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A SYMMETRIC POSITIVE SYSTEM WITH NONUNIFORMLY CHARACTERISTIC BOUNDARY

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Abstract. We study linear symmetric positive systems under maximal nonnegative boundary conditions. First we consider the noncharacteristic boundary and nonhomogeneous boundary conditions; in this case we give sufficient conditions on the boundary data in order to have L^2 and H^1 solutions. The inhomogeneous boundary data are treated directly with the advantage of requiring minimal regularity assumptions. Secondly we consider a boundary value problem with boundary matrix not of constant rank. We assume that the boundary is divided in two parts by an embedded manifold which is the intersection of the reference domain and a noncharacteristic hypersurface. The boundary matrix is negative definite on one side of the boundary with respect to the embedded manifold and is positive semidefinite on the other one. Using also the results of the first part, we discuss the existence of regular solutions.

1. Introduction. Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be a bounded open set with boundary $\partial\Omega$. Let us denote by $(,)$ the inner product in $L^2(\Omega)$ and by $||\cdot||$ its norm, by $(,)_{\partial\Omega}$ the inner product in $L^2(\partial\Omega)$ and by $||\cdot||_{\partial\Omega}$ its norm. Let $H^m(\Omega)$ be the usual Sobolev space of order m, $m = 1, 2, \dots,$ and let $||\cdot||_m$ denote its norm. The norm of $L^\infty(\Omega)$ is denoted by $|\cdot|_\infty$; the norm of $W^{1,\infty}(\Omega)$ is denoted by $|\cdot|_{1,\infty}$. For spaces of functions defined on the boundary we add "$\partial\Omega$" to the symbol of norm: $||\cdot||_{1,\partial\Omega}, |\cdot|_{\infty,\partial\Omega}, |\cdot|_{1,\infty,\partial\Omega}$ are the norms of $H^1(\partial\Omega), L^\infty(\partial\Omega), W^{1,\infty}(\partial\Omega)$. We use the same notations for spaces of scalar, vector-valued or matrix-valued functions. Throughout the paper we will denote by C generic constants which may vary from line to line or even in the same line and by c_i, C_i some specific constants.

In this paper we study some boundary value problems for linear symmetric positive systems. First we consider a boundary value problem with

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noncharacteristic boundary and nonhomogeneous boundary conditions and give sufficient conditions on the boundary data in order to have L^2 and H^1 solutions; see Theorems 1.1 and 1.2. After that, and this is our main concern, we study a boundary value problem with boundary matrix not of constant rank, under maximal nonnegative boundary conditions; see Theorem 1.4. Consider linear operators of first order:

$$L = \sum_{j=1}^{n} A_j \partial_j + B,$$

where $\partial_j = \partial/\partial x_j$. $A_1, \ldots, A_n, B$ are $k \times k$ matrix-valued functions of the space variable $x = (x_1, \ldots, x_n) \in \Omega$. We require that L is symmetric positive in the sense of Friedrichs [1]: the A_j are real symmetric matrices and there exists $\ell_0 > 0$ such that

$$B + B' - \sum_{i=1}^{n} \partial_i A_i \geq 2\ell_0,$$

where B' denotes the transpose of B. Let $\nu = (\nu_1, \ldots, \nu_n)$ be the unit outward normal to $\partial\Omega$; then

$$A_\nu = \sum_{j=1}^{n} A_j \nu_j$$

is the boundary matrix. Pairs of smooth functions u, v satisfy the Green's formula

$$(Lu, v) = (u, L^* v) + (A_\nu u, v)_{\partial\Omega}, \tag{1.1}$$

where L^* denotes the formal adjoint of L. We consider a nonhomogeneous boundary condition

$$M(x)u(x) = g(x) \quad \text{ for } x \in \partial\Omega,$$

where $M(x)$ is a $k \times k$ matrix-valued function of $x \in \partial\Omega$. We study the boundary value problem (b.v.p.)

$$\begin{aligned} Lu &= f \quad \text{in} \quad \Omega, \\ Mu &= g \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.2}$$

Nonhomogeneous b.v.p.'s are usually treated by subtracting a function which takes the boundary data and reduces to a homogeneous b.v.p.; this obliges one to require on the boundary data some excess regularity than is needed. A direct approach was considered by Sarason ([7]), where L^2 solutions were considered. Here we are interested to H^1 solutions under minimal conditions on the boundary data.

Consider the following hypotheses:

(H_1) The boundary is noncharacteristic; i.e., A_ν is invertible on $\partial\Omega$.

(H_2) There exists $t > 0$ such that

$$M + tMA_\nu^{-1}M = 0 \quad \text{ on } \partial\Omega.$$

(H_3) The symmetric part of $\frac{1}{2}A_\nu + tM$ restricted to its range is positive definite on $\partial\Omega$; i.e., there exists $m_0 > 0$ such that

$$\langle (A_\nu + tM + tM')u, u\rangle \geq 2m_0|u|^2 \quad \text{ on } \partial\Omega, \quad \forall u \in R',$$

where R' is the range of $A_\nu + tM + tM'$ (restricted to $\partial\Omega$) and M' denotes the transpose of M.

(H_4) $R \subset R'$, where R denotes the range of M.

The four assumptions together are denoted by (H). It is easily seen that (H) with some $t > 0$ is equivalent to (H) with the particular choice $t = 1$, by replacing tM with M.

Example 1. If A_ν is negative definite on $\partial\Omega$ we need to require the whole vector u on the boundary. Instead of the b.c. $u = g$ we can equivalently consider $-A_\nu u = G$ where $G = -A_\nu g$. With the choice $M = -A_\nu, t = 1$, the assumption (H) holds.

Example 2. If A_ν is positive definite on $\partial\Omega$ no boundary condition is required. Assumption (H) is satisfied with $M = 0$.

These examples are particular cases of the following one. We recall that the boundary condition is maximal nonnegative if for every $x \in \partial\Omega$

$$\langle A_\nu(x)u, u\rangle \geq 0 \quad \forall u \in N(x) = \ker M(x)$$

and in addition the dimension of $N(x)$ is equal to the number of nonnegative eigenvalues of $A_\nu(x)$. Consider the following assumptions:

(H_5) M has constant rank d.

(H_6) $\langle A_\nu(x)u, u\rangle > 0$ for all $u \in N(x)$, $u \neq 0$; $\langle A_\nu(x)u, u\rangle < 0$ for all $u \in N(x)^\perp, u \neq 0$.

Denote (H_1), (H_5), (H_6) by (H$'$). This is a special case of maximal non-negative boundary conditions.

Example 3. Assume (H$'$). After a change of unknowns one can reduce without loss of generality to the case

$$M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where I denotes the $d \times d$ identity matrix. Let us introduce the notation $u = (u^I, u^{II})$, where $u^I = (u_1, \ldots, u_d), u^{II} = (u_{d+1}, \ldots, u_k)$. According to the decomposition of u we write the boundary matrix in the block form

$$A_\nu = \begin{pmatrix} A_\nu^{II} & A_\nu^{I\,II} \\ A_\nu^{II\,I} & A_\nu^{II\,II} \end{pmatrix}.$$

Assumption (H_6) on A_ν is now $A_\nu^{II} < 0$, $A_\nu^{II\,II} > 0$. Since A_ν is invertible, instead of M, we can take as a boundary matrix the matrix $-A_\nu M$. With this particular choice it is easy to see that (H) holds with $t = 1$.

First we discuss the existence of L^2 solutions. Following Sarason [7] we introduce the following definitions.

Suppose $f \in L^2(\Omega)$ and $u \in L^2(\Omega)$, $\overline{u} \in L^2(\partial\Omega)$; we shall say that u is a *weak solution* of $Lu = f$ with *weak boundary values* $\overline{u}$ if

$$(f, v) = (u, L^* v) + (A_\nu \overline{u}, v)_{\partial\Omega} \quad \forall v \in C^\infty(\overline{\Omega}). \tag{1.3}$$

We shall say that u is a *strong solution* of $Lu = f$ with *strong boundary values* $\overline{u}$ if there exists a sequence of functions $\{u^k\} \subset C^\infty(\overline{\Omega})$ with boundary values $\{\overline{u}^k\}$ such that

$$||u^k - u|| + ||\overline{u}^k - \overline{u}||_{\partial\Omega} + ||Lu^k - f|| \to 0.$$

We shall say that $\{u, \overline{u}\}$ is a *weak (strong) solution* of the b.v.p. (1.2) if u with $\overline{u}$ is a weak (strong) solution of $Lu = f$ as defined above and if $\overline{u}$ satisfies $M\overline{u} = g$.

It is clear that strong solutions are also weak.

Theorem 1.1. *Let $\partial\Omega \in Lip$ and assume $A_j \in C^1(\overline{\Omega}), B \in C^0(\overline{\Omega}), A_{\nu\,|\partial\Omega}^{-1}$, $M \in L^\infty(\partial\Omega)$. Assume also* (H). *Let $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega) \cap R$. Then the b.v.p.* (1.2) *has a unique strong solution which satisfies*

$$||u|| + ||\overline{u}||_{\partial\Omega} \le c_1(||f|| + ||g||_{\partial\Omega}), \tag{1.4}$$

where c_1 *depends on* $t, \min\{\ell_0, m_0\}^{-1}, |A_\nu^{-1}|_{\infty,\partial\Omega}, |M|_{\infty,\partial\Omega}$.

Our approach has the advantage that inhomogeneous boundary data may be treated directly and that no differentiability criteria need be imposed on the matrix M or on the boundary data g.

Next we discuss the H^1 regularity of solutions to (1.2) under assumption (H′) on the boundary.

Theorem 1.2. *Assume* $\partial\Omega \in C^\infty$ *and* $A_j, B \in C^1(\overline{\Omega})$, $A^{-1}_{\nu\,|\partial\Omega} \in L^\infty(\partial\Omega)$, $M \in W^{1,\infty}(\partial\Omega)$. *Assume also* (H′). *Let* $f \in H^1(\Omega)$, $g \in H^1(\partial\Omega) \cap R$. *Then the b.v.p.* (1.2) *has a unique solution* $u \in H^1(\Omega)$ *with* $u_{|\partial\Omega} \in H^1(\partial\Omega)$. *Moreover* u *satisfies*

$$||u||_1 + ||u||_{1,\partial\Omega} \leq c_2(||f||_1 + ||g||_{1,\partial\Omega}), \tag{1.5}$$

where c_2 *depends on* $\min\{\ell_0, m_0\}^{-1}, |A_\nu^{-1}|_{\infty,\partial\Omega}, |A_i|_{1,\infty}, |B|_{1,\infty}, |M|_{1,\infty,\partial\Omega}$.

The proofs of Theorems 1.1 and 1.2 are given in Section 2. Theorem 1.2 shows that, given $f \in H^1(\Omega)$, the condition $g \in H^1(\partial\Omega) \cap R$ is sufficient for the H^1 regularity of solutions. We may interpret it by saying that a sufficient condition is that the incoming flow be in $H^1(\partial\Omega)$. A consequence of the theorem is that the outgoing flow is also in $H^1(\partial\Omega)$.

Remark. Under the assumptions of Theorem 1.2, if $u \in H^1(\Omega)$ is a solution of (1.2) then necessarily $u_{|\partial\Omega} \in H^1(\partial\Omega)$. Indeed, from Theorem 1.1 it follows that u with $u_{|\partial\Omega} \in H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$ is the unique solution. Combining with the existence result of Theorem 1.2, we have that $u_{|\partial\Omega} \in H^1(\partial\Omega)$.

Now we introduce a problem with a boundary matrix not of constant rank. We assume that a hypersurface Σ divides Ω into two open sets Ω^+ and Ω^-. Let $h \in C^\infty(\overline{\Omega})$ be a defining function of Σ such that $dh(x) \neq 0$ on Σ and $h(x) = 0$ if and only if $x \in \Sigma$. We suppose that $A_\nu(x)$ changes definiteness simply crossing $\gamma = \Sigma \cap \partial\Omega$. Let us set $\Omega^\pm = \{x \in \Omega : \pm h(x) > 0\}$ and define $\Gamma^+ = \partial\overline{\Omega^+} \setminus \Sigma$, $\Gamma^- = \partial\overline{\Omega^-} \setminus \Sigma$. Our first assumption is

$$A_\nu(x) \begin{cases} \text{is negative definite on } \Gamma^+; \text{ i.e., } \langle A_\nu(x)u, u\rangle < 0 & \forall u, \\ \text{is positive semi-definite on } \Gamma^-; \text{ i.e., } \langle A_\nu(x)u, u\rangle \geq 0 & \forall u. \end{cases} \tag{1.6}$$

Thus Γ^+ is noncharacteristic, while Γ^- is noncharacteristic if $A_\nu(x)$ is actually positive definite; otherwise, it is characteristic. Moreover $A_\nu(x)$ vanishes identically on γ, which is characteristic.

We consider a homogeneous boundary condition that we write in the form

$$u(x) \in N(x) \quad \text{for} \quad x \in \partial\Omega,$$

where $N(x)$ is a linear subspace of C^k. We assume that the boundary condition is maximal nonnegative. Our assumption (1.6) implies that

$$N(x) = \begin{cases} \{0\} & \text{if } x \in \Gamma^+ \\ C^k & \text{if } x \in \Gamma^- \cup \gamma. \end{cases}$$

We study the boundary value problem

$$\begin{aligned} (\lambda + L)u = f \quad & \text{in } \Omega, \\ u \in N \quad & \text{on } \partial\Omega, \end{aligned} \tag{1.7}$$

where for the sake of convenience we put in evidence a coefficient λ so that positivity of the operator is achieved by requiring λ large enough. This allows us to show that the estimates of solutions we are able to obtain are of semigroup type; see (1.10).

Since the boundary condition is maximal nonnegative, the existence of weak solutions is classical. If dim ker A_ν is not constant on $\partial\Omega$, it is well known that weak solutions are not necessarily strong; see [2], [4], [5]. However, our case is covered by the result of Rauch [6]. Thus (1.7) has one and only one strong solution. In this paper we are interested in discussing the regularity of the solution. Nishitani ([3]) has studied the case with boundary matrix negative definite on one side of the boundary with respect to the embedded manifold and positive definite on the other. Thus our result contains in particular his; our proof is completely different and seems to us simpler.

Our starting point is the following result which is a consequence of the identity of weak and strong solutions.

Proposition 1.3. *Assume $\partial\Omega \in C^1$, (1.6) and $A_j \in C^1(\overline{\Omega}), B \in C^0(\overline{\Omega})$; let*

$$\omega = (1/2) \sum_{j=1}^{n} |\partial_j A_j|_\infty + |B|_\infty .$$

Then for every $\lambda > \omega$ and every $f \in L^2(\Omega)$ there exists a unique strong solution $u \in L^2(\Omega)$ of (1.7) in the sense that there is a sequence $\{u_m\} \subset$

$C^\infty(\overline{\Omega})$ with $u_m(x) \in N(x)$ for all $x \in \partial\Omega \setminus \gamma$ and in the $L^2(\Omega)$ norm one has $u_m \to u$ and $(\lambda + L)u_m \to f$. Moreover

$$(\lambda - \omega)||u|| \le ||f||. \tag{1.8}$$

Proof. The result is a direct application of Corollary 2.5 in [6]. □

Now we turn to discussing the regularity of solutions. In general, even for smooth f, the solution u is not necessarily regular. Hence some additional condition on the operator is needed.

Example 4. Consider the boundary value problem

$$\begin{aligned}(\lambda + x_2 a(x)\partial_1 + \partial_2)u = f \quad &\text{in} \quad \Omega,\\ u = 0 \quad &\text{on} \quad \Gamma^+,\end{aligned}$$

where $\Omega = \mathbb{R}^2_+ = \{x_1 > 0\}, \Gamma^+ = \{x_1 = 0, x_2 > 0\}$ and $a(x) \in C^\infty_{(0)}(\Omega)$, $a(x) \ge 0, a(x) = 1$ for $|x| \le R$ for some large $R > 0$. Observe that (1.6) is satisfied with $h = x_2$. Let $f \in C^\infty_{(0)}(\Omega), 0 \le f \le 1$. Assuming $f = 0$ in $D = \{0 < x_1 < \frac{1}{2}x_2^2, x_2 > 0\}$, we immediately see that $u = 0$ in D. Assuming $f = 1$ in the neighborhood of a point on the parabola $x_1 = \frac{1}{2}x_2^2$ with $x_2 < 0$, we see that $u(x) \ge \alpha$ for some $\alpha > 0$ in the right-hand side with respect to the same parabola of a neighborhood of some point with $x_2 > 0$ on the parabola. Because of the discontinuity along the positive branch of the parabola, necessarily $u \notin H^1(\Omega)$.

Let us set

$$C(x) = \sum_{j=1}^{n} \partial_j h(x) A_j(x).$$

From now on we shall assume

$$C(x) \text{ is negative definite on } \overline{\Omega} \cap \Sigma. \tag{1.9}$$

Observe that (1.9) implies that $\overline{\Omega} \cap \Sigma$ is noncharacteristic. The condition does not depend on the defining function h. The sign of $C(x)$ required in (1.9) is the opposite of that considered in the above example. The sign depends on assumption (1.6). For $A_\nu(x)$ positive semi-definite on Γ^+ and negative definite on Γ^-, $C(x)$ must be positive definite.

Now we turn to regularity of solutions. For such regularity one has to impose vanishing conditions on f at Γ^+.

Example 5. As in the remark after Theorem 1.7 of [3], we consider the boundary value problem

$$\begin{aligned}(x_2 a(x)\partial_1 - \partial_2)u = f \quad &\text{in} \quad \Omega,\\ u = 0 \quad &\text{on} \quad \Gamma^+,\end{aligned}$$

with Ω, Γ^+, a as in Example 1, $f \in C^\infty_{(0)}(\Omega)$. Let $f = 1$ in a neighborhood of the origin. Then near the origin $\partial^2_{11}u(x) = -(2x_1 + x_2^2)^{-3/2}$ which is not square integrable. Thus $u \notin H^2(\Omega)$.

The main purpose of this paper is to show the following result.

Theorem 1.4. *Assume $\partial\Omega \in C^\infty$, (1.6), (1.9) and let $m \geq 1$. Let $s = [\frac{n}{2}] + 3$, $A_j \in H^s(\Omega) \cap H^m(\Omega), B \in H^{s-1}(\Omega) \cap H^m(\Omega)$. Then there exists a constant ω_m with the following property. If $\lambda > \omega_m$ and if $f \in H^m(\Omega)$ is such that $\partial^p_\nu f = 0$ on Γ^+ for $p = 0, \ldots, m-1$, then $u \in H^m(\Omega)$ with $\partial^p_\nu u = 0$ on Γ^+ for $p = 0, \ldots, m-1$. Moreover u satisfies*

$$(\lambda - \omega_m)||u||_m \leq ||f||_m. \tag{1.10}$$

ω_m is a nondecreasing function of $||A_j||_m, ||B||_m$ (or $||A_j||_s, ||B||_{s-1}$ if $m < s$). For $m = 1$ we may take

$$\omega_1 = \omega + \sum_{i,j=1}^n |\partial_j A_i|_\infty + \sum_{j=1}^n |\partial_j B|_\infty.$$

We notice that in both theorems it is enough to assume the definiteness of $C(x)$ on $\overline{\Omega} \cap \Sigma \cap \mathcal{O}$, where $\mathcal{O}$ is an arbitrary neighborhood of γ in $\mathbb{R}^n$. The proof of Theorem 1.4 is given in Sections 3 and 4.

2. Proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. That weak solutions are also strong is proved in Sarason [7]. Thus it is enough to show the existence of weak solutions under assumption (H). The Green's formula (1.1) may be written as

$$(Lu, v) + t(Mu, v)_{\partial\Omega} = (u, L^*v) + (u, M^*v)_{\partial\Omega}, \tag{2.1}$$

where $M^* = A_\nu + tM'$. Let us denote by P the orthogonal projection onto R'. Given a smooth $v = u$ we integrate by parts in the left-hand side of (2.1). The assumption on L and (H_3) imply

$$(Lv, v) + t(Mv, v)_{\partial\Omega} \geq \ell_0||v||^2 + m_0||Pv||^2_{\partial\Omega}. \tag{2.2}$$

Thus from (2.1), (2.2) we have

$$||v||^2 + ||Pv||^2_{\partial\Omega} \leq c_0^2(||L^*v||^2 + ||M^*v||^2_{\partial\Omega}), \tag{2.3}$$

where $c_0 = \min\{\ell_0, m_0\}^{-1}$. We denote by H the completion of the subspace $\{V = (L^*v, M^*v) : v \in C^\infty(\overline{\Omega})\}$ with respect to the norm $(||L^*v||^2 + ||M^*v||^2_{\partial\Omega})^{1/2}$. It follows from (2.3) that v is determined uniquely by $V = (L^*v, M^*v)$ and that $(f, v) + t(g, v)_{\partial\Omega}$ is in fact a bounded linear functional on H. Observe that $(g, v)_{\partial\Omega} = (g, Pv)_{\partial\Omega}$ because of (H_4). According to the Riesz representation theorem, there exists $(u, \hat{u}) \in H$ such that

$$(f, v) + t(g, v)_{\partial\Omega} = (u, L^*v) + (\hat{u}, M^*v)_{\partial\Omega} \quad \forall v \in C^\infty(\overline{\Omega}) \tag{2.4}$$

which can be written as

$$(f, v) = (u, L^*v) + (A_\nu\hat{u} + tM\hat{u} - tg, v)_{\partial\Omega}.$$

Thus u is a weak solution of $Lu = f$ with weak boundary values

$$\overline{u} = \hat{u} + tA_\nu^{-1}(M\hat{u} - g) \in L^2(\partial\Omega).$$

There remains to be shown only that $M\overline{u} = g$. Since $g \in L^2(\partial\Omega) \cap R$ we may write $g = Mh$ with $h \in L^2(\partial\Omega)$. Then

$$M\overline{u} = (M + tMA_\nu^{-1}M)\hat{u} - tMA_\nu^{-1}Mh = Mh = g$$

because of (H_2). Now let $v \in L^2(\Omega)$ with $v_{|\partial\Omega} \in L^2(\partial\Omega)$ such that $u = L^*v, \hat{u} = M^*v$. From (2.3), (2.4) we obtain

$$||u||^2 + ||\hat{u}||^2_{\partial\Omega} \leq c_0^2(||f||^2 + t^2||g||^2_{\partial\Omega}).$$

Using the estimate for $\hat{u}$ we obtain that for $\overline{u}$. Thus we have (1.4). As regards the uniqueness of strong solutions, if $\{u, \overline{u}\}$ is the difference of two strong solutions with approximating sequence $\{u^k\}$, passing to the limit in

(2.2) for $v = u^k$ gives $u = 0$ in Ω. From (1.3) it follows $A_\nu \overline{u} = 0$; hence, $\overline{u} = 0$ on $\partial\Omega$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Assume for a while that $f \in H^2(\Omega)$, $g \in H^{5/2}(\partial\Omega)$ $\cap R$ and that the matrix coefficients are sufficiently smooth. Take $G \in H^3(\Omega)$ such that $MG = g$ on $\partial\Omega$. Setting $v = u - G$, (1.2) is reduced to the problem with homogeneous boundary condition $Lv = f - LG$ in $\Omega, Mv = 0$ on $\partial\Omega$. Taking if necessary ℓ_0 larger, the results of Friedrichs ([1]) *et al.* give the existence of a solution $u \in H^2(\Omega)$. In the standard way we derive the a priori estimate in $H^1(\Omega)$. We introduce a suitable covering of $\overline{\Omega}$ and partition of unity subordinate to the covering; thus, we may assume that u has compact support. In the patches that don't intersect the boundary the estimate is standard. The patches at the boundary are taken small enough in such a way that A_ν is there invertible. After extending inside the patches the outer normal and accordingly the normal derivative, we can locally write $\partial_\nu u = A_\nu^{-1}(f - L_{tan}u)$ where L_{tan} is a differential operator of first order containing only tangential differentiation. Next we apply the tangential differentiation to $Lu = f$, substituting everywhere the expression for the normal derivative. We also derive in the tangential directions the boundary condition $Mu = g$. Thus we obtain a system of the form

$$\begin{aligned} \mathcal{L}\partial' u &= f' \quad \text{in} \quad \Omega, \\ \mathcal{M}\partial' u &= g' \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.5}$$

for the vector $\partial' u \in H^1(\Omega)$ of tangential derivatives of u; f' and g' contain also terms of order zero in u. The boundary matrix of $\mathcal{L}$ has the form

$$\begin{pmatrix} A_\nu & & \\ & \dots & \\ & & A_\nu \end{pmatrix} \quad \text{and} \quad \mathcal{M} = \begin{pmatrix} M & & \\ & \dots & \\ & & M. \end{pmatrix}$$

Thus the pair $(\mathcal{L}, \mathcal{M})$ satisfies (H$'$) in a way similar to (L, M). We apply estimate (1.4) to (2.5) and obtain

$$||\partial' u|| + ||\partial' u||_{\partial\Omega} \leq c_1'(||f||_1 + ||g||_{1,\partial\Omega} + ||u|| + ||u||_{\partial\Omega}). \tag{2.6}$$

Using (1.4), (2.6), the estimate in the interior of Ω and estimating $\partial_\nu u$ from the equation we get (1.5). We may show that the dependence on the regularity of the coefficients is as indicated in the statement of the theorem.

Now take $f^m \in H^2(\Omega)$, $g^m \in H^{5/2}(\partial\Omega) \cap R$ such that $f^m \to f$ in $H^1(\Omega)$, $g^m \to g$ in $H^1(\partial\Omega)$ and let u^m be the corresponding solutions. From (1.5) for u^m it follows that $u^m \to u$ in $H^1(\Omega)$ and on the boundary in $H^1(\partial\Omega)$. We pass to the limit in the equations and see that u is the solution of (1.2). Passing to the limit in (1.5) for u^m we get (1.5) for u. To treat the case of matrix coefficients as in the statement we employ another density argument.

3. Proof of Theorem 1.4 for $m = 1$**.** As a first step we show that $u \in H^1(\Omega^+)$ and $u \in H^1(\Sigma \cap \overline{\Omega})$. Take $r(x) \in C^\infty(\overline{\Omega})$ with $dr(x) \neq 0$ on $\partial\Omega$ such that $\Omega = \{r(x) > 0\}$. Cover $\overline{\Omega^+}$ by a finite family of open sets $\{\mathcal{U}_i\}$, $i = 0, \ldots, \ell$. We first take $\mathcal{U}_i, i = 1, \ldots, \kappa$, small enough so that $\{\mathcal{U}_i \cap \gamma\}$ form a covering of γ by coordinate patches. We choose a coordinate system χ_i in $\mathcal{U}_i$ so that $x_1 = r \circ \chi_i^{-1}$ and $x_2 = h \circ \chi_i^{-1}$. We take $\mathcal{U}_i, i = \kappa + 1, \ldots, \kappa'$, small enough again so that $\mathcal{U}_i \cap \Sigma = \emptyset$ and that $\{\mathcal{U}_i \cap \partial\Omega\}, i = 1, \ldots, \kappa'$, form a covering of Γ^+ by coordinate patches; for $i = \kappa + 1, \ldots, \kappa'$, choose a coordinate system with $x_1 = r \circ \chi_i^{-1}$. We also take $\mathcal{U}_i, i = \kappa' + 1, \ldots, \ell$, small enough again so that $\mathcal{U}_i \cap \partial\Omega = \emptyset$ and that $\{\mathcal{U}_i \cap \Sigma\}, i = 1, \ldots, \kappa, \kappa' + 1, \ldots, \ell$, form a covering of $\Sigma \cap \Omega$ by coordinate patches; for $i = \kappa' + 1, \ldots, \ell$, choose a coordinate system with $x_1 = h \circ \chi_i^{-1}$. We finally cover $\overline{\Omega^+} \setminus \cup_{i=1}^{\ell} \mathcal{U}_i$ by $\mathcal{U}_0$. Take a partition of unity $\{\psi_i\}$ subordinate to this covering.

Consider the case $i = 1, \ldots, \kappa$, where $\mathcal{U}_i \cap \gamma \neq \emptyset$. Performing the above change of independent variables we are reduced to the problem

$$\begin{aligned} (\lambda + L^i)v^i = f^i \quad &\text{in} \quad Q, \\ v^i = 0 \quad &\text{on} \quad \Gamma, \end{aligned} \tag{3.1}$$

where $Q = \{x_1 > 0, x_2 > 0\}, \Gamma = \{x_1 = 0, x_2 > 0\}, v^i = \psi_i u$. Because of (1.6) and (1.9) the differential operator

$$L^i = \sum_{j=1}^{n} A_j^i(x)\partial_j + B^i(x)$$

in the local coordinates has boundary matrix $-A_1^i(x) < 0$ on Γ, $A_1^i(x) = 0$ on $\{x_1 = x_2 = 0\}$, $-A_2^i(x) \geq a_2 > 0$ on $\Sigma^+ = \{x_1 > 0, x_2 = 0\}$. Finally $f^i = \psi_i f + \sum_{j=1}^n A_j^i(x)(\partial_j \psi_i)u$. Consider the case $i = \kappa + 1, \ldots, \kappa'$, where $\mathcal{U}_i \cap \Gamma^+ \neq \emptyset, \mathcal{U}_i \cap \Sigma = \emptyset$. With the above change of variables we are reduced to

$$\begin{aligned} (\lambda + L^i)v^i = f^i \quad &\text{in} \quad \mathbb{R}^n_+ = \{x_1 > 0\}, \\ v^i = 0 \quad &\text{on} \quad \{x_1 = 0\}, \end{aligned} \tag{3.2}$$

where v^i and f^i are as before. Here we have $A_1^i(x) \geq a_1 > 0$ on $\{x_1 = 0\}$. Consider the case $i = \kappa' + 1, \ldots, \ell$, where $\mathcal{U}_i \cap \Gamma^+ = \emptyset, \mathcal{U}_i \cap \Sigma \neq \emptyset$. Again we reduce the problem to

$$(\lambda + L^i)v^i = f^i \quad \text{in } \mathbb{R}^n_+. \tag{3.3}$$

In this case $A_1^i(x) \leq -a_1 < 0$ on $\{x_1 = 0\}$ and no boundary condition is required. Finally we have the case $i = 0$ with the problem

$$(\lambda + L^0)v^0 = f^0 \quad \text{in } \mathbb{R}^n. \tag{3.4}$$

The main step is to show the following lemma concerning the case $i = 1, \ldots, \kappa$.

Lemma 3.1. *There exists a constant $\omega' > 0$ depending on Ω, A_j^i, B^i, with the following property. If $\lambda > \omega'$ and if $f^i \in H^1(Q), f^i = 0$ on Γ, then* (3.1) *has a unique solution $v^i \in H^1(Q)$ with $v^i_{|\Sigma^+} \in H^1(\Sigma^+)$. Moreover*

$$(\lambda - \omega')||v^i||_{1,Q} \leq ||f^i||_{1,Q}, \qquad ||v^i||_{1,\Sigma^+} \leq C||f^i||_{1,Q}, \tag{3.5}$$

where $||\cdot||_{1,Q}$ denotes the norm of $H^1(Q)$, $||\cdot||_{1,\Sigma^+}$ denotes the norm of $H^1(\Sigma^+)$.

Assume for the moment Lemma 3.1. Analogous statements for problems (3.2)–(3.4) are well known; see Friedrichs [1]. If ω' is taken sufficiently large it may be chosen for all problems. Given $u \in H^1(\Omega^+)$ and accordingly f^i, let $v^1, \ldots, v^\kappa \in H^1(Q)$ be the solutions of (3.1), $v^{\kappa+1}, \ldots, v^\ell \in H^1(\mathbb{R}^n_+)$ the solutions of (3.2) and (3.3), $v^0 \in H^1(\mathbb{R}^n)$ the solution of (3.4). For $i = \kappa' + 1, \ldots, \ell$, we can show that $v^i \in H^1(\{x_1 = 0\})$; the proof can be performed as for the case $i = 1, \ldots, \kappa$. After the inverse change of variables for each v^i, set $v = \sum_{i=0}^{\ell} v^i \in H^1(\Omega^+)$. This defines a map $\mathcal{S} : H^1(\Omega^+) \mapsto H^1(\Omega^+), v = \mathcal{S}(u)$. From (3.5) and analogous estimates that hold for the solutions of (3.2)–(3.4) we obtain

$$(\lambda - \omega')||v||_{1,\Omega^+} \leq C(\sum_i ||\psi_i f||_{1,\Omega^+} + ||u||_{1,\Omega^+}) = a + C_1||u||_{1,\Omega^+}. \tag{3.6}$$

Take $\lambda \geq \omega' + 2C_1$ and $R \geq 2a(\lambda - \omega')^{-1}$ in this order. Denote by B_R the ball in $H^1(\Omega^+)$ centered in 0 with radius R. If $u \in B_R$ then $v \in B_R$

from (2.12). Thus $\mathcal{S}(B_R) \subset B_R$. Take $u_i \in B_R, i = 1, 2, v_i = \mathcal{S}(u_i), u' = u_1 - u_2, v' = v_1 - v_2$. As for (3.6)

$$(\lambda - \omega')||v'||_{1,\Omega^+} \leq C_1||u'||_{1,\Omega^+}.$$

The above requirement on λ shows that $\mathcal{S}$ is a contractive map. Thus there exists a unique fixed point $u = v$. The inverse change of variables in each problem (3.1)–(3.4) and summation over i from 0 to ℓ show that u is the solution. From $v^i \in H^1(\Sigma^+)$ for $i = 1, \ldots, \kappa$, and $v^i \in H^1(\{x_1 = 0\})$ for $i = \kappa' + 1, \ldots, \ell$, it follows that $u \in H^1(\Sigma \cap \overline{\Omega})$.

Proof of Lemma 3.1. For our convenience we prefer to change notation. We study the problem

$$\begin{aligned} (\lambda + L)u = f \quad \text{in} \quad \Omega^+ = \{x_1 > 0, x_2 > 0\}, \\ u = 0 \quad \text{on} \quad \Gamma^+ = \{x_1 = 0, x_2 > 0\}, \end{aligned} \tag{3.7}$$

under the assumptions $A_1(x) > 0$ on Γ^+, $A_1(x) = 0$ on $\{x_1 = x_2 = 0\}$, $-A_2(x) \geq 2a_2 > 0$ on $\Sigma^+ = \{x_1 > 0, x_2 = 0\}$. To solve (3.7) we use the viscosity method. Consider the problem

$$\begin{aligned} (\lambda + L - \epsilon\Delta)u^\epsilon = f \quad \text{in} \quad \Omega^+, \\ u^\epsilon = 0 \quad \text{on} \quad \Gamma^+, \\ \partial_2 u^\epsilon = 0 \quad \text{on} \quad \Sigma^+, \end{aligned} \tag{3.8}$$

where Δ denotes the Laplacian and ϵ is a small positive parameter. We have

Lemma 3.2. *For all $\lambda > \omega$ and $f \in L^2(\Omega^+)$ problem* (3.8) *has a unique solution $u^\epsilon \in H^2(\Omega^+)$.*

Since (3.8) is a mixed problem in a nonsmooth domain and for such problems the H^2 regularity is not always achieved, we prefer to give a proof of the result in the Appendix.

Given Lemma 3.2 we go on with the proof of Lemma 3.1. We start by showing the L^2-energy estimate. By integration by parts we have for all $u \in H^1(\Omega^+)$

$$2(Lu, u)_{\Omega^+} = (A_\nu u, u)_{\partial\Omega^+} + ((2B - \partial_j A_j)u, u)_{\Omega^+}. \tag{3.9}$$

If we set $u = u^\epsilon$ we obtain

$$(A_\nu u^\epsilon, u^\epsilon)_{\partial\Omega^+} = -(A_2 u^\epsilon, u^\epsilon)_{\Sigma^+} \geq 2a_2 \int_{\Sigma^+} |u^\epsilon|^2. \tag{3.10}$$

Moreover $-\epsilon(\Delta u^\epsilon, u^\epsilon)_{\Omega^+} = \epsilon||\nabla u^\epsilon||^2_{\Omega^+}$. Thus we have

$$\epsilon||\nabla u^\epsilon||^2_{\Omega^+} + a_2 \int_{\Sigma^+} |u^\epsilon|^2 + (\lambda - \omega)||u^\epsilon||^2_{\Omega^+} \leq (f, u^\epsilon)_{\Omega^+}. \tag{3.11}$$

Now the estimate for the derivatives. We multiply (3.8) by $-\Delta u^\epsilon$. Since $f = 0$ on Γ^+ we have

$$-(f, \Delta u^\epsilon)_{\Omega^+} = (\nabla f, \nabla u^\epsilon)_{\Omega^+}.$$

Again by integration by parts

$$\begin{aligned} &-(A_j\partial_j u^\epsilon, \Delta u^\epsilon)_{\Omega^+} = -(A_j\partial_j u^\epsilon, \partial_\nu u^\epsilon)_{\partial\Omega^+} + (\partial_k A_j \partial_j u^\epsilon, \partial_k u^\epsilon)_{\Omega^+} \\ &\quad + (1/2)(A_\nu \partial_k u^\epsilon, \partial_k u^\epsilon)_{\partial\Omega^+} - (1/2)(\partial_j A_j \partial_k u^\epsilon, \partial_k u^\epsilon)_{\Omega^+}. \end{aligned}$$

On the other hand

$$\begin{aligned} &-(A_j\partial_j u^\epsilon, \partial_\nu u^\epsilon)_{\partial\Omega^+} + (1/2)(A_\nu \partial_k u^\epsilon, \partial_k u^\epsilon)_{\partial\Omega^+} \\ &\quad = (1/2)(A_1\partial_1 u^\epsilon, \partial_1 u^\epsilon)_{\Gamma^+} - (1/2)(A_2 \partial_k u^\epsilon, \partial_k u^\epsilon)_{\Sigma^+} \geq a_2 \int_{\Sigma^+} |\nabla u^\epsilon|^2. \end{aligned}$$

Thus we obtain

$$\begin{aligned} &\epsilon||\Delta u^\epsilon||^2_{\Omega^+} + a_2 \int_{\Sigma^+} |\nabla u^\epsilon|^2 + (\lambda - \varpi)||\nabla u^\epsilon||^2_{\Omega^+} \\ &\quad - |\partial_k B|_\infty ||u^\epsilon||_{\Omega^+} ||\partial_k u^\epsilon||_{\Omega^+} \leq (\nabla f, \nabla u^\epsilon)_{\Omega^+}, \end{aligned} \tag{3.12}$$

where $\varpi = \omega + \sum_{j,k} |\partial_k A_j|_\infty$. From (3.11), (3.12) we have the estimate independent of ϵ

$$\sqrt{\epsilon}||\Delta u^\epsilon||_{\Omega^+} + ||u^\epsilon||_{1,\Sigma^+} + (\lambda - \omega')||\nabla u^\epsilon||_{1,\Omega^+} \leq C,$$

where $\omega' = \varpi + \sum_k |\partial_k B|_\infty$. We extract a subsequence weakly convergent to $u \in H^1(\Omega^+)$ with $u_{|\Sigma^+} \in H^1(\Sigma^+)$. Passing to the limit in (3.8) as $\epsilon \to 0$ shows that u is the solution of (3.7).

Second part of the proof. The boundary matrix of L, relative to the part $\Sigma \cap \Omega$ and with respect to $\partial\Omega^-$, is given by

$$d(x)^{-1} \sum_{j=1}^{n} \partial_j h(x) A_j(x) = d(x)^{-1} C(x),$$

with $d(x) = \sqrt{\sum_{j=1}^{n} \partial_j h(x)^2}$, which is negative definite by (1.9). We take an open set $\Omega' \supset \Omega^-$ with smooth boundary $\partial\Omega' \supset \Sigma \cap \overline{\Omega}$. We take an extension $\overline{A}_j$, $\overline{B}$, $\overline{f}$ of A_j, B, f onto Ω' with $\overline{A}_j$, $\overline{B} \in C^1(\overline{\Omega}')$, $\overline{f} \in H^1(\Omega')$ and

$$\sum_{j=1}^{n} |\overline{A}_j|_{1,\infty,\Omega'} + |\overline{B}|_{1,\infty,\Omega'} \leq C(\sum_{j=1}^{n} |A_j|_{1,\infty} + |B|_{1,\infty}), \quad ||f||_{1,\Omega'} \leq C||f||_1.$$

If ν' denotes the exterior unit normal to $\partial\Omega'$, the extension of A_j is taken in such a way that the matrix $\sum_{j=1}^{n} \overline{A}_j(x)\nu_j'(x)$ is still negative definite on $\partial\Omega'$. Take also an extension $\overline{u} \in H^1(\partial\Omega')$ of $u_{|\Sigma\cap\overline{\Omega}}$. We consider the b.v.p. (where $\overline{L}$ denotes the operator with extended coefficients)

$$\begin{aligned} (\lambda + \overline{L})w &= \overline{f} \quad \text{in} \quad \Omega', \\ w &= \overline{u} \quad \text{on} \quad \partial\Omega'. \end{aligned} \tag{3.13}$$

From Theorem 1.2 this problem has a unique solution $w \in H^1(\Omega')$. Setting $\tilde{u} = u$ in Ω^+ and $\tilde{u} = w$ in Ω^-, we have $(\lambda + L)\tilde{u} = f$ in Ω, $\tilde{u} = 0$ on Γ^+ and $\tilde{u} \in H^1(\Omega)$. From Proposition 1.3 $\tilde{u} = u$ in Ω, and hence $u \in H^1(\Omega)$.

Proof of (1.10). We formally apply $\partial_j, j = 1, \ldots, n$, to (1.7) and obtain

$$(\lambda + L)U_j + \sum_{i=1}^{n} \partial_j A_i U_i = F_j, \tag{3.14}$$

where $U_j = \partial_j u, F_j = \partial_j f - \partial_j B u$. We write (3.14), $j = 1, \ldots, n$, as

$$(\lambda + \mathcal{K})U = F, \tag{3.15}$$

where $U = (U_1, \ldots, U_n), F = (F_1, \ldots, F_n)$. Since $u, f \in H^1(\Omega)$ we have $U, F \in L^2(\Omega)$; we easily show that U satisfies (3.15) in the sense of distributions. It is clear that the boundary matrix of $\mathcal{K}$ satisfies (1.6). By means of the localization introduced above we can show that L admits the decomposition

$$L = A_\nu \partial_\nu + L_\tau, \tag{3.16}$$

where L_τ is a first-order operator involving only tangential differentiation with respect to $\partial\Omega$. Thus from $u = 0$ on Γ^+ in the sense of $H^{1/2}(\Gamma^+)$ it follows that $L_\tau u = 0$ on Γ^+ in the sense of $H^{-1/2}(\Gamma^+)$. Since $f = 0$ on Γ^+, it follows from (1.7), (3.16) and the invertibility of A_ν on Γ^+ that also $\partial_\nu u = 0$ in the sense of $H^{-1/2}(\Gamma^+)$. Thus we can show that $U = 0$ on Γ^+ in the sense of $H^{-1/2}(\Gamma^+)$. Denote the corresponding boundary space by $\mathcal{M}$. Then U is a solution of

$$\begin{aligned} (\lambda + \mathcal{K})U &= F \quad \text{in} \quad \Omega, \\ U &\in \mathcal{M} \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.17}$$

We apply Proposition 1.3 to (3.14) and obtain the a priori estimate $(\lambda - \omega)||U_j|| \le ||F_j|| + \sum_i |\partial_j A_i|_\infty ||U_i||$. Combining with (1.8) gives (1.10).

4. Proof of Theorem 1.4 for $m \ge 2$. The theorem has been proved for $m = 1$. Assume that it holds for $\ell = 1, \dots, m-1$. Let $f \in H^m(\Omega)$ with $\partial_\nu^p f = 0$ on Γ^+ for $p = 0, \dots, m-1$. By the induction hypothesis, if λ is sufficiently large, then $u \in H^{m-1}(\Omega)$ with $\partial_\nu^p u = 0$ on Γ^+ for $p = 0, \dots, m-2$. We see that the right-hand side F of (3.14) is in $H^{m-1}(\Omega)$ and such that $\partial_\nu^p F = 0$ on Γ^+ for $p = 0, \dots, m-2$. Then, if λ is sufficiently large, (3.17) has a solution $U \in H^{m-1}(\Omega)$ with $\partial_\nu^p U = 0$ on Γ^+ for $p = 0, \dots, m-2$. Hence we obtain $u \in H^m(\Omega)$ with $\partial_\nu^p u = 0$ on Γ^+ for $p = 0, \dots, m-1$. From (1.10) in the case $m-1$ applied to u and U we obtain (1.10) in case m. This completes the proof of Theorem 1.4.

Appendix.

Proof of Lemma 3.2. For convenience we set $\epsilon = 1$ in the equation and write u instead of u^ϵ. Let us define the space $H = \{u \in H^1(\Omega^+), u = 0 \text{ on } \Gamma^+\}$. The weak formulation of (3.8) is

$$((\lambda + L)u, v)_{\Omega^+} + (\nabla u, \nabla v)_{\Omega^+} = (f, v)_{\Omega^+} \quad \forall v \in H. \tag{A.1}$$

From the Lax–Milgram theorem we have a unique solution $u \in H$ of (A.1) for $\lambda > \omega$; in particular, the coerciveness of the bilinear form in the left-hand side of (A.1) is shown as for (3.11). Set $g = f - (\lambda + L)u \in L^2(\Omega^+)$ and extend u, g to $\mathbb{R}^n_+ = \{x_1 > 0\}$ by $U(x) = u(x_1, -x_2, x_3, \dots, x_n)$, $G(x) = g(x_1, -x_2, x_3, \dots, x_n)$ if $x_2 < 0$. Then $U \in H^1(\mathbb{R}^n_+)$, $G \in L^2(\mathbb{R}^n_+)$. It readily follows that U is a weak solution of the problem

$$\begin{aligned} -\Delta U &= G \quad \text{in} \quad \mathbb{R}^n_+, \\ U &= 0 \quad \text{on} \quad \{x_1 = 0\}. \end{aligned}$$

Since this problem is regular, the solution is in H^2; thus, $U \in H^2(\mathbb{R}^n_+)$ and $u = U_{|\Omega_+} \in H^2(\Omega^+)$.

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