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UNIQUENESS OF POSITIVE RADIAL SOLUTIONS OF $\Delta u + K(|x|)\gamma(u) = 0$*

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Abstract. We investigate the global structure of positive radial solutions of a semilinear elliptic equation $\Delta u + K(|x|)\gamma(u) = 0$, and study the uniqueness of ground state solutions of this equation. Our discussion is based on a Pohozaev-type identity and some detailed investigation for the oscillatory and asymptotic behavior of the solutions and their variational functions.

1. Introduction. In this paper we investigate the structure of positive radial solutions and study the uniqueness of ground state solutions of the semilinear elliptic equation

$$\Delta u + K(t)\gamma(u) = 0, \quad x \in R^n, \tag{1.1}$$

where $t = |x|$, and $n \geq 3$. This problem has been extensively studied by many authors for the nonlinearity independent of t. See, for example, Coffman [1], Kwong [9], Kwong and Zhang [11], McLeod [12], McLeod and Serrin [13], and Peletier and Serrin [17-18], where they treated the nonlinearity $f(u)$ which is negative for small u, and positive for large u. See also Erbe and Tang [3] where $f(u)$ is positive in $u > 0$. Corresponding work on the uniqueness of ground states for quasilinear elliptic equations can be found in Franchi et al. [6], Peletier et al. [19], and most recently Erbe and Tang [4], and Pucci and Serrin [20].

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When $\gamma(u) = u^p$, and $p > 1$, problem (1.1) arises from differential geometry and physics, and the problem of the existence and uniqueness of ground state solutions has been a subject of extensive studies since the first general and systematic study of Ni [14]. See also Kawano et. al. [8], Ni et al. [16], Yanagida [21] and and Yanagida et al. [22]. When $p = \frac{n+2}{n-2}$, the problem is called a *conformal scalar curvature problem*, and is related to the problem of finding conformal Riemannian metrics with prescribed scalar curvature K.

Let Ω be a finite ball in R^n. We shall also be concerned with the uniqueness of radial solutions of the problem

$$\begin{aligned} &\Delta u + K(t)\gamma(u) = 0, \quad u > 0 \quad \text{in} \quad \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.2}$$

This problem was studied by Ni and Nussbaum [15], and recently by Erbe and Tang [5], for more general nonlinearity $f(t,u)$ that is positive for $t > 0$ and $u > 0$, and satisfies some growth conditions. The case when $K(t) = 1$ and $\gamma(u)$ has a *subcritical* growth for large u, was studied in Erbe and Tang [3, 4], and Kwong and Li [10].

Our discussion is based on a Pohozaev-type identity and some detailed investigation for the oscillatory and asymptotic behavior of the solutions and their variational functions. Our paper is organized as follows. In Section 2, we state our main results. In Section 3, we present some general properties of radial solutions. We classify the radial solutions into three types, and give a characteristic description for each type of solution. In Section 4, we complete the proof of the main results.

2. Main results. Since we are interested in radial solutions, we study the initial value problem of an ordinary differential equation

$$\begin{aligned} &u'' + \frac{n-1}{t}u' + K(t)\gamma(u) = 0, \quad t > 0, \\ &u(0) = \alpha > 0, \quad u'(0) = 0. \end{aligned} \tag{2.1}$$

We shall maintain the assumptions that $K(t) \in C^1(0,\infty)$, and $\gamma(u) \in C^1[0,\infty)$, throughout the remainder of this paper. It was proved in [17] that (2.1) has a unique solution when $\alpha > 0$. We denote this solution by $u(t,\alpha)$. If $\alpha > 0$, then $u(t,\alpha)$ is positive for t slightly larger than zero. When it vanishes at some $t > 0$, we define $b(\alpha)$ to be the first zero of $u(t,\alpha)$ in

$(0,\infty)$. Therefore, $u(t,\alpha)>0$ for $t\in(0,b(\alpha))$, and $u(b(\alpha),\alpha)=0$. If $u(t,\alpha)$ is positive in $(0,\infty)$, then we define $b(\alpha)=\infty$.

We introduce some notations from Kawano et al. [8], and Erbe and Tang [3], [4]. We say that

(i) $u(t,\alpha)$ is a *crossing solution* if $b(\alpha)<\infty$;

(ii) $u(t,\alpha)$ is a *slowly decaying solution* if $u(t,\alpha)>0$ in $[0,\infty)$ and $\lim_{t\to\infty} t^{n-2}u(t,\alpha)=\infty$;

(iii) $u(t,\alpha)$ is a *ground state solution*, or a *fast decaying solution* if $u(t,\alpha)>0$ in $[0,\infty)$, $\lim_{t\to\infty} t^{n-2}u(t,\alpha)$ exists, and is finite and positive.

(iv) The structure of positive radial solutions to problem (1.1) is of *Type S* if $u(t,\alpha)$ is a slowly decaying solution for every $\alpha>0$; is of *Type M* if there exists a unique positive number $\alpha^*>0$ such that $u(t,\alpha)$ is a crossing solution for every $\alpha\in(\alpha^*,\infty)$; $u(t,\alpha)$ is a fast decaying solution if and only if $\alpha=\alpha^*$; and $u(t,\alpha)$ is a slowly decaying solution for every $\alpha\in(0,\alpha^*)$.

Note that if u is a ground state solution, then it necessarily holds that $\lim_{t\to\infty} u=0$. But for a slowly decaying solution u, it is possible that $\lim_{t\to\infty} u>0$. Define

$$\begin{aligned} N&=\{\alpha:\alpha>0,\quad u(t,\alpha)\text{ is a crossing solution}\};\\ D_f&=\{\alpha:\alpha>0,\quad u(t,\alpha)\text{ is a fast decaying solution}\};\\ D_s&=\{\alpha:\alpha>0,\quad u(t,\alpha)\text{ is a slowly decaying solution}\}.\end{aligned}$$

Proposition 2.1. *Suppose $K(t)\gamma(u)>0$ for $t>0$ and $u>0$. Then $N\cup D_s\cup D_f=(0,\infty)$.*

Proof. It is well-known that $u(t,\alpha)$ is strictly decreasing in $t>0$ whenever it is positive. Thus, we have either $b(\alpha)<\infty$, or $u(t,\alpha)>0$ in $[0,\infty)$. It remains to show that in the second case $\lim_{t\to\infty} t^{n-2}u(t,\alpha)$ exists, and the limit is either a positive finite number or ∞. We do this by showing that the function $t^{n-2}u(t,\alpha)$ is increasing in $t>0$.

Suppose $u(t)=u(t,\alpha)>0$ in $[0,\infty)$. Let $v(s)=t^{n-2}u(t)$, $s=t^{n-2}$. Then $v(s)>0$ for all $s>0$. By a routine calculation we obtain

$$\frac{dv}{ds}=u(t)+\tfrac{1}{n-2}tu',$$

where $'=\frac{d}{dt}$, and

$$\frac{d^2v}{ds^2}=[\tfrac{1}{n-2}tu''+\tfrac{n-1}{n-2}u']\cdot\tfrac{1}{n-2}\cdot t^{3-n}=-\tfrac{1}{(n-2)^2}t^{4-n}K(t)\gamma(u).$$

Thus,

$$\frac{d^2v}{ds^2} < 0, \quad \text{for} \quad s > 0.$$

Since $v(s) > 0$ in $s > 0$, we must have

$$\frac{dv}{ds} > 0, \quad \text{for} \quad s > 0.$$

Observing that $dv/dt = (n-2)t^{n-3}dv/ds$, it holds that $dv/dt > 0$. This completes the proof. $\square$

To state our main results, we need the following assumptions on K and γ:

- (K1) $K(t) > 0$, and $K'(t) \leq 0$ in $(0, \infty)$.
- (K2) $tK'(t) + (2n-2)K(t) \geq 0$ for large t, and $\int^\infty tK(t)dt = \infty$.
- (Γ1) $0 < \gamma(s) < s\gamma'(s)$, for all $s > 0$.
- (Γ2) Let $\Gamma(u) = \int_0^u \gamma(\tau)d\tau$. There exists an $\eta_\gamma > 0$ such that $(n-2)s\gamma(s) - 2n\Gamma(s) \geq 0$ in $(0, \eta_\gamma)$; and if $(n-2)s\gamma(s) - 2n\Gamma(s) \equiv 0$ on this interval, then $K'(t) < 0$ for some $t \in (0, \infty)$.
- (H) Let $h \geq 1$, and $u = u(t, \alpha)$. Define

$$\psi_h(t) = K(t)(u\gamma'(u) - h\gamma(u)) - \tfrac{h-1}{2}t\gamma(u)K'(t). \tag{2.2}$$

  If there exists a t_0 in $(0, b(\alpha))$ such that $\psi_h(t_0) \geq 0$, then $\psi_h(t) \geq 0$ in $(t_0, b(\alpha))$. If $b(\alpha) = \infty$, then $\psi_h(t)$ is not identically zero in (Ψ, ∞) for any $\Psi > 0$.

Some remarks on these assumptions will be made in Section 3.

Theorem 1. *Assume* (K1–K2), (Γ1)–(Γ2), *and* (H) *hold. Then the structure of positive solutions of* (2.1) *is either of Type S or of Type M. Moreover, problem* (1.1) *has at most one ground state solution, and infinitely many slowly decaying solutions, and problem* (1.2) *has at most one positive radial solution in any finite ball* Ω.

Using Theorem 1, we can easily derive the following two results, which generalizes Theorems 1 and 2 of [3].

Proposition 2.2. *Let* $K(t) = t^l$, $-2 < l \leq 0$, *and* $\bar{\gamma}(u) = u\gamma'(u)/\gamma(u)$. *Assume* ($\Gamma$1) *and* ($\Gamma$2) *hold, and* $\bar{\gamma}(u)$ *is nonincreasing in* u *and is not a* *constant. Then problem* (1.1) *has at most one ground state solution, and problem* (1.2) *has at most one radial solution.*

Proof. Since $K(t) = t^l$, and $-2 < l \le 0$, it is easy to verify that (K1) and (K2) are satisfied. Observe that $\psi_h(t)$ can be simplified to

$$\psi_h(t) = t^l \gamma(u)(\bar{\gamma}(u) - h - \tfrac{h-1}{2} l). \tag{2.3}$$

We see that (H) is satisfied if $\bar{\gamma}(u)$ is nonincreasing in u, and $\bar{\gamma}(u)$ is not a constant. The proof is completed.

Corollary 2.3. *Let* $K(t) = t^l$*. Let* $\gamma(u) = u^p$ *for* $u \ge 1$*; and* $\gamma(u) = u^q$ *for* $0 \le u < 1$*. Assume* $-2 < l \le 0$*, and* $1 < p < \frac{n+2}{n-2} < q$*. Then problem* (1.1) *has at most one ground state solution, and problem* (1.2) *has at most one radial solution.*

Proof. The function $\gamma(u)$ defined here is not continuously differentiable at $u = 1$. However, it is not difficult to see that our proof of Theorem 1 given below remains valid if the condition $\gamma(u) \in C^1[0, \infty)$ is replaced with local Lipschitz continuity. We have seen that (K1) and (K2) are satisfied. Obviously, (Γ1) is fulfilled. If we take $\eta_\gamma = 1$, then

$$(n-2)s\gamma(s) - 2n\Gamma(s) = (n - 2 - \tfrac{2n}{q+1})s^{q+1} > 0,$$

in $(0, \eta_\gamma)$, and (Γ2) is satisfied. Finally, we have $\bar{\gamma}(u) = p$ for $u \ge 1$; and $\bar{\gamma}(u) = q$ for $0 \le u < 1$. So all conditions of Proposition 2.2 are fulfilled, and the proof is completed.

3. General properites of radial solutions. The variation of $u(t, \alpha)$ is defined by $\phi(t, \alpha) = \partial u(t, \alpha)/\partial \alpha$, and satisfies

$$\phi'' + \tfrac{n-1}{t}\phi' + K(t)\gamma'(u)\phi = 0, \quad \phi(0) = 1, \quad \phi'(0) = 0. \tag{3.1}$$

Let L be the linear operator given by

$$L(\phi) = \frac{d^2\phi}{dt^2} + \tfrac{n-1}{t}\frac{d\phi}{dt} + K(t)\gamma'(u)\phi, \quad t \ge 0. \tag{3.2}$$

Then $L(\phi) = 0$. For a given real number $h \ge 1$, we introduce a function

$$G_h(t) = G_h(t, u, \alpha) = u(t, \alpha) + \tfrac{h-1}{2} t u'(t, \alpha). \tag{3.3}$$

Some functions similar to $G_h(t)$ were introduced in Kwong [9], Kwong and Zhang [11], McLeod [12] and McLeod and Serrin [13].

Lemma 3.1. *Let $u = u(t, \alpha)$. We have*

(i) (Erbe and Tang [5])

$$L(G_h(t)) = \psi_h(t), \tag{3.4}$$

where $\psi_h(t)$ is given in (2.2).

(ii) ([5])

$$[t^{n-1}(G_h'(t)\phi(t,\alpha) - G_h(t)\phi'(t,\alpha))]' = t^{n-1}\phi(t,\alpha)\psi_h(t). \tag{3.5}$$

(iii) (Ni and Yotsutani [16]) *Define*

$$Q(t) = Q(t,u,\alpha) = -t^n u^{'2} - (n-2)t^{n-1}uu' - 2t^n K(t)\Gamma(u). \tag{3.6}$$

Then

$$Q(t) = \int_0^t \tau^{n-1}K(\tau)[(n-2)u\gamma(u) - 2n\Gamma(u)]d\tau - 2\int_0^t \tau^n K'(\tau)\Gamma(u)d\tau. \tag{3.7}$$

Proof. The proof of (i) and (ii) was given in [5] for more general cases, and (iii) was given in Proposition 4.3 of [16]. □

The next lemma is an analogue of Lemma 2.8 of Erbe and Tang [4], giving characteristic properties of each type of solution.

Proposition 3.2. *Assume* (Γ1) *holds and $K(t) > 0$ for $t > 0$. We have*

(i) *if $\alpha \in N$, then $Q(b(\alpha)) < 0$;*

(ii) *if $\alpha \in D_s$, then for any $t = T > 0$, there is a $T_1 > T$ such that $Q(T_1) > 0$;*

(iii) *if $\alpha \in D_f$, and*

$$\lim_{s\to\infty} s^2 K(s)\gamma(s^{2-n}) = 0, \tag{3.8}$$

then $\lim_{t\to\infty} Q(t) = 0$.

Proof. The proof of (i) is trivial. Observe that (Γ1) implies $\Gamma(u) < \frac{1}{2}u\gamma(u)$. Therefore the proof of (ii) is the same as that of Lemma 2.8 in [4], except we take $m = 2$ and replace $f(u)$ with $K(t)\gamma(u)$ there. To prove (iii), note that if u is a ground state solution, then

$$\lim_{t\to\infty}[-t^n u^{'2} - (n-2)t^{n-1}uu'] = 0,$$

and (3.8) implies

$$\lim_{t\to\infty} -2t^n K(t)\Gamma(u) = 0.$$

It follows that $\lim_{t\to\infty} Q(t) = 0$.

Remark. Condition (3.8) is mild. It follows from (Γ1) when $n \geq 4$, and it is satisfied for all $n \geq 3$, if (Γ1) holds and $\lim_{t\to\infty} tK(t) = 0$. In any case, it is fulfilled if (K1), (Γ1) and (Γ2) hold.

As an application of Proposition 3.2, we present a structure theorem below for the nonlinearity which is of a simple form.

Proposition 3.3. *Assume $K(t) > 0$ for $t > 0$, and is not a constant. Let $\gamma(u) = u^{(n+2)/(n-2)}$. We have*

(i) *if $K'(t) \leq 0$ for $t > 0$, then $u(t,\alpha)$ is a slowly decaying solution;*
(ii) *if $K'(t) \geq 0$ for $t > 0$, then $u(t,\alpha)$ is a crossing solution.*

Proof. Substituting $\gamma(u)$ by $u^{\frac{n+2}{n-2}}$ in (3.7) yields

$$Q(t) = -\frac{n-2}{n}\int_0^t \tau^n K'(\tau)u(\tau)^{\frac{2n}{n-2}}\,d\tau.$$

If $K'(t) \leq 0$ for $t > 0$, then $Q(t) \geq 0$ in $(0, b(\alpha))$, implying that u is not a crossing solution. Moreover, $\lim_{t\to\infty} Q(t)$ exists and is positive. Thus, u is a slowly decaying solution and proves (i). The proof of (ii) is similar, and we omit it.

Remark a. It is well-known that if $K(t) \equiv 1$, then each positive radial solution of the *conformal scalar curvature problem*

$$\Delta u + K(|x|)u^{\frac{n+2}{n-2}} = 0, \quad x \in R^n, \tag{3.9}$$

is a ground state solution. This observation shows that the second part of condition (Γ2) is necessary in studying uniqueness problem of ground states.

Remark b. Let $\kappa(t)$ be a nonnegative, nonincreasing, and nonconstant function defined in $[0,\infty)$ with $\kappa(0) < 1$. Let u be a positive radial solution of (3.9). It was proved in Ding and Ni [2] that

(i) if $K(t) = 1+\kappa(t)$, then $u(t,\alpha)$ is a slowly decaying solution behaving like $t^{\frac{2-n}{2}}$ as $t \to \infty$,
(ii) if $K(t) = 1 - \kappa(t)$,

then $u(t,\alpha)$ is a crossing solution.

Note that $\kappa(t)$ can be taken arbitrarily small and with compact support. This result, similar to our Proposition 3.3, shows that the structure of solutions of (1.1) is quite delicate and very sensitive to small perturbations.

Next, we prove a result concerning the asymptotic behavior of slowly decaying solutions.

Proposition 3.4. *Assume $K(t) > 0$ for $t > 0$, and $\gamma(u) > 0$ for $u > 0$. Let $u = u(t,\alpha)$ be a slowly decaying solution. Then we have*

(i) *if there exists a $t_K > 0$ such that*

$$tK'(t) + (2n-2)K(t) \leq 0, \quad \textit{for all} \;\; t > t_K, \tag{3.10}$$

then $\lim_{t\to\infty} u > 0$;

(ii) *if $\int^{\infty} tK(t)dt = \infty$, then $\lim_{t\to\infty} u = 0$.*

Proof. (i) Suppose to the contrary that $\lim_{t\to\infty} u = 0$. Then by using L'Hospital's rule we have

$$\lim_{t\to\infty} t^{(n-1)}u' = -(n-2)\lim_{t\to\infty} t^{(n-2)}u = -\infty,$$

since $t^{(n-1)}u'$ is decreasing and $\lim_{t\to\infty} t^{(n-2)}u = \infty$. Define

$$M(t) = M(t,u,\alpha) = t^{2n-2}[u'(t,\alpha)^2/2 + K(t)\Gamma(u(t,\alpha))]. \tag{3.11}$$

Then $\lim_{t\to\infty} M(t) = \infty$. On the other hand, it is easy to verify that

$$M'(t,\alpha) = t^{2n-3}\Gamma(u)[(2n-2)K(t) + tK'(t)]. \tag{3.12}$$

By (3.10) we see that $M' \leq 0$ for $t > t_K$, which contradicts $\lim\limits_{t\to\infty} M(t) = \infty$.

(ii) Let $w(t,\alpha) = (n-2)u + tu'$. It follows from the proof of Proposition 2.1 that $w > 0$ in $t > 0$. A straightforward computation yields

$$w'(t) = -tK(t)\Gamma(u).$$

Now, suppose to the contrary that $\lim_{t\to\infty} u > 0$. Then we can find constants $c > 0$ and $t_\Gamma > 0$ such that

$$w'(t) \leq -ctK(t), \quad \text{for} \;\; t > t_\Gamma$$

Integrating both sides of this inequality from t_Γ to $T > t_\Gamma$, and letting T tend to ∞, we obtain $\lim_{t\to\infty} w = -\infty$. Hence, $\lim_{t\to\infty} tu' = -\infty$. But the last identity implies $\lim_{t\to\infty} u = -\infty$. We obtain a contradiction.

Lemma 3.5. *Under the assumptions of Theorem* 1*, we have*

(i) D_s *is nonempty, and* $(0, \eta_\gamma] \subset D_s$.

ii) N *and* D_s *are open sets, and* D_f *is a closed set.*

Proof. (i) Let $0 < \alpha < \eta_\gamma$ and $u = u(t, \alpha)$. Then $0 < u < \eta_\gamma$, and $(n-2)u\gamma(u) - 2n\Gamma(u) \geq 0$ in $(0, b(\alpha))$. Applying identity (3.7), it follows that $Q(t) > 0$ in $(0, b(\alpha))$. Thus, u is not a crossing function and $b(\alpha) = \infty$. Applying identity (3.7) again, it is easy to see that $\lim_{t\to\infty} Q(t) > 0$, and therefore u must be a slowly decaying solution.

(ii) The continuous dependence of solutions of (2.1) on initial data implies that N is an open set. In view of Proposition 2.1, it remains to show that D_s is an open set. Let $\bar{u} = u(t, \bar{\alpha})$ be a slowly decaying solution. By Proposition 3.4 (ii), we can find a T_η sufficiently large such that

$$u(T_\eta, \bar{\alpha}) < \eta_\gamma/2, \quad \text{and} \quad Q(T_\eta, \bar{\alpha}) > 0.$$

Thus, if α is sufficiently close to $\bar{\alpha}$, then

$$0 < u(T_\eta, \alpha) < \eta_\gamma/2, \quad \text{and} \quad Q(T_\eta, \alpha) > 0.$$

Since $u(t, \alpha)$ is decreasing whenever it is positive, we have

$$u(t, \alpha) < \eta_\gamma/2, \quad \text{if} \quad t > T_\eta, \quad \text{and} \quad u(t, \alpha) > 0.$$

Applying identity (3.7) we obtain

$$Q(t, \alpha) > Q(T_\eta, \alpha) > 0, \quad \text{if} \quad t > T_\eta, \quad \text{and} \quad u(t, \alpha) > 0, \tag{3.13}$$

which implies that $u(t, \alpha) > 0$ for all $t > T_\eta$. Since otherwise, say, at $t = T_0 > T_\eta$, one has $u(T_0, \alpha) = 0$, then $Q(T_0, \alpha) < 0$, contradicting (3.13). It also follows from (3.13) that

$$\lim_{t\to\infty} Q(t, \alpha) > Q(T_\eta, \alpha) > 0.$$

Therefore $u(t, \alpha)$ must be a slowly decaying solution. The proof is completed.

4. Proof of Theorem 1. First we give an outline of the proof. Since $(0, \eta_\gamma) \subset D_s$, we can define

$$\alpha^* = \sup\{\alpha > 0 : \ \alpha' \in D_s, \quad \text{if} \quad 0 < \alpha' < \alpha\} \tag{4.1}$$

If $\alpha^* = \infty$, then the structure of solutions of (2.1) is of Type S, and every solution $u(t, \alpha)$ is a slowly decaying solution. If

$$\alpha^* < \infty, \tag{4.2}$$

then $u(t, \alpha^*)$ is a ground state solution, because D_s and N are open sets. In this case, we shall show that every solution $u(t, \alpha)$ with $\alpha > \alpha^*$ is a crossing solution. Once this assertion is proved, the uniqueness of ground state solutions readily follows.

To complete the proof of Theorem 1, we shall apply the theory of linear second order ordinary differential equations to analyze the oscillatory and asymptotic behavior of $\phi(t, \alpha^*)$. In what follows, we assume that (4.2) holds. For simplicity of notations, we let $u^* = u(t, \alpha^*)$ and $\phi^* = \phi(t, \alpha^*)$. The following two technical lemmas are crucial in our proof.

Lemma 4.1. *Assume* (K1) *and* (Γ1)–(Γ2) *hold. Then* ϕ^* *vanishes exactly once in* $(0, \infty)$.

Lemma 4.2. *Under the assumptions of Theorem* 1, *there exists a constant* $\phi_0^* > 0$ *such that*

$$\lim_{t \to \infty} \phi^* = -\phi_0^*. \tag{4.3}$$

We divide the rest of this section into three subsections. We prove Lemmas 4.1 and 4.2 in Subsections 4.1 and 4.2, respectively. The proof of Theorem 1 is completed in 4.3.

4.1. Proof of Lemma 4.1. The proof is based on the following two claims.

Claim 1. If (K1) and (Γ1) hold, then ϕ^* vanishes at least once in $(0, \infty)$.

Claim 2. If (K1) and (Γ1)–(Γ2) hold, then ϕ^* vanishes at most once in $(0, \infty)$.

Proof of Claim 1. Let $h = 1$ in (3.5), and integrate the resultant identity. We have

$$t^{n-1}(u^{*'}\phi^* - u^*\phi^{*'}) = \int_0^t \tau^{n-1} K(\tau)\phi^*[u^*\gamma'(u^*) - \gamma(u^*)]d\tau. \tag{4.4}$$

Suppose to the contrary that $\phi^* > 0$ for all $t \in (0, \infty)$. Then

$$t^{n-1}(u^{*'}\phi^* - u^*\phi^{*'}) > 0, \quad \text{in} \quad (0, \infty),$$

which, in turn, implies that u^*/ϕ^* (and so also $(t^{n-2}u^*)/(t^{n-2}\phi^*)$) is strictly increasing in $(0,\infty)$. Since $\lim_{t\to\infty} t^{n-2}u^*$ exists and is finite, there is a number $0 \le d^* < \infty$ such that

$$\lim_{t\to\infty} t^{n-2}\phi^* = d^*. \tag{4.5}$$

By L'Hospital's rule one has

$$\lim_{t\to\infty} t^{n-1}\phi^{*'} = (2-n)d^* \le 0. \tag{4.6}$$

By (4.5), (4.6) and the fact that u^* is a ground state solution, we obtain

$$\lim_{t\to\infty} t^{n-1}(u^{*'}\phi^* - u^*\phi^{*'}) = 0.$$

However, if we let $t \to \infty$ in (4.4) and use the assumptions (K1) and (Γ1), then we obtain

$$\lim_{t\to\infty} t^{n-1}(u^{*'}\phi^* - u^*\phi^{*'}) > 0.$$

We get a contradiction, and the claim is proved.

Proof of Claim 2. It suffices to show that u^* intersects every solution $u(t,\alpha)$, $0 < \alpha < \alpha^*$ exactly once in $t \in (0,\infty)$. Since a slowly decaying solution is ultimately larger than the ground state solution u^*, and $u(t,\alpha)$ is a slowly decaying solution when $0 < \alpha < \alpha^*$, we see that u^* and $u(t,\alpha)$ must intersect in $t \in (0,\infty)$ when $0 < \alpha < \alpha^*$.

First we show that u^* and $u(t,\alpha)$ intersect at most once if $\alpha > 0$ is sufficiently small. If it is not true, then we can find a sequence $\{\alpha_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} \alpha_i = 0$, and every $u(t,\alpha_i)$ intersects u^* at least twice. Denote the second intersection point by $(a_i, u^*(a_i))$. Since $\alpha_i \to 0$ as $i \to \infty$, and $u'(t,\alpha_i) < 0$ in $(0,\infty)$, we have $\lim_{i\to\infty} a_i = \infty$. Without loss of generality, we may assume $\alpha_i < \min\{\alpha^*, \eta_\gamma/2\}$, $i = 1,2,\cdots$. It follows that

$$u(a_i,\alpha_i) = u^*(a_i), \text{ and } u'(a_i,\alpha_i) < u^{*'}(a_i) < 0, \quad i = 1,2,\cdots . \tag{4.7}$$

Let $u_i = u(t,\alpha_i)$. By identity (3.7), and conditions (K1) and (Γ2) we get

$$\begin{aligned} Q_i(a_i) = Q(a_i,\alpha_i) = \int_0^{a_i} \tau^{n-1}K(\tau)[(n-2)u_i\gamma(u_i) - 2n\Gamma(u_i)]d\tau \\ - 2\int_0^{a_i} \tau^n K'(\tau)\Gamma(u_i)d\tau > 0. \end{aligned} \tag{4.8}$$

Let $Q^*(t) = Q(t, \alpha^*)$. Recall that $\lim_{t\to\infty} Q^*(t) = 0$. Using identity (3.7) again yields

$$\begin{aligned} Q^*(a_i) = &- \int_{a_i}^{\infty} \tau^{n-1} K(\tau)[(n-2)u^*\gamma(u^*) - 2n\Gamma(u^*)]d\tau \\ &+ 2\int_{a_i}^{\infty} \tau^n K'(\tau)\Gamma(u^*)d\tau < 0. \end{aligned} \tag{4.9}$$

Obviously, (4.8) and (4.9) lead to $Q_i(a_i) > Q^*(a_i)$. Combining this inequality with (3.6) and (4.7) we obtain

$$(n-2)[u^{*'}(a_i) - u_i'(a_i)]u^*(a_i)a_i^{n-1} + [u^{*'}(a_i)^2 - u_i'(a_i)^2]a_i^n > 0.$$

Factoring $[u^{*'}(a_i) - u_i'(a_i)]$ out, and using (4.7) again, one has

$$(n-2)u^*(a_i)a_i^{n-2} + 2u^{*'}(a_i)a_i^{n-1} > 0. \tag{4.10}$$

On the other hand, since u^* is a ground state solution, applying L'Hospital's rule, we have

$$\lim_{t\to\infty}[(n-2)t^{n-2}u^*(t) + 2t^{n-1}u^{*'}(t)] = -(n-2)\lim_{t\to\infty} t^{n-2}u^*(t) < 0.$$

This contradicts (4.10) and shows that there is an ϵ such that $0 < \epsilon \le \alpha^*$, and u^* and $u(t,\alpha)$ intersect at most once if $0 < \alpha < \epsilon$. Let $\bar{\epsilon}$ be the largest number in $(0, \alpha^*]$ so that this assertion is valid. It remains to prove $\bar{\epsilon} = \alpha^*$.

Suppose to the contrary that $\bar{\epsilon} < \alpha^*$. Then there is a sequence $\{\beta_j\}_{j=1}^{\infty}$ such that $\bar{\epsilon} < \beta_j < \alpha^*$, $\lim_{j\to\infty} \beta_j = \bar{\epsilon}$, and $u(t, \beta_j)$ crosses u^* at least twice. Since $\beta_j < \alpha^*$ and $u^* < u(t, \beta_j)$ for large t, $u(t, \beta_j)$ and u^* must intersect a third time, say, at $t = c_j$. Then $\lim_{j\to\infty} c_j = \infty$, and

$$u^*(c_j) = u(c_j, \beta_j), \quad u^{*'}(c_j) < u'(c_j, \beta_j) < 0, \quad j = 1, 2, \cdots. \tag{4.11}$$

For simplicity of notation, let $u_j = u(t, \beta_j)$, $M^*(t) = M(t, u^*, \alpha^*)$ and $M_j(t) = M(t, u_j, \beta_j)$, where M is defined in (3.11). Then

$$M^*(c_j) > M_j(c_j). \tag{4.12}$$

Recall that $\lim_{t\to\infty} M(t) = \infty$, when u is a slowly decaying solution. While

$$\lim_{t\to\infty} M^*(t) < \infty,$$

since $\lim_{t\to\infty} t^{n-1}u^{*'}$ is finite, and $\lim_{t\to\infty} t^{2n-2}\Gamma(u^*(t)) = 0$ by ($\Gamma 2$). Denote $M_* = \lim_{t\to\infty} M^*(t) < \infty$. Choose T_M sufficiently large that $M(T_M, \bar{\epsilon}) > 4M_*$, and $tK'(t) + (2n-2)K(t) \geq 0$ for $t > T_M$ (see (K2)). Without loss of generality, we can assume $c_j > T_M$ and $M_j(T_M) > 2M_*$ for all $j = 1, 2, \cdots$. Since M_j is increasing in (T_M, ∞) by (3.12), we must have $M_j(c_j) > 2M_*$, which contradicts (4.12). The proof is completed.

4.2. Proof of Lemma 4.2. Recall that ϕ^* is a solution of the linear second order equation

$$(t^{n-1}v')' + t^{n-1}K(t)\gamma'(u^*(t))v = 0. \tag{4.13}$$

Let us introduce another equation

$$(t^{n-1}w')' = 0,$$

which has two linearly independent solutions $w_1(t) \equiv 1$, and $w_2(t) \equiv t^{2-n}$.

Since $u^* \sim t^{2-n}$, as $t \to \infty$, in other words, $\lim_{t\to\infty} u^*/t^{2-n}$ is a nonzero finite constant, it follows from ($\Gamma 2$) that the growth of $\gamma'(u^*(t))$ is larger than $4/(n-2)$. Because $K(t)$ is bounded above, we have

$$w_1 w_2 t^{n-1} K(t)\gamma'(u^*(t)) = o(t^{(2-n)+(n-1)-4}) = o(t^{-3}),$$

which, in turn, implies that

$$\int^{\infty} w_1 w_2 t^{n-1} K(t)\gamma'(u^*(t))dt < \infty. \tag{4.14}$$

Applying Theorem 9.1 of Hartman (page 379 of [7]), we conclude that the linear equation (4.13) has two independent solutions, say, $v_1(t)$ and $v_2(t)$, such that $v_i \sim w_i$, $i = 1, 2$, as $t \to \infty$. Note that ϕ^* is a linear combination of $v_1(t)$ and $v_2(t)$. Let ν_1 and ν_2 be two constants such that

$$\phi^* = \nu_1 v_1(t) + \nu_2 v_2(t). \tag{4.15}$$

It suffices to show that $\nu_1 \neq 0$.

Suppose to the contrary that $\nu_1 = 0$. Then $\phi^* \sim t^{2-n}$ as $t \to \infty$. Let $G_h^*(t) = G_h(t, \alpha^*)$. It holds for all $h \geq 1$ that

$$\lim_{t\to\infty} t^{n-1}(G_h^{*'}(t)\phi^*(t) - G_h^*(t)\phi^{*'}(t)) = 0. \tag{4.16}$$

Let τ^* be the unique zero of ϕ^* in $(0, \infty)$. Let ζ_h be the first zero of $G_h^*(t)$, if $G_h^*(t)$ does vanish in $(0, \infty)$. Note that ζ_h may not be defined for all $h > 1$. By the definition of $G_h^*(t)$, we can show that there is a number $\bar{h} > 1$ such that

$$\tau^* = \zeta_{\bar{h}}, \tag{4.17}$$

and

$$t^{n-1}(G_{\bar{h}}'^*(t)\phi^*(t) - G_{\bar{h}}^*(t)\phi^{*'}(t)) = 0, \quad \text{at} \quad t = \tau^*. \tag{4.18}$$

By the condition (H) we see that

$$\psi_{\bar{h}}(\tau^*) \geq 0. \tag{4.19}$$

In fact, if $\psi_{\bar{h}}(\tau^*) < 0$, then $\psi_{\bar{h}}(t) < 0$ in $(0, \tau^*)$. By using Sturm's comparison theorem it follows that $G_{\bar{h}}^*(t)$ vanishes in $(0, \tau^*)$, contradicting (4.17). Combining (4.19) and condition (H) we obtain $\psi_{\bar{h}}(t) \geq 0$, and $\psi_{\bar{h}}(t)$ is not identically zero in (τ^*, ∞). Applying identity (3.5), and using (4.18), we obtain

$$\lim_{t\to\infty} t^{n-1}(G_{\bar{h}}'^*(t)\phi^*(t) - G_{\bar{h}}^*(t)\phi^{*'}(t)) = \int_{\tau^*}^{\infty} t^{n-1}\phi^*(t)\psi_{\bar{h}}(t)dt < 0.$$

But this contradicts (4.16). The proof is completed.

4.3. Proof of Theorem 1. Let $u = u(t, \alpha)$, $u^* = u(t, \alpha^*)$, and $0 < \lambda < n - 2$. Define

$$z(t) = z(t, \alpha, \lambda) = t^\lambda(u(t, \alpha^*) - u(t, \alpha)). \tag{4.20}$$

Then $z(t)$ satisfies

$$z'' + (n - 1 - 2\lambda)\frac{z'}{t} + \lambda(\lambda + 2 - n)\frac{z}{t^2} + t^\lambda K(t)[\gamma(u^*) - \gamma(u)] = 0. \tag{4.21}$$

If there are $\alpha > 0$ and $t_0 > 0$ such that

$$z'(t_0) = 0, \quad \text{and} \quad 0 < u(t_0) < u^*(t_0), \tag{4.22}$$

then

$$\begin{aligned} z''(t_0) &= \{\lambda(n - 2 - \lambda)/t_0^2 - K(t_0)[\gamma(u^*(t_0)) - \gamma(u(t_0))]/[u^*(t_0) - u(t)]\}z \\ &= [\lambda(n - 2 - \lambda)/t_0^2 - K(t_0)\gamma'(\theta(t_0))]z, \end{aligned}$$

where $u(t_0) \le \theta(t_0) \le u^*(t_0)$. Since $\gamma'(u^*(t)) = o(t^{-4})$ as $t \to \infty$, we see that if t_0 is sufficiently large and (4.22) is valid, then

$$z''(t_0) > 0. \tag{4.23}$$

Recall that ϕ^* has a unique zero at $t = \tau^*$ and behaves like a negative constant for t large, we have $t^\lambda \phi^*(t) \to -\infty$ as $t \to \infty$. It follows that when $t = T^* > \tau^*$ is sufficiently large,

$$t^\lambda \phi^*(t) < 0, \quad (t^\lambda \phi^*(t))' < 0,$$

where $t^\lambda \phi^*(t) = \lim_{\alpha \to \alpha^*} z(t)/(\alpha^* - \alpha)$. Now, if $\alpha > \alpha^*$ is sufficiently close to α^* and $u(t, \alpha)$ is not a crossing solution, then $z(T^*) > 0$, $z'(T^*) > 0$. Therefore, $z(t)$ increases in a right neighborhood of T^*. If $z(t)$ decreases for some $t > T^*$, then there exists some $t_0 > T^*$ such that $z(t_0) > 0$, $z'(t_0) = 0$, and $z(t)$ has a local maximum at t_0. But this contradicts (4.22) and (4.23). If $z(t)$ is increasing for all $t > T^*$, then

$$z(t) > z(T^*) > 0 \ \text{ for } \ t > T^*, \quad \text{and} \quad \lim_{t \to \infty} z(t) > z(T^*) > 0,$$

which contradicts the assumption that u is not a crossing solution. This contradiction shows that *there exists an* $\rho > 0$ *such that* $u(t, \alpha)$ *is a crossing solution if* $\alpha^* < \alpha < \alpha^* + \rho$. Thus, for $\alpha^* < \alpha < \alpha^* + \rho$, the first zero $b(\alpha)$ of u is well defined, and

$$\lim_{\alpha \to \alpha^{*+}} b(\alpha) = \infty.$$

Therefore, $b(\alpha)$ is decreasing in a right neighborhood of α^*. By using an argument similar to that in the proof of Theorem 1 of [5] we can show that $b'(\alpha) \neq 0$. Thus, $b'(\alpha) < 0$ whenever $\alpha > \alpha^*$ and $b(\alpha)$ is defined.

Define $\alpha_\infty = \sup\{\alpha > \alpha^* : \quad \alpha' \in N \quad \text{if} \quad \alpha^* < \alpha' < \alpha\}$. Then $b(\alpha)$ is defined and strictly decreasing in $(\alpha^*, \alpha_\infty)$. Therefore, $\lim_{\alpha \to \alpha_\infty^-} b(\alpha) < \infty$, which in turn implies $\alpha_\infty = \infty$. In summary, we have shown that $N = (\alpha^*, \infty)$, $b'(\alpha) < 0$, if $\alpha \in N$, and $D_f = \alpha^*$, and $D_s = (0, \alpha^*)$. So the assertion of Theorem 1 readily follows. The proof is completed.

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