Differential and Integral Equations Volume 11, Number 4, July 1998, pp. 623–640.

ALMOST PERIODIC SOLUTIONS FOR A CLASS OF DUFFING-LIKE SYSTEMS

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(Submitted by: Jean Mawhin)

Abstract. We prove the existence of infinitely many limit periodic solutions for Duffing-like systems of the type $-\ddot{q} = g(t, q)$, where g is limit periodic in time and has a non-degenerate homoclinic solution to the hyperbolic stationary solution $q \equiv 0$. Moreover, we can deal with almost periodic perturbations of the above class of systems. In this case we prove the existence of almost periodic solutions and we show that these solutions are quasi-periodic whenever the perturbation is quasi periodic.

1. Introduction. In this paper we consider the system of differential equations

$$-\ddot{q} = g(t, q) \tag{DF}$$

where we assume:

(g1) $g \in C^3(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$;
(g2) $g(t, 0) = 0$, for all $t \in \mathbb{R}$, and 0 is a hyperbolic rest point for (DF);
(g3) g and $\nabla_q g$ are limit periodic functions in t uniformly on compact sets (for a definition of limit periodic functions, see §2).

Received for publication February 1997.
*Author has been supported by EEC (contract n.ERBCHRXCT 940494).

It is known that, if $g(t,q) = -q + \nabla_q V(t,q)$, where V is superquadratic in q, (DF) has infinitely many homoclinic solutions even for more general time dependencies (see [19,7,20] for the periodic case and [1, 4, 6, 12,13,14, 17, 18, 21] for other type of time dependencies).

Moreover, whenever a suitable nondegeneracy condition holds, it can be shown that system (DF) has rich dynamics, in particular a positive entropy. These results have been proved by variational methods in a series of papers, starting with [20]. For periodic time-dependencies these are classical results which date back to Poincaré [16] and can be proved using perturbation techniques (see, e.g. [9]).

When g depends periodically on time, the classical, perturbative approach, gives much more information on the structure of the solution set. Indeed, one can prove a shadowing lemma which shows the existence of a hyperbolic invariant set on which the dynamics of the system is conjugate to a Bernoulli shift. This, in particular, implies, besides the existence of infinitely many homoclinics, the existence of uncountably many bounded solutions, among which there are periodic and heteroclinic solutions.

When g depends almost-periodically on time (for a definition of almost periodic functions, see §2), as in the periodic case, one expects to find rich dynamics. For this more general time dependence, the classical approach is hardly applicable; indeed, many difficulties arise in the construction of the discrete time-map describing the dynamics (see, however [11, 22]). Using variational methods, as recalled above, the existence of infinitely many homoclinics and of uncountably many bounded solutions have been proved [6, 14, 18]. The aim of this paper is to show that, whenever g is limit periodic and a suitable nondegeneracy condition holds, among this large class of solutions there exist infinitely many limit-periodic ones.

To be more precise, we introduce some notations and recall some results on (DF).

Let us introduce the Banach spaces $X = C_b^2(\mathbb{R}, \mathbb{R}^N)$, $Y = C_b(\mathbb{R}, \mathbb{R}^N)$ of continuous and bounded functions, equipped with the respective uniform norms and denote by X_0 and Y_0 their subspaces of functions vanishing at infinity. We also recall that a function $v \in X$ is a homoclinic solution of (DF) if it solves (DF) and $v(t)$, $v'(t) \to 0$ as $t \to \pm\infty$.

We assume the following nondegeneracy condition:

(g4) There exists a homoclinic solution $\bar{v}$ such that the bounded linear operator $A_0 : X_0 \to Y_0$ defined by $A_0 u = \ddot{u} + \nabla_q g(t, \bar{v})u$ has a bounded inverse.

Remark 1.1. The existence of a homoclinic solution for (DF) follows under some additional assumptions on the structure of the nonlinear term g. See the above quoted references. The existence of the bounded inverse for A_0 is a nondegeneracy assumption on the solution $\bar{v}$. In all the papers where existence of a chaotic dynamics is proved, a condition of this kind is assumed. It can be shown to hold for a class of perturbations of completely integrable systems using the Melnikov function method [10]. Let us point out that in the papers which study the problem using variational techniques a somewhat weaker assumption is assumed, (roughly that $\bar{v}$ is an isolated solution of (DF) in X_0).

We are now in position to state our main theorem.

Theorem 1.2. *If* (g1-4) *hold, then system* (DF) *admits infinitely many limit periodic solutions.*

We cannot prove that almost-periodic solutions exist when g is almost-periodic but not limit-periodic. In particular, our approach does not work if g is quasi-periodic. However, we can consider almost periodic perturbations of (DF). In this case we prove the existence of infinitely many almost periodic solutions. We also show that these solutions are quasi-periodic whenever the perturbation is quasi-periodic. More precisely, we consider the system

$$-\ddot{q} = g(t, q) + \lambda h(t, q, \dot{q}) \tag{DF$_\lambda$}$$

where λ is a small perturbation parameter and we assume:

- **(h)** $h(t, \cdot, \cdot) \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ for any $t \in \mathbb{R}$ and $h(\cdot, x, y) \in C(\mathbb{R}, \mathbb{R}^N)$ is almost periodic, uniformly for (x, y) in compact sets of $\mathbb{R}^N \times \mathbb{R}^N$.

(for a definition of quasi-periodic functions see §2).

We prove

Theorem 1.3. *If* (g1-4) *and* (h) *hold, then there exists* $\bar{\lambda} > 0$ *such that, for any* $\lambda \in (-\bar{\lambda}, \bar{\lambda})$*, the system* (DF$_\lambda$) *has infinitely many almost periodic solutions. Moreover, the solutions are quasi-periodic if* $h(\cdot, x, y)$ *is quasi-periodic.*

To prove the above results we first show that a unique solution u_p of the system (DF) exists in a small X-neighborhood of the function $\sum_{k\in\mathbb{Z}} \bar{v}(\cdot + p_k)$, where p_k are ϵ-periods of g such that $p_k - p_{k-1}$ is sufficiently large and ϵ sufficiently small. Such a result is proved following the shadowing lemma construction made by Angenent in [2, 3] and gives the existence of uncountably many bounded solutions for (DF).

Then we show that the solutions u_p associated with periodic sequences p_k of ϵ-periods (which exist since g is limit periodic), are limit-periodic. This last step uses the fact that u_p is unique in a small X-neighborhood.

We recall here that a "uniqueness" condition has been widely used in the study of existence of almost periodic solutions, see for example [8]. But the situation in which it is usually applied is one in which only one solution exists in $K \times \mathbb{R}$, where K is a compact subset of $\mathbb{R}^{2N}$ condition not verified in our situation.

Other results on the existence of quasi-periodic solutions are contained in [15], where methods of KAM theory are used.

2. Preliminaries on almost-periodic functions. We recall the definition and some properties of almost periodic, limit periodic and quasi-periodic functions (see also [5, 8]).

Definition 2.1. Let $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^m$ be continuous, K be a compact set in $\mathbb{R}^N$ and $\epsilon > 0$. We say that $\tau \in \mathbb{R}$ is an (ϵ, K)-*period* for f if $\sup_{t\in\mathbb{R}} |f(t+\tau, x) - f(t,x)| < \epsilon$ for all $x \in K$. The function f is *almost periodic in t uniformly for x in compact sets*, if for any $\epsilon > 0$ and every compact set K in $\mathbb{R}^N$, the set $P_{\epsilon,K}(f)$ of (ϵ, K)-periods for f is relatively dense, i.e., if there exists $\lambda > 0$ such that any interval of length λ contains at least one element of $P_{\epsilon,K}(f)$. We will write $P_{\epsilon,R}(f)$ for $P_{\epsilon,B_R(0)}(f)$.

The following theorem shows that the class of almost periodic functions can be characterized as the set of continuous functions whose translation in time constitute a precompact set w.r.t. the uniform topology on $\mathbb{R}$.

Theorem 2.2 (Bochner's criterion)**.** *Let $H(f)$ be the closure of the set $\{f(\cdot + \tau, x) : \tau \in \mathbb{R}\}$ with respect to the compact-open topology of $C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^m)$. A function $f(t,x)$ is almost periodic in t uniformly for x in compact sets if and only if $H(f)$ is compact in the compact open topology.*

Note that any $g \in H(f)$ is almost periodic uniformly on compact sets and that $P_{\epsilon,K}(g) = P_{\epsilon,K}(f)$.

Definition 2.3. A function $f \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^m)$ is a *limit*-periodic function if there exists a sequence $(f_k) \subset C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^m)$ of T_k-periodic functions such that $f_k(t,x) \to f(t,x)$, as $k \to +\infty$, uniformly w.r.t. $t \in \mathbb{R}$ and x in compact subsets of $\mathbb{R}^N$.

It is easy to verify that a limit-periodic function is almost-periodic.

The almost periodic functions admit a Fourier analysis. In fact, the Bohr transform of an almost periodic function $f \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^m)$,

$$a(f, \lambda, x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(x,t) e^{-i\lambda t}\, dt$$

exists uniformly for x in compact sets of $\mathbb{R}^N$ for any real λ. Moreover, if $f(t,x)$ is almost periodic in t uniformly for x in compact sets of $\mathbb{R}^N$ then there is a countable set $\exp(f) \subset \mathbb{R}$ such that $a(f,\lambda,x) = 0$ for any $x \in \mathbb{R}^N$ if and only if $\lambda \notin \exp(f)$. If $\lambda \in \exp(f)$ we say that λ is an exponent of f and we say that $a(f,\lambda,x)$ is the Fourier coefficients of f relative to λ.

Definition 2.4. Following M. Bohr we say that a function $f \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^m)$ is *quasi*-periodic if all the exponents of f are of the form

$$\lambda_n = m_1\beta_1 + \cdots + m_k\beta_k$$

where the numbers m_i are integers and $\{\beta_1, \cdots, \beta_k\}$ is a finite subset of $\mathbb{R}$.

We also introduce the modulus of an almost periodic function, $\mathrm{mod}(f)$, defined as the smallest additive group containing $\exp(f)$. It is immediate to recognize that $\mathrm{mod}(f) = \{\sum_{i=1}^{l} m_i\lambda_i \mid m_i \in \mathbb{Z}, l \in \mathbb{N}, \lambda_i \in \exp(f)\}$.

We will say that $(t_n) \subset \mathbb{R}$ is an f-sequence if for any $x \in \mathbb{R}^N$ we have $f(t - t_n, x) \to f(t,x)$ uniformly w.r.t. $t \in \mathbb{R}$.

There is a duality relation between the set of f-sequences and the set of exponents of f. We refer to [8, chapter 4] for a complete discussion of the subject. We recall here the following result:

Lemma 2.5. *Let $f(t,x)$ and $g(t,x)$ be almost periodic functions in t uniformly for x in compact sets of $\mathbb{R}^N$; then the two following statements are equivalent*

(1) $\mathrm{mod}(g) \subset \mathrm{mod}(f)$
(2) *If (t_n) is an f-sequence then $\exists (t_{n_k}) \subset (t_n)$ which is a g-sequence.*

Remark 2.6. f is limit periodic if and only if there exists $\beta \in \mathbb{R}$ such that

$$\mathrm{mod}(f) \subset \{r\beta : r \in \mathbb{Q}\}.$$

f is quasi-periodic if and only if there exists $\beta_1, \ldots, \beta_k \in \mathbb{R}$, such that

$$\mathrm{mod}(f) = \{m_1\beta_1 + \cdots + m_k\beta_k : m_i \in \mathbb{N}\}.$$

3. A shadowing lemma. In this section, following [2], we prove a "shadowing lemma" for (DF) and (DF$_\lambda$). In all the sequel we fix $R = \|\bar{v}\|_X + 1$, $\bar{v}$ given by (g4), and we denote by C a positive constant which may change from time to time.

For all $\tau \in \mathbb{R}$ we define the operator $A_\tau : X_0 \to Y_0$ by $A_\tau u = \ddot{u} + \nabla_q g(t, \bar{v}(t-\tau))u$. Since the set of invertible operators $\mathrm{Inv}(X_0, Y_0)$ is open in the set of bounded linear operators $L(X_0, Y_0)$ and thanks to the almost periodicity of $\nabla_q g$, we have:

Lemma 3.1. *There exists $\bar{\epsilon} > 0$ such that for any $\tau \in P_{\bar{\epsilon},R}(\nabla_q g)$ the operator $A_\tau : X_0 \to Y_0$ is invertible and $\|A_\tau^{-1}\|_{Y_0,X_0} \le C$ (with $C > 0$ independent on $\tau \in P_{\bar{\epsilon},R}(\nabla_q g)$).*

Proof. Let $\tau \in \mathbb{R}$ and introduce an auxiliary operator $\tilde{A}_\tau : X_0 \to Y_0$ defined by $\tilde{A}_\tau u = \ddot{u} + \nabla_q g(t-\tau, \bar{v}(t-\tau))u$. We have $\tilde{A}_\tau = T_\tau A_0 T_{-\tau}$, where $T_\tau u \equiv u(\cdot - \tau)$, hence, by (g4), $\tilde{A}_\tau$ has a bounded inverse given by $\tilde{A}_\tau^{-1} = T_\tau A_0^{-1} T_{-\tau}$ and $\|\tilde{A}_\tau^{-1}\|_{Y_0,X_0} = \|A_0^{-1}\|_{Y_0,X_0}$. Then we have

$$\|A_\tau - \tilde{A}_\tau\|_{X_0,Y_0} \le \sup_{t\in\mathbb{R}} \sup_{|x|\le R} |\nabla_q g(t,x) - \nabla_q g(t-\tau, x)|.$$

Hence, there exists $\bar{\epsilon} > 0$ such that, for any $\tau \in P_{\bar{\epsilon},R}(\nabla_q g)$, we have $\|A_\tau - \tilde{A}_\tau\|_{X_0,Y_0} \le \frac{1}{2}\|A_0^{-1}\|_{Y_0,X_0}^{-1}$ and we get that A_τ is invertible. Moreover, $\|A_\tau^{-1}\|_{Y_0,X_0} \le 2\|A_0^{-1}\|_{Y_0,X_0}$, for any $\tau \in P_{\bar{\epsilon},R}(\nabla_q g)$.

Let us introduce some additional notations. Given $\epsilon > 0$, $N \in \mathbb{N}$ and a sequence $(p_j)_{j\in\mathbb{Z}} \subset P_{\epsilon,R}(\nabla_q g)$ such that $p_{j+1} - p_j \ge N$, we let, for all $j \in \mathbb{Z}$,

$$I_j = \left(\tfrac{p_{j-1}+p_j}{2} - \tfrac{N}{4}, \tfrac{p_j+p_{j+1}}{2} + \tfrac{N}{4}\right),$$

$M_j = I_{j-1} \cap I_j$ and $M = \cup_{j\in\mathbb{Z}} M_j$. Note that $|I_j \setminus M| \ge \frac{N}{2}$.

We set also $X_j = C^2(\bar{I}_j, \mathbb{R}^N)$ and $Y_j = C(\bar{I}_j, \mathbb{R}^N)$, $j \in \mathbb{Z}$ and choose functions $\phi_j \in C_c^\infty(I_j, \mathbb{R}_+)$ such that

(1) $\phi_j(t) = 1$ for any $t \in I_j \setminus M$;
(2) $\sum_{j\in\mathbb{Z}} \phi_j(t) = 1$ for all $t \in \mathbb{R}$;
(3) $\sup_{j\in\mathbb{Z}, t\in I_j} \{\phi_j'(t), \phi_j''(t)\} \le CN^{-1}$.

Letting

$$\psi_j(t) = \frac{\phi_j(t)}{(\sum_{i\in\mathbb{Z}} \phi_i(t)^2)^{1/2}} \in C_c^\infty(I_j, \mathbb{R}_+)$$

we have that

(1) $\psi_j(t) = 1$ for any $t \in I_j \setminus M$;
(2) $\sum_{j\in\mathbb{Z}} \psi_j(t)^2 = 1$ for all $t \in \mathbb{R}$;
(3) $\sup_{j\in\mathbb{Z}, t\in I_j}\{\psi_j'(t), \psi_j''(t)\} \le CN^{-1}$.

In the following, we will consider the functions $\psi_j(t)$ together with its first and second derivatives as multiplication operators from X to X_j^c or from Y to Y_j^c where $X_j^c \subset X_0$ and $Y_j^c \subset Y_0$ denote the subspaces of functions with support contained in I_j.

Finally, for $p = (p_j)_{j\in\mathbb{Z}} \subset P_{\epsilon,R}(\nabla_q g)$ and $\sigma = (\sigma_j)_{j\in\mathbb{Z}} \subset \{0,1\}^{\mathbb{Z}}$ we denote

$$v_{p\sigma}(t) = \sum_{j\in\mathbb{Z}} \sigma_j \phi_j(t)\bar{v}(t - p_j), \qquad t \in \mathbb{R}. \tag{3.1}$$

Note that $v_{p\sigma} \in X$ and let $B_r(v_{p\sigma}) = \{u \in X \ : \ \|u - v_{p\sigma}\|_X \le r\}$. We define the bounded linear operator $A_{p\sigma} : X \to Y$ by

$$A_{p\sigma}u = \ddot{u} + \nabla_q g(t, v_{p\sigma}(t))u, \qquad u \in X. \tag{3.2}$$

Moreover, we consider the bounded linear operator $L_0 : X_0 \to Y_0$ given by $L_0 u = \ddot{u} + \nabla_q g(t, 0)u$. By assumption (g2) we have that L_0 has a bounded inverse.

Starting from the "local" inverses $A_{p_j}^{-1} : Y_0 \to X_0$, $(p_j)_{j\in\mathbb{Z}} \subset P_{\bar{\epsilon},R}(\nabla_q g)$ ($\bar{\epsilon}$ given by Lemma 3.1), we build up the inverse of the operator $A_{p\sigma}$. More precisely, we have

Lemma 3.2. *There exists $\bar{N} \in \mathbb{N}$ such that for any sequence*

$$(p_j) \subset P_{\bar{\epsilon},R}(\nabla_q g), \quad \textit{with } p_{j+1} - p_j \ge \bar{N}, \textit{ and } (\sigma_j)_{j\in\mathbb{Z}} \subset \{0,1\}^{\mathbb{Z}}$$

the operator $A_{p\sigma} : X \to Y$ is invertible and $\|A_{p\sigma}^{-1}\|_{Y,X} \le C$ (with $C > 0$ independent on the sequence $(p_j) \subset P_{\bar{\epsilon},R}(\nabla_q g)$ and $(\sigma_j) \subset \{0,1\}^{\mathbb{Z}}$).

Proof. Given $N \in \mathbb{N}$ and $(p_j) \in P_{\bar{\epsilon},R}(\nabla_q g)$, with $p_{j+1} - p_j \ge N$, we define $A_j \equiv A_{p_j}$ if $\sigma_j = 1$ and $A_j \equiv L_0$ if $\sigma_j = 0$. Then, we consider the auxiliary operator $S_{p\sigma} : Y \to X$ defined by $S_{p\sigma}u = \sum_{j\in\mathbb{Z}} \psi_j A_j^{-1}(\psi_j u)$. By Lemma 3.1, we have

$$\|S_{p\sigma}u\|_X \le 3 \sup_{j\in\mathbb{Z}} \|\psi_j A_j^{-1}(\psi_j u)\|_{X_j} \le C\|u\|_Y.$$

Hence, the operator $S_{p\sigma}$ is well defined and bounded.

We claim that, for $N \in \mathbb{N}$ sufficiently large, both the operators $A_{p\sigma}S_{p\sigma} : Y \to Y$ and $S_{p\sigma}A_{p\sigma} : X \to X$ are invertible. Then, we get immediately that $A_{p\sigma}$ is invertible with inverse given by $A_{p\sigma}^{-1} \equiv (S_{p\sigma}A_{p\sigma})^{-1}S_{p\sigma} = S_{p\sigma}(A_{p\sigma}S_{p\sigma})^{-1} : Y \to X$.

Let us prove the claim. By the definition of $S_{p\sigma}$, we have

$$S_{p\sigma}A_{p\sigma} = I_X + \sum_{j\in\mathbb{Z}} \psi_j A_j^{-1}(A_{p\sigma} - A_j)\psi_j + \sum_{j\in\mathbb{Z}} \psi_j A_j^{-1}[\psi_j, A_{p\sigma}],$$

where $[L, T] = LT - TL$. Using the fact that

$$\sup_{j\in\mathbb{Z}} \|v_{p\sigma} - \sigma_j \bar{v}(\cdot - p_j)\|_{Y_j} \le C \sup_{|t|\ge N/2} |\bar{v}(t)|,$$

let us show that the last two terms are arbitrarily small for $N \in \mathbb{N}$ sufficiently large. Indeed, we have

$$\begin{aligned}\|\sum_{j\in\mathbb{Z}} \psi_j A_j^{-1}(A_{p\sigma} - A_j)\psi_j u\|_X &\le C \sup_{j\in\mathbb{Z}} (\|A_{p\sigma} - A_j\|_{X_j, Y_j} \|u\|_{X_j}) \\ &\le C \sup_{t\in\mathbb{R}} \sup_{|x|\le R} |g_{q,q}(t,x)| (\sup_{j\in\mathbb{Z}} \|v_{p\sigma} - \sigma_j \bar{v}(\cdot - p_j)\|_{Y_j}) \|u\|_X\end{aligned}$$

and

$$\begin{aligned}\|\sum_{j\in\mathbb{Z}} \psi_j A_j^{-1}[\psi_j, A_{p\sigma}]u\|_X &\le C \sup_{j\in\mathbb{Z}} \|[\psi_j, A_{p\sigma}]u\|_{Y_j} \\ &\le C \sup_{j\in\mathbb{Z}} \|2\psi_j' u' + \psi_j'' u\|_{Y_j} \le CN^{-1}\|u\|_X\end{aligned}$$

Hence, for $N \in \mathbb{N}$ sufficiently large, we get $\|S_{p\sigma}A_{p\sigma} - I_X\|_{X,X} \le \frac{1}{2}$. Analogously, we get also $\|A_{p\sigma}S_{p\sigma} - I_Y\|_{Y,Y} \le \frac{1}{2}$, for $N \in \mathbb{N}$ eventually larger. Then, the operators $S_{p\sigma}A_{p\sigma} : X \to X$ and $A_{p\sigma}S_{p\sigma} : Y \to Y$ are invertible and the claim is proved. Moreover, since

$$\|(S_{p\sigma}A_{p\sigma})^{-1}\|_{X,X}, \|(A_{p\sigma}S_{p\sigma})^{-1}\|_{Y,Y} \le 2,$$

we get $\|A_{p\sigma}^{-1}\|_{Y,X} \le 2\|S_{p\sigma}\|_{Y,X}$.

Given $N \ge \bar{N}$, a sequence $(p_j) \subset P_{\bar{\epsilon},R}(\nabla_q g)$, with $p_{j+1} - p_j \ge N$ ($\bar{\epsilon}$ and $\bar{N}$ given by Lemmas 3.1 and 3.2) and a sequence $(\sigma_j) \subset \{0,1\}^{\mathbb{Z}}$, we introduce the nonlinear map $G_{p\sigma} \in C^1(X, X)$ given by

$$G_{p\sigma}(w) = w - A_{p\sigma}^{-1}F(w), \qquad w \in X \tag{3.3}$$

where $F(w) = \ddot{w} + g(t, w)$. Fixed points of $G_{p\sigma}$ are solutions of (DF).

Lemma 3.3. *There exists $\rho > 0$ such that for any $r \in (0, \rho)$ there exists $N_r \in \mathbb{N}$ and $\epsilon_r > 0$ for which for any sequence $(p_j) \subset P_{\epsilon_r,R}(\nabla_q g)$, with $p_{j+1} - p_j \geq N_r$ and any sequence $(\sigma_j) \subset \{0,1\}^{\mathbb{Z}}$, we have $\|G'_{p\sigma}(w)\|_{X,X} \leq \frac{1}{2}$ for all $w \in B_\rho(v_{p\sigma})$ and $\|G_{p\sigma}(v_{p\sigma}) - v_{p\sigma}\|_X \leq \frac{r}{4}$ (where $v_{p\sigma}$ is given by* (3.1)*).*

Proof. Given $\epsilon \in (0, \bar{\epsilon})$, $N \geq \bar{N}$, a sequence $p = (p_j) \subset P_{\epsilon,R}(\nabla_q g)$, with $p_{j+1} - p_j \geq N$, and a sequence $(\sigma_j) \subset \{0,1\}^{\mathbb{Z}}$ let $v_{p\sigma} \in X$, $A_{p\sigma} \in L(X,Y)$ and $G_{p\sigma} \in C^1(X,Y)$ be given, respectively, by (3.1), (3.2) and (3.3).

Since $G'_{p\sigma}(v_{p\sigma}) = 0$, there exists $\rho > 0$ such that $\|G'_{p\sigma}(w)\|_{X,X} \leq \frac{1}{2}$ for any $w \in B_\rho(v_{p\sigma})$. Moreover, for $w \in B_\rho(v_{p\sigma})$, we define $F_\tau(w) = \ddot{w} + g(t-\tau, w)$. Since $F_\tau(T_\tau \bar{v}) = 0$ for any $\tau \in \mathbb{R}$, we have, by Lemma 3.2,

$$\begin{aligned}
\|G_{p\sigma}(v_{p\sigma}) - v_{p\sigma}\|_X &\leq C \sup_{j\in\mathbb{Z}} \|F(v_{p\sigma}) - F_{p_j}(\sigma_j T_{p_j}\bar{v})\|_{Y_j} \\
&\leq C \sup_{j\in\mathbb{Z}} (\|F(v_{p\sigma}) - F(\sigma_j T_{p_j}\bar{v})\|_{Y_j} + \|F(\sigma_j T_{p_j}\bar{v}) - F_{p_j}(\sigma_j T_{p_j}\bar{v})\|_{Y_j}) \\
&\leq C \sup_{j\in\mathbb{Z}} (\|v_{p\sigma} - \sigma_j T_{p_j}\bar{v}\|_{X_j} + \sup_{t\in\mathbb{R}} \sup_{|x|\leq R} |g(t,x) - g(t-p_j,x)|).
\end{aligned}$$

We first observe that, since $g(t,x) = \int_0^1 \nabla_q g(t,sx)x\,ds$, if $\tau \in P_{\epsilon,R}(\nabla_q g)$, then $\tau \in P_{R\epsilon,R}(g)$. Hence $\sup_{t\in\mathbb{R}} \sup_{|x|\leq R} |g(t,x) - g(t-p_j,x)|$ is small for $p_j \in P_{\epsilon,R}(\nabla_q g)$, ϵ sufficiently small.

Then, since $\sup_{j\in\mathbb{Z}} \|v_{p\sigma} - \sigma_j T_{p_j}\bar{v}\|_{X_j} \leq C \sup_{|t|\geq N/2}(|\bar{v}(t)| + |\bar{v}'(t)| + |\bar{v}''(t)|)$, for any $r \in (0,\rho)$, there exists $N_r \geq \bar{N}$ and $\epsilon_r \in (0, \bar{\epsilon})$ such that, for any sequence $(p_j) \subset P_{\epsilon_r,R}(\nabla_q g)$, with $p_{j+1} - p_j \geq N_r$, we have $\|G_{p\sigma}(v_{p\sigma}) - v_{p\sigma}\|_X \leq r/4$.

Finally, as a consequence of the above lemmas, we have

Theorem 3.4. *If* (g1-4) *hold then for any $r \in (0,\rho)$, for any sequence $(p_j) \subset P_{\epsilon_r,R}(\nabla_q g)$, with $p_{j+1} - p_j \geq N_r$, and any sequence $(\sigma_j) \subset \{0,1\}^{\mathbb{Z}}$ there exists a unique $u_{p\sigma} \in B_r(v_{p\sigma})$ solution of* (DF) *(where $\rho > 0$, $N_r \in \mathbb{N}$, $\epsilon_r > 0$ are given by Lemma* 3.3 *and $v_{p\sigma}$ by* (3.1)*).*

Proof. Given $r \in (0,\rho)$, a sequence $p = (p_j) \subset P_{\epsilon_r,R}(\nabla_q g)$, with $p_{j+1} - p_j \geq N_r$, and a sequence $(\sigma_j) \subset \{0,1\}^{\mathbb{Z}}$, let $v_{p\sigma} \in X$, $A_{p\sigma} \in L(X,Y)$ and $G_{p\sigma} \in C^1(X,Y)$ be given, respectively, by (3.1), (3.2) and (3.3).

By Lemma 3.3, the map $G_{p\sigma}$ is a contraction in $B_\rho(v_{p\sigma})$; moreover, for $w \in B_\rho(v_{p\sigma})$ we have that $\|G_{p\sigma}(w) - v_{p\sigma}\|_X \leq \frac{1}{2}\|w - v_{p\sigma}\|_X + \|G_{p\sigma}(v_{p\sigma}) - v_{p\sigma}\|_X$, hence, in particular, we have $G_{p\sigma} : B_r(v_{p\sigma}) \to B_r(v_{p\sigma})$. Therefore, by the contraction mapping principle, there exists unique $u_{p\sigma} \in B_r(v_{p\sigma})$ solution of (DF).

The above shadowing lemma can be generalized to cover perturbations of (DF); namely,

$$-\ddot{q} = g(t, q) + \lambda h(t, q, \dot{q}). \tag{DF$_\lambda$}$$

We recall that h satisfies assumption

(h) $h(t, \cdot, \cdot) \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ for any $t \in \mathbb{R}$ and $h(\cdot, x, y) \in C(\mathbb{R}, \mathbb{R}^N)$ is almost periodic, uniformly for (x, y) in compact sets of $\mathbb{R}^N \times \mathbb{R}^N$.

Together, with (DF$_\lambda$), let us consider the family of systems

$$-\ddot{x} = \bar{g}(t, x) + \lambda \bar{h}(t, x, \dot{x}), \qquad \text{with} \quad \bar{g} \in H(g),\ \bar{h} \in H(h).$$

We have

Corollary 3.5. *If* (g1-4) *and* (h) *hold, then for any $r \in (0, \rho)$ there exists $\lambda_r > 0$ such that if $(p_j) \subset P_{\epsilon_r, R}(\nabla_q g)$, with $p_{j+1} - p_j \geq N_r$ (ρ, ϵ_r and N_r given by Lemma* 3.3), *$(\sigma_j) \subset \{0,1\}^{\mathbb{Z}}$, $\bar{g} \in H(g)$, $\bar{h} \in H(h)$ and $t_n \to +\infty$ is such that $\nabla_q g(t - t_n, x) \to \nabla_q \bar{g}(t, x)$, then there exists $n_r \in \mathbb{N}$ such that for all $n \geq n_r$ and $\lambda \in (-\lambda_r, \lambda_r)$, there exists a unique $u_{\lambda,n} \in B_r(v_{p\sigma}(\cdot - t_n))$ solution of*

$$-\ddot{x} = \bar{g}(t, x) + \lambda \bar{h}(t, x, \dot{x})$$

(*where $v_{p\sigma}$ is given in* (3.1)).

Proof. Given $r \in (0, \rho)$, $p = (p_j) \in P_{\epsilon_r, R}(\nabla_q g)$, with $p_{j+1} - p_j \geq N_r$ and a sequence $(\sigma_j) \subset \{0,1\}^{\mathbb{Z}}$, let $v_{p\sigma} \in X$, $A_{p\sigma} \in L(X, Y)$ and $G_{p\sigma} \in C^1(X, X)$ be given by (3.1), (3.2) and (3.3), respectively. Let $\bar{g} \in H(g)$ and $\bar{h} \in H(h)$, $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$, and consider the maps $\bar{G}_{\lambda n}, G_n \in C^1(X, X)$, given by $\bar{G}_{\lambda n}(w) = w - T_{t_n} A_{p\sigma}^{-1} T_{-t_n} \bar{F}_\lambda(w)$, where $\bar{F}_\lambda(w) = \ddot{w} + \bar{g}(t, x) + \lambda \bar{h}(t, w, \dot{w})$, and $G_n(w) = w - T_{t_n} A_{p\sigma}^{-1} T_{-t_n} F_n(w)$, where $F_n(w) = \ddot{w} + g(t - t_n, w)$. Note that $G_n(T_{t_n} w) = T_{t_n} G_{p\sigma}(w)$ and $F'_n(T_{t_n} w) = T_{t_n} F'(w) T_{-t_n}$. Therefore, we have $G'_n(T_{t_n} v_{p\sigma}) = 0$ and $\|G'_n(w)\|_{X,X} \leq \frac{1}{2}$ for any $w \in B_\rho(T_{t_n} v_{p\sigma})$.

Moreover, by (h), there exists $M > 0$ such that, for any $\bar{h} \in H(h)$, and $|x|, |y| \leq R$, $\sup_{t \in \mathbb{R}}(|\bar{h}(t, x, y)| + |\bar{h}_x(t, x, y)| + |\bar{h}_y(t, x, y)|) \leq M$.

Then,for any $w \in B_r(T_{t_n} v_{p\sigma})$, we have

$$\|\bar{G}'_{\lambda n}(w) - G'_n(w)\|_{X,X} \leq C(\sup_{t \in \mathbb{R}} \sup_{|x| \leq R} |\nabla_q \bar{g}(t, x) - \nabla_q g(t - t_n, x)| + \lambda M).$$

Hence, there exists $\bar{n} \in \mathbb{N}$ and $\bar{\lambda} > 0$ such that $\|\bar{G}'_{\lambda n}(w)\|_{X,X} \leq \frac{2}{3}$ for any $w \in B_r(T_{t_n} v_{p\sigma})$, $n \geq \bar{n}$ and $\lambda \in (-\bar{\lambda}, \bar{\lambda})$. Moreover, since $\|G_n(T_{t_n} v_{p\sigma}) - T_{t_n} v_{p\sigma}\|_X = \|G_{p\sigma}(v_{p\sigma}) - v_{p\sigma}\|_X$, we have

$$\begin{aligned}\|\bar{G}_{\lambda n}(w) - T_{t_n} v_{p\sigma}\|_X \leq \tfrac{2}{3}\|w - T_{t_n} v_{p\sigma}\|_X \\ + \|\bar{G}_{\lambda n}(T_{t_n} v_{p\sigma}) - G_n(T_{t_n} v_{p\sigma})\|_X + \|G_{p\sigma}(v_{p\sigma}) - v_{p\sigma}\|_X.\end{aligned}$$

Then, since

$$\|\bar{G}_{\lambda n}(T_{t_n} v_{p\sigma}) - G_n(T_{t_n} v_{p\sigma})\|_X \leq C(\sup_{t \in \mathbb{R}} \sup_{|x| \leq R} |\bar{g}(t,x) - g(t - t_n, x)| + \lambda M),$$

and thanks to Lemma 3.3, there exists $n_r \geq \bar{n}$ and $\lambda_r \leq \bar{\lambda}$ such that $\bar{G}_{\lambda n} : B_r(T_{t_n} v_{p\sigma}) \to B_r(T_{t_n} v_{p\sigma})$, for any $n \geq n_r$ and $\lambda \in (-\lambda_r, \lambda_r)$. Therefore, by the contraction mapping principle, there exists unique $u_{\lambda,n} \in B_r(T_{t_n} v_{p\sigma})$, solution of

$$-\ddot{x} = \bar{g}(t,x) + \lambda \bar{h}(t, x, \dot{x}).$$

4. Limit periodic solutions. In this section we show that, among the class of solutions of (DF) given by theorem 3.4, there are infinitely many limit periodic solutions.

To begin, let us recall a property of limit periodic functions [5, p. 33].

Lemma 4.1. *Let $\nabla_q g$ be a limit periodic function, and let $(g_k) \subset C^3(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$ be a sequence of periodic functions in t such that $g_k(t,x) \to g(t,x)$ and $\nabla_q g_k(t,x) \to \nabla_q g(t,x)$ as $k \to \infty$ uniformly with respect to $t \in \mathbb{R}$ and with respect to x in compact subsets of $\mathbb{R}^N$. Then there exist $k_0 \in \mathbb{N}$ and $T > 0$ such that, for any $k \geq k_0$, the periods T_k of the functions $g_k(t,x)$ are rational multiples of T, i.e., $T_k = (a_k T)/b_k$, for some a_k, $b_k \in \mathbb{N}$. For any $k \geq k_0$, we set $\bar{T}_k = a_k T = b_k T_k$ (we can always assume $\bar{T}_k \geq 1$).*

Remark 4.2. Note that for any $\epsilon > 0$ there exists $k_\epsilon \in \mathbb{N}$ such that $n\bar{T}_k \in P_{\epsilon,R}(\nabla_q g)$ for any $k \geq k_\epsilon$ and $n \in \mathbb{Z}$.

Given $m \in \mathbb{N}$, consider a periodic sequence $(p_j) \subset \mathbb{Z}(m\bar{T}_{k_\epsilon})$, with $p_{j+l} - p_j = qm\bar{T}_{k_\epsilon}$, for some $l, q \in \mathbb{N}$. In this situation we can take the partition of unity introduced in §3 in the following way: $\phi_j(t) = \phi_{j+l}(t + p_{j+l} - p_j)$. Then, considering a sequence $(\sigma_j) \subset \{0,1\}^{\mathbb{Z}}$, with $\sigma_j = 1$ for all $j \in \mathbb{Z}$ and setting $v_p \equiv v_{p\sigma}$, with $v_{p\sigma}$ given by (3.1) we have $v_p(t + qm\bar{T}_{k_\epsilon}) = v_p(t)$ for all $t \in \mathbb{R}$.

For $k \in \mathbb{N}$, we consider the T_k-periodic second order systems

$$-\ddot{x} = g_k(t, x). \tag{DF$_k$}$$

The problems (DF$_k$) can be seen as perturbations of (DF), and indeed, by Corollary 3.5, it is possible to show that a "shadowing lemma" holds also for (DF$_k$), for $k \in \mathbb{N}$ sufficiently large. Let us prove now how such a shadowing lemma implies the existence of a class of periodic solutions of (DF$_k$), that, as we will prove in Theorem 4.4, converge as $k \to \infty$ (in the uniform norm) to solutions of (DF).

Lemma 4.3. *For any $r \in (0, \rho)$ there exists $k_r \in \mathbb{N}$ such that for any sequence $(p_j) \subset \mathbb{Z}(N_r \bar{T}_{k_r})$, with $p_{j+l} - p_j = qN_r\bar{T}_{k_r}$, for some $l, q \in \mathbb{N}$, (where $\rho > 0$, $N_r \in \mathbb{N}$ are given by Lemma* 3.3 *and $\bar{T}_{k_r}$ is defined in Lemma* 4.1), *there exists unique $u_k \in B_r(v_p)$, periodic solution of (DF$_k$), for all $k \geq k_r$.*

Proof. Let us take a *periodic* sequence (p_j) as in the statement of the lemma.

By Corollary 3.5, we get that for any $r \in (0, \rho)$ there exists $k_r \in \mathbb{N}$ such that (DF$_k$) admits a unique solution u_k in $B_r(v_p)$, for any $k \geq k_r$.

Let $\tilde{T}_k = a_k(qN_r\bar{T}_{k_r}) = (qN_r a_{k_r} b_k)T_k$ (where a_k, b_k and $\bar{T}_{k_r}$ were introduced in Lemma 4.1). Hence, $\tilde{T}_k$ is a multiple of the period $qN_r\bar{T}_{k_r}$ of the sequence (p_j) as well as a multiple of T_k and, by remark 4.2, $v_p(t+\tilde{T}_k) = v_p(t)$ for all $t \in \mathbb{R}$. It is then immediate to check that u_k is periodic: indeed, $u_k(\cdot + \tilde{T}_k)$ is also a solution of (DF$_k$) in $B_r(v_p)$. By uniqueness, $u_k(t) = u_k(t + \tilde{T}_k)$ for all $t \in \mathbb{R}$.

For any $r \in (0, \rho)$, ρ given by Lemma 3.3, we introduce the set of periodic sequences $Z(r) = \{p = (p_j) \subset \mathbb{Z}(N_r\bar{T}_{k_r}) : p_{j+l} - p_j = qN_r\bar{T}_{k_r}$ for some $l, q \in \mathbb{N}\}$, where ϵ_r, N_r are given by Lemma 3.3. and k_r by Lemma 4.3.

Finally, we have the following result:

Theorem 4.4. *If* (g1-4) *hold, then there exist infinitely many limit periodic solutions of* (DF). *More precisely, for any $r \in (0, \rho)$ (ρ given by Lemma* 3.3) *and for any sequence $(p_j) \subset Z(r)$ there exists a limit periodic solution of (DF), $u_p \in B_r(v_p)$.*

Proof. Fix a sequence $(p_j) \subset Z(r)$. By theorem 3.4 there exists unique $u \in B_r(v_p)$ solution of (DF). Take then a sequence (g_k) of periodic functions as in Lemma 4.1. By Lemma 4.3 there is, a unique $u_k \in B_r(v_p)$ periodic solution of (DF_k) for $k \geq k_r$. Using Ascoli's theorem we deduce that, up to a subsequence, (u_k) converges in C^1_{loc} to $\tilde{u} \in B_r(v_p)$. Since $g_k(t,x) \to g(t,x)$ uniformly for $t \in \mathbb{R}$ and $|x| \leq R$, $\tilde{u}$ is a solution of (DF). Then, by uniqueness, $u = \tilde{u}$.

We only have to show that up to subsequences $u_k \to u$ uniformly in $\mathbb{R}$. By the definition of limit-periodic functions this will imply that u is limit periodic (see §2). Suppose that this is not true. Then there exists $\lambda > 0$ such that

$$\sup_{t\in\mathbb{R}} |u_k(t) - u(t)| \geq \lambda \qquad \forall k \geq k_r.$$

Then there exist a sequence $(s_k) \subset \mathbb{R}$, which we can take of the form $n_k q N_r \bar{T}_{k_r}$, $n_k \in \mathbb{Z}$ such that

$$\sup_{t\in[0,qN_r\bar{T}_{k_r}]} |u_k(t+s_k) - u(t+s_k)| \geq \frac{\lambda}{2} \qquad \forall k \geq k_r.$$

Using again Ascoli's theorem we get that (in C^1_{loc}, up to subsequences) $u_k(\cdot + s_k) \to w_1$ and $u(\cdot + s_k) \to w_2$. Since g is almost periodic there exists $\tilde{g} \in H(g)$ such that $g(\cdot + s_k, x) \to \tilde{g}(\cdot, x)$ uniformly in t and for $|x| \leq R$. Then, since for all $t \in \mathbb{R}$ and $|x| \leq R$,

$$\begin{aligned} &|g_k(t+s_k,x) - \tilde{g}(t,x)| \\ &\qquad \leq |g_k(t+s_k,x) - g(t+s_k,x)| + |g(t+s_k,x) - \tilde{g}(t,x)| \\ &\qquad \leq \|g_k(\cdot,x) - g(\cdot,x)\|_Y + \|g(\cdot+s_k,x) - \tilde{g}(\cdot,x)\|_Y \end{aligned}$$

we have also that $g_k(\cdot + s_k, x) \to \tilde{g}(\cdot, x)$ uniformly in $t \in \mathbb{R}$ and for $|x| \leq R$. Finally, both w_1 and w_2 are solutions of the same equation

$$-\ddot{x} = \tilde{g}(t,x).$$

On the other hand, by the choice of the sequence (s_k) we have that w_1 and $w_2 \in B_r(v_p)$ (see remark 4.2). By uniqueness (which holds also for this equation in the same ball $B_r(v_p)$, see corollary 3.5), we deduce that $w_1 = w_2$, a contradiction which proves the theorem.

5. Almost and quasi periodic forcing. In this section we will prove the existence of almost periodic solutions for

$$-\ddot{x} = g(t,x) + \lambda h(t,x,\dot{x}) \qquad (\mathrm{DF}_\lambda,)$$

where $\lambda \in \mathbb{R}$ is a small perturbation parameter, by showing that the solutions of (DF_λ) given by Corollary 3.5 are almost periodic functions.

Theorem 5.1. *Let* (g1-4) *and* (h) *hold. Then, for any* $r \in (0, \rho)$ *and for any sequence* $(p_j) \subset Z(r)$ *there exists an almost periodic solution of (DF*$_\lambda$*),* $u_\lambda \in B_r(v_p)$*, for any* $\lambda \in (-\lambda_r, \lambda_r)$*, (where* ρ *and* λ_r *are given by Lemma* 3.3 *and Corollary* 3.5*, respectively).*

Proof. Given $r \in (0, \rho)$, $\lambda \in (-\lambda_r, \lambda_r)$ and $p \in Z(r)$, we set $P \equiv p_{j+l} - p_j$ (i.e., the period of the sequence (p_j)). Let $u_\lambda \in B_r(v_p)$ be the solution of (DF$_\lambda$) given by Corollary 3.5.

We claim that u_λ is almost periodic. For any sequence $(t_n) \subset \mathbb{R}$, let us consider the sequence $(T_{t_n} u_\lambda) \subset X$. Since it is bounded in X it converges, up to subsequences, in the C^1_{loc} topology by Ascoli's theorem. Now, arguing by contradiction, let us suppose that u_λ is not almost periodic. By the Bochner's criterion, there exists a sequence $(\bar{t}_n) \subset \mathbb{R}$ such that $(T_{\bar{t}_n} u_\lambda)$ is not precompact in $C(\mathbb{R}, \mathbb{R}^N)$. Then, there exist $\mu > 0$ and a subsequence of $(T_{\bar{t}_n} u_\lambda)$, that we denote again by $(T_{\bar{t}_n} u_\lambda)$, such that for any $n \in \mathbb{N}$

$$\|T_{\bar{t}_n} u_\lambda - z_\lambda\|_Y \geq \mu, \tag{5.1}$$

where $z_\lambda \in C^1(\mathbb{R}, \mathbb{R}^N)$ is the C^1_{loc}-limit. Note that, since u_λ is uniformly continuous, we can always assume $(\bar{t}_n) \subset \mathbb{Z}P$.

By the Bochner's criterion, $g(t + \bar{t}_n, x) \to \bar{g}(t, x)$, and $h(t + \bar{t}_n, x, y) \to \bar{h}(t, x, y)$ uniformly w.r.t. $t \in \mathbb{R}$ and $|x|, |y| \leq R$ for some $\bar{g} \in H(g)$, $\bar{h} \in H(h)$. Hence, in particular, $z_\lambda \in X$ is a solution of the system $-\ddot{x} = \bar{g}(t, x) + \lambda \bar{h}(t, x, \dot{x})$. Now, by (5.1), there exists a sequence $(s_n) \subset \mathbb{Z}P$, $s_n \to \infty$ such that, for any $n \in \mathbb{N}$,

$$\sup_{t \in [0, P]} |T_{\bar{t}_n + s_n} u_\lambda(t) - T_{s_n} z_\lambda(t)| \geq \frac{\mu}{2}. \tag{5.2}$$

Again, by Ascoli's theorem, there exist $w_\lambda, \tilde{w}_\lambda \in C^1(\mathbb{R}, \mathbb{R}^N)$ such that, up to subsequences, $T_{\bar{t}_n + s_n} u_\lambda \to w_\lambda$ and $T_{s_n} z_\lambda \to \tilde{w}_\lambda$, in C^1_{loc}. Then, arguing as above, we have that there exists $\tilde{g} \in H(g)$ and $\tilde{h} \in H(h)$ such that, up to subsequences, $g(\cdot + \bar{t}_n + s_n, x) \to \tilde{g}(\cdot, x)$ and $h(\cdot + \bar{t}_n + s_n, x, y) \to \tilde{h}(\cdot, x, y)$ in $C(\mathbb{R}, \mathbb{R}^N)$, uniformly for $|x|, |y| \leq R$. Moreover, since all the convergences are uniform, we get also $\bar{g}(\cdot + s_n, x) \to \tilde{g}(\cdot, x)$ and $\bar{h}(\cdot + s_n, x, y) \to \tilde{h}(\cdot, x, y)$ in $C(\mathbb{R}, \mathbb{R}^N)$, uniformly for $|x|, |y| \leq R$. Therefore, w_λ and $\tilde{w}_\lambda$ are both solutions of $-\ddot{x} = \tilde{g}(t, x) + \lambda \tilde{h}(t, x, \dot{x})$. Furthermore, since $p_{j+l} - p_j = P$ and $(\bar{t}_n), (s_n) \subset \mathbb{Z}P$, we have that $T_{\bar{t}_n + s_j} u_\lambda \in B_r(v_p)$, for any n, $j \in \mathbb{Z}$. Then, passing to the limit in C^1_{loc}, we get $w_\lambda \in B_r(v_p)$ as $j = n \to +\infty$ and, as

$n \to +\infty$ first, and then $j \to +\infty$, we get also $\tilde{w}_\lambda \in B_r(v_p)$ (recall that, by remark 4.2, $v_p(t) = v_p(t-P))$). Finally, w_λ and $\tilde{w}_\lambda$ are both solutions of

$$-\ddot{x} = \tilde{g}(t,x) + \lambda\tilde{h}(t,x,\dot{x}),$$

hence, by uniqueness we get $w_\lambda = \tilde{w}_\lambda$ in contradiction with (5.2).

Theorem 5.1 states the existence of a class of almost periodic solutions of (DF_λ). Now we want to show that there is a close relation between the time recurrence properties of the field $g_\lambda(t,x,y) = g(t,x) + \lambda h(t,x,y)$ and the ones of these solutions.

First of all, we prove a technical lemma. Setting, for $t \in \mathbb{R}$, $d(t, \mathbb{Z}T) = \inf_{k\in\mathbb{Z}} |t - kT|$, we have

Lemma 5.2. *For any $T > 0$ there exists $f \in C(\mathbb{R}, \mathbb{R}^N)$, periodic with period T such that if $(t_n) \subset \mathbb{R}$ is an $(f + g_\lambda)$-sequence then (t_n) is a g_λ-sequence and $d(t_n, \mathbb{Z}T) \to 0$ (see §2 for the definitions).*

Proof. If for every g_λ-sequence (t_n) we have that $d(t_n, \mathbb{Z}T) \to 0$, we can take $f = 0$. If not, we have that

$$\exists \text{ a } g_\lambda\text{-sequence } (s_n) \text{ such that } d(s_n, \mathbb{Z}T) \geq \xi > 0, \quad \forall n \in \mathbb{N}. \tag{5.3}$$

Let $\delta = \xi/12$, $M > 0$ and $\rho \in C([0,T], \mathbb{R}^N)$ be such that $\operatorname{supp}\rho \subset ((T/2) - \delta, (T/2) + \delta)$ and

$$|\rho(\tfrac{T}{2})| = 3 \sup_{t\in\mathbb{R}} \sup_{|x|,|y|\leq M} |g_\lambda(t,x,y)|. \tag{5.4}$$

We define $f : \mathbb{R} \to \mathbb{R}^N$ extending ρ by periodicity. It is immediate to recognize that f is continuous and periodic with minimal period T.

We claim now that if (t_n) is an $f + g_\lambda$-sequence, then (t_n) is a g_λ-M-sequence, i.e., $g_\lambda(t - t_n, x, y) \to g_\lambda(t,x,y)$ uniformly with respect to $t \in \mathbb{R}$ and $|x|, |y| \leq M$. Assuming by contradiction that this is not true, there exists a sequence $(t_n) \subset \mathbb{R}$ which is an $f + g_\lambda$-sequence but not a g_λ-M-sequence. Up to a subsequence we can assume that $f(t - t_n) \to f(t - t_0)$, for some $t_0 \in (-T/2, T/2)$, and $g_\lambda(t - t_n, x, y) \to \tilde{g}(t,x,y)$, uniformly with respect to $t \in \mathbb{R}$ and $|x|, |y| \leq M$ for some $\tilde{g} \in H(g_\lambda)$ with $\tilde{g}(t,x,y) \neq g_\lambda(t,x,y)$ for some $t \in \mathbb{R}$, $|x|, |y| \leq M$. We then obtain

$$f(t - t_0) - f(t) = g_\lambda(t,x,y) - \tilde{g}(t,x,y) \quad \forall t \in \mathbb{R},\ |x|, |y| \leq M. \tag{5.5}$$

From (5.4) and (5.5) we get $|t_0| \le 2\delta$. Letting

$$N_{3\delta} = \{t \in \mathbb{R} : |t - \tfrac{T}{2} + kT| \le 3\delta \quad \text{for some} \quad k \in \mathbb{Z}\},$$

we have that $f(t-t_0) - f(t) = 0\ \forall t \in \mathbb{R} \setminus N_{3\delta}$ and hence, $g_\lambda(t,x,y) = \tilde{g}(t,x,y)$ for any $t \in \mathbb{R} \setminus N_{3\delta}$ and $|x|, |y| \le M$. Then there exists $\bar{t} \in N_{3\delta}$ and $|\bar{x}|, |\bar{y}| \le M$ for which $\tilde{g}(\bar{t},\bar{x},\bar{y}) \ne g_\lambda(\bar{t},\bar{x},\bar{y})$.

Since (s_n) given by (5.3) is a g_λ (and hence also $\tilde{g}$)-sequence, there exists $\bar{n} \in \mathbb{N}$ such that $\tilde{g}(\bar{t} - s_n, \bar{x}, \bar{y}) \ne g_\lambda(\bar{t} - s_n, \bar{x}, \bar{y})$ for all $n \ge \bar{n}$. This implies that $\bar{t} - s_n \in N_{3\delta}$ for any $n \ge \bar{n}$ from which we derive that

$$d(s_n, \mathbb{Z}T) \le d(\bar{t} - \tfrac{T}{2}, \mathbb{Z}T) + d(s_n - \bar{t} + \tfrac{T}{2}, \mathbb{Z}T) \le 6\delta = \tfrac{\xi}{2},$$

a contradiction which proves our claim. From the claim follows that if (t_n) is an $(f + g_\lambda)$-sequence, then it is a g_λ-M-sequence. This plainly implies that (t_n) is an f-sequence. Then, since (t_n) is an f-sequence and an $(f + g_\lambda)$-sequence, it is a g_λ-sequence too. Moreover, since f has minimal period T and (t_n) is an f-sequence, we have $d(t_n, \mathbb{Z}T) \to 0$ and the lemma follows.

Let us now consider the set of almost periodic solutions of (DF$_\lambda$) given by Theorem 5.1. If $(p_j) \subset \mathbb{Z}(N_r\bar{T}_{k_r})$ and $p_{j+l} - p_j = P$ for a certain $l \in \mathbb{N}$, then the corresponding solution of (DF$_\lambda$) $u_\lambda \in B_r(v_p)$ is almost periodic. We note that if (t_n) is a g_λ-sequence and $d(t_n, \mathbb{Z}P) \to 0$, then along a subsequence $u_\lambda(t - t_n) \to u_\lambda(t)$ uniformly for $t \in \mathbb{R}$. Indeed, by the Bochner's criterion, along a subsequence $u_\lambda(\cdot - t_n) \to v$. Since (t_n) is a g_λ-sequence we have that v is a solution of (DF$_\lambda$). Since $d(t_n, \mathbb{Z}P) \to 0$ we have that $v \in B_r(v_p)$ and by uniqueness we deduce that $v = u_\lambda$. By Lemma 5.2 we can choose a periodic function f with period P, such that if (t_n) is an $(f + g_\lambda)$-sequence, then (t_n) is a g_λ-sequence and $d(t_n, \mathbb{Z}P) \to 0$. Therefore, we have the following chain of implications:

$$\begin{aligned} (t_n) \text{ is a } (f + g_\lambda)\text{-sequence} \quad &\Rightarrow \quad (t_n) \text{ is a } g_\lambda\text{-sequence and } d(t_n, \mathbb{Z}P) \to 0 \\ &\Rightarrow \quad \exists (t_{n_k}) \subset (t_n) \text{ which is a } u_\lambda\text{-sequence.} \end{aligned}$$

By Lemma 2.5 we conclude:

Theorem 5.3. *Let $(p_j) \subset \mathbb{Z}(N_r\bar{T}_{k_r})$ be such that $p_{j+l} - p_j = P\ \forall j \in \mathbb{Z}$ for a given $l \in \mathbb{N}$. If u_λ is the unique solution of (DF$_\lambda$) in $B_r(v_p)$ given by Theorem* 5.1, *then there exists $f \in C(\mathbb{R}, \mathbb{R}^N)$ periodic with period T such that* $\mathrm{mod}(u_{\lambda,p}) \subset \mathrm{mod}(g_\lambda) + \mathrm{mod}(f)$ (*see* §2 *for the definitions*).

This implies in particular that if g_λ is quasi-periodic, then also u_λ is quasi-periodic. Indeed, since $\mathrm{mod}(g_\lambda) \subset \{m_1\beta_1 + \cdots + m_k\beta_k \mid m_1, \ldots, m_k \in \mathbb{Z}\}$

for a certain finite set of rationally independent real numbers $\{\beta_1, \dots, \beta_k\}$, then $\mathrm{mod}(u_\lambda) \subset \{m_1\beta_1 + \cdots + m_k\beta_k + m_{k+1}\frac{2\pi}{P} \mid m_1, \dots, m_{k+1} \in \mathbb{Z}\}$.

REFERENCES

[1] S. Alama and Y. Y. Li, *On "multibump" bound states for certain semilinear elliptic equations*, Indiana Univ. Math. J. **41** (1993), 983–1026.

[2] S. Angenent, *The shadowing lemma for elliptic pde*, Dynamics of infinite dimensional systems (S. Chaow and J. Hale, eds.), vol. F37, Springer-Verlag, Berlin Heidelberg New York, 1987, pp. 7–22 (NATO ASI).

[3] ———, *A variational interpretation of Melnikov's function and exponentially small separatrix splitting*, Symplectic Geometry (Cambridge) (J. Mierczyński, ed.), vol. 192, Cambridge University Press, 1993, pp. 5–35 (London Math. Soc. Lecture Note).

[4] M. L. Bertotti and S. V. Bolotin, *Homoclinic solutions of quasiperiodic Lagrangian systems*, Differential Integral Equations **8** (1995), 1733–1760.

[5] A. S. Besicovitch, *Almost Periodic Functions*, Dover Publications, Inc., New York, 1954.

[6] V. Coti Zelati, P. Montecchiari, and M. Nolasco, *Multibump homoclinic solutions for a class of second order, almost periodic hamiltonian systems*, NoDEA Nonlinear Differential Equations Appl. **4** (1997), 77–99.

[7] V. Coti Zelati and P. H. Rabinowitz, *Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials*, J. Amer. Math. Soc. **4** (1991), 693–727.

[8] A. M. Fink, *Almost periodic functions*, vol. 37, Springer Verlag, New York, 1974 (Lecture Notes in Mathematics).

[9] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector fields*, vol. 42, Springer-Verlag, Berlin Heidelberg New York, 1983 (Appl. Math. Sci.).

[10] V. Melnikov, *On the stability of the center for periodic perturbations*, Trans. Moscow Math. Soc. **12** (1963), 1–57.

[11] K. Meyer and G. Sell, *Melnikov transforms, Bernoulli bundles, and almost periodic perturbations*, Trans. Amer. Math. Soc. **314** (1989), 63–105.

[12] P. Montecchiari, *Existence and multiplicity of homoclinic orbits for a class of asymptotically periodic second order Hamiltonian systems*, Ann. Mat. Pura Appl. (4) **168** (1995), 317–354.

[13] P. Montecchiari and M. Nolasco, *Multibump solutions for perturbations of periodic second order Hamiltonian systems*, Nonlinear Anal. **27** (1996), 1355–1372.

[14] P. Montecchiari, M. Nolasco, and S. Terracini, *Multiplicity of homoclinics for a class of time recurrent second order Hamiltonian systems*, Calc. Var. Partial Differential Equations (to appear).

[15] J. Moser, *Combination tones for Duffing's equations*, Comm. Pure Appl. Math. **18** (1965), 167–181.

[16] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Gauthier-Villars, Paris, 1897–1899.

[17] P. H. Rabinowitz, *Homoclinics for an almost periodically forced singular Hamiltonian system*, Topol. Meth. in Non. Anal. **6** (1995), 49–66.

[18] P. H. Rabinowitz, *Multibump solutions for an almost periodically forced singular Hamiltonian system* (1995) (Preprint, Madison).

[19] E. Séré, *Existence of infinitely many homoclinic orbits in Hamiltonian systems*, Math. Z. **209** (1992), 27–42.

[20] ———, *Looking for the Bernoulli shift*, Ann. Inst. H. Poincaré. Anal. Non Linéaire **10** (1993 no. 5,), 561–590.

[21] E. Serra, M. Tarallo, and S. Terracini, *On the existence of homoclinic solutions for almost periodic second order systems*, Ann. Inst. H. Poincaré. Anal. Non Linéaire **13** (1996), 783–812.

[22] S. Wiggins, *Chaotic transport in dynamical systems*, Interdisciplinary Applied Mathematics, vol. 2, Springer-Verlag, Berlin Heidelberg New York, 1992.