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SOME RESULTS FOR SECOND ORDER ELLIPTIC OPERATORS HAVING UNBOUNDED COEFFICIENTS

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Abstract. By proving some estimates for the derivatives of the transition semigroup relative to a suitable stochastic equation, we prove Schauder estimates for a class of second order linear differential operators with unbounded coefficients, both in the drift and in the diffusion term. We also study asymptotic behaviour of the transition semigroup and under suitable dissipative conditions we prove existence and uniqueness of invariant measures and exponentially mixing property, uniform with respect to initial datum.

1. Introduction. We are here concerned with the elliptic operator

$$\begin{aligned}\mathcal{A}_0 u(x) &= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x) D_{ij}u(x) + \sum_{i=1}^{d} F_i(x) D_i u(x) \\ &= \frac{1}{2}\mathrm{Tr}[A(x)D^2u(x)] + < F(x), Du(x) >, \quad x \in \mathbb{R}^d\end{aligned} \tag{1.1}$$

and with the associated stochastic differential equation

$$dX_t = F(X_t)\,dt + G(X_t)\,dW_t, \tag{1.2}$$

where $F = (F_1, \dots, F_d)$ is a vector field in $\mathbb{R}^d$ of class C^3, $G : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^r; \mathbb{R}^d)$ is of class C^3 as well and $(a_{ij}) = A = GG^\star$. F and G have polynomial growth and F satisfies some dissipativity conditions (cf. Hypotheses of Section 2). W_t is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with values in $\mathbb{R}^r$.

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The present work covers the case of the following class of second order linear operators

$$\mathcal{A}_0 u(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x) D_{ij} u(x) + \sum_{i=1}^{d} F_i(x) D_i u(x), \tag{1.3}$$

where $F_i(x) = P_i(\sqrt{1+|x|^2})$ and P_i are polynomials of odd degree $2m-1$ having the coefficient of the highest order term strictly negative, i.e.,

$$P_i(t) = -a^i t^{2m-1} + a^i_{2(m-1)} t^{2(m-1)} + \ldots + a^i_0, \quad a^i > 0$$

and where a_{ij} have polynomial growth together with their derivatives up to the third order, that is,

$$\sup_{x\in\mathbb{R}^d} \frac{|D^\beta a_{ij}(x)|}{1+|x|^{2k-l}} < +\infty, \quad |\beta| = l, \quad l = 0, \ldots, 3,$$

with $k \leq m-1$. A simple example is given by

$$\mathcal{A}_0 u(x) = \tfrac{1}{2}(1+x^2)u''(x) - x^3 u'(x), \quad x \in \mathbb{R}. \tag{1.4}$$

Our aim is to prove existence and uniqueness of solutions for the elliptic and the parabolic problem associated to operator $\mathcal{A}_0$ and optimal regularity results in spaces of Hölderian functions (Schauder estimates). Moreover we want to prove existence and uniqueness of invariant measures for the semigroup P_t and under stronger dissipativity assumptions for F we show that

$$\|P_t\varphi - < \varphi, \mu > \|_0 \leq C e^{-\eta t}\|\varphi\|_0,$$

where μ is the unique invariant measure.

All the results of the present paper are classical in the case of bounded coefficients. Unbounded coefficients have been considered by Aronson and Besala (see [1] and [2]) and by Cannarsa and Vespri (see [5]) in weighted spaces of functions. In a series of papers Da Prato, Lunardi and Vespri (see [9], [17], [19] and [20]) studied Ornstein-Uhlenbeck operator and some generalizations in space of continuous and bounded functions and in L^2 spaces with respect to invariant measures (for more details see introduction of [7]).

In the first part of this paper we study the elliptic and the parabolic equations associated to the operator $\mathcal{A}_0$

$$\lambda\varphi(x) - \mathcal{A}_0\varphi(x) = f(x), \quad x \in \mathbb{R}^d, \quad \lambda > 0 \tag{1.5}$$

and

$$\begin{cases} u_t(t,x) = \mathcal{A}_0 u(t,x) + f(t,x), & t \in (0,T], \quad x \in \mathbb{R}^d \\ u(0,x) = \varphi(x), & x \in \mathbb{R}^d. \end{cases} \tag{1.6}$$

After proving existence and uniqueness of solutions, we give Schauder estimates, both for problem (1.5) and for problem (1.6). This means for example that if in (1.5) $f \in C_b^\theta(\mathbb{R}^d)$ with $\theta \in (0,1)$ then the strict solution $\varphi \in C_b^{2+\theta}(\mathbb{R}^d)$. As in [7] the preliminary fundamental result is given by the following estimates for the transition semigroup associated to equation (1.2)

$$\sup_{x \in \mathbb{R}^d} |D^j P_t\varphi(x)| \leq C(t \wedge 1)^{-j/2}\|\varphi\|_0, \quad j = 0, \dots, 3. \tag{1.7}$$

We recall that P_t is defined for any $\varphi \in B_b(\mathbb{R}^d)$ by

$$P_t\varphi(x) = \mathbb{E}(\varphi(X_t(x))), \quad t \geq 0, \quad x \in \mathbb{R}^d,$$

where $X_t(x)$ is the solution of equation (1.2). We remark that in [7] the coefficients of the diffusion term are constant, while in the present case they have polynomial growth. Nevertheless the same kind of estimates can be proved. The main tool in order to get (1.7) is Bismut-Elworthy formula which gives the derivatives of $P_t\varphi$ in terms of the derivatives of $X_t(x)$ with respect to the initial datum x. Therefore the first important step is to study some boundedness properties for the solution $X_t(x)$ and its derivatives, up to the third order. To this purpose, the main estimate that we prove is the following

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}|X_t(x)|^p \leq C_p(t \wedge 1)^{-\frac{p}{2(m-1)}}, \quad t > 0, \tag{1.8}$$

for any $p > 0$. Actually, such an estimate is the fundamental tool in order to estimate the second and the third derivatives of $X_t(x)$ with respect to x which appear in Bismut-Elworthy formula for the second and the third derivatives of $P_t\varphi$.

Once proved (1.7), we show that the transition semigroup P_t is *weakly continuous* in the sense introduced in [6], so that we can introduce the infinitesimal generator $\mathcal{A}$. Then existence and uniqueness of solutions, both for the elliptic and the parabolic problem, are proved. In particular in the elliptic case we prove a Maximum Principle and we give a characterization of $D(\mathcal{A})$. We conclude with Schauder estimates by using a more general method due to Lunardi (see [18]) and based upon interpolation theory. A first consequence of Schauder estimates is the characterization of the domain of the infinitesimal generator of the semigroup P_t, considered as acting on $C_b^\theta(\mathbb{R}^d)$.

In the second part of the paper we study asymptotic behaviour of the solution of the stochastic equation $X_t(x)$. In particular we prove existence and uniqueness of the invariant measure μ for system (1.2) and we prove that the following rate of convergence hold in the strictly dissipative case

$$\|P_t\varphi - < \varphi, \mu > \|_0 \leq Ce^{-\eta t}\|\varphi\|_0, \quad t \geq 0, \tag{1.9}$$

for a suitable positive constant η. This implies ergodicity and strongly mixing property. As a preliminary step strong Feller property is proved in the case of not differentiable coefficients.

2. Notation and main assumptions. Throughout the paper $C_b(\mathbb{R}^d)$ is the Banach space of uniformly continuous and bounded functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ endowed with the norm

$$\|\varphi\|_0 = \sup_{x \in \mathbb{R}^d} |\varphi(x)|.$$

$C_b^k(\mathbb{R}^d)$ is the Banach space of k-times differentiable real functions defined on $\mathbb{R}^d$ and having bounded and continuous derivatives up the k-th order. $C_b^k(\mathbb{R}^d)$ is endowed with the norm

$$\|\varphi\|_k = \|\varphi\|_0 + \sum_{i=1}^k [\varphi]_i,$$

where

$$[\varphi]_i =: \sup_{x \in \mathbb{R}^d} |D^i\varphi(x)|, \quad i = 1, \dots, k.$$

$C_b^\theta(\mathbb{R}^d)$ is the Banach space of θ-Hölder continuous and bounded functions $\varphi \in C_b(\mathbb{R}^d)$ with the norm

$$\|\varphi\|_{C_b^\theta(\mathbb{R}^d)} =: \|\varphi\|_0 + [\varphi]_\theta,$$

where

$$[\varphi]_\theta =: \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\theta}.$$

Moreover, for any $k \in \mathbb{N}$, $C_b^{k+\theta}(\mathbb{R}^d)$ is the Banach space of all functions $\varphi \in C_b^k(\mathbb{R}^d)$ such that

$$[\varphi]_{k,\theta} = \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|D^k\varphi(x)-D^k\varphi(y)|}{|x-y|^\theta} < +\infty.$$

$C_b^{k+\theta}(\mathbb{R}^d)$ is endowed with the norm

$$\|\varphi\|_{C_b^{k+\theta}(\mathbb{R}^n)} = \|\varphi\|_{C_b^k(\mathbb{R}^n)} + [\varphi]_{k,\theta}.$$

Hypothesis 2.1. 1. *There exists* $m \geq 2$ *such that* $\forall\, j = 0,\dots,3$

$$\sup_{x\in\mathbb{R}^d} \frac{|D^\beta F(x)|}{1+|x|^{2m-1-j}} < +\infty, \quad \beta = (i_1,\dots,i_j).$$

2. *There exist* $a > 0$ *and* $b, c \in \mathbb{R}$ *such that*

$$< F(x+h) - F(x), h > \leq -a|h|^{2m} + b|x|^{2m} + c, \quad \forall\, x, h \in \mathbb{R}^d.$$

Hypothesis 2.2. 1. *There exists* $k \leq m-1$ *such that* $\forall\, j = 0,\dots,3$

$$\sup_{x\in\mathbb{R}^d} \frac{\|D^\beta G(x)\|}{1+|x|^{k-j}} < +\infty, \quad \beta = (i_1,\dots,i_j).$$

2. *For any* $x \in \mathbb{R}^d$ *there exists* $G^{-1}(x) \in \mathcal{L}(\mathbb{R}^d;\mathbb{R}^r)$ *and for a suitable* $N > 0$ *it holds*

$$\sup_{x\in\mathbb{R}^d} \|G^{-1}(x)\| \leq N.$$

We remark that in the present paper for any $T \in \mathcal{L}(\mathbb{R}^r;\mathbb{R}^d)$ we denote

$$\|T\|^2 = \mathrm{Tr}\,[TT^\star].$$

Hypothesis 2.3. *For all $p>0$ there exists $\omega_p \in \mathbb{R}$ such that for all $x, y \in \mathbb{R}^d$*

$$< DF(x)y, y > +p\,\|DG(x)y\|^2 \leq \omega_p |y|^2.$$

In particular this implies that for any $x, y \in \mathbb{R}^d$

$$< F(x) - F(y), x - y > +p\,\|G(x) - G(y)\|^2 \leq \omega_p |x-y|^2. \tag{2.1}$$

Throughout the paper we will denote with C (without any index) any positive constant appearing in inequalities, whose dependence on some parameters is not important. Such constants may change even in the same chain of inequalities. We will denote with C_p any positive constant whose dependence on a parameter p we want to emphasize.

3. An estimate for the solution of the SDE. Let us consider the problem

$$dX_t = F(X_t)\,dt + G(X_t)\,dW_t, \quad X_0 = x \tag{3.1}$$

and assume that F and G satisfy Hypotheses 2.1, 2.2 and 2.3. It is easy to show that F and G satisfy the *monotonicity condition* and the *growth condition* of Theorem V.1.1 by Krylov [14] where existence and uniqueness of solutions for equations of the same class of (3.1) are proved. Then we have

Theorem 3.1. *Under Hypotheses* 2.1, 2.2 *and* 2.3 *there exists a unique solution for problem* (3.1). *That is, there exists an a.s. unique d-dimensional process $X_t(x)$ $\mathcal{F}_t$-measurable for any $t \geq 0$ and continuous in t which satisfies equation* (3.1). *Moreover, for any $p \geq 1$ we have*

$$\sup_{t\geq 0} \mathbb{E}|X_t(x)|^p \leq C_p(|x|^p + 1). \tag{3.2}$$

The main result of this section is given by the following estimate which will be essential in order to prove any further estimate.

Proposition 3.2. *Assume that Hypotheses* 2.1, 2.2 *and* 2.3 *hold. Then for any $p \geq 1$*

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}|X_t(x)|^p \leq C_p (t \wedge 1)^{-\frac{p}{2(m-1)}}, \quad \forall\, t > 0. \tag{3.3}$$

Proof. Let $p = 2q$ with $q \geq 1$ (the case $p \in [1, 2)$ follows from Hölder inequality). We have

$$|X_t(x)|^{2q} = |x|^{2q} + 2q \int_0^t |X_s(x)|^{2(q-2)} \big(|X_s(x)|^2 < X_s(x), F(X_s(x)) > + (q-1)\mathrm{Tr}\left[A_s(x)\big(X_s(x) \otimes X_s(x)\big)\right] + \frac{1}{2}|X_s(x)|^2 \mathrm{Tr}\left[A_s(x)\right]\big) ds + 2q \int_0^t |X_s(x)|^{2(q-1)} < X_s(x), G(X_s(x))\, dW_s >,$$

where for any $t \geq 0$ and $x \in \mathbb{R}^d$ we defined

$$A_t(x) =: A(X_t(x)) = G(X_t(x))G^\star(X_t(x)).$$

Now, by taking expectation we get

$$\mathbb{E}|X_t(x)|^{2q} = |x|^{2q} + 2q\, \mathbb{E} \int_0^t |X_s(x)|^{2(q-2)} \big(|X_s(x)|^2 < X_s(x), F(X_s(x)) > + \frac{1}{2}|X_s(x)|^2 \mathrm{Tr}\left[A_s(x)\right] + (q-1)\mathrm{Tr}\left[A_s(x)\big(X_s(x) \otimes X_s(x)\big)\right]\big)\, ds.$$

Thus from strict dissipativity of F (see Hypothesis 2.1-2.) and the condition on the growth of G, by deriving with respect to t we have

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}|X_t(x)|^{2q} &= 2q\mathbb{E}\big(|X_t(x)|^{2(q-1)} < X_t(x), F(X_t(x)) >\big) \\
&\quad + q\mathbb{E}\big(|X_t(x)|^{2(q-1)}\mathrm{Tr}\left[A_t(x)\right]\big) \\
&\quad + 2q(q-1)\mathbb{E}\big(|X_t(x)|^{2(q-2)}\mathrm{Tr}\left[A_t(x)\big(X_t(x) \otimes X_t(x)\big)\right]\big) \\
&\leq 2q\big(-a\,\mathbb{E}|X_t(x)|^{2(q+m-1)} + b\,|F(0)|\,\mathbb{E}|X_t(x)|^{2q-1} + c\,\mathbb{E}|X_t(x)|^{2(q-1)}\big) \\
&\quad + (q - \frac{1}{2})\,\mathbb{E}\big(|X_t(x)|^{2(q-1)}\|G(X_t(x))\|^2\big) \\
&\leq 2q\big(-a\,\mathbb{E}|X_t(x)|^{2(q+m-1)} + b\,|F(0)|\,\mathbb{E}|X_t(x)|^{2q-1} + c\mathbb{E}|X_t(x)|^{2(q-1)}\big) \\
&\quad + C(q - \frac{1}{2})\mathbb{E}\big(|X_t(x)|^{2(q-1)}\big(1 + |X_t(x)|^{2k}\big)\big) \\
&\leq -C\big(\mathbb{E}|X_t(x)|^{2(q+m-1)} - 1\big) \leq -C\big(\big(\mathbb{E}|X_t(x)|^{2q}\big)^{\frac{q+m-1}{q}} - 1\big).
\end{aligned}$$

Therefore for any $q \geq 1$ we get

$$\frac{d}{dt}\mathbb{E}|X_t(x)|^{2q} \leq C_q\big(1 - \big(\mathbb{E}|X_t(x)|^{2q}\big)^{\frac{q+m-1}{q}}\big). \tag{3.4}$$

Now, by using a result on ordinary differential equations proved in [7] (see Lemma 3.4) we have that (3.3) follows.

4. Some boundedness results for the derivatives of $X_t(x)$ up to the third order. We assumed that F and G are differentiable up to the third order. Then as in the classical case of Lipschitz continuous coefficients the process $X_t(x)$ is differentiable (in this case not in mean square, but only in probability) with respect to the initial datum x, up to the third order and the derivatives are solutions of some stochastic differential equations. For a proof see [14]. In this section we will prove some estimates for such derivatives.

4.1. First derivative. Let $Y_t^h(x)$ be the solution of the derivative equation

$$dY_t= DF(X_t(x))Y_t\,dt + DG(X_t(x))Y_t\,dW_t, \quad Y_0= h. \tag{4.1}$$

It is possible to show (see [14]) that $Y_t^h(x)$ is the first order derivative in probability of $X_t(x)$ with respect to x along the direction $h \in \mathbb{R}^d$. In the sequel we will denote

$$Y_t^i(x) =: Y_t^{e_i}(x).$$

Proposition 4.1. *Under Hypotheses* 2.1, 2.2 *and* 2.3 *for any* $p \geq 1$

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\big(|Y_t^i(x)|^p\big) \leq f_p(t), \quad t \geq 0, \tag{4.2}$$

for a suitable increasing continuous function $f_p : \mathbb{R}^+ \to \mathbb{R}^+$.

Proof. As before we reduce to consider the case $p \geq 2$ and we set $p = 2q$. By using Ito's formula we have

$$\begin{aligned}\frac{d}{dt}\mathbb{E}|Y_t^i(x)|^{2q} &= 2q\mathbb{E}\big(|Y_t^i(x)|^{2(q-1)} < Y_t^i(x), DF(X_t(x))Y_t^i(x) >\big)\\ &\quad +(q-1)\mathbb{E}\big(|Y_t^i(x)|^{2(q-2)}\mathrm{Tr}\big[A_t^i(x)\big(Y_t^i(x)\otimes Y_t^i(x)\big)\big]\big)\\ &\quad +q\mathbb{E}\big(|Y_t^i(x)|^{2(q-1)}\mathrm{Tr}\big[A_t^i(x)\big]\big),\end{aligned}$$

where we denoted

$$A_t^i(x) =: \big(DG(X_t(x))Y_t^i(x)\big)\big(DG(X_t(x))Y_t^i(x)\big)^\star.$$

Then from Hypothesis 2.3 we get

$$\frac{d}{dt}\mathbb{E}|Y_t^i(x)|^{2q} \leq C_q \mathbb{E}|Y_t^i(x)|^{2q},$$

so that

$$\sup_{x\in \mathbb{R}^d} \mathbb{E}\big(|Y_t^i(x)|^p\big) \leq \exp(C_p t) =: f_p(t), \quad t \geq 0. \tag{4.3}$$

4.2. Second derivative. For any $i, j = 1, \dots, d$ we denote by $Z_t^{ij}(x)$ the solution of the problem

$$d\,Z_t = DF(X_t(x))Z_t\,dt + DG(X_t(x))Z_t\,dW_t + d\,R_t^{ij}(x), \quad Z_0 = 0,$$

where $R_t^{ij}(x)$ is the Ito's process defined by

$$\begin{aligned} d\,R_t^{ij}(x) &= D^2F(X_t(x))(Y_t^i(x), Y_t^j(x))\,dt + D^2G(X_t(x))(Y_t^i(x), Y_t^j(x))\,dW_t,\\ R_0^{ij}(x) &= 0. \end{aligned}$$

As for the first derivative, it is possible to show that $Z_t^{ij}(x)$ is the second order derivative in probability of $X_t(x)$ with respect to x and along directions e_i and e_j.

Proposition 4.2. *Under Hypotheses* 2.1, 2.2 *and* 2.3, *for any* $p \geq 1$

$$\sup_{x\in \mathbb{R}^d} \mathbb{E}|Z_t^{ij}(x)|^p \leq g_p(t), \quad t \geq 0, \tag{4.4}$$

where $g_p : \mathbb{R}^+ \to \mathbb{R}^+$ *is an increasing continuous function.*

Proof. As above let us consider $p = 2q$, $q \geq 1$ and let us apply Ito's formula to the process $|Z_t^{ij}(x)|^{2q}$. We get

$$\begin{aligned} \frac{d}{dt}\mathbb{E}|Z_t^{ij}(x)|^{2q} = {} & 2q\mathbb{E}\big(|Z_t^{ij}(x)|^{2(q-1)} < Z_t^{ij}(x), DF(X_t(x))Z_t^{ij}(x) >\big)\\ & + 2q\mathbb{E}\big(|Z_t^{ij}(x)|^{2(q-1)} < Z_t^{ij}(x), D^2F(X_t(x))(Y_t^i(x), Y_t^j(x)) >\big)\\ & + 2q(q-1)\mathbb{E}\big(|Z_t^{ij}(x)|^{2(q-2)}\mathrm{Tr}\left[C_t^{ij}(x)\big(Z_t^{ij}(x)\otimes Z_t^{ij}(x)\big)\right]\big)\\ & + q(q-1)\mathbb{E}\big(|Z_t^{ij}(x)|^{2(q-1)}\mathrm{Tr}\left[C_t^{ij}(x)\right]\big), \end{aligned}$$

with $C_t^{ij}(x)$ defined by

$$C_t^{ij}(x) =:\big(DG(X_t(x))Z_t^{ij}(x) + D^2G(X_t(x))(Y_t^i(x), Y_t^j(x))\big)$$
$$\big(DG(X_t(x))Z_t^{ij}(x) + D^2G(X_t(x))(Y_t^i(x), Y_t^j(x))\big)^\star.$$

Then from Hypothesis 2.2-1. we have

$$\begin{aligned}\operatorname{Tr}\left[C_t^{ij}(x)\right] &\leq C\big\|DG(X_t(x))Z_t^{ij}(x)\big\|^2 + C\big\|D^2G(X_t(x))(Y_t^i(x), Y_t^j(x))\big\|^2\\ &\leq C\big\|DG(X_t(x))Z_t^{ij}(x)\big\|^2 + C\big(1+|X_t(x)|^{2(k-2)}\big)\big(|Y_t^i(x)|\,|Y_t^j(x)|\big)^2.\end{aligned}$$

By using Hypothesis 2.3 it follows

$$\begin{aligned}\frac{d}{dt}\mathbb{E}|Z_t^{ij}(x)|^{2q} &\leq C\mathbb{E}|Z_t^{ij}(x)|^{2q}\\ &+ C\mathbb{E}\big(|Z_t^{ij}(x)|^{2q-1}\big(1+|X_t(x)|^{2m-3}\big)|Y_t^i(x)|\,|Y_t^j(x)|\big)\\ &+C\mathbb{E}\big(|Z_t^{ij}(x)|^{2(q-1)}\big(1+|X_t(x)|^{2(k-2)}\big)|Y_t^i(x)|^2\,|Y_t^j(x)|^2\big),\end{aligned}$$

so that we have

$$\begin{aligned}\frac{d}{dt}\mathbb{E}|Z_t^{ij}(x)|^{2q} &\leq C\mathbb{E}|Z_t^{ij}(x)|^{2q}\\ &+ C\big(\mathbb{E}|Z_t^{ij}(x)|^{2q}\big)^{1-\frac{1}{2q}}\left(\mathbb{E}\big(|Y_t^i(x)|\,|Y_t^j(x)|\big)^{2q}\big(1+|X_t(x)|^{2m-3}\big)^{2q}\right)^{\frac{1}{2q}}\\ &+C\big(\mathbb{E}|Z_t^{ij}(x)|^{2q}\big)^{1-\frac{1}{q}}\left(\mathbb{E}\big(|Y_t^i(x)|\,|Y_t^j(x)|\big)^{2q}\big(1+|X_t(x)|^{2(k-2)}\big)^{q}\right)^{\frac{1}{q}}\\ &\leq C\mathbb{E}|Z_t^{ij}(x)|^{2q}\\ &+C\big(\mathbb{E}|Z_t^{ij}(x)|^{2q}+1\big)\left(\mathbb{E}\big(|Y_t^i(x)|\,|Y_t^j(x)|\big)^{2q}\big(1+|X_t(x)|^{2m-3}\big)^{2q}\right)^{\frac{1}{2q}}\\ &+C\big(\mathbb{E}|Z_t^{ij}(x)|^{2q}+1\big)\left(\mathbb{E}\big(|Y_t^i(x)|\,|Y_t^j(x)|\big)^{2q}\big(1+|X_t(x)|^{2(k-2)}\big)^{q}\right)^{\frac{1}{q}}.\end{aligned}$$

Now, by using (3.3) and (4.2) and generalized Hölder inequality it is easy to check that for any $\alpha, \beta, \gamma \geq 1$ and $i_1, \ldots, i_n \in \{1, \ldots, d\}$

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\left(\big(1+|X_t(x)|^\alpha\big)^\beta\big(|Y_t^{i_1}(x)|\cdots|Y_t^{i_n}(x)|\big)^\gamma\right) \leq C(t\wedge 1)^{-\frac{\alpha\beta}{2(m-1)}}\tilde{f}(t), \tag{4.5}$$

where $\tilde{f} = \tilde{f}_{\alpha,\beta,\gamma} : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function depending only on α, β and γ. Since $k \leq m-1$ we get

$$\frac{d}{dt}\mathbb{E}|Z_t^{ij}(x)|^p \leq C\big(1+\big(t\wedge 1\big)^{\frac{1}{2(m-1)}-1}\tilde{f}_p(t)\big)\big(\mathbb{E}|Z_t^{ij}(x)|^p+1\big),$$

and then, by setting

$$\lambda_p(t) =: \int_0^t (s\wedge 1)^{\frac{1}{2(m-1)}-1}\tilde{f}_p(s)\,ds, \quad t>0,$$

we have

$$\sup_{x\in\mathbb{R}^d}\mathbb{E}|Z_t^{ij}(x)|^p \le \exp\big(C_p(\lambda_p(t)+t)\big)-1 =: g_p(t). \tag{4.6}$$

4.3. Third derivative. For any $i,j,k=1,\dots,d$ let $V_t^{ijk}(x)$ be the solution of the following problem

$$dV_t = DF(X_t(x))V_t\,dt + DG(X_t(x))V_t\,dW_t + dS_t^{ijk,1}(x) + dS_t^{ijk,2}(x), \;\; V_0 = 0,$$

where the processes $S_t^{ijk,1}(x)$ and $S_t^{ijk,2}(x)$ are respectively defined by

$$dS_t^{ijk,1}(x) =: D^3F(X_t(x))(Y_t^i(x),Y_t^j(x),Y_t^k(x))\,dt \\ + D^3G(X_t(x))(Y_t^i(x),Y_t^j(x),Y_t^k(x))\,dW_t$$

and

$$dS_t^{ijk,2}(x) =: \frac{1}{4}\sum_{\sigma\in\mathcal{S}_3} D^2F(X_t(x))\big(Y_t^{\sigma(i)}(x), Z_t^{\sigma(j)\sigma(k)}(x)\big)\,dt \\ + \frac{1}{4}\sum_{\sigma\in\mathcal{S}_3} D^2G(X_t(x))\big(Y_t^{\sigma(i)}(x), Z_t^{\sigma(j)\sigma(k)}(x)\big)\,dW_t$$

(here $\mathcal{S}_3$ denotes the set of all permutations of three elements). As for the first two derivatives we have

Proposition 4.3. *Under Hypotheses* 2.1, 2.2 *and* 2.3, *for any* $p\ge 1$ *it holds*

$$\sup_{x\in\mathbb{R}^d}\mathbb{E}\big|V_t^{ijk}(x)\big|^p \le h_p(t), \quad t\ge 0 \tag{4.7}$$

where $h_p:\mathbb{R}^+\to\mathbb{R}^+$ *is an increasing continuous function.*

Proof. Proceeding as for the first and the second derivatives we apply Ito's formula to the process $|V_t^{ijk}(x)|^{2q}$, with $2q=p$, and we get

$$\frac{d}{dt}\mathbb{E}|V_t^{ijk}(x)|^{2q} = 2q\mathbb{E}\big(|V_t^{ijk}(x)|^{2(q-1)} < V_t^{ijk}(x), DF(X_t(x))V_{ijk}(t,x) >\big) \\ + \frac{q}{2}\sum_{\sigma\in\mathcal{S}_3}\mathbb{E}\big(|V_t^{ijk}(x)|^{2(q-1)} < V_t^{ijk}(x), D^2F(X_t(x))\big(Y_t^{\sigma(i)}(x), Z_t^{\sigma(j)\sigma(k)}(x)\big) >\big) \\ + 2q\mathbb{E}\big(|V_t^{ijk}(x)|^{2(q-1)} < V_t^{ijk}(x), D^3F(X_t(x))(Y_t^i(x),Y_t^j(x),Y_t^k(x)) >\big) \\ + 2q(q-1)\mathbb{E}\big(|V_t^{ijk}(x)|^{2(q-2)}\mathrm{Tr}\big[D_t^{ijk}(x)\big(V_t^{ijk}(x)\otimes V_t^{ijk}(x)\big)\big]\big) \\ + q(q-1)\mathbb{E}\big(|V_t^{ijk}(x)|^{2(q-1)}\mathrm{Tr}\big[D_t^{ijk}(x)\big]\big) = I_1+I_2+I_3+I_4+I_5,$$

where

$$D_t^{ijk}(x) =:\Big(DG(X_t(x))V_t^{ijk}(x) + \tfrac{1}{4}\sum_{\sigma\in\mathcal{S}_3} D^2G(X_t(x))\big(Y_t^{\sigma(i)}(x), Z_t^{\sigma(j)\sigma(k)}(x)\big)$$
$$+D^3G(X_t(x))(Y_t^i(x), Y_t^j(x), Y_t^k(x))\Big)$$
$$\Big(DG(X_t(x))V_t^{ijk}(x) + \frac{1}{4}\sum_{\sigma\in\mathcal{S}_3} D^2G(X_t(x))\big(Y_t^{\sigma(i)}(x), Z_t^{\sigma(j)\sigma(k)}(x)\big)$$
$$+D^3G(X_t(x))(Y_t^i(x), Y_t^j(x), Y_t^k(x))\Big)^\star.$$

It holds

$$\begin{aligned}
\mathrm{Tr}\big[D_t^{ijk}(x)\big] &\le C\big\|DG(X_t(x))V_t^{ijk}(x)\big\|^2 \\
&\quad +C\sum_{\sigma\in\mathcal{S}_3}\big\|D^2G(X_t(x))\big(Y_t^{\sigma(i)}(x), Z_{\sigma(j)\sigma(k)}(t,x)\big)\big\|^2 \\
&\quad +C\big\|D^3G(X_t(x))\big(Y_t^i(x), Y_t^j(x), Y_t^k(x)\big)\big\|^2 \\
&\le C\big\|DG(X_t(x))V_t^{ijk}(x)\big\|^2 \\
&\quad +C\sum_{\sigma\in\mathcal{S}_3}\big(1+|X_t(x)|^{2(k-2)}\big)\big(|Y_t^{\sigma(i)}(x)||Z_t^{\sigma(j)\,\sigma(k)}(x)|\big)^2 \\
&\quad +C\big(1+|X_t(x)|^{2(k-3)}\big)\big(|Y_t^i(x)|\,|Y_t^j(x)|\,|Y_t^k(x)|\big)^2.
\end{aligned}$$

Now, let us estimate $I_j,\ j=1,\dots,5$. By using Hypothesis 2.3 we have

$$\begin{aligned}
&I_1+I_4+I_5 \le C\mathbb{E}|V_t^{ijk}(x)|^{2q} \\
&+C\sum_{\sigma\in\mathcal{S}_3}\mathbb{E}\Big(\big(1+|X_t(x)|^{2(k-2)}\big)\big(|Y_t^{\sigma(i)}(x)|\,|Z_{\sigma(j)\,\sigma(k)}(t,x)|\big)^2|V_t^{ijk}(x)|^{2(q-1)}\Big) \\
&+C\mathbb{E}\Big(\big(1+|X_t(x)|^{2(k-3)}\big)\big(|Y_t^i(x)|\,|Y_t^j(x)|\,|Y_t^k(x)|\big)^2|V_t^{ijk}(x)|^{2(q-1)}\Big) \\
&\le C\mathbb{E}|V_t^{ijk}(x)|^{2q} + C\big(\mathbb{E}|V_t^{ijk}(x)|^{2q}+1\big)\big(t\wedge 1\big)^{\frac{m-k+1}{m-1}-1}\tilde{g}_q(t).
\end{aligned}$$

Last inequality follows from (3.3), (4.2) and (4.4) by using the same arguments used in (4.5), with $\tilde{g}_q : \mathbb{R}^+ \to \mathbb{R}^+$ an increasing continuous function depending only on q. For I_2 we get

$$I_2 \leq C \sum_{\sigma \in \mathcal{S}_3} \mathbb{E}\big(|V_t^{ijk}(x)|^{2q-1}\big(1+|X_t(x)|^{2m-3}\big)|Y_t^{\sigma(i)}(x)|\,|Z_t^{\sigma(j)\,\sigma(k)}(x)|\big)$$
$$\leq C \sum_{\sigma \in \mathcal{S}_3} \Big(\mathbb{E}\big(\big(1+|X_t(x)|^{2m-3}\big)|Y_t^{\sigma(i)}(x)|\,|Z_t^{\sigma(j)\,\sigma(k)}(x)|\big)^{2q}\Big)^{\frac{1}{2q}}$$
$$\times \big(\mathbb{E}|V_t^{ijk}(x)|^{2q}\big)^{1-\frac{1}{2q}} \leq C\big(1+\mathbb{E}|V_t^{ijk}(x)|^{2q}\big)\big(t\wedge 1\big)^{\frac{1}{2(m-1)}-1}\tilde{g}_q(t).$$

For I_3 it holds

$$I_3 \leq C\mathbb{E}\big(|V_t^{ijk}(x)|^{2q-1}\big(1+|X_t(x)|^{2(m-2)}\big)|Y_t^i(x)|\,|Y_t^j(x)|\,|Y_t^k(x)|\big)$$
$$\leq C\Big(\mathbb{E}\big(\big(1+|X_t(x)|^{2(m-2)}\big)|Y_t^i(x)|\,|Y_t^j(x)|\,|Y_t^k(x)|\big)^{2q}\Big)^{\frac{1}{2q}}\big(\mathbb{E}|V_t^{ijk}(x)|^{2q}\big)^{1-\frac{1}{2q}}$$

and as for I_2 we get

$$I_3 \leq C\big(1+\mathbb{E}|V_t^{ijk}(x)|^{2q}\big)\big(t\wedge 1\big)^{\frac{1}{m-1}-1}\tilde{f}_p(t).$$

Then we have

$$\frac{d}{dt}E|V_t^{ijk}(x)|^p \leq C_p\Big(1+\big(t\wedge 1\big)^{\frac{1}{2(m-1)}-1}\Big)\tilde{g}_p(t)\big(1+E|V_t^{ijk}(x)|^p\big),$$

and as for the second derivative this implies that

$$\sup_{x\in\mathbb{R}^d} E|V_t^{ijk}(x)|^p \leq \exp\big(C_p(\mu_p(t)+t)\big)-1 =: h_p(t), \quad t\geq 0, \tag{4.8}$$

where

$$\mu_p(t) =: \int_0^t (s\wedge 1)^{\frac{1}{2(m-1)}-1}\tilde{g}_p(s)\,ds.$$

5. The transition semigroup. For any $\varphi \in B_b(\mathbb{R}^d)$ and $t\geq 0$ the function $P_t\varphi$ is defined by

$$P_t\varphi(x) = \mathbb{E}\big(\varphi(X_t(x))\big), \quad x\in\mathbb{R}^d. \tag{5.1}$$

The semigroup P_t, $t\geq 0$ is the transition semigroup associated to equation (3.1). P_t is a contraction semigroup and, as we show in next Proposition, for any λ large enough $e^{-\lambda t}P_t$ is *weakly continuous*, in the sense introduced in [6].

Proposition 5.1. *P_t, $t \geq 0$, is a semigroup of bounded linear operators on $C_b(\mathbb{R}^d)$ such that*

1) *for every $T > 0$ the family of functions $\{P_t\varphi\,;\, t \in [0,T]\}$ is equi-uniformly continuous,*
2) *for every $R > 0$ we have*

$$\lim_{t\to 0} \sup_{|x|\leq R} |P_t\varphi(x) - \varphi(x)| = 0, \tag{5.2}$$

3) *if (φ_n) is a bounded sequence in $C_b(\mathbb{R}^d)$ converging to φ uniformly on compact subsets of $\mathbb{R}^d$, then for any $R > 0$ we have*

$$\lim_{n\to\infty} \sup_{|x|\leq R} |P_t\varphi_n(x) - P_t\varphi(x)| = 0 \tag{5.3}$$

and the limit is uniform in $t \in [0,T]$, for any $T > 0$.

Proof. Let us fix $\varphi \in \mathrm{Lip}_b(\mathbb{R}^d)$. We have

$$\begin{aligned}|P_t\varphi(x) - P_t\varphi(y)| &= |\mathbb{E}\big(\varphi(X_t(x)) - \varphi(X(t,y))\big)|\\ &\leq \mathbb{E}|\varphi(X_t(x)) - \varphi(X_t(y))| \leq \|\varphi\|_{\mathrm{Lip}}\mathbb{E}|X_t(x) - X_t(y)|.\end{aligned}$$

Then we have

$$\begin{aligned}|P_t\varphi(x) - P_t\varphi(y)| &\leq \|\varphi\|_{\,\mathrm{Lip}}\mathbb{E}|\int_0^1 Y_t^{x-y}(\theta x + (1-\theta)y)\,d\theta|\\ &\leq \|\varphi\|_{\,\mathrm{Lip}}\int_0^1 \mathbb{E}|Y_t^{x-y}(\theta x + (1-\theta)y)|\,d\theta,\end{aligned}$$

so that by using from (4.2) it follows

$$|P_t\varphi(x) - P_t\varphi(y)| \leq \|\varphi\|_{\mathrm{Lip}} f_1(t)|x-y|.$$

Now, if $\varphi \in C_b(\mathbb{R}^d)$ for any $\epsilon > 0$ there exists $\varphi_\epsilon \in \mathrm{Lip}_b(\mathbb{R}^d)$ such that $\|\varphi - \varphi_\epsilon\|_0 < \epsilon$ and then for any $x, y \in \mathbb{R}^d$ and $t \in [0,T]$ we have

$$\begin{aligned}&|P_t\varphi(x) - P_t\varphi(y)|\\ &\quad\leq |P_t\varphi(x) - P_t\varphi_\epsilon(x)| + |P_t\varphi_\epsilon(x) - P_t\varphi_\epsilon(y)| + |P_t\varphi_\epsilon(y) - P_t\varphi(y)|\\ &\quad\leq 2\|\varphi - \varphi_\epsilon\|_0 + |P_t\varphi_\epsilon(x) - P_t\varphi_\epsilon(y)|\\ &\quad\leq 2\epsilon + \|\varphi_\epsilon\|_{\mathrm{Lip}} f_1(t)|x-y| \leq 2\epsilon + \|\varphi_\epsilon\|\mathrm{Lip} f_1(T)|x-y|,\end{aligned}$$

which implies first statement.

Let us prove statement 2). For any $\varphi \in C_b^2(\mathbb{R}^d)$ by Itô's formula we have

$$\begin{aligned}\varphi(X_t(x)) = \varphi(x) + \int_0^t &< F(X_s(x)), \varphi_x(X_s(x)) > ds \\ &+ \int_0^t \tfrac{1}{2}\mathrm{Tr}\big[G(X_s(x))G^\star(X_s(x))\varphi_{xx}(X_s(x))\big]ds \\ &+ \int_0^t < \varphi_x(X_s(x)), G(X_s(x))\,dW_s >\end{aligned}$$

and taking expectation we get

$$\begin{aligned}P_t\varphi(x) - \varphi(x) = \frac{1}{2}\int_0^t &\mathbb{E}\big(\mathrm{Tr}\big[G(X_s(x))G^\star(X_s(x))\varphi_{xx}(X_s(x))\big]\big)\,ds \\ &+ \int_0^t \mathbb{E}\big(< F(X_s(x)), \varphi_x(X_s(x)) >\big)\,ds.\end{aligned}$$

Now, let us fix $R, T > 0$. We have

$$\begin{aligned}&\mathrm{Tr}\big[G(X_s(x))G^\star(X_s(x))\varphi_{xx}(X_s(x))\big] \\ &\quad \le \|\varphi\|_2 \|G(X_s(x))\|^2 \le C\|\varphi\|_2\big(1 + |X_s(x)|^{2k}\big)\end{aligned}$$

and then from (3.2), we get

$$\sup_{|x|\le R} \mathbb{E}\,\mathrm{Tr}\big[G(X_s(x))G^\star(X_s(x))\varphi_{xx}(X_s(x))\big] \le C_R\|\varphi\|_2. \tag{5.4}$$

Moreover, by using Hypothesis 2.1-1. we have

$$|F(X_s(x))| \le C\big(1 + |X_s(x)|^{2m-1}\big) \tag{5.5}$$

and then

$$\sup_{|x|\le R} \mathbb{E}|F(X_s(x))| \le C_R, \quad \forall\, s \le T.$$

Therefore, for any $\varphi \in C_b^2(\mathbb{R}^d)$ it holds $\sup_{|x|\le R}|P_t\varphi(x) - \varphi(x)| \le C_R\|\varphi\|_2 t$, which implies

$$\lim_{t\to 0}\sup_{|x|\le R}|P_t\varphi(x) - \varphi(x)| = 0, \quad \forall\, R > 0. \tag{5.6}$$

Now, as $C_b^2(\mathbb{R}^d)$ is dense in $C_b(\mathbb{R}^d)$, statement 2) follows.

Statement 3) can be proved as in [7], Proposition 5.1. □

An easy consequence of previous Theorem is that for any $\varphi \in C_b(\mathbb{R}^d)$ the function $u : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ defined by $u(t, x) = P_t\varphi(x)$, $t \geq 0$ $x \in \mathbb{R}^d$ is continuous.

As in [6], it is possible to define the *infinitesimal generator* of the semigroup P_t, $t \geq 0$, as the unique closed operator $\mathcal{A} : D(\mathcal{A}) \subset C_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ such that for any $\lambda > 0$ and $\varphi \in C_b(\mathbb{R}^d)$ it holds

$$R(\lambda, \mathcal{A})\varphi(x) = \int_0^{+\infty} e^{-\lambda t} P_t\varphi(x)\, dt, \quad x \in \mathbb{R}^d. \tag{5.7}$$

In Section 6 we will describe $D(\mathcal{A})$.

6. Derivatives of the semigroup P_t. We assumed both F and G to be of class C^3. Therefore, as in the case of Lipschitz continuous coefficients for any $\varphi \in B_b(\mathbb{R}^d)$ the function $P_t\varphi \in C^3(\mathbb{R}^d)$, $\forall t > 0$. Indeed as proved in [14] for any $\varphi \in C_b^1(\mathbb{R}^d)$ the function $P_t\varphi$ is differentiable and for any $x, h \in \mathbb{R}^d$ and $t \geq 0$ it holds

$$< D(P_t\varphi)(x), h >= \mathbb{E}\big(< D\varphi(X_t(x)), Y_t^h(x) >\big), \tag{6.1}$$

where the process $Y_t^h(x)$ is defined as the solution of the derivative equation (4.1). This allows to prove Bismut-Elworthy formula (see [12] or [2] in a more general form) even in the case of locally Lipschitz coefficients. Existence of second and third order derivatives of P_t follows as in [12] from (6.1), semigroup law, Itô formula and Markov property.

In this section by using estimates proved in third section we will prove that for all $t > 0$

$$\|D^j(P_t\varphi)\|_0 \leq C(t \wedge 1)^{-j/2}\|\varphi\|_0. \tag{6.2}$$

6.1. First derivative.

Proposition 6.1. *For every $\varphi \in B_b(\mathbb{R}^d)$ and $t > 0$ it holds*

$$\|D(P_t\varphi)\|_0 \leq C(t \wedge 1)^{-1/2}\|\varphi\|_0. \tag{6.3}$$

Proof. For any $i = 1, \dots, d$, we have

$$< D(P_t\varphi)(x), e_i >= \frac{1}{t}\mathbb{E}\big(\varphi(X_t(x)) \int_0^t < Y_s^i(x), d\,W_s^G(x) >\big),$$

where for sake of simplicity we define

$$W_t^G(x) =: (G^{-1})^\star(X_t(x))W_t, \quad t \geq 0, \quad x \in \mathbb{R}^d. \tag{6.4}$$

Therefore, from Hypothesis 2.2-1, and from (4.2) it follows

$$\begin{aligned}
|D_i(P_t\varphi)(x)| &\leq \frac{\|\varphi\|_0}{t}\mathbb{E}\Big|\int_0^t < G^{-1}(X_s(x))Y_s^i(x), dW_s >\Big| \\
&\leq \frac{\|\varphi\|_0}{t}\big(\mathbb{E}\int_0^t |G^{-1}(X_s(x))Y_s^i(x)|^2\, ds\big)^{1/2} \\
&\leq \frac{\|\varphi\|_0 N}{t}\big(\int_0^t f_2^2(s)\, ds\big)^{1/2} \leq \frac{\|\varphi\|_0 N}{t^{1/2}} f_2(t)
\end{aligned}$$

so that if $t \leq 1$ we get

$$\|D_i(P_t\varphi)\|_0 \leq \|\varphi\|_0 N t^{-1/2} f_2(1).$$

Now, if $t > 0$ we have

$$\|D_i(P_t\varphi)\|_0 = \|D_i(P_1(P_{t-1}\varphi))\|_0 \leq N f_2(1)\|P_{t-1}\varphi\|_0 \leq N f_2(1)\|\varphi\|_0 = C\|\varphi\|_0$$

and then (6.3) follows. □

Remark 6.2. From (6.3) we have that for arbitrary $\varphi \in B_b(\mathbb{R}^d)$ and $t > 0$

$$|P_t\varphi(x) - P_t\varphi(y)| \leq C(t \wedge 1)^{-1/2}\|\varphi\|_0|x - y|, \quad \forall\, x, y \in \mathbb{R}^d, \tag{6.5}$$

so that $P_t\varphi$ is Lipschitz continuous and the semigroup P_t is strongly Feller.

6.2. Second and third derivatives. Bismut-Elworthy formula for the second derivative of $P_t\varphi$ along the directions e_i, e_j at any point $x \in \mathbb{R}^d$, gives for every $\varphi \in B_b(\mathbb{R}^d)$ (see [12])

$$\begin{aligned}
&D^2(P_t\varphi)(x)(e_i, e_j) \qquad (6.6)\\
&= +\frac{4}{t^2}\mathbb{E}\Big(\phi(X_t(x))\int_{t/2}^t < Y_s^j(x), dW_s^G(x) > \int_0^{t/2} < Y_s^i(x), dW_s^G(x) >\Big) \\
&\quad - \frac{2}{t}\mathbb{E}\int_0^{t/2}\big\langle D(P_{t-s}\varphi)(X_s(x)), DG(X_s(x))Y_s^j(x)\big(G^{-1}(X_s(x))Y_s^i(x)\big)\big\rangle\, ds \\
&\quad + \frac{2}{t}\mathbb{E}\int_0^{t/2}\big\langle D(P_{t-s}\varphi)(X_s(x)), Z_s^{ij}(x)\big\rangle\, ds.
\end{aligned}$$

As for the first derivative we can prove the following estimate for $D^2(P_t\varphi)$.

Proposition 6.3. *Let $\varphi \in B_b(\mathbb{R}^d)$. Then for any $t > 0$ it holds*

$$\|D^2(P_t\varphi)\|_0 \leq C(t \wedge 1)^{-1}\|\varphi\|_0. \tag{6.7}$$

In the proof we proceed as in the proof of Proposition 6.5 of [7], just recalling that the process $W_t^Q = Q^{-1/2}W_t$ introduced in [7] is now substituted by the process $W_t^G(x)$ defined in (6.4).

Indeed, as in the proof of Proposition 6.1, in order to get (6.7) we use Hypothesis 2.2-2, (3.3), (4.2), (4.4), and Proposition 6.1 itself.

In the same way, we have the following:

Proposition 6.4. *For every $\varphi \in B_b(\mathbb{R}^d)$ and $t > 0$ we have*

$$\|D^3(P_t\varphi)\|_0 \leq C(t \wedge 1)^{-3/2}\|\varphi\|_0 \quad t > 0. \tag{6.8}$$

In the proof we proceed as for the second derivative. Indeed from (6.6) we get a formula for the third derivative of $P_t\varphi$ and by using the estimates proved for $X_t(x)$ and for its derivatives up to the third order, (6.8) follows.

7. Existence and uniqueness of solutions.

7.1. The parabolic case. Let us consider the parabolic problem associated to operator $\mathcal{A}_0$

$$\begin{cases} v_t(t,x) = \mathcal{A}_0 v(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ v(0,x) = \varphi(x) & x \in \mathbb{R}^d. \end{cases} \tag{7.1}$$

Definition 7.1. We say that a function $v : [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$ is a classical solution of (7.1) if

(1) v is continuous on $[0,+\infty) \times \mathbb{R}^d$.
(2) $v(t,\cdot) \in C_b^2(\mathbb{R}^d)$ for all $t > 0$.
(3) $v(\cdot,x) \in C^1((0,+\infty))$ for all $x \in \mathbb{R}^d$.
(4) v_t, v_x and v_{xx} are continuous on $(0,+\infty) \times \mathbb{R}^d$.
(5) v satisfies (7.1).

As in [7] we can prove existence and uniqueness of classical solutions for problem (7.1) by using Ito's formula.

Theorem 7.2. *For any $\varphi \in C_b(\mathbb{R}^n)$ the function $u : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ defined by $u(t,x) = P_t\varphi(x)$ is the unique classical solution of problem* (1.6).

7.2. The elliptic case. As in [7], by using techniques similar to the one used in [9] and [20] we describe the domain of $\mathcal{A}$ in $C_b(\mathbb{R}^n)$. First, we adapt the Maximum Principle to our operator $\mathcal{A}_0$.

Lemma 7.3 (Maximum Principle). *Let us fix* $\varphi \in \cap_{p\geq 1} W^{2,p}_{\text{loc}}(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ *and* $\lambda > 0$. *If we assume that* $\lambda\varphi - \mathcal{A}_0\varphi \in C_b(\mathbb{R}^d)$, *then we have*

$$\|\varphi\|_0 \leq \frac{1}{\lambda}\|\lambda\varphi - \mathcal{A}_0\varphi\|_0. \tag{7.2}$$

Proof. As for any second order linear differential operator with real uniformly continuous and bounded coefficients, also for the operator $\mathcal{A}_0$ it is possible to show that if there exists $x_0 \in \mathbb{R}^d$ such that $|\varphi(x_0)| = \|\varphi\|_0$, then if x_0 is a maximum point it holds $\mathcal{A}_0\varphi(x_0) \leq 0$ and if x_0 is minimum point it holds $\mathcal{A}_0\varphi(x) \geq 0$. So, let us assume that $|\varphi(x)| \neq \|\varphi\|_0$, for any $x \in \mathbb{R}^d$ and $\|\varphi\|_0 = \sup_{x\in\mathbb{R}^d} \varphi(x)$. For any $n \in \mathbb{N}$ we define

$$\varphi_n(x) = \varphi(x) - \frac{1}{n}\big(|x|^2 + k\big),$$

where k is a nonnegative constant, independent of n, to be determined. We have $\varphi_n(x) \leq \varphi(x) \leq \|\varphi\|_0$, for any $x \in \mathbb{R}^d$. Moreover, for each $\epsilon > 0$ there exists a compact set $K_\epsilon \subset \mathbb{R}^d$ such that

$$\sup_{x\in K_\epsilon} \varphi(x) \geq \|\varphi\|_0 - \epsilon.$$

Now, we chose n_0 such that

$$\sup_{x\in K_\epsilon} \frac{1}{n}\big(|x|^2 + k\big) \leq \epsilon, \quad \forall\, n \geq n_0.$$

It follows that for any $n \geq n_0$

$$\sup_{x\in\mathbb{R}^d} \varphi_n(x) \geq \sup_{x\in K_\epsilon} \varphi_n(x) \geq \|\varphi\|_0 - 2\epsilon$$

so that φ_n attains its maximum at a certain point y_n and

$$\varphi_n(y_n) = \max_{x\in\mathbb{R}^d} \varphi_n(x) \to \|\varphi\|_0, \quad \text{as } n \to +\infty. \tag{7.3}$$

We have

$$\begin{aligned}\lambda\varphi_n(x) - \mathcal{A}_0\varphi_n(x) &= f(x) - \frac{1}{n}\Big(\lambda\big(|x|^2 + k\big) - \mathcal{A}_0\big(|x|^2 - k\big)\Big)\\ &= f(x) - \frac{1}{n}\Big(\lambda\big(|x|^2 + k\big) - \|G(x)\|^2 - 2 < F(x), x >\Big).\end{aligned}$$

By using Hypotheses 2.1-2 and 2.2-1, it is easy to verify that for a suitable nonnegative constant C it holds $\|G(x)\|^2 + 2 < F(x), x > \leq C$ and then

$$\lambda\varphi_n - \mathcal{A}_0\varphi_n \leq f - \frac{1}{n}(\lambda k - C).$$

Therefore if we chose $k = C/\lambda$ it follows that

$$\varphi_n(y_n) \leq \frac{1}{\lambda}\big(\mathcal{A}_0\varphi_n(y_n) + f(y_n)\big) \leq \frac{1}{\lambda}\|f\|_0$$

and by taking the limit as $n \to +\infty$ (7.2) follows. □

As in [7] from previous Proposition we derive the following results.

Proposition 7.4. *It holds*

$$\begin{cases} D(\mathcal{A}) = \left\{\varphi \in \cap_{p\geq 1} W^{2,p}_{\text{loc}}(\mathbb{R}^d) \cap C_b(\mathbb{R}^d) : \mathcal{A}_0\varphi \in C_b(\mathbb{R}^d)\right\} \\ \mathcal{A}\varphi = \mathcal{A}_0\varphi, \quad \forall\, \varphi \in D(\mathcal{A}). \end{cases} \tag{7.7}$$

Theorem 7.5. *For any $f \in C_b(\mathbb{R}^d)$ and $\lambda > 0$ there exists an unique $\varphi \in W^{2,p}_{\text{loc}}(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ $\forall\, p \geq 1$, such that*

$$\lambda\varphi - \mathcal{A}_0\varphi = f. \tag{7.5}$$

Moreover, $\varphi = R(\lambda, \mathcal{A})f$.

8. Schauder estimates. From the estimates proved for the first three derivatives of the semigroup P_t we derive the following interpolation result

Proposition 8.1. *Let $\alpha \in (0,3]$ and $\theta \in (0,1)$, with $\theta < \alpha$. Then for any $t > 0$ it holds*

$$\|P_t\|_{L(\mathcal{C}^\theta_b(\mathbb{R}^d), \mathcal{C}^\alpha_b(\mathbb{R}^d))} \leq C(t \wedge 1)^{-\frac{\alpha-\theta}{2}}. \tag{8.1}$$

The proof is the same as the proof of Proposition 8.1 of [7], just by substituting as before the process W^Q with process W^G and by using estimates (4.2), (4.4) and (4.7).

Now we apply the general method due to Lunardi [18] based upon interpolation theory and Schauder estimates.

Theorem 8.2. *Let $f \in C_b^\theta(\mathbb{R}^d)$, with $\theta \in (0,1)$ and let $\lambda > 0$. If*

$$\lambda\varphi - \mathcal{A}_0\varphi = f,$$

then $\varphi \in C_b^{2+\theta}(\mathbb{R}^d)$ and for a constant C independent of f

$$\|\varphi\|_{C_b^{2+\theta}(\mathbb{R}^n)} \le C\|f\|_{C_b^\theta(\mathbb{R}^n)}. \tag{8.2}$$

Theorem 8.3. *For any $T > 0$, let f be a continuous function from $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}$ such that $f(t,\cdot) \in C_b^\theta(\mathbb{R}^d)$ and $\sup_{t\in[0,T]} \|f(t,\cdot)\|_{C_b^\theta(\mathbb{R}^d)} < \infty$. Let $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be the unique bounded classical solution of the parabolic problem*

$$\begin{cases} u_t(t,x) = \mathcal{A}_0 u(t,x) + f(t,x), & t > 0, \quad x \in \mathbb{R}^d, \\ u(0,x) = 0, & x \in \mathbb{R}^d. \end{cases} \tag{8.3}$$

Then $u(t,\cdot) \in C_b^{2+\theta}(\mathbb{R}^n)$ for every $t \in (0,T]$ and it holds

$$\|u(t,\cdot)\|_{C_b^{2+\theta}(\mathbb{R}^d)} \le C \sup_{t\in[0,T]} \|f(t,\cdot)\|_{C_b^\theta(\mathbb{R}^d)}, \tag{8.4}$$

for a constant $C > 0$ independent of f.

Remark 8.4. The semigroup P_t is well defined from $C_b(\mathbb{R}^d)$ to $C_b(\mathbb{R}^d)$ and from $C_b^1(\mathbb{R}^d)$ to $C_b^1(\mathbb{R}^d)$. Then by interpolation P_t is well defined in $C_b^\theta(\mathbb{R}^d)$. Moreover, for every $\lambda > 0$ the operator

$$F(\lambda)\varphi(x) = \int_0^{+\infty} e^{-\lambda t} P_t\varphi(x)\, dt$$

is well defined and bounded in $C_b^\theta(\mathbb{R}^d)$ and as in the case of $C_b(\mathbb{R}^d)$ it is injective and satisfies resolvent law. Then it is possible to define the infinitesimal generator of P_t as the unique closed operator

$$\mathcal{A} : D(\mathcal{A}) \subset C_b^\theta(\mathbb{R}^d) \to C_b^\theta(\mathbb{R}^d)$$

such that for any $\lambda > 0$ and $\varphi \in C_b^\theta(\mathbb{R}^d)$ it holds

$$R(\lambda, \mathcal{A})\varphi(x) = F(\lambda)\varphi(x) = \int_0^{+\infty} e^{-\lambda t} P_t\varphi(x)\, dt.$$

As a first consequence of Theorem 8.2, we have the following characterization of the domain of the infinitesimal generator of P_t in $C_b^\theta(\mathbb{R}^d)$

$$D(\mathcal{A}) = \{\ \varphi \in C_b^{2+\theta}(\mathbb{R}^d) : \mathcal{A}_0\varphi \in C_b^\theta(\mathbb{R}^d)\ \}. \tag{8.5}$$

At present it is not clear if for any $\varphi \in D(\mathcal{A})$ the stronger property holds

$$\begin{cases} \frac{1}{2}Tr\,[A(x)D^2\varphi(x)] & \in C_b^\theta(\mathbb{R}^d) \\ < F(x), D\varphi(x) > & \in C_b^\theta(\mathbb{R}^d). \end{cases}$$

9. Asymptotic behaviour of P_t in the strictly dissipative case. In this section we want to prove existence , uniqueness, ergodicity and strongly mixing property for the invariant measure relative to the transition semigroup P_t associated to the stochastic problem

$$dX_t = F(X_t)\,dt + G(X_t)\,dW_t, \quad X_0 = x. \tag{9.1}$$

We assume that F and G satisfy the following conditions:

Hypothesis 9.1. 1. *There exists $m \geq 2$ such that*

$$\sup_{x\in\mathbb{R}^d} \frac{|F(x)|}{1+|x|^{2m-1}} < +\infty.$$

2. *There exists $k \leq m-1$ such that*

$$\sup_{x\in\mathbb{R}^d} \frac{|G(x)|}{1+|x|^{k}} < +\infty.$$

3. *There exist $\omega \in \mathbb{R}$ such that $\forall\, x, y \in \mathbb{R}^d$*

$$2 < F(x) - F(y), h > + \|G(x) - G(y)\|^2 \leq \omega|x-y|^2.$$

Existence of an invariant measure is an immediate consequence of (3.2). Indeed, if for some $x_0 \in \mathbb{R}^d$ there exists a constant $M > 0$ such that

$$\sup_{t\geq 0} \mathbb{E}|X_t(x_0)|^2 \leq M,$$

then the family of probability measures $\{\,P_t(x_0,\cdot)\,\}_{t\geq 0}$ is tight and then there exists at least one invariant measure (for a proof see [11]).

In Section 6 we proved that if F and G are regular then $P_t\varphi$ is regular, so that strong Feller property holds. Now, proceeding as in [21] and [8], we show that even if F and G are only continuous, nevertheless the semigroup P_t is strongly Feller.

Proposition 9.2. *Assume that Hypothesis* 9.1 *holds. Then the semigroup* P_t *is strongly Feller.*

Proof. As in [21] and [8] we approximate F and G by means of two sequences of functions $(F_n) \subset C^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and $(G_n) \subset C^\infty(\mathbb{R}^d;\mathcal{L}(\mathbb{R}^r,\mathbb{R}^d))$ defined respectively by

$$F_n(x) =: \int_{\mathbb{R}^d} \rho_n(x-\xi)F(\xi)\,d\xi, \qquad G_n(x) =: \int_{\mathbb{R}^d} \rho_n(x-\xi)G(\xi)\,d\xi,$$

where ρ_n are usual mollificators in $\mathbb{R}^d$. Then, since F_n and G_n satisfy Hypothesis 9.1-3. for any n, we can introduce the semigroup P_t^n relative to the problem

$$dX_t^n = F_n(X_t^n)\,dt + G_n(X_t^n)\,dW_t, \quad X_0^n = x.$$

For such a semigroup strong Feller property holds. Then by taking the limit as $n \to +\infty$ strong Feller property of P_t follows, provided

$$\lim_{n\to+\infty} \mathbb{E}|X_t^n(x) - X_t(x)|^2 = 0, \tag{9.2}$$

for any fixed $t \geq 0$ and $x \in \mathbb{R}^d$ (see [8] and [11] for more details). In order to prove (9.2), we remark that F_n and G_n satisfy Hypothesis 9.1 with the same constants for any n, so that we have

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}|X_t^n(x) - X_t(x)|^2 = \mathbb{E}\|G_n(X_t^n(x)) - G(X_t(x))\|^2\\
&+2\mathbb{E} < F_n(X_t^n(x)) - F(X_t(x)), X_t^n(x) - X_t(x) >\\
&\leq C\mathbb{E}|X_t^n(x) - X_t(x)|^2 + C\mathbb{E}\|G_n(X_t(x)) - G(X_t(x))\|^2\\
&+C\big(\mathbb{E}|F_n(X_t(x)) - F(X_t(x))|^2\big)^{1/2}\big(\mathbb{E}|X_t^n(x) - X_t(x)|^2\big)^{1/2}\\
&\leq C\mathbb{E}|X_t^n(x) - X_t(x)|^2 + C\mathbb{E}\|G_n(X_t(x)) - G(X_t(x))\|^2\\
&+C\mathbb{E}|F_n(X_t(x)) - F(X_t(x))|^2.
\end{aligned}$$

Now, we conclude the proof as in [8]. □

In order to prove uniqueness, ergodicity and strongly mixing property we assume a stronger dissipativity condition for F, i.e.,

Hypothesis 9.3. *For suitable $\rho > 0$ and $c_\rho > 0$ we have for all $x, y \in \mathbb{R}^d$*

$$2 < F(x) - F(y), x - y > +\|G(x) - G(y)\|^2 \leq -c_\rho |x - y|^{2+\rho}. \tag{9.3}$$

Let V_t, $t \geq 0$, be another Wiener process independent of W_t and define

$$\widetilde{W}_t = \begin{cases} W_t & \text{if } \ t \geq 0 \\ V_{-t} & \text{if } \ t < 0. \end{cases} \tag{9.4}$$

For any $\lambda > 0$ let $X_t(\lambda, x)$ be the solution of the problem

$$dX_t = F(X_t)\, dt + G(X_t)\, d\widetilde{W}_t, \quad X_{-\lambda} = x. \tag{9.5}$$

As for $X_t(x)$ it is possible to show that for any $p \geq 1$

$$\sup_{t \geq -\lambda} \mathbb{E}|X_t(\lambda, x)|^p \leq C\big(|x|^p + 1\big). \tag{9.6}$$

Theorem 9.4. *Assume that Hypotheses* 9.1 *and* 9.3 *hold. Then there exists an unique invariant measure μ and for any $\varphi \in B_b(\mathbb{R}^d)$, we have*

$$\|P_t\varphi - < \varphi, \mu > \|_0 \leq C\big(t \vee 1\big)^{-1/\rho} \|\varphi\|_0, \tag{9.7}$$

so that μ is ergodic and strongly mixing.

Proof. For any $x \in \mathbb{R}^d$ and $\lambda > 0$ we set

$$Z_t(\lambda, x) =: X_t(\lambda, x) - X_t(\lambda, 0), \quad t \geq 0. \tag{9.8}$$

We have

$$|Z_t(\lambda, x)|^2 = |x|^2 + 2\int_{-\lambda}^{t} < F(X_s(\lambda, x)) - F(X_s(\lambda, 0)), Z_s(\lambda, x) > \, ds$$
$$+ \int_{-\lambda}^{t} \operatorname{Tr} A_s(\lambda, x)\, ds + \int_{-\lambda}^{t} < G(X_s(\lambda, x)) - G(X_s(\lambda, 0)), Z_s(\lambda, x) > \, dW_s,$$

where for any $\lambda > 0$ and $x \in \mathbb{R}^d$

$$A_t(\lambda, x) =: \big(G(X_t(\lambda, x)) - G(X_t(\lambda, 0))\big)\big(G(X_t(\lambda, x)) - G(X_t(\lambda, 0))\big)^\star.$$

By taking expectation and deriving with respect to t we get

$$\frac{d}{dt}\mathbb{E}|Z_t(\lambda,x)|^2 = 2\mathbb{E}\big(< F(X_t(\lambda,x)) - F(X_t(\lambda,0)), Z_t(\lambda,x) >\big)$$
$$+\mathbb{E}\|G(X_t(\lambda,0)) - G(X_t(\lambda,x))\|^2 \le -c_\rho\mathbb{E}|Z_t(\lambda,x)|^{2+\rho}$$
$$\le -c_\rho\big(\mathbb{E}|Z_t(\lambda,x)|^2\big)^{1+\rho/2},$$

so that by comparison we get

$$\mathbb{E}|Z_t(\lambda,x)|^2 \le |x|^2\big(1+|x|^\rho c_\rho(t+\lambda)\big)^{-2/\rho}.$$

This implies that

$$\lim_{\lambda\to+\infty}\mathbb{E}|X_0(\lambda,x) - X_0(\lambda,0)|^2 = \lim_{\lambda\to+\infty}\mathbb{E}|Z_0(\lambda,x)|^2 = 0. \tag{9.9}$$

Now we want to show that the sequence $\{X_0(\lambda,x)\,;\,\lambda\ge 0\}$ is a Cauchy sequence in $L^2(\Omega;\mathbb{R}^d)$, for every $x\in\mathbb{R}^d$. This should imply that

$$\exists \lim_{\lambda\to+\infty} X_0(\lambda,x) = \eta, \quad \text{in } L^2(\Omega;\mathbb{R}^d),$$

for a suitable random variable η (independent of x because of (9.9)). Then the law of η

$$\mathcal{L}(\eta) =: \mu$$

is the unique invariant measure for (9.1) (for a proof see [10] or [11]).

For any $0<\gamma<\delta$ we define

$$Z_t(\gamma,\delta,x) =: X_t(\gamma,x) - X_t(\delta,x), \quad t\ge-\gamma,\ x\in\mathbb{R}^d.$$

We have

$$\mathbb{E}|Z_t(\gamma,\delta,x)|^2 = \mathbb{E}|x - X_{-\gamma}(\delta,x)|^2$$
$$+\mathbb{E}\Big(\int_{-\gamma}^t 2 < F(X_s(\gamma,x)) - F(X_s(\delta,x)), Z_s(\gamma,\delta,x) > +\mathrm{Tr}\left[A_s(\gamma,\delta,x)\right] ds\Big),$$

where

$$A_t(\gamma,\delta,x) =: \big(G(X_t(\gamma,x)) - G(X_t(\delta,x))\big)\big(G(X_t(\gamma,x)) - G(X_t(\delta,x))\big)^\star.$$

By using again Hypothesis 9.3, we get

$$\mathbb{E}|X_0(\gamma,x)-X_0(\delta,x)|^2 \leq \frac{\mathbb{E}|x-X_{-\gamma}(\delta,x)|^2}{\big(1+c_\rho\gamma\big(\mathbb{E}|x-X_{-\gamma}(\delta,x)|^2\big)^{\rho/2}\big)^{2/\rho}},$$

from which it follows that for any $0<\gamma<\delta$

$$\mathbb{E}|X_0(\gamma,x)-X_0(\delta,x)|^2 \leq \big(c_\rho\gamma\big)^{-2/\rho}. \tag{9.10}$$

Let us fix $\varphi \in B_b(\mathbb{R}^d)$ and $1<t\leq s$. As proved in Proposition 9.2, the semigroup P_t is *strongly Feller* and it holds $\|P_1\varphi\|_{\text{Lip}} \leq C\|\varphi\|_0$, so that we have

$$\begin{aligned}&|P_t\varphi(x)-P_s\varphi(x)|^2 = |\mathbb{E}\big(P_1\varphi(X_0(t-1,x))\big)-\mathbb{E}\big(P_1\varphi(X_0(s-1,x))\big)|^2\\ &\leq \|P_1\varphi\|^2_{\text{Lip}}\mathbb{E}|X_0(t-1,x)-X_0(s-1,x)|^2 \leq C\big(c_\rho(t-1)\big)^{-2/\rho}\|\varphi\|_0^2.\end{aligned} \tag{9.11}$$

Therefore, since

$$\|P_t\varphi - <\varphi,\mu>\|_0 \leq 2\|\varphi\|_0, \quad t\geq 0$$

by taking the limit as $s\to+\infty$ we get

$$\|P_t\varphi - <\varphi,\mu>\|_0 \leq C(t\vee 1)^{-1/\rho}\|\varphi\|_0, \quad t\geq 0. \tag{9.12}$$

If we assume that F satisfies a stronger dissipativity condition, then a stronger rate of convergence holds.

Theorem 9.5. *Assume that Hypothesis* 9.1 *holds. Moreover, assume that there exist* $\rho>0$ *and* $c_\rho,\ d_\rho>0$ *such that*

$$<F(x)-F(y),x-y> + \|G(x)-G(y)\|^2 \leq -c_\rho|x-y|^{2+\rho}-d_\rho|x-y|^2. \tag{9.13}$$

Then for any $\varphi\in B_b(\mathbb{R}^d)$*, we have*

$$\|P_t\varphi - <\varphi,\mu>\|_0 \leq Ce^{-d_\rho t}\|\varphi\|_0, \quad t\geq 0. \tag{9.14}$$

Proof. We have to proceed as in the proof of previous Theorem (see also Theorem 5.2 of [8]). □

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REFERENCES

[1] D.G. Aronson and P. Besala, *Parabolic equations with unbounded coefficients*, J. Differential Equations, 3 (1967), 1–14.

[2] P. Besala, *On the existence of a fundamental solution for a parabolic equation with unbounded coefficients*, Ann. Polon. Math., 29 (1975), 403–409.

[3] J.M. Bismut, *Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions*, Z. Wahrscheinlichkeitstheorie verw. Gebiete, 56 (1981), 469–505.

[4] P. Cannarsa and G. Da Prato, *Schauder estimates for elliptic equations in infinite dimensions*, preprint Scuola Normale Superiore, Pisa (1994).

[5] P. Cannarsa and V. Vespri, *Generation of analytic semigroups by elliptic operators with unbounded coefficients*, SIAM J. Math. Analysis, 18 (1987), 857–872.

[6] S. Cerrai, *A Hille Yosida theorem for weakly continuous semigroups*, Semigroup Forum, 49 (1994), 349–367.

[7] S. Cerrai, *Elliptic and parabolic equations in* $\mathbb{R}^n$ *with coefficients having polynomial growth*, Commun. in Partial Differential Equations, 21 (1996), 281–317.

[8] S. Cerrai, *Invariant measures for a class of SDE's with Drift term having polynomial growth*, Preprint Scuola Normale Superiore, Pisa (1995), to appear in Dynamic Systems and Applications.

[9] G. Da Prato and A. Lunardi, *On the Ornstein–Uhlenbeck operator in spaces of continuous functions*, J. Functional Analysis, 131 (1995), 94–114.

[10] G. Da Prato and J. Zabczyk, "Stochastic Equations in Infinite Dimensions," Cambridge University Press, Cambridge (1992).

[11] G. Da Prato and J. Zabczyk, "Ergodicity for Infinite Dimensional Systems," Cambridge University Press, (to appear) (1996).

[12] K.D. Elworthy and X.M. Li, *Formulae for the derivatives of heat semigroups*, Warwick Preprints 45 (1993) J. Functional Analysis (to appear).

[13] M.I. Freidlin and A.D. Wentzell, "Random Perturbations of Dynamical Systems," Springer Verlag, Berlin (1983).

[14] N.V. Krylov, "Introduction to the Theory of Diffusion Processes," American Mathematical Society, Providence (1995).

[15] S. Kusuoka and D.W. Strook, *Some boundedness properties of certain stationary diffusion processes*, J. Functional Analysis, 60 (1985), 243–264.

[16] A. Lunardi, "Analytic Semigroups and Optimal Regularity in Parabolic Problems," Birkäuser Verlag, Basel (1995).

[17] A. Lunardi, *Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in* $\mathbb{R}^n$, Preprint Scuola Normale Superiore, Pisa (1994).

[18] A. Lunardi, *An interpolation method to characterize domains of generators of semigroups*, Preprint Scuola Normale Superiore, Pisa (1995), Semigroup Forum (to appear).

[19] A. Lunardi, *On the Ornstein–Uhlenbeck operator in L^2 spaces with respect to invariant measures*, Preprint Scuola Normale Superiore, Pisa (1995), TAMS (to appear).

[20] A. Lunardi and V. Vespri, *Optimal L^∞ and Schauder estimates for elliptic and parabolic operators with unbounded coefficients*, Preprint Scuola Normale Superiore, Pisa (1995).

[21] S. Peszat and J. Zabczyk, *Strong Feller property and irriducibility for diffusion processes on Hilbert spaces*, Preprint Institute of Mathematics, Polish Academy of Sciences (1993).

[22] H. Triebel, "Interpolation Theory, Function Spaces, Differential Operators," North-Holland Amsterdam (1986).