. Notice that the homogeneous multipliers P, Q, and R have to be suitably modified in order to fit the boundary conditions on $\partial\Omega_0$.

5. The case of constant μ. In this section, we focus on the case when the fluids in presence have the same viscosity $\mu > 0$. The density ρ and the conductivity σ are assumed to be piecewise constant at the initial time, and the initial velocity and magnetic field satisfy (22). In order to simplify the presentation, we consider the case $B \equiv 0$, $u_3 \equiv 0$ in (23, 24), which is the most relevant framework in physical applications. The system under consideration here is therefore the following:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \operatorname{div}\ u = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu\Delta u + \nabla\big(p + \frac{B_3^2}{2}\big) = \rho f, \\ \partial_t B_3 + (u.\nabla)B_3 - \operatorname{div}\big(\frac{\nabla B_3}{\sigma(\rho)}\big) = 0. \end{cases} \tag{75}$$

Global existence and stability of weak solutions for (75) satisfying the energy bounds of Theorem 1 can be easily obtained adapting the proof of [11]. Moreover, the additional regularity of the velocity and pressure fields (31) holds since μ is constant.

We want now to investigate regularity properties of the interface and of the third component of the magnetic field. We already know that (31) holds for the vector field u and the pressure p. Let Γ_0 be the interface between the two fluids, parametrized by $\gamma_0 : I \to \mathbb{T}^2$, C^∞, where I is a closed interval of $[0, \infty)$ or a disjoint finite union of such intervals. Then, denoting by Γ_t the interface at time t, and by γ_t a parametrization of Γ_t, we can formally write

$$\gamma_t(s) = X(t, 0, \gamma_0(s)). \tag{76}$$

Therefore, regularity properties of γ_t can be deduced from the analysis of singularities of the flow X. As proven in [9], $L^2((0,T); H^2(\mathbb{T}^2))$ velocity fields have integral curves X with the following regularity:

$$X \in C^{1/2}([0,T]; W^{2,2-\varepsilon}(\mathbb{T}^2))^N \text{ for all small enough } \varepsilon > 0. \tag{77}$$

It follows that using the trace theorem, we have

$$\gamma_t \in W^{3/2-\varepsilon,2-\varepsilon}(I). \tag{78}$$

As a result, Γ_t has finite length, even though we did not take into account capillary effects. Let us briefly recall the proof of (77). First, taking the first derivatives of the ordinary differential equation, we obtain

$$\partial_t \partial_i X_k = \partial_i X_l \partial_l u_k(t, X); \tag{79}$$

hence,

$$|\nabla X(t,0,x)| \leq C \exp\big(C \int_0^t |d(s, X(s,0,x))|\, ds\big). \tag{80}$$

Similarly, we obtain

$$\partial_t \partial^2_{ij} X_k = \partial^2_{ij} X_l\ \partial_l u_k(t,X) + \partial_i X_l\ \partial_j X_m\ \partial^2_{lm} u_k(t,X); \tag{81}$$

hence¡

$$|\nabla^2 X(t,0,x)| \leq C \exp\big(C \int_0^t |d(s, X(s,0,x))|\, ds\big) \int_0^t \big|\nabla^2 u(s, X(s,0,x))\big|\, ds, \tag{82}$$

so that (77) is now obvious, using the incompressibility of u, Trudinger's inequality (58) and Jensen's inequality as in [7].

Next, we wish to prove that for all positive T, $\partial_t B_3 \in L^2((0,T)\times \mathbb{T}^2)$ and $\nabla B_3 \in L^\infty(0,T; L^2(\mathbb{T}^2))$. Multiplying the equation on B_3 by $\partial_t B_3$, we obtain after space and time integration

$$\int_0^t |\partial_t B_3|^2\, ds + \int_0^t \int_{\mathbb{T}^2} \frac{1}{2\sigma(\rho)} \partial_t |\nabla B_3|^2\, dx\, ds \leq C \int_0^t |u.\nabla B_3|^2_{L^2(\mathbb{T}^2)}\, ds,$$

from which we deduce that

$$\begin{aligned} \int_0^t |\partial_t B_3|^2\, ds + \int_{\mathbb{T}^2} \frac{|\nabla B_3|^2}{2\sigma(\rho)}\, dx \leq C + C \int_0^t |u|_{L^\infty(\mathbb{T}^2)} |\nabla B_3|_{L^2(\mathbb{T}^2)}\, ds \\ + \big| \int_0^t \int_{\mathbb{T}^2} \operatorname{div}\big(\frac{u}{\sigma(\rho)}\big) \frac{|\nabla B_3|^2}{2}\, dx\, ds \big|. \end{aligned} \tag{83}$$

The incompressibility condition then yields

$$\operatorname{div}\big(\frac{u}{\sigma(\rho)}\big) = -\frac{1}{\sigma(\rho)^2} \operatorname{div}(u\sigma(\rho));$$

hence, we have

$$\begin{aligned}
&\int_{\mathbb{T}^2} \operatorname{div}\big(\frac{u}{\sigma(\rho)}\big)\frac{|\nabla B_3|^2}{2}\,dx = \int_{\mathbb{T}^2} u_i\sigma(\rho)\partial_i\big(\frac{\partial_k B_3}{\sigma(\rho)}\big)\frac{\partial_k B_3}{\sigma(\rho)}\,dx\\
&= -\int_{\mathbb{T}^2}\big\{B_3\partial_k u_i\partial_i\big(\frac{\partial_k B_3}{\sigma(\rho)}\big) + B_3 u_i\partial_i\big(\operatorname{div}\big(\frac{\nabla B_3}{\sigma(\rho)}\big)\big)\big\}\,dx\\
&= \int_{\mathbb{T}^2}\big\{\frac{1}{\sigma(\rho)}\partial_k u_i\partial_i B_3\partial_k B_3 + (u.\nabla B_3)\big(\partial_t B_3 - u.\nabla B_3\big)\big\}\,dx.
\end{aligned}$$

Inserting the above identity in (83), we deduce that

$$\begin{aligned}
&\int_0^t |\partial_t B_3|^2_{L^2(\mathbb{T}^2)}\,ds + |\nabla B_3(t,.)|^2_{L^2(\mathbb{T}^2)}\\
&\qquad\le C + C\int_0^t (|u|_{L^\infty(\mathbb{T}^2)} + |\nabla u|_{L^\infty(\mathbb{T}^2)})|\nabla B_3|^2_{L^2(\mathbb{T}^2)}\,ds,
\end{aligned}$$

so that the claimed regularity on B_3 is proven, using Gronwall's lemma combined with the $L^2(0,T;H^2(\mathbb{T}^2))$ bound on u.

As a consequence, we can rewrite the equation on B_3 as follows:

$$\operatorname{div}\big(\frac{1}{\sigma(\rho)}\nabla B_3\big) = F, \tag{84}$$

where F belongs to $L^2((0,T)\times\mathbb{T}^2)$ for all $T>0$.

In the sequel, we shall consider piecewise-constant σ such that (5) holds with $\kappa_0^{-1} = 8C_1$. This provides an $r^* > 4$ defined by (74) such that

$$\nabla B_3 \in L^2(0,T;L^r(\mathbb{T}^2)) \quad \text{for all } r < r^*.$$

We now concentrate on the static problem (84), omitting for a while the time variable. Then we claim that ∇B_3 is more regular than $L^{r^*-\varepsilon}$. Transforming (84) into

$$-\Delta B_3 = -\nabla B_3.\nabla\log(\sigma(\rho)) + \sigma(\rho)F, \tag{85}$$

we first estimate the $W^{k,p}$ norm of ∇B_3 in terms of the right-hand side of (85) in measure norm for small enough $\varepsilon>0$. By elliptic regularity, we have

$$|\nabla B_3|_{W^{k,p}(\mathbb{T}^2)} \le C|\Delta B_3|_{\mathcal{M}(\mathbb{T}^2)} \tag{86}$$

for

$$0<k<1 \text{ and } k-\frac{2}{p} < -1, \tag{87}$$

using the fact that there exists a Green's function for the Laplacian on the torus that has the same singularities, thus shares the same local integrability properties as the Green's function on the whole space $\log|x|$.

We next use the regularity of the interface and the standard trace theorem (see [1]) to evaluate $|\Delta B_3|_{\mathcal{M}(\mathbb{T}^2)}$ in view of the norm $|\nabla B_3|_{W^{k,p}(\mathbb{T}^2)}$. First observe that classical interpolation arguments yield for $r < r^*$

$$|\nabla B_3|_{W^{\alpha,q}(\mathbb{T}^2)} \leq C|\nabla B_3|^{1-\theta}_{L^r(\mathbb{T}^2)}|\nabla B_3|^{\theta}_{W^{k,p}(\mathbb{T}^2)}, \tag{88}$$

where α, q, and $\theta \in (0,1)$ satisfy

$$\alpha = \theta k \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{p}. \tag{89}$$

In view of (78), γ_t has a bounded Lipschitz norm. Thus, we deduce from the trace theorem on Γ_t that

$$|\nabla B_3|_{L^1(\Gamma_t)} \leq \Lambda(|\gamma_t|_{Lip(I)})|\nabla B_3|_{W^{\alpha,q}(\mathbb{T}^2)}, \tag{90}$$

as soon as

$$\alpha - \frac{2}{q} \geq -1. \tag{91}$$

Notice that the constant in (90) can be expressed as a locally bounded function Λ of $|\gamma_t|_{Lip(I)}$. Inserting (89) into (91), we see that to make our argument valid it suffices to show that there exists some $\theta \in (0,1)$ such that

$$\theta\big(k - \frac{2}{p} + \frac{2}{r}\big) + \big(1 - \frac{2}{r}\big) \geq 0. \tag{92}$$

Since $k - \frac{2}{p} < -1$, thus $k - \frac{2}{p} + \frac{2}{r} < 0$, and $1 - \frac{2}{r} > 0$, the assertion (92) holds true by choosing θ small enough.
Gathering the estimates (88, 90) to (86), we obtain

$$|\nabla B_3|_{W^{k,p}(\mathbb{T}^2)} \leq |\sigma(\rho)F|_{L^2} + C\Lambda(|\gamma_t|_{Lip(I)})|\nabla B_3|^{\theta}_{W^{k,p}(\mathbb{T}^2)}|\nabla B_3|^{1-\theta}_{L^r(\mathbb{T}^2)}.$$

Using now that $t \mapsto \gamma_t$ is bounded in $C([0,T];Lip(I))$, we finally obtain from the time dependence of the above functions

$$B_3 \in L^2(0,T;W^{k+1,p}(\mathbb{T}^2)), \tag{93}$$

for (k,p) satisfying (87).

6. Further results and remarks.

Remark 5. We could have proven as in [9] that the regularity obtained in Theorem 2 also holds in the case when the density ρ may vanish in some region of $\mathbb{T}^2$. However, it is not stated this way in Theorem 2 since Maxwell's equations have to be modified in vacuum. As a matter of fact, the physical model is no longer valid as such.

Remark 6. Let us observe that for the original 3 dimensional model (1), global existence of weak solutions can be proven in unbounded domains $\Omega \subset \mathbb{R}^3$ (see [8]). In the case when $f \equiv 0$, the main ingredient of the proof relies upon the following a priori estimate: there exists $\alpha > 0$ such that for all $T > 0$, there exists $C_T > 0$ such that

$$\int_{(0,T)\times\Omega} \big(\rho(t,x)|u(t+h,x) - u(t,x)|^2 + |B(t+h,x) - B(t,x)|^2\big)\, dt\, dx \le C_T h^\alpha. \tag{94}$$

Remark 7. It is an interesting open question to know whether weak solutions of the 3-dimensional system (1) in $\mathbb{T}^3$ associated to initial data independent of x_3 are in fact solutions of (23, 24) in $\mathbb{T}^2$.

Remark 8. Uniqueness of weak solutions is not known for the 3-d problem, even when $M = 1$. However, using a theorem we proved in a previous work ([7]) in the case of 2-dimensional nonhomogeneous Navier–Stokes equations, we claim that weak solutions of (21) are unique if μ and σ are constant, and $\rho_0 \in W^{1,4+\varepsilon}(\mathbb{T}^2)$ for some positive ε.

Remark 9. From a practical standpoint, it would be interesting to evaluate at least a bound from below for the constant κ_0 in (5) (or equivalently for the constant C_1 appearing in (70) and defined in (69)). Indeed, for the applications we have in mind, the conductivities differ from one fluid to the other from many orders of magnitude, and therefore it is an open question to know whether the regularity shown in Section 5 holds or not.

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