Differential and Integral Equations Volume 11, Number 4, July 1998, pp. 589–603.

DIFFERENTIABILITY WITH RESPECT TO DELAY*

Veli-Matti Hokkanen
University of Jyväskylä, Department of Mathematics, FIN-40351 Jyväskylä, Finland

Gheorghe Moroşanu
University of Jassy, Faculty of Mathematics, R-6600 Iaşi, Romania

(Submitted by: Viorel Barbu)

Abstract. The differentiability with respect to the delay of the solution of the retarded differential equation (1.1) below will be shown by using elementary arguments. The investigation is extended to the infinite-dimensional case. An application to heat conduction is also given.

1. Introduction. In the paper [9] Hale and Ladeira considered the initial value problem

$$x'(t) = f(x(t), x(t-\tau)), \quad t > 0,$$
$$x(t) = \phi(t), \quad -r \le t \le 0,$$

where $f : \mathbf{R}^{2n} \mapsto \mathbf{R}^n$, r is a fixed positive number and $\tau \in (0, r]$. They were mainly concerned with the differentiability of the solution of the above problem with respect to the delay τ.

Here, using elementary arguments, we prove a result of differentiability for the more general problem

$$x'(t) = f(t, x(t), x(t-\tau)), \quad t > 0, \tag{1.1a}$$
$$x(t) = \phi(t), \quad -r \le t \le 0, \tag{1.1b}$$

where $f : \Omega \mapsto \mathbf{R}^n$ and Ω is open in $\mathbf{R}_+ \times \mathbf{R}^{2n}$. An infinite-dimensional version of (1.1) is also investigated and an example is given.

Received for publication May 1997.
AMS Subject Classifications: 34K15, 34K30, 47H15.
* This research is supported by the Academy of Finland and by the Romanian Academy and carried out while the second author was visiting University of Jyväskylä.

2. Existence and uniqueness. Assume that

(2.1) $f : (t, x, y) \mapsto f(t, x, y)$ belongs to $C(\Omega; \mathbf{R}^n)$ and it is locally Lipschitzian in Ω with respect to x;

(2.2) $\phi \in C([-r, 0]; \mathbf{R}^n)$ and $\tau \in (0, r]$ is fixed such that $(0, \phi(0), \phi(-\tau)) \in \Omega$.

If (2.1) and (2.2) hold, then, by the classical "method of steps," it follows that problem (1.1) has a unique noncontinuable C^1 solution, defined on an interval $[0, T_{max})$ and, moreover, $(t, x(t), x(t-\tau))$ leaves every compact set of Ω as $t \nearrow T_{max}$.

3. Continuous dependence. Let $\tau_0 \in (0, r)$ and $\phi_0 \in C([-r, 0]; \mathbf{R}^n)$ be such that $(0, \phi_0(0), \phi_0(-\tau_0)) \in \Omega$. We suppose, in addition, that $f \in C(\Omega; \mathbf{R}^n)$ and $f : (t, x, y) \mapsto f(t, x, y)$ is locally Lipschitzian in Ω with respect to (x, y).

Denote by $x(t, \phi_0, \tau_0)$ the noncontinuable solution of problem (1.1) with $\phi = \phi_0$ and $\tau = \tau_0$, defined on $[0, T^0_{max})$, $T^0_{max} = T_{max}(\phi_0, \tau_0)$. Let $T \in (0, T^0_{max})$ be fixed. Let $\|\cdot\|_C$ be the norm of $C([-r, 0]; \mathbf{R}^n)$. Obviously, for $\phi \in C([-r, 0]; \mathbf{R}^n)$ and $\tau \in (0, r)$ such that

$$\|\phi - \phi_0\|_C < \delta, \quad |\tau - \tau_0| < \delta; \tag{3.1}$$

where $\delta > 0$ is small enough, we have that $(0, \phi(0), \phi(-\tau)) \in \Omega$. Moreover, there exists a compact set $K \subset \Omega$ such that $(t, x(t, \phi, \tau), x(t - \tau, \phi, \tau)) \in K$, for every $t \in [0, T]$ and for every ϕ, τ satisfying (3.1). Hence the application $(\phi, \tau) \mapsto x(\cdot, \phi, \tau), V \mapsto C([0, T]; \mathbf{R}^n)$ is continuous at $(\phi, \tau) = (\phi_0, \tau_0)$, where $V \subset C([-r, 0]; \mathbf{R}^n) \times \mathbf{R}$ is a neighborhood of (ϕ_0, τ_0). The reasoning is classical (see, e.g., [2]). We can do it first on a small interval $[0, \alpha]$, and then the proof is completed by successively stepping intervals of length α until the interval $[0, T]$ is covered. Of course, the tool used for this purpose is Gronwall's lemma. In fact, the application $(\phi, \tau) \mapsto x(\cdot, \phi, \tau)$ is locally Lipschitzian.

This continuity is not surprising, because on a small interval $[0, \alpha]$ we have

$$f(t, x(t, \phi, \tau), x(t - \tau, \phi, \tau)) = f(t, x(t, \phi, \tau), \phi(t - \tau)) =: g_{\phi,\tau}(t, x(t, \phi, \tau)),$$

where ϕ, τ satisfy relations (3.1). So, in fact, on $[0, \alpha]$, the continuity on (ϕ, τ) reduces to the problem of continuity on the right-hand side for an ordinary differential equation, and this is essentially solved in [8] (see also [6]).

Moreover, this remark suggests to us that the differentiability of the solution with respect to (ϕ, τ) also holds under appropriate assumptions.

Remark 3.1. Similar results concerning the existence, continuous dependence, and even differentiability with respect to (τ, p) can be stated for the equation

$$x'(t) = f(t, x(t), x(t-\tau), \tau, p).$$

However, we restrict ourselves to equation (1.1).

4. Differentiability with respect to delay. It is well known that the solution of (1.1) is smooth in ϕ (see [8]). So, for the sake of simplicity, we shall discuss only the differentiability with respect to τ.

Suppose that

$$\phi \in C^1([-r, 0]; \mathbf{R}^n), \tag{4.1}$$

$$A := \{\tau \in (0, r) : (0, \phi(0), \phi(-\tau)) \in \Omega\} \neq \emptyset. \tag{4.2}$$

Obviously, A is open. Suppose, in addition, that

$$\begin{array}{l} f \in C(\Omega; \mathbf{R}^n) \text{ and there exist Jacobi matrices } f_x,\ f_y \\ \text{and their components are continuous in } \Omega. \end{array} \tag{4.3}$$

Theorem 4.1. *Let assumptions* (4.1)–(4.3) *be satisfied. Let* $[0, T^0_{max})$ *be the maximal interval of the noncontinuable solution* $x = x(t, \tau_0)$ *of problem* (1.1) *with* $\tau = \tau_0 \in A$. *Then, for every* $T \in (0, T^0_{max})$, *there exists* $\delta > 0$ *such that for* $|\tau - \tau_0| < \delta$ *the solution* $x(t, \tau)$ *is defined on* $[0, T]$ *and the application* $\tau \mapsto x(\cdot, \tau), (\tau_0 - \delta, \tau_o + \delta) \mapsto C([0, T]; \mathbf{R}^n)$ *is continuously differentiable. Moreover,* $z(t, \tau) = \frac{\partial}{\partial \tau} x(t, \tau)$ *satisfies the variational system*

$$\frac{\partial}{\partial t} z(t, \tau) = \begin{cases} f_x(t, x(t, \tau), \phi(t-\tau)) z(t, \tau) \\ \quad -\phi'(t-\tau) f_y(t, x(t, \tau), \phi(t-\tau)), \ \text{if } 0 \le t < \tau, \\ f_x(t, x(t, \tau), x(t-\tau, \tau)) z(t, \tau) \\ \quad + f_y(t, x(t, \tau), x(t-\tau, \tau))(z(t-\tau, \tau) - \frac{\partial}{\partial t} x(t-\tau, \tau)), \\ \qquad\qquad \text{if } \tau \le t \le T, \end{cases} \tag{4.4}$$

$$z(t, \tau) = 0, \quad -r \le t \le 0, \tag{4.5}$$

where $\tau \in (\tau_0 - \delta, \tau_0 + \delta)$.

Proof. Let $T \in (0, T^0_{max})$ be fixed. There exists $\delta > 0$ such that for every τ, $|\tau - \tau_0| < \delta$, the solution $x(t, \tau)$ is defined on $[0, T]$ and the application $\tau \mapsto x(\cdot, \tau), (\tau_0 - \delta, \tau_0 + \delta) \mapsto C([0, T]; \mathbf{R}^n)$ is continuous. For $\delta > 0$ small, there is a compact set $K \subset \Omega$ such that $(t, x(t, \tau), x(t - \tau, \tau)) \in K$, for $0 \le t \le T$, $|\tau - \tau_0| < \delta$. If $T^0_{max} \le \tau_0$, then $T < \tau_0$, and we can assume that $T < \tau$, for $|\tau - \tau_0| < \delta$. In this case, for $0 \le t \le T$,

$$f(t, x(t, \tau), x(t - \tau, \tau)) = f(t, x(t, \tau), \phi(t - \tau)) =: g(t, x(t, \tau), \tau), \qquad (4.6)$$

and so the problem reduces to the problem of differentiability of the solution of an ordinary differential equation with respect to a parameter. In this case our result is already proved and we have the first part of (4.4), for $0 \le t \le T$ (see, e.g., [1], [2]). Now, let us consider the case $\tau_0 < T < T^0_{max}$. For a small interval $I_1 = [0, \alpha]$, $0 < \alpha < \tau_0 - \delta$, equation (4.6) still holds. Assuming that the result is true on the interval $I_k = [0, k\alpha]$, does the same result hold on the next interval of length α, say I_{k+1}? To answer this question, we have just to remark that $\tau - t$ is located to the left of I_{k+1}, for $t \in I_{k+1}$ and $|\tau - \tau_0| < \delta$. Hence τ intervenes in

$$g(t, x, \tau) := f(t, x, x(t - \tau, \tau)), \quad t \in I_{k+1},$$

by means of $x(t - \tau, \tau)$. So to apply the classical differentiability result in I_{k+1}, it suffices to prove that the application $(t, \tau) \mapsto x(t, \tau)$ is continuously differentiable on $D_k := [0, k\alpha] \times (\tau_0 - \delta, \tau_0 + \delta)$. For (t, τ), $(t_1, \tau_1) \in D_k$ we have

$$\|x(t, \tau) - x(t_1, \tau_1)\| \le \|x(t, \tau) - x(t, \tau_1)\| + \|x(t, \tau_1) - x(t_1, \tau_1)\|,$$

and hence, by the continuity with respect to τ, the application $(t, \tau) \mapsto x(t, \tau)$ is continuous on D_k. As $x(t, \tau)$ satisfies (1.1), it follows that $\frac{\partial x}{\partial t}$ is also continuous on D_k. On the other hand, a continuity result holds for the linear system (4.4), (4.5) (see also Remark 3.1), and hence $z = \frac{\partial x}{\partial \tau}$ is also continuous on D_k (even $\frac{\partial z}{\partial t}$ is continuous on D_k). Therefore, the application $(t, \tau) \mapsto x(t, \tau)$ is indeed continuously differentiable on D_k. The proof of the theorem is now complete.

Remark 4.1. Following an idea suggested by V. Barbu, let us change the variables:

$$s = \frac{t}{\tau}, \quad y(s) = x(t) = x(s\tau). \qquad (4.7)$$

So, the system (1.1) becomes

$$y'(s) = \tau f(s\tau, y(s), y(s-1)), \quad s \geq 0, \tag{4.8a}$$

$$y(s) = \phi(s\tau), \quad -1 \leq s \leq 0, \tag{4.8b}$$

which is a system with fixed delay.

Clearly, a new result of differentiability can be formulated, but we have to assume, in addition, that f is C^1. In this case $\frac{\partial y}{\partial \tau}$ involves the derivative of y with respect to the initial function on $[-1, 0]$. Formally, we have

$$\frac{\partial}{\partial \tau} x(t,\tau) = \frac{\partial}{\partial \tau} y(s,\tau) = \frac{\partial}{\partial \tau} y(\frac{t}{\tau},\tau) = -\frac{t}{\tau^2}\frac{\partial y}{\partial s}(\frac{t}{\tau},\tau) + \frac{\partial y}{\partial \tau}(\frac{t}{\tau},\tau). \tag{4.9}$$

We leave to the reader to formulate and prove such a result. Thus the problem of dependence on delay can be reduced to the problem of dependence on a simple parameter in the equation and on the initial function. Anyway, the autonomous case, as considered by Hale and Ladeira ([9]), does not require additional assumptions upon f. However, the approach of Hale and Ladeira ([9]) could be very appropriate for neutral equations (see [5], [11], [12]).

Remark 4.2. Similar results can be obtained for an equation with finitely many delays, including the case of time-dependent delays. Obviously, in this case, the device described in Remark 4.1 cannot be applied. Higher-order differentiability with respect to the delay can also be investigated.

5. The infinite-dimensional case. Let $T > 0$ and let V be a real reflexive Banach space with the dual V^*. Let V be embedded densely and continuously in a real Hilbert space H, which is identified with its dual H^*, and hence $V \subset H = H^* \subset V^*$. We denote the norms of V, H and V^* by $\|\cdot\|$, $|\cdot|$, and $\|\cdot\|_*$, respectively. Denote by $\langle u, v\rangle$ the value of $v \in V^*$ at $u \in V$. If $u, v \in H$, this is their scalar product in H.

Consider the following delay problem:

$$x'(t) + A(t)x(t) = B(t)x(t-\tau), \quad 0 < t < T, \tag{5.1a}$$

$$x(0) = h, \tag{5.1b}$$

$$x(t) = \phi(t), \text{ for a.e. } t \in (-r, 0), \tag{5.1c}$$

where $r > 0$ is fixed and $\tau \in (0, r]$.

5.1. Existence and uniqueness. Let $2 \leq p < \infty$. We suppose that the operators $A(t), B(t) : V \mapsto V^*$ satisfy the following hypotheses:

(H.1) For every $\psi \in L^p(0,T;V)$, the functions $t \mapsto A(t)\psi(t)$ and $t \mapsto B(t)\psi(t)$ are V^*-measurable in $(0,T)$. Moreover, there exists $C > 0$ such that

$$\|A(t)y\|_* + \|B(t)y\|_* \leq C(1 + \|y\|^{p-1}), \tag{5.2}$$

for all $y \in V$ and almost every $t \in (0,T)$;

(H.2) For almost every $t \in (0,T)$, $A(t)$ is monotone and hemicontinuous;

(H.3) There exist α, $\omega > 0$ such that

$$\langle y, A(t)y\rangle + \alpha|y|^2 \geq \omega\|y\|^p, \text{ for all } y \in V \text{ and a.e. } t \in (0,T). \tag{5.3}$$

Theorem 5.1. *If* (H.1)–(H.3) *hold, then for every* $\phi \in L^p(-r,0;V)$, $\tau \in (0,r]$ *and* $h \in H$, *problem* (5.1) *has a unique solution* $x \in L^p(0,T;V) \cap C([0,T];H)$ *with* $x' \in L^q(0,T;V^*)$, *where* q *satisfies* $1/p + 1/q = 1$.

Proof. On $[0,\alpha]$, $\alpha > 0$ small enough, problem (5.1) can be written as

$$x'(t) + A(t)x(t) = f(t), \quad 0 < t < \alpha, \tag{5.4a}$$

$$x(0) = h, \tag{5.4b}$$

where $f(t) = B(t)\phi(t-\tau)$. According to (H.1) $f \in L^q(0,\alpha;V^*)$. So, by [3, Chapter I, Section 4.2] problem (5.4) has a unique solution $x \in L^p(0,\alpha;V) \cap C([0,\alpha];H)$ with $x' \in L^q(0,\alpha;V^*)$. We can pass to the interval $[\alpha, 2\alpha]$ with the initial value $x(\alpha)$. The proof is completed by stepping the intervals of length α until the interval $[0,T]$ is covered.

5.2. Continuous dependence on ϕ, h, and τ. We assume again (H.1), (H.2) and, in addition,

(H.4) There exist α, $\omega > 0$ satisfying, for each $y_1, y_2 \in V$ and almost every $t \in (0,T)$,

$$\langle y_1 - y_2, A(t)y_1 - A(t)y_2\rangle + \alpha|y_1 - y_2|^2 \geq \omega\|y_1 - y_2\|^p; \tag{5.5}$$

(H.5) The operator $\psi \mapsto B(\cdot)\psi(\cdot)$, $L^p(0,T;V) \mapsto L^q(0,T;V^*)$ is continuous.

Remark 5.1. If $A(t)$ are linear operators, then (H.4) is equivalent to (H.3). If V is finite dimensional, $B(\cdot)$ satisfies the Carathéodory conditions and (5.2) holds, then (H.5) is valid; cf. [10]. Since the measurability condition in (H.1) is a weaker one, (H.5) is not superfluous. Indeed, (H.1), but not (H.5), is satisfied by

$$V = \mathbf{R} \quad \text{and} \quad B(t)u = \begin{cases} 1, & \text{if} \quad u \geq 0 \\ 0, & \text{if} \quad u < 0. \end{cases}$$

Theorem 5.2. *Assume that* (H.1), (H.2), (H.4) *and* (H.5) *hold. Then the solution of* (5.1) *depends continuously on* (ϕ, h, τ) *as a function from* $L^p(-r, 0; V) \times H \times (0, r)$ *to* $L^p(0, T; V) \cap C([0, T]; H)$.

Proof. We may suppose that $A(t)0 = 0$, and hence (H.4) implies (H.3). Otherwise we can replace $A(t)$ and $B(t)$ by $\tilde{A}(t)$ and $\tilde{B}(t)$, defined by $\tilde{A}(t)y := A(t)y - A(t)0$ and $\tilde{B}(t)y := B(t)y - B(t)0$. Let us fix (ϕ_0, h_0, τ_0) in $L^p(-r, 0; V) \times H \times (0, r)$ and denote by $\bar{x}(t)$ the corresponding solution of (5.1). Also, let (ϕ, h, τ) belong to the same product space such that

$$\|\phi - \phi_0\|_{L^p(-r,0;V)} < \delta, \quad |h - h_0| < \delta, \quad |\tau - \tau_0| < \delta, \tag{5.5}$$

with $\delta > 0$ small enough, and denote by $x(t) = x(t, \phi, h, \tau)$ the corresponding solution of (5.1). In a small interval $[0, \alpha]$ we have

$$x'(t) + A(t)x(t) = B(t)\phi(t - \tau) \text{ a.e. } t \in (0, \alpha), \tag{5.6a}$$

$$x(0) = h, \tag{5.6b}$$

and

$$\bar{x}'(t) + A(t)\bar{x}(t) = B(t)\phi_0(t - \tau_0) \text{ a.e. } t \in (0, \alpha), \tag{5.7a}$$

$$\bar{x}(0) = h_0. \tag{5.7b}$$

By our assumptions it follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|x(t) - \bar{x}(t)|^2 + \omega\|x(t) - \bar{x}(t)\|^p - \alpha_1|x(t) - \bar{x}(t)|^2 \\ \leq \langle x(t) - \bar{x}(t), B(t)\phi(t - \tau) - B(t)\phi_0(t - \tau_0)\rangle, \end{aligned}$$

for almost every $t \in (0, \alpha)$ and for some $\alpha_1 > 0$. Using the well-known inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \text{ for } a, b \geq 0, \tag{5.8}$$

we can easily obtain

$$
\begin{aligned}
&|x(t)-\bar{x}(t)|^2+\int_0^t \|x(s)-\bar{x}(s)\|^p\,ds \\
&\quad \le C_1\bigg(\int_0^t \|B(s)\phi(s-\tau)-B(s)\phi_0(s-\tau_0)\|_*^q\,ds+|h-h_0|^2\bigg),
\end{aligned}
\tag{5.9}
$$

for $0 \le t \le \alpha$, where C_1 is a positive constant. Let us notice that

$$
\begin{aligned}
&\bigg(\int_0^\alpha \|\phi(s-\tau)-\phi_0(s-\tau_0)\|^p\,ds\bigg)^{1/p} \\
&\quad \le \bigg(\int_0^\alpha \|\phi_0(s-\tau)-\phi_0(s-\tau_0)\|^p\,ds\bigg)^{1/p}+\|\phi-\phi_0\|_{L^p(-r,0;V)}.
\end{aligned}
\tag{5.10}
$$

By (5.5), (5.9), (5.10) and (H.5), for each $\epsilon > 0$, there is $\delta > 0$ such that

$$
\|x-\bar{x}\|_{C([0,\alpha];H)}+\|x-\bar{x}\|_{L^p(0,\alpha;V)}<\epsilon. \tag{5.11}
$$

We can continue the above reasoning on intervals of length α, obtaining inequalities similar to (5.11). For this purpose we can use the following inequalities:

$$
\begin{aligned}
&\bigg(\int_{k\alpha}^{(k+1)\alpha} \|x(s-\tau)-\bar{x}(s-\tau_0)\|^p\,ds\bigg)^{1/p} \\
&\quad \le \bigg(\int_{k\alpha}^{(k+1)\alpha} \|\bar{x}(s-\tau)-\bar{x}(s-\tau_0)\|^p\,ds\bigg)^{1/p} \\
&\quad + \bigg(\int_{k\alpha}^{(k+1)\alpha} \|x(s-\tau)-\bar{x}(s-\tau)\|^p\,ds\bigg)^{1/p}.
\end{aligned}
$$

The whole interval $[0, T]$ will be covered in a finite number of steps.

5.3. Differentiability with respect to τ. For the sake of simplicity we shall assume that $A(t) = A \in L(V, V^*)$ and $B(t) = B \in L(V, V^*)$ and that there exist α, $\omega > 0$ such that

$$
\langle y, Ay\rangle+\alpha|y|^2 \ge \omega\|y\|^2,\ \text{for every } y \in V. \tag{5.12}
$$

We are interested in the problem of differentiability with respect to τ of the solution $x = x(t,\tau)$ of the problem

$$\frac{\partial}{\partial t}x(t,\tau) + Ax(t,\tau) = Bx(t-\tau,\tau), \quad 0 < t < T, \tag{5.13a}$$

$$x(0,\tau) = h, \tag{5.13b}$$

$$x(t,\tau) = \phi(t), \text{ for a.e. } t \in (-r,0), \tag{5.13c}$$

where $\tau \in (0,r)$. The existence and continuous dependence of solutions for (5.13) are guaranteed by the above assumptions. In order to prove the differentiability of $x(t,\tau)$ with respect to τ let us change the variables (see (4.7))

$$s = \frac{t}{\tau}, \quad y(s,\tau) = x(t,\tau) = x(s\tau,\tau). \tag{5.14}$$

So problem (5.13) becomes

$$\frac{\partial}{\partial s}y(s,\tau) + \tau Ay(s,\tau) = \tau By(s-1,\tau), \quad 0 < s < \frac{T}{\tau}, \tag{5.15a}$$

$$y(0,\tau) = h \tag{5.15b}$$

$$y(s,\tau) = \phi(s\tau), \quad -\frac{r}{\tau} \le s \le 0. \tag{5.15c}$$

Taking into account (4.9) it suffices to investigate the existence of $\frac{\partial}{\partial\tau}y(s,\tau)$. Let us fix $\tau_0 \in (0,r)$. On a small interval $[0,\alpha]$ problem (5.15) can be rewritten as

$$\frac{\partial}{\partial s}y(s,\tau) + \tau Ay(s,\tau) = \tau B\phi((s-1)\tau)), \quad 0 < s < \alpha, \tag{5.16a}$$

$$y(0,\tau) = h, \tag{5.16b}$$

with $|\tau - \tau_0|$ sufficiently small. We assume that $\phi \in W^{1,2}(-r,0;V)$ and consider the problem

$$\begin{aligned} z'(s) + \tau_0 Az(s) = (s-1)\tau_0 B\phi'((s-1)\tau_0) - Ay(s,\tau_0) \\ + B\phi((s-1)\tau_0), \quad 0 < s < \alpha, \end{aligned} \tag{5.17a}$$

$$z(0) = 0. \tag{5.17b}$$

Obviously, the right-hand side of (5.17a) is a function in $L^2(0,\alpha;V^*)$, and hence (5.17) has a unique solution $z \in L^2(0,\alpha;V) \cap C([0,\alpha];H)$ with $z' \in L^2(0,\alpha;V^*)$.

Denote $\theta_h(s) := \frac{1}{h}\big(y(s,\tau_0+h) - y(s,\tau_0)\big) - z(s)$ with $|h| > 0$ sufficiently small. By (5.16) and (5.17) we can see that

$$\begin{aligned} \frac{d}{ds}\theta_h(s) + \tau_0 A\theta_h(s) &= \tau_0 B\frac{1}{h}\big(\phi\big((s-1)(\tau_0+h)\big) - \phi\big((s-1)\tau_0\big)\big) \\ &- Ay(s,\tau_0+h) + B\phi\big((s-1)(\tau_0+h)\big) - (s-1)\tau_0 B\phi'\big((s-1)\tau_0\big) \\ &+ Ay(s,\tau_0) - B\phi\big((s-1)\tau_0\big), \quad 0 < s < \alpha, \end{aligned} \tag{5.18a}$$

$$\theta_h(0) = 0. \tag{5.18b}$$

By the result of continuous dependence, the right-hand side of (5.18a) converges to zero, as $h \to 0$, in $L^2(0,\alpha;V^*)$. Multiplying (5.18a) by $\theta_h(s)$ and using the properties of A, B we can see that $\theta_h \to 0$, as $h \to 0$, in $L^2(0,\alpha;V) \cap C([0,\alpha];H)$. This procedure can be continued until the interval $[0,T/\tau]$ is covered. So we have proved the following theorem.

Theorem 5.3. *If $A, B \in L(V,V^*)$, $\phi \in W^{1,2}(-r,0;V)$, $h = \phi(0)$, (5.12) holds and $y(s,\tau)$ is the solution of (5.15), then there exists $z(s,\tau) = \frac{\partial}{\partial\tau}y(s,\tau)$, and it satisfies the variational equation*

$$\begin{aligned} &\frac{\partial}{\partial s}z(s,\tau) + \tau Az(s,\tau) + Ay(s,\tau) = \\ &= \begin{cases} \tau(s-1)B\phi'((s-1)\tau) + B\phi((s-1)\tau), & \text{for } 0 < s < 1, \\ \tau Bz(s-1,\tau) + By(s-1,\tau), & \text{for } s \geq 1, \end{cases} \end{aligned} \tag{5.19a}$$

$$z(0,\tau) = 0. \tag{5.19b}$$

Remark 5.2. The above theorem still holds if instead of (5.13a) we consider a nonhomogeneous equation with a nonhomogeneous term $f \in L^2(0,T,V^*)$.

Other results can also be obtained for the nonlinear and nonautonomous case for which we state the following theorem. We need the following hypothesis. Let us denote by $D(A(t))$ the domain of $A(t)$.

(H.6) There exist α, $\omega > 0$ such that, for each $y_1, y_2 \in D(A(t))$ and for almost every $t \in (0,T)$,

$$\langle y_1 - y_2, A(t)y_1 - A(t)y_2\rangle + \alpha|y_1 - y_2|^2 \geq \omega\|y_1 - y_2\|^2.$$

Theorem 5.4. *Let $T, r > 0$, $\phi \in W^{1,2}(-r, 0; V)$ and $h = \phi(0)$. Assume that the operators $A(t) : D(A(t)) \subset V \mapsto V^*$ and $B(t) : D(B(t)) \subset V \mapsto V^*$, $t \in [0, T]$, satisfy* (H.4) *and $B(t)$ are uniformly Lipschitzian and that* (5.13) *has a solution $x(\cdot, \tau) \in W^{1,2}(-r, 0; V^*) \cap L^2(-r, T; V)$, for each $\tau \in (0, r)$. Then the function $\tau \mapsto x(\cdot, \tau)$ belongs to $W^{1,2}(\delta, r; L^2(0, T; V))$ and $x(t, \cdot) \in W^{1,2}(\delta, r; H)$, for each $\delta \in (0, r)$ and $t \in [0, T]$. Thus, for each $t \in (0, T)$ and for almost every $\tau \in (0, r)$, there is $\frac{\partial x}{\partial \tau}(t, \tau) \in V$ satisfying*

$$\lim_{k \to 0} \Big| \frac{x(t, \tau + k) - x(t, \tau)}{k} - \frac{\partial x}{\partial \tau}(t, \tau) \Big| = 0,$$
$$\lim_{k \to 0} \Big\| \frac{x(\cdot, \tau + k) - x(\cdot, \tau)}{k} - \frac{\partial x}{\partial \tau}(\cdot, \tau) \Big\|_{L^2(0,T;V)} = 0.$$

Proof. Let $\tau \in (0, r)$ and $k \in (0, r - \tau)$. Then by (5.13a), (H.6) and by the uniform Lipschitz continuity of the $B(t)$'s,

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} |x(t, \tau + k) - x(t, \tau)|^2 &= \langle x(t, \tau + k) - x(t, \tau), x'(t, \tau + k) - x'(t, \tau) \rangle \\ &\le -\omega \|x(t, \tau + k) - x(t, \tau)\|^2 + \alpha |x(t, \tau + k) - x(t, \tau)|^2 \\ &\quad + \|x(t, \tau + k) - x(t, \tau)\| M \|x(t - \tau - k, \tau + k) - x(t - \tau, \tau)\|, \end{aligned}$$

for almost every $t \in (0, T)$. By integrating over $[s, t]$ and by inequality (5.8) we obtain

$$\begin{aligned} &\frac{1}{2} |x(t, \tau + k) - x(t, \tau)|^2 + \frac{\omega}{2} \int_s^t \|x(\sigma, \tau + k) - x(\sigma, \tau)\|^2 \, d\sigma \\ &\quad \le \frac{1}{2} |x(s, \tau + k) - x(s, \tau)|^2 + \alpha \int_s^t |x(\sigma, \tau + k) - x(\sigma, \tau)|^2 \, d\sigma \\ &\quad\quad + \frac{M^2}{2\sqrt{\omega}} \int_s^t \|x(\sigma - \tau - k, \tau + k) - x(\sigma - \tau, \tau)\|^2 \, d\sigma; \end{aligned}$$

whence, by Gronwall's inequality,

$$\begin{aligned} &\frac{1}{2} |x(t, \tau + k) - x(t, \tau)|^2 + \frac{\omega}{2} \int_s^t \|x(\sigma, \tau + k) - x(\sigma, \tau)\|^2 \, d\sigma \\ &\quad \le e^{M_1 (t - s)} \Big(\frac{1}{2} |x(s, \tau + k) - x(s, \tau)|^2 \\ &\quad\quad + M_2 \int_{s - \tau}^{t - \tau} \|x(\sigma - k, \tau + k) - x(\sigma, \tau)\|^2 \, d\sigma \Big), \end{aligned} \tag{5.20}$$

for each $0 \le s \le t \le T$.

Let $\delta \in (0, r)$. We choose first in (5.20) $s = 0$ and $t = \tau$. Then by (5.13c)

$$\frac{1}{2}|x(\tau, \tau + k) - x(\tau, \tau)|^2 + \frac{\omega}{2}\int_0^\tau \|x(\sigma, \tau + k) - x(\sigma, \tau)\|^2 \, d\sigma$$
$$\le e^{M_1\tau}\Big(\frac{1}{2}|\phi(0) - \phi(0)|^2 + M_2\int_{-\tau}^0 \|\phi(\sigma - k) - \phi(\sigma)\|^2 \, d\sigma\Big) \le M_3k^2,$$

since $\phi \in W^{1,2}(-r, 0; V)$ (cf. [4, pp. 156, 145]). Next we choose in (5.20) $s = \tau$ and $t = 2\tau$. Then

$$\frac{1}{2}|x(2\tau, \tau + k) - x(2\tau, \tau)|^2 + \frac{\omega}{2}\int_\tau^{2\tau} \|x(\sigma, \tau + k) - x(\sigma, \tau)\|^2 \, d\sigma$$
$$\le e^{M_1\tau}\Big(\frac{1}{2}|x(\tau, \tau + k) - x(\tau, \tau)|^2 + M_2\int_0^\tau \|x(\sigma, \tau + k) - x(\sigma, \tau)\|^2 \, d\sigma\Big)$$
$$\le e^{M_1\tau}(M_3k^2 + \frac{2M_2}{\omega}M_3k^2) \le M_4M_3k^2.$$

After n steps we have

$$\frac{1}{2}|x(n\tau, \tau + k) - x(n\tau, \tau)|^2 + \frac{\omega}{2}\int_{n\tau}^{n\tau+\tau} \|x(\sigma, \tau + k) - x(\sigma, \tau)\|^2 \, d\sigma \le M_4^nM_3k^2. \tag{5.21}$$

The number of steps needed to reach T is the smallest natural number greater than T/τ. Let us denote it by N. Hence

$$\int_0^T \|x(\sigma, \tau + k) - x(\sigma, \tau)\|^2 \, d\sigma = \sum_{j=0}^{N-1}\int_{j\tau}^{j\tau+\tau} \|x(\sigma, \tau + k) - x(\sigma, \tau)\|^2 \, d\sigma$$
$$+ \int_{T-\tau}^T \|x(\sigma, \tau + k) - x(\sigma, \tau)\|^2 d\sigma \le \sum_{j=0}^N \frac{2}{\omega}M_4^jM_3k^2 \le M_6M_4^{T/\tau}M_3k^2.$$

Thus

$$\int_\delta^{r-k}\int_0^T \|x(\sigma, \tau + k) - x(\sigma, \tau)\|^2 \, d\sigma d\tau \le M_\delta k^2,$$

where $M_\delta < \infty$. Since the space $L^2(0,T;V)$ is reflexive, the function $\tau \mapsto x(\cdot,\tau)$ belongs to $W^{1,2}(\delta, r; L^2(0,T;V))$, for each $\delta \in (0,r)$ (cf. [4, pp. 156, 145]).

Let $t \in (0,T)$ and $n \in \mathbf{N}$ be such that $t \in [n\tau, n\tau+\tau]$. Then by (5.20) with $s = n\tau$,

$$\begin{aligned}
\tfrac{1}{2}|x(t,\tau+k) - x(t,\tau)|^2 &\le e^{M_1(t-n\tau)}\Big(\tfrac{1}{2}|x(n\tau,\tau+k) - x(n\tau,\tau)|^2 \\
&\quad + M_2\int_{n\tau-\tau}^{t-\tau} \|x(\sigma-k,\tau+k) - x(\sigma,\tau)\|^2\,d\sigma\Big) \\
&\le e^{M_1 t}\Big(\tfrac{1}{2}\|x(n\tau,\tau+k) - x(n\tau,\tau)|^2 \\
&\quad + M_2\int_{n\tau-\tau}^{n\tau} \|x(\sigma-k,\tau+k) - x(\sigma,\tau)\|^2\,d\sigma\Big) \\
&\le e^{M_1 t}(M_4^n M_3 k^2 + M_4^{n-1} M_3 k^2) \le M_7 M_4^n k^2,
\end{aligned}$$

by (5.21). Thus

$$\int_\delta^{r-k} |x(t,\tau+k) - x(t,\tau)|^2\,d\tau \le M_8 M_\delta k^2,$$

which implies that $x(t,\cdot) \in W^{1,2}(\delta,r;H)$.

6. An example. Let us consider the heat conduction in an open bounded domain $\Omega \subset \mathbf{R}^3$. The continuity equation for energy reads as

$$\frac{\partial}{\partial t}\rho(x,t) + \nabla\cdot\mathbf{j}(x,t) = 0, \quad x \in \Omega,\ t \in \mathbf{R}, \tag{6.1}$$

if there are no heat sources. Let $u(x,t)$, $\rho(x,t)$ and $\mathbf{j}(x,t)$ represent the temperature, energy density and the heat flow density, respectively, at place $x \in \Omega$ and at time $t \in \mathbf{R}$. Let $\tau, c, k_1, k_2 > 0$ be constants, and assume that

$$\rho(x,t) = cu(x,t), \quad x \in \Omega, \quad t \in \mathbf{R}, \tag{6.2}$$

$$\mathbf{j}(x,t) = -k_1\nabla u(x,t) - k_2\nabla u(x,t-\tau), \quad x \in \Omega, \quad t \in \mathbf{R}. \tag{6.3}$$

Instead of the usual Fourier's law (i.e., (6.3) with $k_2 = 0$) we thus assume that the heat flow density has a retarded term; (6.3) is a special case of the general formula of [7] (i.e., the convolution kernel is chosen to be $k_1\delta(t) + k_2\delta(t-\tau)$ where δ is Dirac's delta function). Hence we have the following system:

$$\frac{\partial u}{\partial t}(x,t) - \frac{k_1}{c}\triangle u(x,t) = \frac{k_2}{c}\triangle u(x,t-\tau), \quad x \in \Omega, \quad t \in (0,T), \tag{6.4}$$

$$\mathbf{j}(x,t)\cdot \mathbf{n}(x) = \xi(u(x,t)), \quad x \in \partial\Omega, \quad t \in [0,T], \tag{6.5}$$

$$u(x,t) = \phi(x,t), \quad x \in \Omega, \quad -1 \le t \le 0, \tag{6.6}$$

where $\phi \in W^{1,2}(-1,0;H^1(\Omega))$ and $\mathbf{n}(x)$ is the outward normal unit vector at the point $x \in \partial\Omega$ (the boundary of Ω), and $\xi : \mathbf{R} \mapsto \mathbf{R}$ is continuous and monotone satisfying $|\xi(q)| \le M(1+|q|)$, for each $q \in \mathbf{R}$. The boundary condition introduced by (6.5) represents the convective and radiative cooling. Define $V = H^1(\Omega)$, $H = L^2(\Omega)$ and $A, B : V \mapsto V^*$ by

$$\langle y, Az\rangle = \frac{k_1}{c}\int_\Omega \nabla y(x)\cdot\nabla z(x)\,dx + \frac{1}{c}\int_{\partial\Omega} y(x)\xi(z(x)),$$
$$\langle y, Bz\rangle = \frac{k_2}{c}\int_\Omega \nabla y(x)\cdot\nabla z(x)\,dx, \quad \text{for each} \quad y,z \in V.$$

Then the problem of finding a weak solution for (6.4)–(6.6) is equivalent to (5.13). The assumptions of Theorems 5.1, 5.2 and 5.4 are satisfied. If, in addition, ξ is linear, then by Theorem 5.3 the derivative $\frac{\partial u}{\partial \tau}$ can be found from equations (5.19).

REFERENCES

[1] H. Amann, "Ordinary Differential Equations. An Introduction to Nonliner Analysis," Walter de Gruyter, Berlin, 1990.
[2] V. Barbu, "Ecuaţii diferenţiale," Editura Junimea, Jassy, 1985.
[3] V. Barbu and Th. Precupanu, "Convexity and Optimization in Banach Spaces," Editura Academiei and D. Reidel, Bucharest and Dordrecht, 1986.
[4] H. Brézis, "Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert," North–Holland, Amsterdam, 1973.
[5] R.D. Driver, *Existence and continuous dependence of solutions of a neutral functional differential equation,* Arch. Rat. Mech. Anal., 19 (1965), 149–166.
[6] G. Gripenberg, S.-O. Londen, and O. Staffans, "Volterra Integral and Functional Equations," Cambridge University Press, Cambridge, 1990.
[7] M.E. Gurtin and A.C. Pipkin, *A general theory of heat conduction with finite wave speeds,* Arch. Rat. Mech. Anal., 31 (1968), 113–126.
[8] J.K. Hale, "Theory of Functional Differential Equations," Springer, Berlin, 1977.

[9] J.K. Hale and Luiz A.C. Ladeira, *Differentiability with respect to delays,* J. Diff. Equs., 92 (1991), 14–26.

[10] M.A. Krasnoselskiĭ and P.P. Zabreĭko, "Geometrical Methods of Nonlinear Analysis," Springer, Berlin, 1985.

[11] W.R. Melvin, *A class of neutral functional differential equations,* J. Diff. Equs., 12 (1972), 524–534.

[12] W.R. Melvin *Topologies for neutral functional differential equations,* J. Diff. Equs., 13 (1973), 24–34.